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# Rota-type operators on 3-dimensional nilpotent associative algebras

*N.G. Abdujabborov, I.A. Karimjanov and M.A. Kodirova*

**Abstract.** We give the description of Rota–Baxter operators, Reynolds operators, Nijenhuis operators and average operators on 3-dimensional nilpotent associative algebras over  $\mathbb{C}$ .

## 1 Introduction

Rota–Baxter algebra appeared from the probability theory and has found applications in many fields of mathematics and physics, such as number theory, quasi-symmetric functions, Lie algebras, and Yang–Baxter equations. Rota–Baxter operators were defined by Baxter to solve an analytic formula in probability [2],[7], [9], [10]. It has been related to other areas in mathematics and mathematical physics [1], [3], [5], [15], [21]

A Rota–Baxter operator on an associative algebra  $A$  over a field  $F$  is defined to be a linear map  $P : A \rightarrow A$  satisfying

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \quad \forall x, y \in A, \quad \lambda \in F. \quad (1)$$

Note that, if  $P$  is a Rota–Baxter operator of weight  $\lambda \neq 0$ , then  $\lambda^{-1}P$  is a Rota–Baxter operator of weight 1. Therefore, it is sufficient to consider Rota–Baxter operators of weight 0 and 1.

At the moment there are descriptions of all Rota–Baxter operators on 3-dimensional simple Lie algebra [19], [20], on 4-dimensional simple associative algebra [21] and

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*Key words:* Rota–Baxter operator; Reynolds operator; Nijenhuis operator; average operator; nilpotent; associative algebras

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some other algebras [4], [11], [16]. The study of some particular cases of Rota–Baxter operators was initiated in [22], [23] and [8]. Recently, in [13] the authors presented all homogeneous Rota-type operators on null-filiform associative algebras.

Here we give also the definition of operators which we consider in the present paper.

- Reynolds operator:  $P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))$
- Nijenhuis operator:  $P(x)P(y) = P(xP(y) + P(x)y - P(xy))$
- average operator:  $P(x)P(y) = P(xP(y))$

Low dimensional associative algebras have been classified in several works. Hazlett classified nilpotent algebras of dimension less than or equal to 4 over the complex numbers [12]. Afterwards, Kruse and Price classified nilpotent associative algebras of dimension less than or equal to 4 over any field [14]. Mazzola published his results on associative unitary algebras of dimension 5 over algebraically closed fields of characteristic different from 2, and on nilpotent commutative associative algebras of dimension less than or equal to 5 over algebraically closed fields of characteristic different from 2,3 [17], [18].

Throughout this paper algebras are considered over the field of complex numbers.

**Theorem 1.** [6] *Any three-dimensional complex nilpotent associative algebra  $A$  is isomorphic to one of the following pairwise non-isomorphic algebras with a basis  $\{e_1, e_2, e_3\}$ :*

$$\begin{aligned}
 A_1 : e_1e_2 &= e_2e_1 = e_3, \\
 A_2 : e_1^2 &= e_2, e_1e_2 = e_2e_1 = e_3, \\
 A_3 : e_1e_2 &= -e_2e_1 = e_3, \\
 A_4^\alpha : e_1^2 &= e_3, e_2^2 = \alpha e_3, e_1e_2 = e_3, \quad \text{with } \alpha \in \mathbb{C}, \\
 A_5 : e_1e_1 &= e_2.
 \end{aligned}$$

and the omitted products vanish.

Now let  $P$  be a linear operator on  $A$  such that

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

## 2 Main result

### 2.1 Rota–Baxter operator

**Theorem 2.** *There are five types of Rota-Baxter operators of weight 0 for the 3-dimensional associative algebra  $A_1$ , which are as follows:*

$$\begin{aligned}
 P_1 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix} && \text{where } a_{22} \neq -a_{11} \\
 P_2 &= \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \\
 P_3 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} && \text{where } a_{21} \neq 0 \\
 P_4 &= \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} && \text{where } a_{12} \neq 0 \\
 P_5 &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix} && \text{where } a_{11}a_{12} \neq 0
 \end{aligned}$$

*Proof.* Since  $P$  is a linear operator, we only need to consider the base elements. We have the equations:

$$\begin{aligned}
 a_{31} &= a_{32} = 0, \\
 a_{12}(a_{33} - a_{11}) &= 0, \\
 a_{21}(a_{33} - a_{22}) &= 0, \\
 a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) &= 0
 \end{aligned} \tag{2}$$

from

$$\begin{aligned}
 P(e_2)P(e_3) &= P(e_2P(e_3) + P(e_2)e_3) \Rightarrow a_{31} = 0 \\
 P(e_1)P(e_3) &= P(e_1P(e_3) + P(e_1)e_3) \Rightarrow a_{32} = 0 \\
 P(e_1)P(e_1) &= P(e_1P(e_1) + P(e_1)e_1) \Rightarrow a_{12}(a_{33} - a_{11}) = 0 \\
 P(e_2)P(e_2) &= P(e_2P(e_2) + P(e_2)e_2) \Rightarrow a_{21}(a_{33} - a_{22}) = 0 \\
 P(e_1)P(e_2) &= P(e_1P(e_2) + P(e_1)e_2) \Rightarrow a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) = 0.
 \end{aligned}$$

The missing cases  $P(e_3)P(e_1)$ ,  $P(e_3)P(e_3)$  and etc., lead to equations equivalent to the ones in the system (2).

Now we consider all possible cases:

**Case 1.** If  $a_{12} = 0$ , then the system of equations (2) becomes

$$\begin{aligned}
 a_{31} &= a_{32} = a_{12} = 0, \\
 a_{21}(a_{33} - a_{22}) &= 0, \\
 a_{11}a_{22} - a_{33}(a_{11} + a_{22}) &= 0.
 \end{aligned} \tag{3}$$

**Case 1.1.** • If  $a_{21} = 0$  and  $a_{22} \neq -a_{11}$ , then we obtain

$$P_1 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix}, \quad a_{22} \neq -a_{11}$$

- If  $a_{21} = 0$  and  $a_{22} = -a_{11}$ , then from system of equation (3) we get  $a_{22} = a_{11} = 0$  and we have

$$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

**Case 1.2.** If  $a_{21} \neq 0$ , then from system (3) we derive  $a_{33} = a_{22} = 0$  and

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{21} \neq 0.$$

**Case 2.** If  $a_{12} \neq 0$ , then system (2) becomes

$$\begin{aligned} a_{31} &= a_{32} = 0, & a_{12} &\neq 0, \\ a_{33} &= a_{11}, \\ a_{21}(a_{11} - a_{22}) &= 0, \\ a_{12}a_{21} - a_{11}^2 &= 0. \end{aligned} \tag{4}$$

**Case 2.1.** If  $a_{21} = 0$ , then we get  $a_{33} = a_{11} = 0$  and deduce

$$P_4 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{12} \neq 0.$$

**Case 2.2.** If  $a_{21} \neq 0$ , then system of equations (4) becomes

$$\begin{aligned} a_{31} &= a_{32} = 0, & a_{12}a_{21} &\neq 0, \\ a_{33} &= a_{22} = a_{11}, \\ a_{12}a_{21} - a_{11}^2 &= 0 \end{aligned}$$

which will yield us to the Rota-Baxter operator

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, \quad a_{11}a_{12} \neq 0.$$

□

**Theorem 3.** *The Rota-Baxter operators of weight 1 on the 3-dimensional associative algebra  $A_1$  are the following:*

$$\begin{aligned}
 P_1 &= \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & -1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \\
 P_2 &= \begin{pmatrix} -1 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \\
 P_3 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{1+a_{11}+a_{22}} \end{pmatrix}, & a_{11} + a_{22} \neq -1 \\
 P_4 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, & a_{21} \neq 0 \\
 P_5 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & -1 & a_{23} \\ 0 & 0 & -1 \end{pmatrix}, & a_{21} \neq 0 \\
 P_6 &= \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, & a_{12} \neq 0 \\
 P_7 &= \begin{pmatrix} -1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & -1 \end{pmatrix}, & a_{12} \neq 0 \\
 P_8 &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}(a_{11}+1)}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, & a_{11}a_{12}(1 + a_{11}) \neq 0.
 \end{aligned}$$

*Proof.* Since  $P$  is a linear operator, we only need to consider the base elements which  $P$  satisfying equation (1) with  $\lambda = 1$ . We get the equations:

$$\begin{aligned}
 a_{31} &= a_{32} = 0, \\
 a_{12}(a_{33} - a_{11}) &= 0, \\
 a_{21}(a_{33} - a_{22}) &= 0, \\
 a_{12}a_{21} + a_{11}a_{22} - a_{33}(1 + a_{11} + a_{22}) &= 0.
 \end{aligned} \tag{5}$$

So, the solutions of the system of equations (5) give us all Rota-Baxter operators of weight 1 for  $A_1$ . □

We present the list of Rota-Baxter operators of weights 0 and 1 on the  $A_2, A_3, A_4^\alpha$  and  $A_5$  algebras.

Algebra	Rota-Baxter Operators of weight 0	Restrictions
$A_2$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$ $P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{1}{2}a_{11} & \frac{2}{3}a_{12} \\ 0 & 0 & \frac{1}{3}a_{11} \end{pmatrix}$	$a_{33} \neq 0$  $a_{11} \neq 0$
$A_3$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}^2}{a_{12}} & -a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_3 = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{22} \neq -a_{11}$  $a_{12} \neq 0$
$A_4^\alpha$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2 + a_{11}^2 + a_{11}a_{12}$ $= (2a_{11} + a_{12})a_{33};$ $\alpha a_{12}a_{22} + a_{11}(a_{21} + a_{22})$ $= (a_{11} + a_{21} + a_{22} + \alpha a_{12})a_{33};$ $a_{21}(a_{11} + a_{12}) + \alpha a_{12}a_{22}$ $= (\alpha a_{12} + a_{21})a_{33};$ $\alpha a_{22}^2 + a_{21}^2 + a_{21}a_{22}$ $= (2\alpha a_{22} + a_{21})a_{33}$
$A_5$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{1}{2}a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11} \neq 0$

Algebra	Rota-Baxter Operators of weight 1	Restrictions
$A_2$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}^2}{2a_{11}+1} & \frac{2a_{11}a_{12}(a_{11}+1)}{3a_{11}^2+3a_{11}+1} \\ 0 & 0 & \frac{a_{11}^3}{3a_{11}^2+3a_{11}+1} \end{pmatrix}$	$a_{11} \notin \{-\frac{1}{2}, \frac{1}{6}(-3 \pm i\sqrt{3})\}$
$A_3$	$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & -1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_2 = \begin{pmatrix} -1 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}(1+a_{11})}{a_{12}} & -1 - a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{1+a_{11}+a_{22}} \end{pmatrix}$	$a_{12} \neq 0$  $a_{11} + a_{22} \neq -1$
$A_4^\alpha$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2 + a_{11}^2 + a_{11}a_{12}$ $= (1 + 2a_{11} + a_{12})a_{33};$ $\alpha a_{12}a_{22} + a_{11}(a_{21} + a_{22})$ $= (1 + a_{11} + a_{21} + a_{22} + \alpha a_{12})a_{33};$ $a_{21}(a_{11} + a_{12}) + \alpha a_{12}a_{22}$ $= (\alpha a_{12} + a_{21})a_{33};$ $\alpha a_{22}^2 + a_{21}^2 + a_{21}a_{22}$ $= (\alpha(1 + 2a_{22}) + a_{21})a_{33}$
$A_5$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}^2}{1+2a_{11}} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$1 + 2a_{11} \neq 0$



## 2.2 Reynolds operator

**Theorem 4.** All Reynolds operators on the 3-dimensional associative algebra  $A_1$  are listed below:

$$\begin{aligned}
 P_1 &= \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \\
 P_2 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}-a_{11}a_{22}} \end{pmatrix} && \text{where } a_{11} + a_{22} - a_{11}a_{22} \neq 0 \\
 P_3 &= \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} && \text{where } a_{12}a_{22} \neq 0 \\
 P_4 &= \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} && \text{where } a_{21} \neq 0 \\
 P_5 &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & \frac{a_{11}}{1-a_{11}} \end{pmatrix} && \text{where } (1 - a_{11})a_{12} \neq 0
 \end{aligned}$$

*Proof.* Consider

$$P(e_3)P(e_3) = (a_{31}e_1 + a_{32}e_2 + a_{33}e_3)(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) = 2a_{31}a_{32}e_3.$$

On the other hand,

$$\begin{aligned}
 P(e_3)P(e_3) &= P(e_3P(e_3)) + P(e_3)e_3 - P(e_3)P(e_3) = -2a_{31}a_{32}P(e_3) \\
 &= -2a_{31}^2a_{32}e_1 - 2a_{31}a_{32}^2e_2 - 2a_{31}a_{32}a_{33}e_3.
 \end{aligned}$$

Comparing the coefficients of the basic elements we derive

$$a_{31}a_{32} = 0.$$

Now we consider the next cases:

**Case 1.** If  $a_{31} = 0$ , then identity

$$P(e_3)P(e_2) = P(e_3P(e_2)) + P(e_3)e_2 - P(e_3)P(e_2))$$

implies that  $a_{21}a_{32} = 0$ .

**Case 1.1.** If  $a_{21} = 0$ , from

$$P(e_3)P(e_1) = P(e_3P(e_1)) + P(e_3)e_1 - P(e_3)P(e_1))$$

we obtain  $a_{32}(a_{11} - 1) = 0$ .

**Case 1.1.1.** If  $a_{32} = 0$ , then from

$$P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1 - P(e_1)P(e_1))$$

we have  $a_{12}(a_{11} - a_{33} + a_{11}a_{33}) = 0$ .

1. If  $a_{12} = 0$ , considering  $P(e_2)P(e_1) = P(e_2P(e_1) + P(e_2)e_1 - P(e_2)P(e_1))$  we deduce that  $a_{11}a_{22} = (a_{11} + a_{22} - a_{11}a_{22})a_{33}$ .

- If  $a_{11} + a_{22} - a_{11}a_{22} = 0$ , then we deduce the operator  $P_1$ ;
- If  $a_{11} + a_{22} - a_{11}a_{22} \neq 0$ , then we derive the operator  $P_2$ .

2. If  $a_{12} \neq 0$ , then we get operator  $P_3$ .

**Case 1.1.2.** If  $a_{32} \neq 0$ , then we have  $a_{11} = 1$ . Moreover, from

$$P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2 - P(e_1)P(e_2))$$

we derive that  $a_{32} = 0$ , that is a contradiction.

**Case 1.2.** If  $a_{21} \neq 0$ , then we have  $a_{32} = 0$  and

$$P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1 - P(e_1)P(e_1)) \Rightarrow a_{12}(a_{11} - a_{33} + a_{11}a_{33}) = 0;$$

$$P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2 - P(e_1)P(e_2)) \Rightarrow \begin{aligned} & a_{12}a_{21}(1 + a_{33}) + a_{11}a_{22} \\ & = (a_{11} + a_{22} - a_{11}a_{22})a_{33}; \end{aligned}$$

$$P(e_2)P(e_2) = P(e_2P(e_2) + P(e_2)e_2 - P(e_2)P(e_2)) \Rightarrow a_{22} - a_{33} + a_{22}a_{33} = 0.$$

Therefore,

- If  $a_{12} = 0$ , we have operator  $P_4$ ;
- If  $a_{12} \neq 0$ , we obtain operator  $P_5$ .

**Case 2.** If  $a_{31} \neq 0$ , then we have  $a_{32} = 0$  and from identities

$$P(e_3)P(e_1) = P(e_3P(e_1) + P(e_3)e_1 - P(e_3)P(e_1)) \Rightarrow a_{12} = 0$$

$$P(e_3)P(e_2) = P(e_3P(e_2) + P(e_3)e_2 - P(e_3)P(e_2)) \Rightarrow a_{22} = 1, a_{31} = 0$$

we obtain contradiction with  $a_{31} \neq 0$ . □

The Reynolds operators on algebras  $A_2, A_3, A_4^\alpha$  and  $A_5$  listed below.

Algebra	Reynolds Operators	Restrictions
$A_2$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	$a_{33} \neq 0$
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}}{2-a_{11}} & \frac{2a_{12}}{3-2a_{11}} \\ 0 & 0 & \frac{a_{11}}{3-2a_{11}} \end{pmatrix}$	
		$a_{11}(a_{11} - 2)(2a_{11} - 3) \neq 0$

Algebra	Reynolds Operators	Restrictions
$A_3$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{11} & a_{23} \\ 0 & 0 & -1 \end{pmatrix}$	$a_{12}a_{21}(a_{33} + 1)$ $= a_{11}a_{22} + (a_{11}a_{22} - a_{11} - a_{22})a_{33}$
$A_4^\alpha$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2 + a_{11}^2 + a_{11}a_{12}$ $= (2a_{11} + a_{12} - a_{11}^2 - a_{11}a_{12} - \alpha a_{12}^2)a_{33};$ $\alpha a_{12}a_{22} + a_{11}(a_{21} + a_{22})$ $= (a_{11} + a_{21} + a_{22} + \alpha a_{12}$ $- a_{11}(a_{21} + a_{22}) - \alpha a_{12}a_{22})a_{33};$ $a_{21}(a_{11} + a_{12}) + \alpha a_{12}a_{22}$ $= (\alpha a_{12} + a_{21} - a_{11}a_{21}$ $- a_{12}a_{21} - \alpha a_{12}a_{22})a_{33};$ $\alpha a_{22}^2 + a_{21}^2 + a_{21}a_{22}$ $= (2\alpha a_{22} + a_{21} - a_{21}^2$ $- a_{21}a_{22} - \alpha a_{22}^2)a_{33}$
$A_5$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}}{2-a_{11}} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11}(a_{11} - 2) \neq 0$

### 2.3 Nijenhuis operator

**Theorem 5.** *There are three types of Nijenhuis operators on the 3-dimensional associative algebra  $A_1$ , which are as follows:*

$$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$$

where  $a_{21} \neq 0$

where  $a_{22} \neq a_{11}$

*Proof.* Since  $P$  is a linear operator, we only need to consider the base elements which are satisfying in the equation

$$P(x)P(y) = P(xP(y) + P(x)y - P(xy))$$

and we obtain the equations:

$$\begin{aligned} a_{31} &= a_{32} = 0, \\ a_{12}(a_{33} - a_{11}) &= 0, \\ a_{21}(a_{33} - a_{22}) &= 0, \\ a_{12}a_{21} + (a_{11} - a_{33})(a_{22} - a_{33}) &= 0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} P(e_3)P(e_2) &= P(e_3P(e_2) + P(e_3)e_2 - P(e_3e_2)) \Rightarrow a_{31} = 0 \\ P(e_3)P(e_1) &= P(e_3P(e_1) + P(e_3)e_1 - P(e_3e_1)) \Rightarrow a_{32} = 0 \\ P(e_1)P(e_1) &= P(e_1P(e_1) + P(e_1)e_1 - P(e_1e_1)) \Rightarrow a_{12}(a_{33} - a_{11}) = 0 \\ P(e_2)P(e_2) &= P(e_2P(e_2) + P(e_2)e_2 - P(e_2e_2)) \Rightarrow a_{21}(a_{33} - a_{22}) = 0 \\ P(e_1)P(e_2) &= P(e_1P(e_2) + P(e_1)e_2 - P(e_1e_2)) \Rightarrow a_{12}a_{21} + (a_{11} - a_{33})(a_{22} - a_{33}) = 0. \end{aligned}$$

We have the following cases:

**Case 1.** If  $a_{33} = a_{11}$ , then (6) becomes

$$\begin{aligned} a_{21}(a_{33} - a_{22}) &= 0, \\ a_{12}a_{21} &= 0 \end{aligned}$$

- If  $a_{21} = 0$ , then we have the operator  $P_1$
- If  $a_{21} \neq 0$ , then we obtain the operator  $P_2$ .

**Case 2.** If  $a_{33} \neq a_{11}$ , we obtain  $P_3$ . □

Now we present the list of Nijenhuis operators on algebras  $A_2, A_3, A_4^\alpha$  and  $A_5$ .

Algebra	Nijenhuis Operators	Restrictions
$A_2$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	
$A_3$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$ $P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{(a_{11}-a_{33})(a_{22}-a_{33})}{a_{12}} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{22} \neq a_{11}$  $a_{12} \neq 0$

Algebra	Nijenhuis Operators	Restrictions
$A_4^0$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{33} - a_{11} & a_{13} \\ a_{33} - a_{22} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
$A_4^\alpha, \alpha \neq 0$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12} = \frac{(a_{33} - a_{11})(1 \pm \sqrt{1 - 4\alpha})}{2\alpha}$ $a_{21} = \frac{(a_{33} - a_{22})(1 \mp \sqrt{1 - 4\alpha})}{2}$
$A_5$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & 2a_{11} - a_{22} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{23}a_{32} = -(a_{11} - a_{22})^2$ $a_{33} \neq a_{11}$

### 2.4 Average operator

**Theorem 6.** *The average operators on the 3-dimensional associative algebra  $A_1$  are the following:*

$$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{22} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$$

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & 2a_{11} \end{pmatrix}$$

where  $a_{12} \neq 0$ .

*Proof.* Since  $P$  is a linear operator, we only need to consider the base elements which are satisfying in the equation

$$P(x)P(y) = P(xP(y))$$

and also we have the equations:

$$\begin{aligned} a_{31} &= a_{32} = 0, \\ a_{12}(a_{33} - 2a_{11}) &= 0, \\ a_{21}(a_{33} - 2a_{22}) &= 0, \\ a_{12}a_{21} + a_{22}(a_{11} - a_{33}) &= 0, \\ a_{12}a_{21} + a_{11}(a_{22} - a_{33}) &= 0. \end{aligned} \tag{7}$$

The solutions of the system of equations (7) give us all average operators on  $A_1$  algebras. □

Finally, we give the list of average operators on algebras  $A_2, A_3, A_4^\alpha$  and  $A_5$ .

Algebra	average Operators	Restrictions
$A_2$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$ $P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{11} \end{pmatrix}$	$a_{33} \neq 0$  $a_{11} \neq 0$
$A_3$	$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$ $P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	$a_{33} \neq 0$  $a_{11} \neq 0$  $a_{12}a_{21} = a_{11}a_{22}$
$A_4^0$	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$ $P_2 = \begin{pmatrix} a_{11} & -a_{11} & a_{13} \\ -a_{22} & a_{22} & a_{23} \\ -a_{32} & a_{32} & a_{33} \end{pmatrix}$	

Algebra	average Operators	Restrictions
$A_4^\alpha, \alpha \neq 0$	$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ -a_{22} & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$	$a_{22} \neq 0$
	$P_3 = \begin{pmatrix} -\frac{a_{12}(1 \pm \sqrt{1-4\alpha})}{2} & a_{12} & a_{13} \\ -\frac{a_{22}(1 \pm \sqrt{1-4\alpha})}{2} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	
	$P_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\alpha a_{11} a_{12}}{a_{11} + a_{12}} & \frac{\alpha a_{12}^2}{a_{11} + a_{22}} & a_{23} \\ 0 & 0 & \frac{a_{11}^2 + a_{11} a_{12} + \alpha a_{12}^2}{a_{11} + a_{12}} \end{pmatrix}$	$a_{12} \neq -a_{11}$ $a_{11}^2 + a_{11} a_{12} + \alpha a_{12}^2 \neq 0$
	$P_5 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	$a_{11} \neq 0$
$A_5$	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11} \neq 0$

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