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► **To cite this version:**

Vsevolod Gubarev. Rota-Baxter operators and Bernoulli polynomials. Communications in Mathematics, 2021, Volume 29 (2021), Issue 1 (Special Issue: Ostrava Mathematical Seminar) (1), pp.1 - 14. 10.2478/cm-2021-0001 . hal-03665009

HAL Id: hal-03665009

<https://hal.science/hal-03665009>

Submitted on 11 May 2022

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Rota-Baxter operators and Bernoulli polynomials

Vsevolod Gubarev

Abstract. We develop the connection between Rota-Baxter operators arisen from algebra and mathematical physics and Bernoulli polynomials. We state that a trivial property of Rota-Baxter operators implies the symmetry of the power sum polynomials and Bernoulli polynomials. We show how Rota-Baxter operators equalities rewritten in terms of Bernoulli polynomials generate identities for the latter.

1 Introduction

Given an algebra A and a scalar $\lambda \in F$, where F is a ground field, a linear operator $R: A \rightarrow A$ is called a Rota-Baxter operator (RB-operator) on A of weight λ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \quad (1)$$

holds for all $x, y \in A$. The algebra A is called Rota-Baxter algebra. By algebra we mean a vector space endowed a bilinear not necessarily associative product.

The notion of Rota-Baxter operator was introduced by G. Baxter [6] in 1960 as formal generalization of integration by parts formula (when $\lambda = 0$) and then developed by G.-C. Rota [30] and others [5], [9].

In 1980s, the deep connection between constant solutions of the classical Yang-Baxter equation from mathematical physics and Rota-Baxter operators of weight zero on a semisimple finite-dimensional Lie algebra was discovered [7], [31]. Further, the connection of Rota-Baxter operators with the associative Yang-Baxter equation was found [4], [12], [28].

To the moment, applications of Rota-Baxter operators in symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra etc. were found [4], [5], [10], [11], [17], [18], [19]. The notion of Rota-Baxter operator is useful in such branch of number theory as multiple zeta function [13], [35].

2020 MSC: 11B68, 16W99

Key words: Rota-Baxter operator, Bernoulli number, Bernoulli polynomial

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In 1966, J. Miller found an interesting connection between Rota-Baxter operators and the power sum polynomials [25] over a field of characteristic zero. We start with an algebra A which is unital and power-associative (it means that every one-generated subalgebra is associative). Let R be a Rota-Baxter operator on A of weight -1 . Denote by 1 the unit of A and put $a = R(1)$. For each $n \in \mathbb{N}$, define a polynomial $F_n(x) \in \mathbb{Q}[x]$ by the equalities

$$F_n(m) = \sum_{j=1}^m j^n.$$

Then $R(a^n) = F_n(a)$.

In 2010, O. Ogievetsky and V. Schechtman restated this connection to find a new proof of the Schlömilch-Ramanujan formula [29]. In 2017, the author reproved the connection formula to apply for Rota-Baxter operators of nonzero weight on the matrix algebra [16].

Our goal is to develop this connection. There exist several different proofs of the symmetry of the power sum polynomials

$$F_n(y) = (-1)^{n+1} F_n(-1 - y)$$

and the symmetry of Bernoulli polynomials

$$B_n(x) = (-1)^n B_n(1 - x)$$

involving infinite series, generating functions or some special identities [22], [26], [33]. In section 2, we prove that both symmetries follow from the trivial property of Rota-Baxter operators: let P be an RB-operator of weight -1 , then the operator $(\text{id} - P)$ is so.

In section 3, we show how identities concerned Rota-Baxter operators rewritten in terms of Bernoulli polynomials and Bernoulli numbers generate a plenty of identities for both of them. In particular, we find a symmetric expression for the product $B_i(x)B_j(x)B_k(x)$ and count the sum

$$\sum_{\substack{i+j+k=n \\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x),$$

where $\mathcal{B}_s(x) = B_s(x)/s$ is the divided Bernoulli polynomial. The approach for counting the same sum for usual (not divided) Bernoulli polynomials was developed in [20]. About the products of Bernoulli polynomials and Bernoulli numbers see also [3], [8], [14].

2 Symmetry of the power sum polynomials

Statement 1 ([18]). *Let P be an RB-operator of weight λ . Then*

- a) *the operator $-P - \lambda \text{id}$ is an RB-operator of weight λ ,*
- b) *the operator $\lambda^{-1}P$ is an RB-operator of weight 1 , provided $\lambda \neq 0$.*

Given an algebra A , let us define a map ϕ on the set of all RB-operators on A as $\phi(P) = -P - \lambda(P) \text{id}$. It is clear that ϕ^2 coincides with the identity map.

Let $F_n(m) = \sum_{j=1}^m j^n$ for natural n, m . Bernoulli polynomials $B_n(x)$ are connected with the power sum polynomials in the following way:

$$F_n(m) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}. \quad (2)$$

Statement 2 ([16], [25], [29]). *Let A be a unital power-associative algebra, R be an RB-operator on A of weight λ , $a = R(1)$. Then $R(a^n) = (-\lambda)^{n+1} F_n(-a/\lambda)$ for all $n \in \mathbb{N}$. In particular, $R(a^n) = F_n(a)$ for all $n \in \mathbb{N}$ provided that $\lambda = -1$.*

Let us show how the trivial property of Rota-Baxter operators from Statement 1 a) implies the symmetry of the power sum polynomials and the symmetry of Bernoulli polynomials.

Lemma 1. *Let A be a unital power-associative algebra, R be an RB-operator on A of weight -1 , $a = R(1)$ and $b = \phi(R)(1) = 1 - a$. For all positive natural n , we have*

$$R(a^n) - a^n = (-1)^{n+1}(\phi(R)(b^n) - b^n). \quad (3)$$

Proof. From

$$(-1)^{n+1}(\phi(R)(b^n) - b^n) = (-1)^{n+1}(-R)(b^n) = R((-b)^n) = R((a-1)^n),$$

we conclude that it is enough to state $R((a-1)^n) = R(a^n) - a^n$. We prove the last equality by induction on n . For $n = 1$, we get the true equality $\frac{a^2-a}{2} = \frac{a^2-a}{2}$. Suppose that this holds true for all natural numbers less than n . Now we rewrite $R((a-1)^{n+1})$ by (1) and the induction hypothesis

$$\begin{aligned} R((a-1)^{n+1}) &= R((a-1)^n(a-1)) = R((a-1)^n R(1)) - R((a-1)^n) \\ &= R((a-1)^n) R(1) - R(R((a-1)^n)) + R((a-1)^n) - R((a-1)^n) \\ &= (R(a^n) - a^n)a - R(R(a^n) - a^n). \end{aligned} \quad (4)$$

Again by (1), we calculate

$$\begin{aligned} R(R(a^n)) &= R(R(a^n) \cdot 1) = R(a^n)R(1) - R(a^n R(1)) + R(a^n \cdot 1) \\ &= R(a^n)a - R(a^{n+1}) + R(a^n). \end{aligned} \quad (5)$$

Substituting (5) in (4) gives us the proof of the inductive step. \square

Theorem 1. *Let n be a positive natural number. Then*

- a) $F_n(y) = (-1)^{n+1} F_n(-1-y)$ for all y ,
- b) $B_n(x) = (-1)^n B_n(1-x)$ for all x .

Proof. a) Let us consider a unital power-associative algebra A with a Rota-Baxter operator R on A of weight $\lambda = -1$. Put $a = R(1)$. We may consider the free (unital) associative RB-algebra of weight -1 generated by 1 [18] instead of A . Actually it is the polynomial algebra $F[x]$ with $x = a$. Define $Q = \phi(R) = \text{id} - R$ and $b = Q(1) = 1 - a$. Applying Statement 2 to the formula (3), we get

$$F_n(a - 1) = (-1)^{n+1} F_n(-a),$$

which gives a).

b) It follows from a) via (2). \square

3 Product of two Bernoulli polynomials

For any n ,

$$F_n(m) = \frac{1}{n+1} \sum_{j=0}^n (-1)^j \binom{n+1}{j} B_j m^{n+1-j}, \quad (6)$$

where B_j is Bernoulli number.

Let us show how Rota-Baxter operators could generate a plenty of identities for Bernoulli numbers and Bernoulli polynomials. Let A be a power-associative algebra and R be a Rota-Baxter operator of weight -1 on A , $a = R(1)$. Consider the equality

$$R(a^n)R(a^m) = R(R(a^n)a^m + a^n R(a^m) - a^{n+m}), \quad n, m \in \mathbb{N}. \quad (7)$$

The left-hand side of (7) by (2) and Statement 2 is equal to

$$\begin{aligned} R(a^n)R(a^m) &= \mathcal{B}_{n+1}(a+1)\mathcal{B}_{m+1}(a+1) - \mathcal{B}_{m+1}\mathcal{B}_{n+1}(a+1) \\ &\quad - \mathcal{B}_{n+1}\mathcal{B}_{m+1}(a+1) + \mathcal{B}_{n+1}\mathcal{B}_{m+1}, \end{aligned} \quad (8)$$

where $\mathcal{B}_n(x) = B_n(x)/n$ and $\mathcal{B}_n = B_n/n$.

Let us write down the right-hand side of (7) by (2), (6), and Statement 2,

$$\begin{aligned} &R(R(a^n)a^m + a^n R(a^m) - a^{n+m}) \\ &= \sum_{i=0}^n (-1)^{n-i} \frac{1}{n+1} \binom{n+1}{n-i} B_{n-i} (\mathcal{B}_{m+2+i}(a+1) - \mathcal{B}_{m+2+i}) \\ &\quad + \sum_{j=0}^m (-1)^{m-j} \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j} (\mathcal{B}_{n+2+j}(a+1) - \mathcal{B}_{n+2+j}) \\ &\quad - \mathcal{B}_{n+m+1}(a+1) + \mathcal{B}_{n+m+1}. \end{aligned} \quad (9)$$

Comparing (8) and (9), we get the identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) - \mathcal{B}_i\mathcal{B}_j = \sum_{l \geq 0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} (\mathcal{B}_{i+j-2l}(x) - \mathcal{B}_{i+j-2l}). \quad (10)$$

Here $i = n+1 \geq 1$, $j = m+1 \geq 1$ and $x = a+1$.

Up to constant, the equality (10) coincides with the famous identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) = \sum_{l \geq 0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} \mathcal{B}_{i+j-2l}(x) + \frac{(-1)^{i-1}(i-1)!(j-1)!}{(i+j)!} B_{i+j} \quad (11)$$

known at least since 1923 [27].

Remark 1. Writing down (7) on the first power of a , we get the identity

$$B_{n+m} + \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{n-k} B_{n-k} B_{m+k+1} + \frac{1}{m+1} \sum_{l=0}^m \binom{m+1}{m-l} B_{m-l} B_{n+l+1} = 0 \quad (12)$$

discovered by T. Agoh in 1988 [1].

Remark 2. Let us sum (11) for $i+j = N \geq 2$ and $i = 1, \dots, N-1$:

$$\begin{aligned} \sum_{\substack{i+j=N \\ i,j>0}} \mathcal{B}_i(x)\mathcal{B}_j(x) &= \sum_{i=1}^{N-1} \left(\frac{1}{i} + \frac{1}{N-i} \right) \mathcal{B}_N(x) \\ &+ \sum_{\substack{i+j=N \\ i,j>0}} \sum_{l>0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} \mathcal{B}_{N-2l}(x) \\ &+ \frac{B_N}{N(N-1)} \sum_{i=1}^{N-1} \frac{(-1)^{i-1}(i-1)!(N-1-i)!}{(N-2)!} \\ &= 2H_{N-1} \mathcal{B}_N(x) + 2 \sum_{l=1}^{\lfloor \frac{N-1}{2} \rfloor} B_{2l} \mathcal{B}_{N-2l}(x) \frac{1}{2l} \sum_{i=1}^{N-1} \binom{i-1}{2l-1} \\ &+ \frac{B_N}{N(N-1)} \sum_{p=0}^{N-2} \frac{(-1)^p}{\binom{N-2}{p}} \\ &= 2H_{N-1} \mathcal{B}_N(x) + \frac{2}{N} \sum_{l=1}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N}{2l} \mathcal{B}_{2l} B_{N-2l}(x) + \frac{2\mathcal{B}_N}{N} \\ &= 2H_{N-1} \mathcal{B}_N(x) + \frac{2}{N} \sum_{k=1}^N \binom{N}{k} \mathcal{B}_k B_{N-k}(x) \\ &+ B_{N-1}(x), \end{aligned} \quad (13)$$

where $H_i = 1 + 1/2 + \dots + 1/i$. We have used the equality (14) from [32]

$$\sum_{r=0}^n (-1)^r / \binom{n}{r} = (1 + (-1)^n) \frac{n+1}{n+2}. \quad (14)$$

Thus, we got in (13) the identity found by I. Gessel in 2005 [14] (see also [34]),

$$\begin{aligned} \frac{N}{2} \left(-B_{N-1}(x) + \sum_{k=1}^{N-1} \mathcal{B}_k(x) \mathcal{B}_{N-k}(x) \right) \\ = \sum_{k=1}^N \binom{N}{k} \mathcal{B}_k \mathcal{B}_{N-k}(x) + H_{N-1} B_N(x). \end{aligned} \quad (15)$$

By the same strategy, we can compute

$$\sum_{k=0}^N B_k(x) B_{N-k}(x) = \frac{2}{N+2} \sum_{t \geq 0} \binom{N+2}{2t+2} B_{2t} B_{N-2t}(x), \quad (16)$$

which is the identity obtained by D. Kim et al. in 2012 [21] (see also [2]).

The case $x = 0$ for (15) implies the famous identity of H. Miki [24] (1978)

$$\sum_{k=2}^{N-2} \mathcal{B}_k \mathcal{B}_{N-k} = \sum_{k=2}^{N-2} \binom{N}{k} \mathcal{B}_k \mathcal{B}_{N-k} + 2H_N \mathcal{B}_N \quad (17)$$

and for (16) it implies the identity of Yu. Matiyasevich [23] (1997)

$$(N+2) \sum_{k=2}^{N-2} B_k B_{N-k} = 2 \sum_{k=2}^{N-2} \binom{N+2}{k} B_k B_{N-k} + N(N+1) B_N. \quad (18)$$

4 Product of three Bernoulli polynomials

We may also produce other identities involving the products of three, four etc. Bernoulli numbers. To do this, it is enough to consider the equality

$$\begin{aligned} R(a^n) R(a^m) R(a^l) &= R(R(a^n) R(a^l) a^m + R(a^m) R(a^l) a^n + R(a^n) R(a^m) a^l \\ &\quad - R(a^n) a^{m+l} - R(a^m) a^{n+l} - R(a^l) a^{n+m} + a^{n+m+l}) \end{aligned} \quad (19)$$

and the same equalities for four, five etc. multipliers (see the formulas in [17]).

Let us derive the explicit identity which follows from (19).

Theorem 2. *The following identity holds for all $i, j, k > 0$,*

$$\begin{aligned} \mathcal{B}_i(x) \mathcal{B}_j(x) \mathcal{B}_k(x) &= \sum_{q, t \geq 0} B_{2q} B_{2t-2q} \left[\binom{i+j-2q}{2t-2q} \frac{1}{i+j-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} \right) \right. \\ &\quad + \binom{i+k-2q}{2t-2q} \frac{1}{i+k-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{k} \binom{k}{2q} \right) \\ &\quad \left. + \binom{j+k-2q}{2t-2q} \frac{1}{j+k-2q} \left(\frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right] \\ &\quad \times \mathcal{B}_{i+j+k-2t}(x) \\ &\quad - \frac{(-1)^j}{ij \binom{i+j}{i}} B_{i+j} \mathcal{B}_k(x) - \frac{(-1)^k}{ik \binom{i+k}{i}} B_{i+k} \mathcal{B}_j(x) \\ &\quad - \frac{(-1)^k}{jk \binom{j+k}{j}} B_{j+k} \mathcal{B}_i(x) - \frac{1}{2} \mathcal{B}_{i+j+k-2}(x) + \text{const.} \end{aligned} \quad (20)$$

Proof. Let $i = n + 1 \geq 2$, $j = m + 1 \geq 2$, $k = l + 1 \geq 2$ and $x = a + 1$. We calculate the left-hand side of (19) as

$$\begin{aligned}
 (\mathcal{B}_i(x) - \mathcal{B}_i)(\mathcal{B}_j(x) - \mathcal{B}_j)(\mathcal{B}_k(x) - \mathcal{B}_k) &= \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) - (\mathcal{B}_i\mathcal{B}_j(x)\mathcal{B}_k(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_i(x)\mathcal{B}_k(x) + \mathcal{B}_k\mathcal{B}_i(x)\mathcal{B}_j(x)) \\
 &\quad + (\mathcal{B}_i\mathcal{B}_j\mathcal{B}_k(x) + \mathcal{B}_i\mathcal{B}_k\mathcal{B}_j(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_k\mathcal{B}_i(x)) - \mathcal{B}_i\mathcal{B}_j\mathcal{B}_k. \tag{21}
 \end{aligned}$$

The last term on the right-hand side of (19) equals

$$\mathcal{B}_{i+j+k-2}(x) - \mathcal{B}_{i+j+k-2}. \tag{22}$$

We also have

$$\begin{aligned}
 -R(R(a^n)a^{m+l} + R(a^m)a^{n+l} + R(a^l)a^{n+m}) \\
 &= -\sum_{q \geq 0} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) B_{2q}\mathcal{B}_{i+j+k-1-2q}(x) + \mathcal{B}_i\mathcal{B}_{j+k-1}(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_{i+k-1}(x) + \mathcal{B}_k\mathcal{B}_{i+j-1}(x) - \frac{3}{2}\mathcal{B}_{i+j+k-2}(x) + \text{const}. \tag{23}
 \end{aligned}$$

We write down

$$\begin{aligned}
 R(R(R(a^n)a^m)a^l) &= \frac{1}{n+1} \sum_{p=0}^n (-1)^{n-p} \binom{n+1}{n-p} B_{n-p} \frac{1}{m+p+2} \\
 &\times \sum_{s=0}^{p+m+1} (-1)^{m+p+1-s} \binom{m+p+2}{m+p+1-s} B_{m+p+1-s} (\mathcal{B}_{l+s+2}(x) - \mathcal{B}_{l+s+2}). \tag{24}
 \end{aligned}$$

We want to transform (24) to the form

$$\sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} \sum_{t \geq 0} \frac{1}{i+j-2q} \binom{i+j-2q}{2t} B_{2t}\mathcal{B}_{i+j+k-2t-2q}(x) + \text{const}. \tag{25}$$

By exchange $n - p = 2q$ in (24), we get the summand

$$\frac{1}{2} R(R(a^{n+m})a^l). \tag{26}$$

Further, by exchange $m + n - 2q + 1 - s = 2t$, we have the additional summands

$$\begin{aligned}
 \frac{1}{2} \sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} (\mathcal{B}_{i+j+k-1-2q}(x) - \mathcal{B}_{i+j+k-1-2q}) \\
 - \frac{1}{2} \mathcal{B}_i (\mathcal{B}_{j+k-1}(x) - \mathcal{B}_{j+k-1}). \tag{27}
 \end{aligned}$$

If we let $2q$ be equal i , up to a constant we get the summand

$$\begin{aligned} -\mathcal{B}_i \sum_{t=0}^{[(j-1)/2]} \frac{1}{j} \binom{j}{2t} B_{2t} \mathcal{B}_{j+k-2t}(x) \\ = -\mathcal{B}_i \sum_{t \geq 0} \frac{1}{j} \binom{j}{2t} B_{2t} \mathcal{B}_{j+k-2t}(x) + \mathcal{B}_i \mathcal{B}_j \mathcal{B}_k(x). \end{aligned} \quad (28)$$

Finally, letting $2t$ be equal j , we get (25) and the additional summand

$$(\mathcal{B}_k - \mathcal{B}_k(x)) \sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} \mathcal{B}_{i+j-2q}. \quad (29)$$

Applying the formula

$$R(R(a^n)R(a^m)a^l) = R(R(R(a^n)a^m)a^l) + R(R(R(a^m)a^n)a^l) - R(R(a^{n+m})a^l),$$

summing all such six expressions, by the formulas (21)–(29) we prove the statement. We have rewritten the sum of (29) and the analogue of (29) for j by (11), the sum equals

$$(\mathcal{B}_k - \mathcal{B}_k(x)) \left(\mathcal{B}_i \mathcal{B}_j + \frac{(-1)^i (i-1)! (j-1)!}{(i+j)!} B_{i+j} \right).$$

Theorem is proved. \square

Corollary 1. For all $i, j, k > 0$, we have

$$\begin{aligned} \int_0^x \mathcal{B}_i(y) \mathcal{B}_j(y) \mathcal{B}_k(y) dy \\ = \sum_{q, t \geq 0} B_{2q} B_{2t-2q} \left[\binom{i+j-2q}{2t-2q} \frac{1}{i+j-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} \right) \right. \\ \left. + \binom{i+k-2q}{2t-2q} \frac{1}{i+k-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right. \\ \left. + \binom{j+k-2q}{2t-2q} \frac{1}{j+k-2q} \left(\frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right] \frac{\mathcal{B}_{i+j+k+1-2t}(x)}{i+j+k-2t} \\ - \frac{1}{ijk} \left(\frac{(-1)^j}{\binom{i+j}{i}} B_{i+j} \mathcal{B}_{k+1}(x) + \frac{(-1)^k}{\binom{i+k}{i}} B_{i+k} \mathcal{B}_{j+1}(x) + \frac{(-1)^k}{\binom{j+k}{j}} B_{j+k} \mathcal{B}_{i+1}(x) \right) \\ - \frac{\mathcal{B}_{i+j+k-1}(x)}{2(i+j+k-2)} + \text{const}. \end{aligned} \quad (30)$$

Proof. It follows from (20) and the formula $\int_0^x B_n(y) dy = \mathcal{B}_{n+1}(x) - \mathcal{B}_{n+1}$. \square

Let us denote by $H_{n,s} = 1 + 1/2^s + \dots + 1/n^s$. So, $H_n = H_{n,1}$.

Corollary 2. For all $N \geq 3$, we have

$$\begin{aligned}
& \frac{1}{3!} \sum_{\substack{i+j+k=N \\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) \\
&= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(\mathcal{B}_{2t}(H_{N-1} - H_{2t}) + \frac{1}{2!} \sum_{\substack{i+j=2t; \\ i,j>0}} \mathcal{B}_i\mathcal{B}_j \right) \\
&\quad - \frac{1}{12} \binom{N-1}{2} \mathcal{B}_{N-2}(x) + \frac{H_n^2 - H_{n,2}}{2} \mathcal{B}_N(x) + \text{const.} \quad (31)
\end{aligned}$$

Proof. We apply the formula (20) for all $i + j + k + N \geq 3$ and $i, j, k > 0$. Let us compute the summation of the right-hand side of (20) divided by 6 for $q > 0$:

$$\begin{aligned}
A &= \sum_{\substack{t \geq 0 \\ q > 0}} B_{2q} B_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2q} \sum_{j=1}^{N-i-1} \frac{1}{i+j-2q} \binom{i+j-2q}{2t-2q} \mathcal{B}_{N-2t}(x) \\
&= \sum_{t>q>0} B_{2q} \mathcal{B}_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2q} \left(\binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q} \right) \mathcal{B}_{N-2t}(x) \\
&\quad + \sum_{t>0} B_{2t} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2t} (H_{N-1-2q} - H_{i-2t}) \mathcal{B}_{N-2t}(x) \\
&= \sum_{t>q>0} B_{2q} \mathcal{B}_{2t-2q} \mathcal{B}_{N-2t}(x) \left(\binom{N-1-2q}{2t-2q} \binom{N-2}{2q} - \binom{2t-1}{2q-1} \binom{N-2}{2t} \right) \\
&\quad + \sum_{t>0} \mathcal{B}_{2t} \left[\binom{N-2}{2t} H_{N-1-2t} - \binom{N-2}{2t} H_{N-2t-2} \right. \\
&\quad \quad \left. + \frac{1}{2t} \binom{N-2}{2t} - \frac{1}{2t} \right] \mathcal{B}_{N-2t}(x) \\
&= \sum_{t>q>0} \binom{N-1}{2t} \binom{2t-1}{2q} B_{2q} \mathcal{B}_{2t-2q} \mathcal{B}_{N-2t}(x) \\
&\quad + \sum_{t>0} \frac{1}{2t} \binom{N-1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) \\
&= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(\frac{\mathcal{B}_{2t}}{2t} + \sum_{t>q>0} \binom{2t-1}{2q} B_{2q} \mathcal{B}_{2t-2q} \right) \\
&\quad - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) \\
&= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \sum_{q>0} \frac{1}{2t} \binom{2t}{2q} B_{2q} \mathcal{B}_{2t-2q} - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x). \quad (32)
\end{aligned}$$

By the Miki identity (17), we may rewrite (32) as

$$A = \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(-H_{2t-1} \mathcal{B}_{2t} + \frac{1}{2!} \sum_{\substack{i+j=2t, \\ i,j>0}} \mathcal{B}_i \mathcal{B}_j \right) - \sum_{q>0} \frac{1}{2q} \mathcal{B}_{2q} \mathcal{B}_{N-2q}(x). \quad (33)$$

Above we have used the equalities

$$\begin{aligned} \sum_{j=1}^{N-i-1} \frac{1}{i+j-2q} \binom{i+j-2q}{2t-2q} &= \frac{1}{2t-2q} \sum_{j=1}^{N-i-1} \binom{i+j-2q-1}{2t-2q-1} \\ &= \frac{1}{2t-2q} \sum_{s=i-2q}^{N-2q-2} \binom{s}{2t-2q-1} \\ &= \frac{1}{2t-2q} \left(\binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q} \right); \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{N-2} \binom{i-1}{2q-1} H_{i-2q} &= \sum_{j=1}^{N-2-2q} \frac{1}{j} \sum_{i=j+2q}^{N-2} \binom{i-1}{2q-1} \\ &= \sum_{j=1}^{N-2-2q} \frac{1}{j} \left(\binom{N-2}{2q} - \binom{j+2q-1}{2q} \right) \\ &= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \sum_{j=1}^{N-2-2q} \binom{j+2q-1}{2q-1} \\ &= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \left(\binom{N-2}{2q} - 1 \right). \end{aligned}$$

The summation of the right-hand side of (20) for $q = 0$ gives us

$$\begin{aligned} &3 \sum_{t \geq 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \sum_{\substack{i+j=N-k \\ i,j>0}} \binom{i+j}{2t} \frac{1}{i+j} \left(\frac{1}{i} + \frac{1}{j} \right) \mathcal{B}_{N-2t}(x) \\ &= 3 \sum_{t \geq 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \sum_{i=1}^{N-k-1} \frac{1}{N-k} \left(\frac{1}{i} + \frac{1}{N-k-i} \right) \mathcal{B}_{N-2t}(x) \\ &= 6 \sum_{t > 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \frac{H_{N-k-1}}{N-k} \mathcal{B}_{N-2t}(x) + 6 \mathcal{B}_N(x) \sum_{k=1}^{N-2} \frac{H_{N-k-1}}{N-k} \\ &= 6 \sum_{t > 0} \mathcal{B}_{2t} \binom{N-1}{2t} \left(H_{N-1} - \frac{1}{2t} \right) \mathcal{B}_{N-2t}(x) \\ &\quad + 3(H_{N-1}^2 - H_{N-1,2}) \mathcal{B}_N(x). \end{aligned} \quad (34)$$

Here we have used the formulas (see [15, p. 279–280])

$$\sum_{s=1}^{n-1} \binom{s}{m} H_s = \binom{n}{m+1} \left(H_n - \frac{1}{m+1} \right), \quad \sum_{s=1}^n \frac{H_s}{s} = \frac{1}{2} (H_n^2 + H_{n,2}).$$

The sum of the first three summands from the fourth line of (20) by (14) gives

$$\begin{aligned} -3 \sum_{k=1}^{N-2} B_{N-k} \mathcal{B}_k(x) \frac{1}{(N-k)(N-k-1)} \sum_{i=1}^{N-k-1} \frac{(-1)^i}{\binom{N-k-2}{i-1}} \\ = 3 \sum_{k=1}^{N-2} \frac{(1+(-1)^{N-k})}{(N-k)(N-k-1)} \frac{N-k-1}{N-k} B_{N-k} \mathcal{B}_k(x) \\ = 6 \sum_{0 < t < N/2} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x). \end{aligned} \quad (35)$$

Finally, for the last non-constant term of (20) we have

$$\sum_{i=1}^{N-2} (N-1-i) \mathcal{B}_{N-2}(x) = \binom{N-1}{2} \mathcal{B}_{N-2}(x). \quad (36)$$

The formulas (33)–(36) imply (31). \square

In [20], authors studied the expressions

$$S_{\geq 0}(r, n) = \sum_{i_1 + \dots + i_r = n} B_{i_1}(x) \dots B_{i_r}(x), \quad (37)$$

where the sum runs over all nonnegative integers i_1, \dots, i_r and $r \geq 1$. Moreover, in [20] it was shown that

$$\begin{aligned} S_{\geq 0}(r, n) = \sum_{k=1}^n B_k(x) \left(\frac{\binom{n+r}{k}}{n+r} \sum_{i_1 + \dots + i_r = n-k+1} (B_{i_1}(1) \dots B_{i_r}(1) \right. \\ \left. - B_{i_1} \dots B_{i_r}) \right) + \text{const}, \end{aligned} \quad (38)$$

and the formula for the last constant term was given. The simple idea lies behind such kind of decomposition: polynomials $1 = B_0(x), B_1(x), \dots, B_n(x)$ form a linear basis of the vector space $\text{Span}(1, x, x^2, \dots, x^n)$. Also, $(B_k(x))' = kB_{k-1}(x)$, the property which is not fulfilled for the polynomials $1, \mathcal{B}_1(x), \dots, \mathcal{B}_n(x)$, and so the approach from [20] could not be applied directly for calculating the analogues of $S_{\geq 0}(r, n)$ or $S_{> 0}(r, n)$ for the sum of products $\mathcal{B}_{i_1}(x) \dots \mathcal{B}_{i_r}(x)$.

Let us decompose $S_{\geq 0}(r, n) = \sum_{k=0}^n \alpha_k B_{n-k}(x)$. In [20], it was shown that $\alpha_1 = 0$. It is easy to see from (38) that all odd coefficients α_{2s+1} are zero.

Remark 3. The formula (38) applied for $r = 2$ up to a constant coincides with (16). Due to (38), we have

$$\begin{aligned}
S_{\geq 0}(3, n) &= \sum_{k=1}^n B_k(x) \left(\frac{\binom{n+3}{k}}{n+3} \sum_{a+b+c=n-k+1} B_a(1)B_b(1)B_c(1) - B_a B_b B_c \right) \\
&= -\frac{2}{n+3} \sum_{t \geq 0} \binom{n+3}{n-2t} \left(\sum_{a+b+c=2t+1} B_a B_b B_c \right) B_{n-2t}(x) \\
&= \binom{n+2}{2} B_n(x) + \frac{1}{4} \binom{n+2}{4} B_{n-2}(x) \\
&\quad + \frac{3}{n+3} \sum_{t \geq 2} \binom{n+3}{n-2t} B_{n-2t}(x) \sum_{q=0}^t B_{2q} B_{2t-2q}. \tag{39}
\end{aligned}$$

If we calculate the sum $S_{\geq 0}(3, n)$ due to the formula (20), we will get the same as (39) modulo the equality (16).

Acknowledgments

Author is supported by the Austrian Science Foundation FWF, grant P28079.

Author is grateful to Oleg Ogievetsky and Li Guo for pointing out the references [29] and [25] respectively.

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Received: 15 July 2019

Accepted for publication: 18 February 2020

Communicated by: Pasha Zusmanovich