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Rota-Baxter operators and Bernoulli polynomials

Vsevolod Gubarev

Abstract. We develop the connection between Rota-Baxter operators arisen from algebra and mathematical physics and Bernoulli polynomials. We state that a trivial property of Rota-Baxter operators implies the symmetry of the power sum polynomials and Bernoulli polynomials. We show how Rota-Baxter operators equalities rewritten in terms of Bernoulli polynomials generate identities for the latter.

1 Introduction

Given an algebra A and a scalar $\lambda \in F$, where F is a ground field, a linear operator $R: A \rightarrow A$ is called a Rota-Baxter operator (RB-operator) on A of weight λ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \quad (1)$$

holds for all $x, y \in A$. The algebra A is called Rota-Baxter algebra. By algebra we mean a vector space endowed a bilinear not necessarily associative product.

The notion of Rota-Baxter operator was introduced by G. Baxter [6] in 1960 as formal generalization of integration by parts formula (when $\lambda = 0$) and then developed by G.-C. Rota [30] and others [5], [9].

In 1980s, the deep connection between constant solutions of the classical Yang-Baxter equation from mathematical physics and Rota-Baxter operators of weight zero on a semisimple finite-dimensional Lie algebra was discovered [7], [31]. Further, the connection of Rota-Baxter operators with the associative Yang-Baxter equation was found [4], [12], [28].

To the moment, applications of Rota-Baxter operators in symmetric polynomials, quantum field renormalization, Loday algebras, shuffle algebra etc. were found [4], [5], [10], [11], [17], [18], [19]. The notion of Rota-Baxter operator is useful in such branch of number theory as multiple zeta function [13], [35].

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Affiliation:

University of Vienna, Oskar-Morgenstern-Platz 1, 1090, Vienna, Austria & Sobolev
 Institute of Mathematics, Acad. Koptyug ave. 4, 630090 Novosibirsk, Russia
 E-mail: wsevolod89@gmail.com

In 1966, J. Miller found an interesting connection between Rota-Baxter operators and the power sum polynomials [25] over a field of characteristic zero. We start with an algebra A which is unital and power-associative (it means that every one-generated subalgebra is associative). Let R be a Rota-Baxter operator on A of weight -1 . Denote by 1 the unit of A and put $a = R(1)$. For each $n \in \mathbb{N}$, define a polynomial $F_n(x) \in \mathbb{Q}[x]$ by the equalities

$$F_n(m) = \sum_{j=1}^m j^n.$$

Then $R(a^n) = F_n(a)$.

In 2010, O. Ogievetsky and V. Schechtman restated this connection to find a new proof of the Schlömilch-Ramanujan formula [29]. In 2017, the author reproved the connection formula to apply for Rota-Baxter operators of nonzero weight on the matrix algebra [16].

Our goal is to develop this connection. There exist several different proofs of the symmetry of the power sum polynomials

$$F_n(y) = (-1)^{n+1} F_n(-1 - y)$$

and the symmetry of Bernoulli polynomials

$$B_n(x) = (-1)^n B_n(1 - x)$$

involving infinite series, generating functions or some special identities [22], [26], [33]. In section 2, we prove that both symmetries follow from the trivial property of Rota-Baxter operators: let P be an RB-operator of weight -1 , then the operator $(\text{id} - P)$ is so.

In section 3, we show how identities concerned Rota-Baxter operators rewritten in terms of Bernoulli polynomials and Bernoulli numbers generate a plenty of identities for both of them. In particular, we find a symmetric expression for the product $B_i(x)B_j(x)B_k(x)$ and count the sum

$$\sum_{\substack{i+j+k=n \\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x),$$

where $\mathcal{B}_s(x) = B_s(x)/s$ is the divided Bernoulli polynomial. The approach for counting the same sum for usual (not divided) Bernoulli polynomials was developed in [20]. About the products of Bernoulli polynomials and Bernoulli numbers see also [3], [8], [14].

2 Symmetry of the power sum polynomials

Statement 1 ([18]). *Let P be an RB-operator of weight λ . Then*

- a) *the operator $-P - \lambda \text{id}$ is an RB-operator of weight λ ,*
- b) *the operator $\lambda^{-1}P$ is an RB-operator of weight 1 , provided $\lambda \neq 0$.*

Given an algebra A , let us define a map ϕ on the set of all RB-operators on A as $\phi(P) = -P - \lambda(P)\text{id}$. It is clear that ϕ^2 coincides with the identity map.

Let $F_n(m) = \sum_{j=1}^m j^n$ for natural n, m . Bernoulli polynomials $B_n(x)$ are connected with the power sum polynomials in the following way:

$$F_n(m) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}. \quad (2)$$

Statement 2 ([16], [25], [29]). *Let A be a unital power-associative algebra, R be an RB-operator on A of weight λ , $a = R(1)$. Then $R(a^n) = (-\lambda)^{n+1}F_n(-a/\lambda)$ for all $n \in \mathbb{N}$. In particular, $R(a^n) = F_n(a)$ for all $n \in \mathbb{N}$ provided that $\lambda = -1$.*

Let us show how the trivial property of Rota-Baxter operators from Statement 1 a) implies the symmetry of the power sum polynomials and the symmetry of Bernoulli polynomials.

Lemma 1. *Let A be a unital power-associative algebra, R be an RB-operator on A of weight -1 , $a = R(1)$ and $b = \phi(R)(1) = 1 - a$. For all positive natural n , we have*

$$R(a^n) - a^n = (-1)^{n+1}(\phi(R)(b^n) - b^n). \quad (3)$$

Proof. From

$$(-1)^{n+1}(\phi(R)(b^n) - b^n) = (-1)^{n+1}(-R)(b^n) = R((-b)^n) = R((a-1)^n),$$

we conclude that it is enough to state $R((a-1)^n) = R(a^n) - a^n$. We prove the last equality by induction on n . For $n = 1$, we get the true equality $\frac{a^2-a}{2} = \frac{a^2-a}{2}$. Suppose that this holds true for all natural numbers less than n . Now we rewrite $R((a-1)^{n+1})$ by (1) and the induction hypothesis

$$\begin{aligned} R((a-1)^{n+1}) &= R((a-1)^n(a-1)) = R((a-1)^n R(1)) - R((a-1)^n) \\ &= R((a-1)^n)R(1) - R(R((a-1)^n)) + R((a-1)^n) - R((a-1)^n) \\ &= (R(a^n) - a^n)a - R(R(a^n) - a^n). \end{aligned} \quad (4)$$

Again by (1), we calculate

$$\begin{aligned} R(R(a^n)) &= R(R(a^n) \cdot 1) = R(a^n)R(1) - R(a^n R(1)) + R(a^n \cdot 1) \\ &= R(a^n)a - R(a^{n+1}) + R(a^n). \end{aligned} \quad (5)$$

Substituting (5) in (4) gives us the proof of the inductive step. \square

Theorem 1. *Let n be a positive natural number. Then*

- a) $F_n(y) = (-1)^{n+1}F_n(-1-y)$ for all y ,
- b) $B_n(x) = (-1)^n B_n(1-x)$ for all x .

Proof. a) Let us consider a unital power-associative algebra A with a Rota-Baxter operator R on A of weight $\lambda = -1$. Put $a = R(1)$. We may consider the free (unital) associative RB-algebra of weight -1 generated by 1 [18] instead of A . Actually it is the polynomial algebra $F[x]$ with $x = a$. Define $Q = \phi(R) = \text{id} - R$ and $b = Q(1) = 1 - a$. Applying Statement 2 to the formula (3), we get

$$F_n(a - 1) = (-1)^{n+1} F_n(-a),$$

which gives a).

b) It follows from a) via (2). \square

3 Product of two Bernoulli polynomials

For any n ,

$$F_n(m) = \frac{1}{n+1} \sum_{j=0}^n (-1)^j \binom{n+1}{j} B_j m^{n+1-j}, \quad (6)$$

where B_j is Bernoulli number.

Let us show how Rota-Baxter operators could generate a plenty of identities for Bernoulli numbers and Bernoulli polynomials. Let A be a power-associative algebra and R be a Rota-Baxter operator of weight -1 on A , $a = R(1)$. Consider the equality

$$R(a^n)R(a^m) = R(R(a^n)a^m + a^n R(a^m) - a^{n+m}), \quad n, m \in \mathbb{N}. \quad (7)$$

The left-hand side of (7) by (2) and Statement 2 is equal to

$$\begin{aligned} R(a^n)R(a^m) &= \mathcal{B}_{n+1}(a+1)\mathcal{B}_{m+1}(a+1) - \mathcal{B}_{m+1}\mathcal{B}_{n+1}(a+1) \\ &\quad - \mathcal{B}_{n+1}\mathcal{B}_{m+1}(a+1) + \mathcal{B}_{n+1}\mathcal{B}_{m+1}, \end{aligned} \quad (8)$$

where $\mathcal{B}_n(x) = B_n(x)/n$ and $\mathcal{B}_n = B_n/n$.

Let us write down the right-hand side of (7) by (2), (6), and Statement 2,

$$\begin{aligned} &R(R(a^n)a^m + a^n R(a^m) - a^{n+m}) \\ &= \sum_{i=0}^n (-1)^{n-i} \frac{1}{n+1} \binom{n+1}{n-i} B_{n-i} (\mathcal{B}_{m+2+i}(a+1) - \mathcal{B}_{m+2+i}) \\ &\quad + \sum_{j=0}^m (-1)^{m-j} \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j} (\mathcal{B}_{n+2+j}(a+1) - \mathcal{B}_{n+2+j}) \\ &\quad - \mathcal{B}_{n+m+1}(a+1) + \mathcal{B}_{n+m+1}. \end{aligned} \quad (9)$$

Comparing (8) and (9), we get the identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) - \mathcal{B}_i\mathcal{B}_j = \sum_{l \geq 0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} (\mathcal{B}_{i+j-2l}(x) - \mathcal{B}_{i+j-2l}). \quad (10)$$

Here $i = n+1 \geq 1$, $j = m+1 \geq 1$ and $x = a+1$.

Up to constant, the equality (10) coincides with the famous identity

$$\mathcal{B}_i(x)\mathcal{B}_j(x) = \sum_{l \geq 0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} \mathcal{B}_{i+j-2l}(x) + \frac{(-1)^{i-1}(i-1)!(j-1)!}{(i+j)!} B_{i+j} \quad (11)$$

known at least since 1923 [27].

Remark 1. Writing down (7) on the first power of a , we get the identity

$$B_{n+m} + \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{n-k} B_{n-k} B_{m+k+1} + \frac{1}{m+1} \sum_{l=0}^m \binom{m+1}{m-l} B_{m-l} B_{n+l+1} = 0 \quad (12)$$

discovered by T. Agoh in 1988 [1].

Remark 2. Let us sum (11) for $i+j = N \geq 2$ and $i = 1, \dots, N-1$:

$$\begin{aligned} \sum_{\substack{i+j=N \\ i,j>0}} \mathcal{B}_i(x)\mathcal{B}_j(x) &= \sum_{i=1}^{N-1} \left(\frac{1}{i} + \frac{1}{N-i} \right) \mathcal{B}_N(x) \\ &+ \sum_{\substack{i+j=N \\ i,j>0}} \sum_{l>0} \left(\frac{1}{i} \binom{i}{2l} + \frac{1}{j} \binom{j}{2l} \right) B_{2l} \mathcal{B}_{N-2l}(x) \\ &+ \frac{B_N}{N(N-1)} \sum_{i=1}^{N-1} \frac{(-1)^{i-1}(i-1)!(N-1-i)!}{(N-2)!} \\ &= 2H_{N-1} \mathcal{B}_N(x) + 2 \sum_{l=1}^{\lfloor \frac{N-1}{2} \rfloor} B_{2l} \mathcal{B}_{N-2l}(x) \frac{1}{2l} \sum_{i=1}^{N-1} \binom{i-1}{2l-1} \\ &+ \frac{B_N}{N(N-1)} \sum_{p=0}^{N-2} \frac{(-1)^p}{\binom{N-2}{p}} \\ &= 2H_{N-1} \mathcal{B}_N(x) + \frac{2}{N} \sum_{l=1}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N}{2l} \mathcal{B}_{2l} B_{N-2l}(x) + \frac{2\mathcal{B}_N}{N} \\ &= 2H_{N-1} \mathcal{B}_N(x) + \frac{2}{N} \sum_{k=1}^N \binom{N}{k} \mathcal{B}_k B_{N-k}(x) \\ &+ B_{N-1}(x), \end{aligned} \quad (13)$$

where $H_i = 1 + 1/2 + \dots + 1/i$. We have used the equality (14) from [32]

$$\sum_{r=0}^n (-1)^r / \binom{n}{r} = (1 + (-1)^n) \frac{n+1}{n+2}. \quad (14)$$

Thus, we got in (13) the identity found by I. Gessel in 2005 [14] (see also [34]),

$$\begin{aligned} \frac{N}{2} \left(-B_{N-1}(x) + \sum_{k=1}^{N-1} \mathcal{B}_k(x) \mathcal{B}_{N-k}(x) \right) \\ = \sum_{k=1}^N \binom{N}{k} \mathcal{B}_k \mathcal{B}_{N-k}(x) + H_{N-1} B_N(x). \end{aligned} \quad (15)$$

By the same strategy, we can compute

$$\sum_{k=0}^N B_k(x) B_{N-k}(x) = \frac{2}{N+2} \sum_{t \geq 0} \binom{N+2}{2t+2} B_{2t} B_{N-2t}(x), \quad (16)$$

which is the identity obtained by D. Kim et al. in 2012 [21] (see also [2]).

The case $x = 0$ for (15) implies the famous identity of H. Miki [24] (1978)

$$\sum_{k=2}^{N-2} \mathcal{B}_k \mathcal{B}_{N-k} = \sum_{k=2}^{N-2} \binom{N}{k} \mathcal{B}_k \mathcal{B}_{N-k} + 2H_N \mathcal{B}_N \quad (17)$$

and for (16) it implies the identity of Yu. Matiyasevich [23] (1997)

$$(N+2) \sum_{k=2}^{N-2} B_k B_{N-k} = 2 \sum_{k=2}^{N-2} \binom{N+2}{k} B_k B_{N-k} + N(N+1) B_N. \quad (18)$$

4 Product of three Bernoulli polynomials

We may also produce other identities involving the products of three, four etc. Bernoulli numbers. To do this, it is enough to consider the equality

$$\begin{aligned} R(a^n) R(a^m) R(a^l) &= R(R(a^n) R(a^l) a^m + R(a^m) R(a^l) a^n + R(a^n) R(a^m) a^l \\ &\quad - R(a^n) a^{m+l} - R(a^m) a^{n+l} - R(a^l) a^{n+m} + a^{n+m+l}) \end{aligned} \quad (19)$$

and the same equalities for four, five etc. multipliers (see the formulas in [17]).

Let us derive the explicit identity which follows from (19).

Theorem 2. *The following identity holds for all $i, j, k > 0$,*

$$\begin{aligned} \mathcal{B}_i(x) \mathcal{B}_j(x) \mathcal{B}_k(x) &= \sum_{q, t \geq 0} B_{2q} B_{2t-2q} \left[\binom{i+j-2q}{2t-2q} \frac{1}{i+j-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} \right) \right. \\ &\quad + \binom{i+k-2q}{2t-2q} \frac{1}{i+k-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{k} \binom{k}{2q} \right) \\ &\quad \left. + \binom{j+k-2q}{2t-2q} \frac{1}{j+k-2q} \left(\frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right] \\ &\quad \times \mathcal{B}_{i+j+k-2t}(x) \\ &\quad - \frac{(-1)^j}{ij \binom{i+j}{i}} B_{i+j} \mathcal{B}_k(x) - \frac{(-1)^k}{ik \binom{i+k}{i}} B_{i+k} \mathcal{B}_j(x) \\ &\quad - \frac{(-1)^k}{jk \binom{j+k}{j}} B_{j+k} \mathcal{B}_i(x) - \frac{1}{2} \mathcal{B}_{i+j+k-2}(x) + \text{const.} \end{aligned} \quad (20)$$

Proof. Let $i = n + 1 \geq 2$, $j = m + 1 \geq 2$, $k = l + 1 \geq 2$ and $x = a + 1$. We calculate the left-hand side of (19) as

$$\begin{aligned}
 (\mathcal{B}_i(x) - \mathcal{B}_i)(\mathcal{B}_j(x) - \mathcal{B}_j)(\mathcal{B}_k(x) - \mathcal{B}_k) &= \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) - (\mathcal{B}_i\mathcal{B}_j(x)\mathcal{B}_k(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_i(x)\mathcal{B}_k(x) + \mathcal{B}_k\mathcal{B}_i(x)\mathcal{B}_j(x)) \\
 &\quad + (\mathcal{B}_i\mathcal{B}_j\mathcal{B}_k(x) + \mathcal{B}_i\mathcal{B}_k\mathcal{B}_j(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_k\mathcal{B}_i(x)) - \mathcal{B}_i\mathcal{B}_j\mathcal{B}_k. \tag{21}
 \end{aligned}$$

The last term on the right-hand side of (19) equals

$$\mathcal{B}_{i+j+k-2}(x) - \mathcal{B}_{i+j+k-2}. \tag{22}$$

We also have

$$\begin{aligned}
 -R(R(a^n)a^{m+l} + R(a^m)a^{n+l} + R(a^l)a^{n+m}) \\
 &= -\sum_{q \geq 0} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) B_{2q}\mathcal{B}_{i+j+k-1-2q}(x) + \mathcal{B}_i\mathcal{B}_{j+k-1}(x) \\
 &\quad + \mathcal{B}_j\mathcal{B}_{i+k-1}(x) + \mathcal{B}_k\mathcal{B}_{i+j-1}(x) - \frac{3}{2}\mathcal{B}_{i+j+k-2}(x) + \text{const}. \tag{23}
 \end{aligned}$$

We write down

$$\begin{aligned}
 R(R(R(a^n)a^m)a^l) &= \frac{1}{n+1} \sum_{p=0}^n (-1)^{n-p} \binom{n+1}{n-p} B_{n-p} \frac{1}{m+p+2} \\
 &\times \sum_{s=0}^{p+m+1} (-1)^{m+p+1-s} \binom{m+p+2}{m+p+1-s} B_{m+p+1-s} (\mathcal{B}_{l+s+2}(x) - \mathcal{B}_{l+s+2}). \tag{24}
 \end{aligned}$$

We want to transform (24) to the form

$$\sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} \sum_{t \geq 0} \frac{1}{i+j-2q} \binom{i+j-2q}{2t} B_{2t}\mathcal{B}_{i+j+k-2t-2q}(x) + \text{const}. \tag{25}$$

By exchange $n - p = 2q$ in (24), we get the summand

$$\frac{1}{2} R(R(a^{n+m})a^l). \tag{26}$$

Further, by exchange $m + n - 2q + 1 - s = 2t$, we have the additional summands

$$\begin{aligned}
 \frac{1}{2} \sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} (\mathcal{B}_{i+j+k-1-2q}(x) - \mathcal{B}_{i+j+k-1-2q}) \\
 - \frac{1}{2} \mathcal{B}_i (\mathcal{B}_{j+k-1}(x) - \mathcal{B}_{j+k-1}). \tag{27}
 \end{aligned}$$

If we let $2q$ be equal i , up to a constant we get the summand

$$\begin{aligned} -\mathcal{B}_i \sum_{t=0}^{[(j-1)/2]} \frac{1}{j} \binom{j}{2t} B_{2t} \mathcal{B}_{j+k-2t}(x) \\ = -\mathcal{B}_i \sum_{t \geq 0} \frac{1}{j} \binom{j}{2t} B_{2t} \mathcal{B}_{j+k-2t}(x) + \mathcal{B}_i \mathcal{B}_j \mathcal{B}_k(x). \end{aligned} \quad (28)$$

Finally, letting $2t$ be equal j , we get (25) and the additional summand

$$(\mathcal{B}_k - \mathcal{B}_k(x)) \sum_{q \geq 0} \frac{1}{i} \binom{i}{2q} B_{2q} \mathcal{B}_{i+j-2q}. \quad (29)$$

Applying the formula

$$R(R(a^n)R(a^m)a^l) = R(R(R(a^n)a^m)a^l) + R(R(R(a^m)a^n)a^l) - R(R(a^{n+m})a^l),$$

summing all such six expressions, by the formulas (21)–(29) we prove the statement. We have rewritten the sum of (29) and the analogue of (29) for j by (11), the sum equals

$$(\mathcal{B}_k - \mathcal{B}_k(x)) \left(\mathcal{B}_i \mathcal{B}_j + \frac{(-1)^i (i-1)! (j-1)!}{(i+j)!} B_{i+j} \right).$$

Theorem is proved. \square

Corollary 1. For all $i, j, k > 0$, we have

$$\begin{aligned} \int_0^x \mathcal{B}_i(y) \mathcal{B}_j(y) \mathcal{B}_k(y) dy \\ = \sum_{q, t \geq 0} B_{2q} B_{2t-2q} \left[\binom{i+j-2q}{2t-2q} \frac{1}{i+j-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{j} \binom{j}{2q} \right) \right. \\ + \binom{i+k-2q}{2t-2q} \frac{1}{i+k-2q} \left(\frac{1}{i} \binom{i}{2q} + \frac{1}{k} \binom{k}{2q} \right) \\ \left. + \binom{j+k-2q}{2t-2q} \frac{1}{j+k-2q} \left(\frac{1}{j} \binom{j}{2q} + \frac{1}{k} \binom{k}{2q} \right) \right] \frac{\mathcal{B}_{i+j+k+1-2t}(x)}{i+j+k-2t} \\ - \frac{1}{ijk} \left(\frac{(-1)^j}{\binom{i+j}{i}} B_{i+j} \mathcal{B}_{k+1}(x) + \frac{(-1)^k}{\binom{i+k}{i}} B_{i+k} \mathcal{B}_{j+1}(x) + \frac{(-1)^k}{\binom{j+k}{j}} B_{j+k} \mathcal{B}_{i+1}(x) \right) \\ - \frac{\mathcal{B}_{i+j+k-1}(x)}{2(i+j+k-2)} + \text{const}. \end{aligned} \quad (30)$$

Proof. It follows from (20) and the formula $\int_0^x B_n(y) dy = \mathcal{B}_{n+1}(x) - \mathcal{B}_{n+1}$. \square

Let us denote by $H_{n,s} = 1 + 1/2^s + \dots + 1/n^s$. So, $H_n = H_{n,1}$.

Corollary 2. For all $N \geq 3$, we have

$$\begin{aligned}
 & \frac{1}{3!} \sum_{\substack{i+j+k=N \\ i,j,k>0}} \mathcal{B}_i(x)\mathcal{B}_j(x)\mathcal{B}_k(x) \\
 &= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(\mathcal{B}_{2t}(H_{N-1} - H_{2t}) + \frac{1}{2!} \sum_{\substack{i+j=2t; \\ i,j>0}} \mathcal{B}_i\mathcal{B}_j \right) \\
 & \quad - \frac{1}{12} \binom{N-1}{2} \mathcal{B}_{N-2}(x) + \frac{H_n^2 - H_{n,2}}{2} \mathcal{B}_N(x) + \text{const.} \quad (31)
 \end{aligned}$$

Proof. We apply the formula (20) for all $i + j + k + N \geq 3$ and $i, j, k > 0$. Let us compute the summation of the right-hand side of (20) divided by 6 for $q > 0$:

$$\begin{aligned}
 A &= \sum_{\substack{t \geq 0 \\ q > 0}} B_{2q} B_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2q} \sum_{j=1}^{N-i-1} \frac{1}{i+j-2q} \binom{i+j-2q}{2t-2q} \mathcal{B}_{N-2t}(x) \\
 &= \sum_{t>q>0} B_{2q} \mathcal{B}_{2t-2q} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2q} \left(\binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q} \right) \mathcal{B}_{N-2t}(x) \\
 & \quad + \sum_{t>0} B_{2t} \sum_{i=1}^{N-2} \frac{1}{i} \binom{i}{2t} (H_{N-1-2q} - H_{i-2t}) \mathcal{B}_{N-2t}(x) \\
 &= \sum_{t>q>0} \mathcal{B}_{2q} \mathcal{B}_{2t-2q} \mathcal{B}_{N-2t}(x) \left(\binom{N-1-2q}{2t-2q} \binom{N-2}{2q} - \binom{2t-1}{2q-1} \binom{N-2}{2t} \right) \\
 & \quad + \sum_{t>0} \mathcal{B}_{2t} \left[\binom{N-2}{2t} H_{N-1-2t} - \binom{N-2}{2t} H_{N-2t-2} \right. \\
 & \quad \left. + \frac{1}{2t} \binom{N-2}{2t} - \frac{1}{2t} \right] \mathcal{B}_{N-2t}(x) \\
 &= \sum_{t>q>0} \binom{N-1}{2t} \binom{2t-1}{2q} \mathcal{B}_{2q} \mathcal{B}_{2t-2q} \mathcal{B}_{N-2t}(x) \\
 & \quad + \sum_{t>0} \frac{1}{2t} \binom{N-1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) \\
 &= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(\frac{\mathcal{B}_{2t}}{2t} + \sum_{t>q>0} \binom{2t-1}{2q} \mathcal{B}_{2q} \mathcal{B}_{2t-2q} \right) \\
 & \quad - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x) \\
 &= \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \sum_{q>0} \frac{1}{2t} \binom{2t}{2q} \mathcal{B}_{2q} \mathcal{B}_{2t-2q} - \sum_{t>0} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x). \quad (32)
 \end{aligned}$$

By the Miki identity (17), we may rewrite (32) as

$$A = \sum_{t>0} \binom{N-1}{2t} \mathcal{B}_{N-2t}(x) \left(-H_{2t-1} \mathcal{B}_{2t} + \frac{1}{2!} \sum_{\substack{i+j=2t, \\ i,j>0}} \mathcal{B}_i \mathcal{B}_j \right) - \sum_{q>0} \frac{1}{2q} \mathcal{B}_{2q} \mathcal{B}_{N-2q}(x). \quad (33)$$

Above we have used the equalities

$$\begin{aligned} \sum_{j=1}^{N-i-1} \frac{1}{i+j-2q} \binom{i+j-2q}{2t-2q} &= \frac{1}{2t-2q} \sum_{j=1}^{N-i-1} \binom{i+j-2q-1}{2t-2q-1} \\ &= \frac{1}{2t-2q} \sum_{s=i-2q}^{N-2q-2} \binom{s}{2t-2q-1} \\ &= \frac{1}{2t-2q} \left(\binom{N-2q-1}{2t-2q} - \binom{i-2q}{2t-2q} \right); \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{N-2} \binom{i-1}{2q-1} H_{i-2q} &= \sum_{j=1}^{N-2-2q} \frac{1}{j} \sum_{i=j+2q}^{N-2} \binom{i-1}{2q-1} \\ &= \sum_{j=1}^{N-2-2q} \frac{1}{j} \left(\binom{N-2}{2q} - \binom{j+2q-1}{2q} \right) \\ &= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \sum_{j=1}^{N-2-2q} \binom{j+2q-1}{2q-1} \\ &= \binom{N-2}{2q} H_{N-2q-2} - \frac{1}{2q} \left(\binom{N-2}{2q} - 1 \right). \end{aligned}$$

The summation of the right-hand side of (20) for $q = 0$ gives us

$$\begin{aligned} &3 \sum_{t \geq 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \sum_{\substack{i+j=N-k \\ i,j>0}} \binom{i+j}{2t} \frac{1}{i+j} \left(\frac{1}{i} + \frac{1}{j} \right) \mathcal{B}_{N-2t}(x) \\ &= 3 \sum_{t \geq 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \sum_{i=1}^{N-k-1} \frac{1}{N-k} \left(\frac{1}{i} + \frac{1}{N-k-i} \right) \mathcal{B}_{N-2t}(x) \\ &= 6 \sum_{t > 0} \mathcal{B}_{2t} \sum_{k=1}^{N-2} \binom{N-k}{2t} \frac{H_{N-k-1}}{N-k} \mathcal{B}_{N-2t}(x) + 6 \mathcal{B}_N(x) \sum_{k=1}^{N-2} \frac{H_{N-k-1}}{N-k} \\ &= 6 \sum_{t > 0} \mathcal{B}_{2t} \binom{N-1}{2t} \left(H_{N-1} - \frac{1}{2t} \right) \mathcal{B}_{N-2t}(x) \\ &\quad + 3(H_{N-1}^2 - H_{N-1,2}) \mathcal{B}_N(x). \end{aligned} \quad (34)$$

Here we have used the formulas (see [15, p. 279–280])

$$\sum_{s=1}^{n-1} \binom{s}{m} H_s = \binom{n}{m+1} \left(H_n - \frac{1}{m+1} \right), \quad \sum_{s=1}^n \frac{H_s}{s} = \frac{1}{2} (H_n^2 + H_{n,2}).$$

The sum of the first three summands from the fourth line of (20) by (14) gives

$$\begin{aligned} -3 \sum_{k=1}^{N-2} B_{N-k} \mathcal{B}_k(x) \frac{1}{(N-k)(N-k-1)} \sum_{i=1}^{N-k-1} \frac{(-1)^i}{\binom{N-k-2}{i-1}} \\ = 3 \sum_{k=1}^{N-2} \frac{(1+(-1)^{N-k})}{(N-k)(N-k-1)} \frac{N-k-1}{N-k} B_{N-k} \mathcal{B}_k(x) \\ = 6 \sum_{0 < t < N/2} \frac{1}{2t} \mathcal{B}_{2t} \mathcal{B}_{N-2t}(x). \end{aligned} \quad (35)$$

Finally, for the last non-constant term of (20) we have

$$\sum_{i=1}^{N-2} (N-1-i) \mathcal{B}_{N-2}(x) = \binom{N-1}{2} \mathcal{B}_{N-2}(x). \quad (36)$$

The formulas (33)–(36) imply (31). \square

In [20], authors studied the expressions

$$S_{\geq 0}(r, n) = \sum_{i_1 + \dots + i_r = n} B_{i_1}(x) \dots B_{i_r}(x), \quad (37)$$

where the sum runs over all nonnegative integers i_1, \dots, i_r and $r \geq 1$. Moreover, in [20] it was shown that

$$\begin{aligned} S_{\geq 0}(r, n) = \sum_{k=1}^n B_k(x) \left(\frac{\binom{n+r}{k}}{n+r} \sum_{i_1 + \dots + i_r = n-k+1} (B_{i_1}(1) \dots B_{i_r}(1) \right. \\ \left. - B_{i_1} \dots B_{i_r}) \right) + \text{const}, \end{aligned} \quad (38)$$

and the formula for the last constant term was given. The simple idea lies behind such kind of decomposition: polynomials $1 = B_0(x), B_1(x), \dots, B_n(x)$ form a linear basis of the vector space $\text{Span}(1, x, x^2, \dots, x^n)$. Also, $(B_k(x))' = kB_{k-1}(x)$, the property which is not fulfilled for the polynomials $1, \mathcal{B}_1(x), \dots, \mathcal{B}_n(x)$, and so the approach from [20] could not be applied directly for calculating the analogues of $S_{\geq 0}(r, n)$ or $S_{> 0}(r, n)$ for the sum of products $\mathcal{B}_{i_1}(x) \dots \mathcal{B}_{i_r}(x)$.

Let us decompose $S_{\geq 0}(r, n) = \sum_{k=0}^n \alpha_k B_{n-k}(x)$. In [20], it was shown that $\alpha_1 = 0$. It is easy to see from (38) that all odd coefficients α_{2s+1} are zero.

Remark 3. The formula (38) applied for $r = 2$ up to a constant coincides with (16). Due to (38), we have

$$\begin{aligned}
S_{\geq 0}(3, n) &= \sum_{k=1}^n B_k(x) \left(\frac{\binom{n+3}{k}}{n+3} \sum_{a+b+c=n-k+1} B_a(1)B_b(1)B_c(1) - B_a B_b B_c \right) \\
&= -\frac{2}{n+3} \sum_{t \geq 0} \binom{n+3}{n-2t} \left(\sum_{a+b+c=2t+1} B_a B_b B_c \right) B_{n-2t}(x) \\
&= \binom{n+2}{2} B_n(x) + \frac{1}{4} \binom{n+2}{4} B_{n-2}(x) \\
&\quad + \frac{3}{n+3} \sum_{t \geq 2} \binom{n+3}{n-2t} B_{n-2t}(x) \sum_{q=0}^t B_{2q} B_{2t-2q}. \tag{39}
\end{aligned}$$

If we calculate the sum $S_{\geq 0}(3, n)$ due to the formula (20), we will get the same as (39) modulo the equality (16).

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References

- [1] T. Agoh: On Bernoulli numbers, I. *C. R. Math. Rep. Acad. Sci. Canada* 10 (1988) 7–12.
- [2] T. Agoh: Convolution identities for Bernoulli and Genocchi polynomials. *Electron. J. Comb.* 21 (1) (2014) 1–14.
- [3] T. Agoh, K. Dilcher: Integrals of products of Bernoulli polynomials. *J. Math. Anal. Appl.* 381 (1) (2011) 10–16.
- [4] M. Aguiar: Pre-Poisson algebras. *Lett. Math. Phys.* 54 (4) (2000) 263–277.
- [5] F.V. Atkinson: Some aspects of Baxter’s functional equation. *J. Math. Anal. Appl.* 7 (1) (1963) 1–30.
- [6] G. Baxter: An analytic problem whose solution follows from a simple algebraic identity. *Pacific J. Math* 10 (3) (1960) 731–742.
- [7] A.A. Belavin, V.G. Drinfel’d: Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funct. Anal. Appl.* 16 (3) (1982) 159–180.
- [8] L. Carlitz: Note on the integral of the product of several Bernoulli polynomials. *J. London Math. Soc.* s1-34 (3) (1959) 361–363.
- [9] P. Cartier: On the structure of free Baxter algebras. *Adv. Math.* 9 (2) (1972) 253–265.
- [10] A. Connes, D. Kreimer: Renormalization in quantum field theory and the Riemann–Hilbert problem I: The Hopf algebra structure of graphs and the main theorem. *Commun. Math. Phys.* 210 (1) (2000) 249–273.
- [11] K. Ebrahimi-Fard: Loday-type algebras and the Rota-Baxter relation. *Lett. Math. Phys.* 61 (2) (2002) 139–147.

- [12] K. Ebrahimi-Fard: Rota-Baxter algebras and the Hopf algebra of renormalization. Ph.D. Thesis, University of Bonn (2006)
- [13] K. Ebrahimi-Fard, L. Guo: Multiple zeta values and Rota-Baxter algebras. *Integers* 8 (2-A4) (2008) 1–18.
- [14] I.M. Gessel: On Miki's identity for Bernoulli numbers. *J. Number Theory* 110 (1) (2005) 75–82.
- [15] R.L. Graham, D.E. Knuth, O. Patashnik: *Concrete Mathematics: a foundation for computer science*. Addison-Wesley Professional, Reading (MA, USA) (1994). Second ed.
- [16] V. Gubarev: Rota-Baxter operators on unital algebras. *Mosc. Math. J.* (Accepted) Preprint arXiv:1805.00723v3
- [17] V. Gubarev, P. Kolesnikov: Embedding of dendriform algebras into Rota-Baxter algebras. *Cent. Eur. J. Math. – Open Mathematics* 11 (2) (2013) 226–245.
- [18] L. Guo: *An introduction to Rota-Baxter algebra*. International Press Somerville, Higher Education Press, Beijing (2012). Surveys of Modern Mathematics, vol. 4.
- [19] L. Guo, W. Keigher: Baxter algebras and shuffle products. *Adv. Math.* 150 (1) (2000) 117–149.
- [20] D.S. Kim, T. Kim: Bernoulli basis and the product of several Bernoulli polynomials. *Int. J. Math. Math. Sci.* 2012 (2012) 12 pp.
- [21] D.S. Kim, T. Kim, S.-H. Lee, Y.-H. Kim: Some identities for the product of two Bernoulli and Euler polynomials. *Adv. Differ. Equ.* 2012 (95) (2012) 14 pp.
- [22] D.H. Lehmer: A new approach to Bernoulli polynomials. *Am. Math. Mon.* 95 (10) (1988) 905–911.
- [23] Yu. Matiyasevich: Identities with Bernoulli numbers. <http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernoulli.htm> (1997)
- [24] H. Miki: A relation between Bernoulli numbers. *J. Number Theory* 10 (3) (1978) 297–302.
- [25] J.B. Miller: Some properties of Baxter operators. *Acta Math. Hung.* 17 (3-4) (1966) 387–400.
- [26] N.J. Newsome, M.S. Nogin, A.H. Sabuwala: A proof of symmetry of the power sum polynomials using a novel Bernoulli number identity. *J. Integer Seq.* 20 (2) (2017) 10 pp.
- [27] N. Nielsen: *Traité élémentaire des nombres de Bernoulli*. Gauthier-Villars (1923).
- [28] O. Ogievetsky, T. Popov: *R*-matrices in rime. *Adv. Theor. Math. Phys.* 14 (2) (2010) 439–505.
- [29] O.V. Ogievetskii, V.V. Schechtman: Nombres de Bernoulli et une formule de Schlömilch-Ramanujan. *Mosc. Math. J.* 10 (4) (2010) 765–788.
- [30] G.-C. Rota: Baxter algebras and combinatorial identities. I. *Bull. Am. Math. Soc.* 75 (2) (1969) 325–329.
- [31] M.A. Semenov-Tyan-Shanskii: What is a classical *r*-matrix?. *Funct. Anal. its Appl.* 17 (1983) 259–272.
- [32] B. Sury, T. Wang, F.-Z. Zhao: Identities involving reciprocals of binomial coefficients. *J. Integer Seq.* 7 (2) (2004) 12 pp.
- [33] H.J.H. Tuenter: A symmetry of power sum polynomials and Bernoulli numbers. *Am. Math. Mon.* 108 (3) (2001) 258–261.

- [34] D. Zagier: Curious and exotic identities for Bernoulli numbers (Appendix). In: *Bernoulli numbers and zeta functions*. Springer (2014) 239–262.
- [35] J. Zhao: *Multiple zeta functions, multiple polylogarithms and their special values*. World Scientific (2016). Series on Number Theory and Its Applications, vol. 12.

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