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► **To cite this version:**

Mikhail V. Ignatev. Gradedness of the set of rook placements in  $A_{n-1}$ . Communications in Mathematics, 2021, Volume 29 (2021), Issue 2 (Special Issue: 3rd International Workshop on Nonassociative Algebras in Málaga) (2), pp.171 - 182. 10.2478/cm-2021-0016 . hal-03665022

**HAL Id: hal-03665022**

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Submitted on 11 May 2022

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## Gradedness of the set of rook placements in $A_{n-1}$

Mikhail V. Ignatev

**Abstract.** A rook placement is a subset of a root system consisting of positive roots with pairwise non-positive inner products. To each rook placement in a root system one can assign the coadjoint orbit of the Borel subgroup of a reductive algebraic group with this root system. Degenerations of such orbits induce a natural partial order on the set of rook placements. We study combinatorial structure of the set of rook placements in  $A_{n-1}$  with respect to a slightly different order and prove that this poset is graded.

### 1 Introduction

Denote by  $G = \mathrm{GL}_n(\mathbb{C})$  the group of all invertible  $n \times n$  matrices over the complex numbers. Let  $B$  be the Borel subgroup of  $G$  consisting of all invertible upper-triangular matrices,  $U$  be the unipotent radical of  $B$  (it consists of all upper-triangular matrices with 1's on the diagonal), and  $T$  be the subgroup of all invertible diagonal matrices (it is the maximal torus of  $G$  contained in  $B$ ). Next, let  $\mathfrak{b}$  and  $\mathfrak{n}$  be the Lie algebras of  $B$  and  $U$  respectively.

Let  $\Phi$  be the root system of  $G$  with respect to  $T$ ,  $\Phi^+$  be the set of positive roots with respect to  $B$ ,  $\Delta$  be the set of simple roots, and  $W$  be the Weyl group of  $\Phi$  (for basic facts on algebraic groups and root systems, see [3], [4] and [5]). The root system  $\Phi$  is of type  $A_{n-1}$ ; as usual, we identify the set of positive roots with the subset of the Euclidean space  $\mathbb{R}^n$  of the form

$$A_{n-1}^+ = \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\}.$$

Under this identification,  $\Delta$  consists of the roots  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ,  $1 \leq i \leq n-1$  ( $\{\epsilon_i\}_{i=1}^n$  is the standard basis of  $\mathbb{R}^n$ ).

2020 MSC: 06A07, 17B22, 17B08.

Key words: Root system, rook placement, Borel subgroup, coadjoint orbit, graded poset

During the work on the final version of the paper I was partially supported by FAPESP grant no. 19/00563–1. A part of this work was done during my stay at the Oberwolfach Research Institute for Mathematics in February–March 2018 via the program “Research in pairs” together with Alexey Petukhov. I thank Alexey for fruitful discussions. I also thank the Oberwolfach Research Institute for Mathematics for hospitality and financial support.

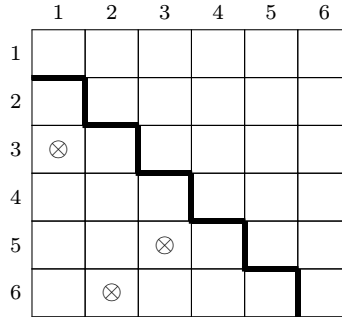
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**Definition 1.** A rook placement is a subset  $D \subseteq \Phi^+$  such that  $(\alpha, \beta) \leq 0$  for all distinct  $\alpha, \beta \in D$ . (Here  $(\cdot, \cdot)$  denotes the standard inner product on  $\mathbb{R}^n$ .)

**Example 1.** Let  $n = 6$ . Below we draw the rook placement  $D = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_6, \epsilon_3 - \epsilon_5\}$ . If a root  $\epsilon_i - \epsilon_j$  is contained in  $D$ , then we put the symbol  $\otimes$  in the  $(j, i)$ th entry of the  $n \times n$  chessboard. If we interpret these symbols as rooks, then it follows from the definition that the rooks do not hit each other.



We denote the set of all rook placement in  $A_{n-1}$  by  $\mathcal{R}(n)$ . Further, let  $\mathcal{I}(n)$  be the set of all orthogonal rook placements. Below we describe two closely related partial orders on these sets.

The Lie algebra  $\mathfrak{n}$  has the basis  $\{e_\alpha, \alpha \in \Phi^+\}$  consisting of the root vectors: for  $\alpha = \epsilon_i - \epsilon_j$ ,  $e_\alpha$  is nothing but the elementary matrix  $e_{i,j}$ . Denote by  $\{e_\alpha^*, \alpha \in \Phi^+\}$  the dual basis of the dual space  $\mathfrak{n}^*$ . Given a rook placement  $D$ , put

$$f_D = \sum_{\beta \in D} e_\beta^* \in \mathfrak{n}^*.$$

The group  $B$  acts on its Lie algebra  $\mathfrak{b}$  by the adjoint action, and  $\mathfrak{n}$  is an invariant subspace. Hence one has the dual action of the groups  $B$  and  $U$  on the space  $\mathfrak{n}^*$ ; we call this action *coadjoint*. We say that the  $B$ -orbit  $\Omega_D \subset \mathfrak{n}^*$  of the linear form  $f_D$  is *associated* with the rook placement  $D$ .

Such orbits play an important role in the A.A. Kirillov’s orbit method [14], [15] describing representations of  $B$  and  $U$ . For  $D \in \mathcal{I}(n)$ , such orbits were studied by A.N. Panov in [18] and by me in [6]. One can define analogues of such orbits for other root systems, see [7], [8], [9] for the case of  $\mathcal{I}(n)$ . For arbitrary rook placements in  $\mathcal{R}(n)$ , such orbits were considered in [10]; see also [1], [2], where C. Andre and A. Neto used rook placements to construct so-called supercharacter theory for the group  $U$ . Note that in [16], [17], A. Melnikov studied the adjoint  $B$ -orbits of elements of the form  $\sum_{\beta \in D} e_\beta$ ,  $D \in \mathcal{I}(n)$ .

Given a subset  $A \subseteq \mathfrak{n}^*$ , we will denote by  $\overline{A}$  its closure with respect to the Zariski topology. There exists a natural partial order on the set  $\mathcal{R}(n)$  induced by the degenerations of associated orbits: we will write  $D_1 \leq_B D_2$  if  $\Omega_{D_1} \subseteq \overline{\Omega_{D_2}}$ . We need to introduce one more partial order on the set of rook placements. Namely, given an arbitrary  $D \in \mathcal{R}(n)$ , denote by  $R_D$  the  $n \times n$  matrix defined by

$$(R_D)_{i,j} = \begin{cases} \#\{\epsilon_a - \epsilon_b \in D \mid a \leq j, b \geq i\}, & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

Put  $D_1 \leq D_2$  if  $(R_{D_1})_{i,j} \leq (R_{D_2})_{i,j}$  for all  $i, j$ .

**Example 2.** Let  $n = 4$ ,  $D_1 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_4\}$ ,  $D_2 = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4\}$ . Then

$$\begin{array}{c}
 \begin{array}{cccc}
 & 1 & 2 & 3 & 4 \\
 1 & \square & \square & \square & \square \\
 2 & \otimes & \square & \square & \square \\
 3 & \square & \square & \square & \square \\
 4 & \square & \otimes & \square & \square
 \end{array} \\
 D_1 =
 \end{array}
 , \quad
 R_{D_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$
  

$$\begin{array}{c}
 \begin{array}{cccc}
 & 1 & 2 & 3 & 4 \\
 1 & \square & \square & \square & \square \\
 2 & \square & \square & \square & \square \\
 3 & \otimes & \square & \square & \square \\
 4 & \square & \otimes & \square & \square
 \end{array} \\
 D_2 =
 \end{array}
 , \quad
 R_{D_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We conclude that  $D_1 \leq D_2$ . On the other hand, it is easy to check that  $D_1 \not\leq_B D_2$ , see [10, Remark 1.6 (iii)], so these two partial orders on  $\mathcal{R}(n)$  do not coincide.

Nevertheless, it turns out that these orders are closely related to each other. Precisely, given rook placements  $D_1, D_2 \in \mathcal{R}(n)$ , it follows from  $D_1 \leq_B D_2$  that  $D_1 \leq D_2$  [10, Theorem 1.5]. Furthermore, if  $D_1, D_2 \in \mathcal{I}(n)$  then the conditions  $D_1 \leq_B D_2$  and  $D_1 \leq D_2$  are equivalent [6, Theorem 1.7]. Besides, given a rook placement

$$D = \{\epsilon_{i_1} - \epsilon_{j_1}, \dots, \epsilon_{i_l} - \epsilon_{j_l}\},$$

we denote by  $w_D \in S_n$  the permutation, which is equal to the product of transpositions

$$w_D = (i_1, j_1) \dots (i_l, j_l).$$

Now, both of the conditions above (for orthogonal rook placements  $D_1, D_2$ ) are equivalent to the condition that  $w_{D_1}$  is less or equal to  $w_{D_2}$  with respect to the Bruhat order [6, Theorem 1.1]. Similar facts are true for orthogonal rook placements in the root system  $C_n$ , see [7]. Note that these results are in some sense “dual” to A. Melnikov’s results.

In the paper [12], F. Incitti studied the order on  $\mathcal{I}(n)$  induced by the Bruhat order on the elements  $w_D, D \in \mathcal{I}(n)$ , from purely combinatorial point of view (see also [11] for other classical root systems). In particular, given an orthogonal rook placement  $D$ , he explicitly described the set of its immediate predecessors (it consists of  $D' \in \mathcal{I}(n)$  such that there exists an edge from  $D'$  to  $D$  in the Hasse diagram of this poset). The set of immediate predecessors for the partial order  $\leq$  on  $\mathcal{I}(n)$  and  $\mathcal{R}(n)$  was described by me in [6, Lemmas 3.6, 3.7, 3.8] and by A.S. Vasyukhin and me in [10, Theorem 3.3] respectively. (In the case of  $\mathcal{I}(n)$ , the set of immediate predecessors for  $\leq$  coincides with the set described by F. Incitti, which implies that those two partial orders coincide.)

Furthermore, F. Incitti proved that the poset  $\mathcal{I}(n)$  is graded and calculated its Möbius function. Recall that a finite poset  $X$  is called *graded* if it has the greatest and the lowest elements and all maximal chains in  $X$  have the same length. Gradedness is equivalent to the existence of a rank function. By definition, it is a (unique) function  $\rho$  on  $X$ , which value on the lowest element is zero, such that if  $x$  is an immediate predecessor of  $y$  then  $\rho(y) = \rho(x) + 1$ . In [12, Theorem 5.2], F. Incitti constructed the rank function on  $\mathcal{I}(n)$ . As the main result of this paper, we prove the gradedness of the poset  $\mathcal{R}(n)$ .

The main tool used in the proof is so-called Kerov placements (see [13]). To each rook placement  $D \in \mathcal{R}(n)$  one can assign a certain orthogonal rook placement  $K(D) \in \mathcal{I}(2n-2)$ . We prove that if rook placements  $D_1$  is an immediate predecessor of  $D_2$  in  $\mathcal{R}(n)$  then  $K(D_1)$  is an immediate predecessor of  $K(D_2)$  in  $\mathcal{I}(2n-2)$  (and vice versa), see Theorem 3. As a corollary, we construct a rank function on  $\mathcal{R}(n)$  and prove the gradedness of this poset, see Corollary 1.

The structure of the paper is as follows. In the next section we describe the set of immediate predecessors of a given rook placement for  $\mathcal{I}(n)$  and  $\mathcal{R}(n)$ . In the third section we introduce the Kerov map

$$K: \mathcal{R}(n) \rightarrow \mathcal{I}(2n-2)$$

and show that it preserves the property “to be an immediate predecessor”. This allows us to use F. Incitti’s results to construct a rank function on  $\mathcal{R}(n)$ , which implies the gradedness of this poset.

## 2 Immediate predecessors

To prove that the set  $\mathcal{R}(n)$  is graded with respect to the partial order introduced above, we need to describe the set of immediate predecessors of a given rook placement in  $\mathcal{R}(n)$  and  $\mathcal{I}(n)$ . Such a description for  $\mathcal{R}(n)$  was provided in [10], while for  $\mathcal{I}(n)$  it was presented in F. Incitti’s work [12]. Recall that a rook placement  $D \in \mathcal{R}(n)$  is called an *immediate predecessor* of a rook placement  $T \in \mathcal{R}(n)$  if  $D < T$  and there are no  $S \in \mathcal{R}(n)$  such that  $D < S < T$ . (As usual,  $D < T$  means that  $D \leq T$  and  $D \neq T$ .) In other words, there exists an oriented edge from  $D$  to  $T$  in the Hasse diagram of the poset  $\mathcal{R}(n)$ . The definition of immediate predecessors for  $\mathcal{I}(n)$  is literally the same.

We denote the set of all immediate predecessors in  $\mathcal{R}(n)$  (respectively, in  $\mathcal{I}(n)$ ) of a rook placement  $D \in \mathcal{R}(n)$  (respectively, of an orthogonal rook placement  $D \in \mathcal{I}(n)$ ) by  $L_{\mathcal{R}}(D)$  (respectively, by  $L_{\mathcal{I}}(D)$ ). This set consists of rook placements of several types, which we will describe now. First, we will consider the set  $L_{\mathcal{R}}(D)$  in details.

It is convenient to introduce the following notation. We will write simply  $(i, j)$  instead of  $\epsilon_j - \epsilon_i$ ,  $i > j$ . Besides, for each  $k$  from 1 to  $n$ , we put

$$\mathcal{R}_k = \{(k, s) \in \Phi^+ \mid 1 \leq s < k\}, \quad \mathcal{C}_k = \{(r, k) \in \Phi^+ \mid k < r \leq n\}.$$

**Definition 2.** The sets  $\mathcal{R}_k, \mathcal{C}_k$  are called the  $k$ th row and the  $k$ th column of  $\Phi^+$  respectively. We will write  $\text{row}(\alpha) = k$  and  $\text{col}(\alpha) = k$  if  $\alpha \in \mathcal{R}_k$  and  $\alpha \in \mathcal{C}_k$  respectively. Note that, for  $D \in \mathcal{R}(n)$ , one has

$$|D \cap \mathcal{R}_k| \leq 1 \text{ and } |D \cap \mathcal{C}_k| \leq 1 \text{ for all } 1 \leq k \leq n.$$

Furthermore, if  $D \in \mathcal{I}(n)$  then

$$|D \cap (\mathcal{R}_k \cup \mathcal{C}_k)| \leq 1 \text{ for all } 1 \leq k \leq n.$$

There exists a natural partial order on the set of positive roots: given  $\alpha, \beta \in \Phi^+$ , by definition,  $\alpha \leq \beta$  if  $\beta - \alpha$  is a (probably, empty) sum of positive roots. In the other words,

$$(a, b) \leq (c, d) \text{ if } c \geq a \text{ and } d \leq b.$$

Given a rook placement  $D \in \mathcal{R}(n)$ , denote by  $\widetilde{M}(D)$  the set of minimal roots from  $D$  (with respect to  $\leq$ ). Now, we set

$$M_{\mathcal{R}}(D) = \{(i, j) \in \widetilde{M}(D) \mid D \cap \mathcal{R}_k \neq \emptyset \text{ and } D \cap \mathcal{C}_k \neq \emptyset \text{ for all } j < k < i\},$$

$$N_{\mathcal{R}}^-(D) = \{D_{(i,j)}^-, (i, j) \in M_{\mathcal{R}}(D)\},$$

where  $D_{(i,j)}^- = D \setminus \{(i, j)\}$ .

Next, fix a root  $(i, j) \in D$ . Denote

$$m = \min\{k \mid j < k < i \text{ and } D \cap \mathcal{C}_k = \emptyset\}.$$

Suppose that such a number  $m$  exists. Assume that  $D \cap \mathcal{R}_k \neq \emptyset$  for all  $k$  from  $j+1$  to  $m$ . Assume, in addition, that there are no  $(p, q) \in D$  such that  $(i, j) > (p, q)$  and  $(i, m) \not> (p, q)$ . The set of all roots  $(i, j) \in D$  satisfying these conditions is denoted by  $A_{\rightarrow}^{\mathcal{R}}(D)$ ; given  $(i, j) \in A_{\rightarrow}^{\mathcal{R}}(D)$ , we put

$$D_{(i,j)}^{\rightarrow, \mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(i, m)\}.$$

Similarly, suppose that there exists a number

$$m' = \max\{k \mid j < k < i \text{ and } D \cap \mathcal{R}_k = \emptyset\}.$$

Assume also that  $D \cap \mathcal{C}_k \neq \emptyset$  for  $m'+1 \leq k \leq i-1$  and that there are no  $(p, q) \in D$  such that  $(i, j) > (p, q)$  and  $(m', j) \not> (p, q)$ . Denote the set of all such  $(i, j)$ 's by  $A_{\uparrow}^{\mathcal{R}}$ ; given  $(i, j) \in A_{\uparrow}^{\mathcal{R}}$ , we put

$$D_{(i,j)}^{\uparrow, \mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(m', j)\}.$$

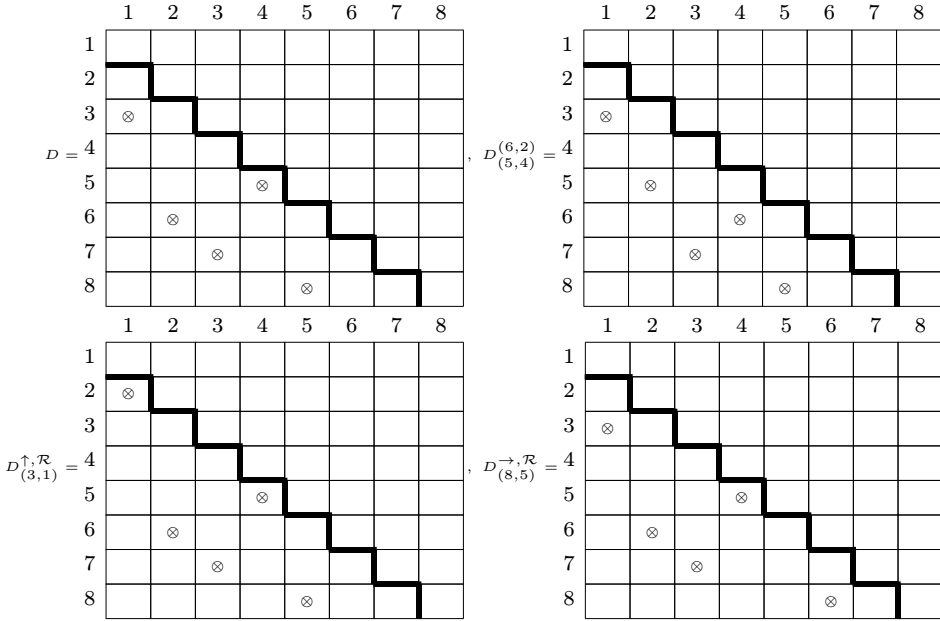
Now, let  $B_{(i,j)}^{\mathcal{R}}(D)$  be the set of roots  $(\alpha, \beta) \in D$  such that  $(\alpha, \beta) > (i, j)$  and there are no  $(p, q) \in D$  satisfying  $(i, j) < (p, q) < (\alpha, \beta)$ . For each  $(\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D)$  we set

$$D_{(i,j)}^{(\alpha, \beta), \mathcal{R}} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(i, \beta), (\alpha, j)\}.$$

By definition, let

$$N_{\mathcal{R}}^0(D) = \left\{ D_{(i,j)}^{\uparrow, \mathcal{R}}, (i, j) \in A_{\uparrow}^{\mathcal{R}} \right\} \cup \left\{ D_{(i,j)}^{\rightarrow, \mathcal{R}}, (i, j) \in A_{\rightarrow}^{\mathcal{R}} \right\} \\ \cup \bigcup_{(i,j) \in D} \left\{ D_{(i,j)}^{(\alpha, \beta), \mathcal{R}}, (\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D) \right\}.$$

**Example 3.** Let  $n = 8$  and  $D = \{(3, 1), (6, 2), (7, 3), (5, 4), (8, 5)\}$ . Clearly,  $M_{\mathcal{R}}(D) = \{(5, 4)\}$ ,  $(8, 5) \in A_{\rightarrow}^{\mathcal{R}}$ ,  $(3, 1) \in A_{\uparrow}^{\mathcal{R}}$  and  $(6, 2) \in B_{(5,4)}^{\mathcal{R}}(D)$ . On the picture below we draw the rook placements  $D$ ,  $D_{(5,4)}^{(6,2),\mathcal{R}}$ ,  $D_{(3,1)}^{\uparrow,\mathcal{R}}$  and  $D_{(8,5)}^{\rightarrow,\mathcal{R}}$ .



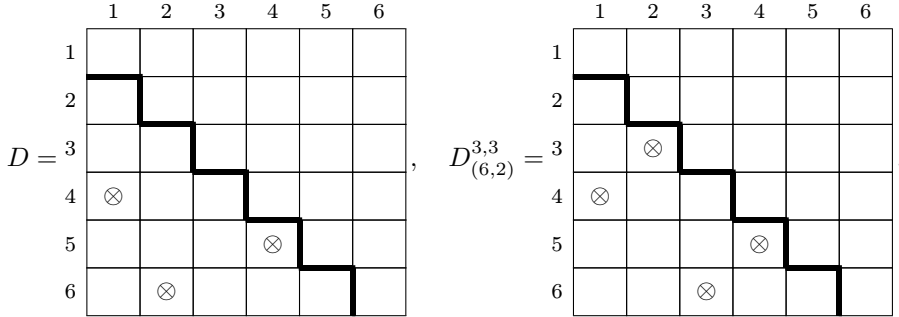
Next, fix a root  $(i, j) \in D$ , and consider a pair  $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$ . Suppose that  $i > \beta \geq \alpha > j$ ,  $D \cap \mathcal{R}_{\alpha} = D \cap \mathcal{C}_{\beta} = \emptyset$ ,  $D \cap \mathcal{R}_k \neq \emptyset$ ,  $D \cap \mathcal{C}_k \neq \emptyset$  for all  $\alpha < k < \beta$ , and the conditions  $(p, q) \in D$ ,  $(i, j) > (p, q)$ ,  $(\alpha, j) \not> (p, q)$  imply  $(i, \beta) > (p, q)$ . Moreover, assume that if  $\alpha \neq \beta$  then  $D \cap \mathcal{R}_{\beta} \neq \emptyset$  and  $D \cap \mathcal{C}_{\alpha} \neq \emptyset$ . Denote the set of all such pairs  $(\alpha, \beta)$  by  $C_{(i,j)}^{\mathcal{R}}(D)$ . For an arbitrary pair  $(\alpha, \beta) \in C_{(i,j)}^{\mathcal{R}}(D)$ , we put

$$D_{(i,j)}^{\alpha,\beta,\mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(i, \beta), (\alpha, j)\}.$$

By definition, let

$$N_{\mathcal{R}}^+(D) = \bigcup_{(i,j) \in D} \left\{ D_{(i,j)}^{\alpha,\beta,\mathcal{R}}, (\alpha, \beta) \in C_{(i,j)}^{\mathcal{R}}(D) \right\}.$$

**Example 4.** Let  $n = 6$  and  $D = \{(4, 1), (6, 2), (5, 4)\}$ , then  $(3, 3) \in C_{(6,2)}^{\mathcal{R}}(D)$ . On the picture below we draw the rook placements  $D$  and  $D_{(6,2)}^{3,3,\mathcal{R}}$ .



Finally, we set

$$N_{\mathcal{R}}(D) = N_{\mathcal{R}}^-(D) \cup N_{\mathcal{R}}^0(D) \cup N_{\mathcal{R}}^+(D).$$

The set of immediate predecessors of a given rook placement from  $\mathcal{R}(n)$  is described as follows.

**Theorem 1 ([10, Theorem 3.3]).** Let  $D \in \mathcal{R}(n)$ . Then  $L_{\mathcal{R}}(D) = N(D)$ .

Now we turn to the description of immediate predecessors for  $\mathcal{I}(n)$ . Given an orthogonal rook placement  $D \in \mathcal{R}(n)$ , put

$$M_{\mathcal{I}}(D) = \{(i, j) \in \widetilde{M}(D) \mid D \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset \text{ for all } j < k < i\},$$

$$N_{\mathcal{I}}^-(D) = \{D_{(i,j)}^-, (i, j) \in M_{\mathcal{I}}(D)\},$$

where  $D_{(i,j)}^- = D \setminus \{(i, j)\}$ , as above.

Let  $D \in \mathcal{I}(n)$ ,  $(i, j) \in D$ . Denote

$$m = \min\{k \mid j < k < i \text{ and } D \cap \mathcal{C}_k = D \cap \mathcal{R}_k = \emptyset\}.$$

Suppose that such a number  $m$  exists. Assume that there are no  $(p, q) \in D$  such that  $(i, j) > (p, q)$  and  $(i, m) \not> (p, q)$ . The set of all  $(i, j) \in D$  satisfying these conditions is denoted by  $A_{\rightarrow}^{\mathcal{I}}(D)$ ; given  $(i, j) \in A_{\rightarrow}^{\mathcal{I}}(D)$ , we set

$$D_{(i,j)}^{\rightarrow,\mathcal{I}} = (D \setminus \{(i, j)\}) \cup \{(i, m)\}.$$

Similarly, suppose that there exists

$$m' = \max\{k \mid j < k < i \text{ and } D \cap \mathcal{R}_k = D \cap \mathcal{C}_k = \emptyset\},$$

and there are no  $(p, q) \in D$  such that  $(i, j) > (p, q)$  and  $(m', j) \not> (p, q)$ . The set of all such  $(i, j)$ 's is denoted by  $A_{\uparrow}^{\mathcal{I}}$ ; given  $(i, j) \in A_{\uparrow}^{\mathcal{I}}$ , we set

$$D_{(i,j)}^{\uparrow,\mathcal{I}} = (D \setminus \{(i, j)\}) \cup \{(m', j)\}.$$



Next, let  $B_{(i,j)}^{\mathcal{I}}(D)$  be the set of roots  $(\alpha, \beta) \in D$  such that  $j < \beta < i < \alpha$ ,

$$D \cap (\mathcal{R}_r \cup \mathcal{C}_r) \neq \emptyset$$

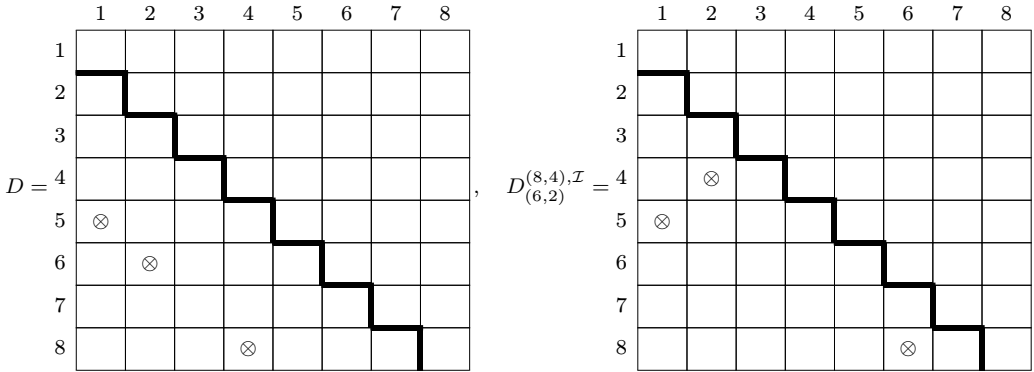
for all  $\beta < r < i$  and there are no  $(p, q) \in D$  for which  $j < q < \beta < p < i$  or  $\beta < q < i < p < \alpha$  (in other words, for which  $(i, j) > (p, q)$  and  $(\beta, j) \not> (p, q)$ , or  $(\alpha, \beta) > (p, q)$  and  $(\alpha, i) \not> (p, q)$ ). To each  $(\alpha, \beta) \in B_{(i,j)}^{\mathcal{I}}(D)$  we assign the set

$$D_{(i,j)}^{(\alpha,\beta),\mathcal{I}} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(\beta, j), (\alpha, i)\}.$$

Now, let

$$N_{\mathcal{I}}^0(D) = \left\{ D_{(i,j)}^{\uparrow,\mathcal{I}}, (i, j) \in A_{\uparrow}^{\mathcal{I}} \right\} \cup \left\{ D_{(i,j)}^{\rightarrow,\mathcal{I}}, (i, j) \in A_{\rightarrow}^{\mathcal{I}} \right\} \\ \cup \bigcup_{(i,j) \in D} \left\{ D_{(i,j)}^{(\alpha,\beta),\mathcal{R}}, (\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D) \right\} \cup \bigcup_{(i,j) \in D} \left\{ D_{(i,j)}^{(\alpha,\beta),\mathcal{I}}, (\alpha, \beta) \in B_{(i,j)}^{\mathcal{I}}(D) \right\}.$$

**Example 5.** If  $n = 8$ ,  $D = \{(5, 1), (6, 2), (8, 4)\}$ , then  $(8, 4) \in B_{6,2}^{\mathcal{I}}(D)$ , hence



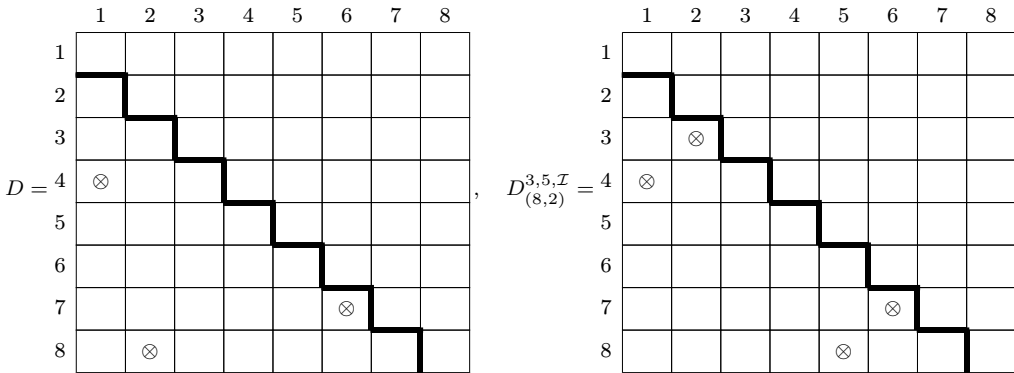
Besides, denote by  $C_{i,j}^{\mathcal{I}}(D)$  the set of pairs  $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$  such that  $i > \beta > \alpha > j$ ,

$$D \cap (\mathcal{R}_\alpha \cup \mathcal{C}_\alpha) = D \cap (\mathcal{R}_\beta \cup \mathcal{C}_\beta) = \emptyset,$$

$D \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset$  for all  $\beta > k > \alpha$ , and if  $(p, q) \in D$ ,  $(i, j) > (p, q)$ ,  $(\alpha, j) \not> (p, q)$  then  $(i, \beta) > (p, q)$ . For each pair  $(i, j) \in C_{(i,j)}^{\mathcal{I}}(D)$ , we put

$$D_{(i,j)}^{\alpha,\beta,\mathcal{I}} = (D \setminus \{(i, j)\}) \cup \{(i, \beta), (\alpha, j)\}.$$

**Example 6.** Let  $n = 8$ ,  $D = \{(4, 1), (8, 2), (7, 6)\}$ , then  $(3, 5) \in C_{(8,2)}^{\mathcal{I}}(D)$ , so



Finally, we denote

$$N_{\mathcal{I}}^+(D) = \bigcup_{(i,j) \in D} \{D_{(i,j)}^{\alpha,\beta,\mathcal{I}}, (\alpha, \beta) \in C_{(i,j)}^{\mathcal{I}}(D)\},$$

$$N_{\mathcal{I}}(D) = N_{\overline{\mathcal{R}}}(D) \cup N_{\mathcal{I}}^0(D) \cup N_{\mathcal{I}}^+(D).$$

Immediate predecessors in  $\mathcal{I}(n)$  are described by the following F. Incitti’s theorem (see also [6, Subsection 2.4]).

**Theorem 2 ([12, Theorem 5.1]).** Let  $D \in \mathcal{I}(n)$ . Then  $L_{\mathcal{I}}(D) = N_{\mathcal{I}}(D)$ .

### 3 Kerov map and the main result

In this section, we introduce our main technical tool, Kerov orthogonal rook placements, and, using them, prove that  $\mathcal{R}(n)$  is graded.

**Definition 3.** Let  $n \geq 3$ , and  $D$  be a rook placement from  $\mathcal{R}(n)$ . A *Kerov rook placement* corresponding to  $D$  is, by definition, the orthogonal rook placement  $K(D) \in \mathcal{I}(2n - 2)$  constructed by the following rule: if

$$D = \{(i_1, j_1), \dots, (i_s, j_s)\},$$

then

$$K(D) = (2i_1 - 2, 2j_1 - 1) \dots (2i_s - 2, 2j_s - 1).$$

(Kerov rook placements were introduced in the paper [13]). We call the map  $K: \mathcal{R}(n) \rightarrow \mathcal{I}(2n - 2)$  given by the rule  $D \mapsto K(D)$  the *Kerov map*.

**Example 7.** If  $n = 8$  and  $D = \{(3, 1), (6, 2), (7, 3), (5, 4), (8, 6)\} \in \mathcal{R}(8)$ , then

$$K(D) = (4, 1) \cdot (10, 3) \cdot (12, 5) \cdot (8, 7) \cdot (14, 11)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 4 & 2 & 10 & 1 & 12 & 6 & 8 & 7 & 9 & 3 & 14 & 5 & 13 & 11 \end{pmatrix} \in \mathcal{I}(14).$$

The following proposition is evident.

**Proposition 1.** *Let  $D, T \in \mathcal{R}(n)$ . Then the conditions  $T \leq D$  and  $K(T) \leq K(D)$  are equivalent.*

The following theorem plays the crucial role in the proof of the main result.

**Theorem 3.** *Let  $D, T \in \mathcal{R}(n)$  be rook placements. Then the conditions  $T \in L_{\mathcal{R}}(D)$  and  $K(T) \in L_{\mathcal{I}}(K(D))$  are equivalent.*

*Proof.* Clearly,  $K(T) \in L_{\mathcal{I}}(D)$  implies  $T \in L_{\mathcal{R}}(D)$ . Indeed, since there are no orthogonal involutions from  $\mathcal{I}(2n-2)$  between  $K(T)$  and  $K(D)$ , we conclude that, in particular, there are no Kerov involutions between them. It remains to prove that the converse is also true.

Assume that  $T \in L_{\mathcal{R}}(D)$ . By Theorem 1, this is equivalent to

$$T \in N_{\mathcal{R}}(D) = N^-(D) \cup N_{\mathcal{R}}^0(D) \cup N_{\mathcal{R}}^+(D).$$

We will consider these variants case-by-case.

First, suppose that  $T \in N_{\mathcal{R}}^-(D)$ . This means that  $T = D_{(i,j)}^-$  for a certain root  $(i, j) \in M(D)$ . Automatically,  $K(T) = K(D) \setminus \{(2i-2, 2j-1)\}$ . It follows immediately from  $(i, j) \in \widetilde{M}(D)$  that  $(2i-2, 2j-1) \in \widetilde{M}(K(D))$ . Since  $(i, j) \in M(D)$ , we see that  $D \cap \mathcal{R}_k$  and  $D \cap \mathcal{C}_k$  are nonempty if  $i < k < j$ . This shows that  $K(D \cap \mathcal{R}_{2k-2})$  and  $K(D) \cap \mathcal{C}_{2k-1}$  are nonempty for all such  $k$ . Thus,

$$(2i-2, 2j-1) \in M(K(D)),$$

i.e.,  $K(T) \in N_{\mathcal{I}}^-(K(D))$ . By Theorem 2,  $K(T) \in L_{\mathcal{I}}(K(D))$ .

Next, assume that  $T \in N_{\mathcal{R}}^0(D)$ . If  $T = D_{(i,j)}^{(\alpha,\beta),\mathcal{R}}$  for some  $(i, j) \in D$ ,  $(\alpha, \beta) \in \mathcal{B}_{(i,j)}^{\mathcal{R}}(D)$ , then it is easy to see that

$$(2\alpha-2, 2\beta-1) \in \mathcal{B}_{(2i-2, 2j-1)}^{\mathcal{R}}(K(D))$$

and

$$K(T) = K(D)_{(2i-2, 2j-1)}^{(2\alpha-2, 2\beta-1), \mathcal{R}} \in N_{\mathcal{R}}^0(K(D)),$$

hence

$$K(T) \in N_{\mathcal{I}}^0(D) \subset L_{\mathcal{I}}(K(D)).$$

Now consider the case when  $T = D_{(i,j)}^{\rightarrow, \mathcal{R}}$  for some  $(i, j) \in A_{\rightarrow}^{\mathcal{R}}$ . (The case  $T = D_{(i,j)}^{\uparrow, \mathcal{R}}$ ,  $(i, j) \in A_{\uparrow}^{\mathcal{R}}$  can be considered similarly.) Let  $T = (D \setminus \{(i, j)\}) \cup \{(i, m)\}$ , then

$$K(T) = (K(D) \setminus \{(2i-2, 2j-1)\}) \cup \{(2i-2, 2m-1)\}.$$

Since there are no root in  $D$  which is less than  $(i, j)$  but not less than  $(i, m)$ , we have a similar condition for  $K(D)$ . Since  $D \cap \mathcal{C}_k \neq \emptyset$  for P. Heymans: Pfaffians and skew-symmetric matrices  $j < k < m$ , one has  $K(D) \cap \mathcal{C}_{2k-1} \neq \emptyset$  for such  $k$ . On the other hand,  $D \cap \mathcal{R}_k$  is nonempty for  $j < k \leq m$ , so  $K(D) \cap \mathcal{R}_{2k-2}$  is also nonempty for such  $k$ . Thus,  $K(D) \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset$  for  $2j-1 < k < 2m-1$ , which means that  $(2i-2, 2j-1) \in A_{\rightarrow}^{\mathcal{I}}$  and  $K(T) = K(D)_{(2i-2, 2j-1)}^{\rightarrow, \mathcal{I}}$ . Hence, by Theorem 2,  $K(T) \in L_{\mathcal{I}}(K(D))$ , as required.

Finally, suppose that  $T \in N_{\mathcal{R}}^+(D)$ , i.e.,  $T = D_{(i,j)}^{\alpha,\beta,\mathcal{R}}$  for certain  $(i, j) \in D$  and  $(\alpha, \beta) \in C_{(i,j)}^{\mathcal{R}}(D)$ . Since  $i > \beta \geq \alpha > j$ , we have

$$2i - 2 > 2\beta - 1 > 2\alpha - 2 > 2j - 1.$$

It follows from  $D \cap \mathcal{R}_\alpha = D \cap \mathcal{C}_\beta = \emptyset$  that

$$K(D) \cap \mathcal{R}_{2\alpha-2} = K(D) \cap \mathcal{C}_{2\beta-1} = \emptyset.$$

Since  $K(D)$  is a Kerov rook placement, the condition

$$K(D) \cap \mathcal{C}_{2\alpha-2} = K(D) \cap \mathcal{R}_{2\beta-1} = \emptyset$$

is satisfied automatically. If  $\alpha = \beta$  then there is nothing to prove. If  $\beta > \alpha$  then  $D \cap \mathcal{R}_k \neq \emptyset$  and  $D \cap \mathcal{C}_k \neq \emptyset$  for all  $k$  from  $\alpha+1$  to  $\beta-1$ , hence  $K(D) \cap \mathcal{R}_{2k-2} \neq \emptyset$  and  $K(D) \cap \mathcal{C}_{2k-1} \neq \emptyset$  for all such  $k$ . Furthermore,  $D \cap \mathcal{R}_\beta$  and  $D \cap \mathcal{C}_\alpha$  are nonempty, which implies that  $K(D) \cap \mathcal{R}_{2\beta-2}$  and  $D \cap \mathcal{C}_{2\alpha-1}$  are also nonempty. Thus, we obtain  $K(D) \cap (\mathcal{R}_k \cap \mathcal{C}_k) \neq \emptyset$  for all  $k$  from  $2\alpha - 1$  to  $2\beta - 2$ , sa required. We conclude that  $(2\alpha - 2, 2\beta - 1) \in C_{(2i-2, 2j-1)}^{\mathcal{I}}(D)$  and  $K(T) = K(D)_{(2i-2, 2j-1)}^{2\alpha-2, 2\beta-1, \mathcal{I}}$ . Theorem 2 guarantees that  $K(T) \in L_{\mathcal{I}}(K(D))$ . The proof is complete. □

**Corollary 1.** *For each  $n \geq 2$  the poset  $\mathcal{R}(n)$  is graded with the rank function*

$$\rho(D) = \frac{l(w_{K(D)}) + |D|}{2},$$

where  $l(w)$  is the length of a permutation  $w$  in the corresponding symmetric group.

*Proof.* As we mentioned in the introduction, F. Incitti showed that the set  $\mathcal{I}(2n-2)$  of orthogonal rook placements is graded. Precisely [11, Theorem 5.3.2], the rank function on this poset has the form

$$\rho(D) = \frac{l(w_D) + |D|}{2}.$$

Applying Theorem 3, we see that the restriction of this rank function to  $K(\mathcal{R}(n))$  in fact provided the rank function of the required form on  $\mathcal{R}(n)$ . This concludes the proof. □

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Received: 11 October 2019

Accepted for publication: 7 November 2019

Communicated by: Ivan Kaygorodov