



**HAL**  
open science

# $(\Phi, \phi)$ -derivations on semiprime rings and Banach algebras

Bilal Ahmad Wani

► **To cite this version:**

Bilal Ahmad Wani.  $(\Phi, \phi)$ -derivations on semiprime rings and Banach algebras. Communications in Mathematics, 2021, Volume 29 (2021), Issue 3 (3), pp.371 - 383. 10.2478/cm-2021-0013. hal-03665003

**HAL Id: hal-03665003**

**<https://hal.science/hal-03665003>**

Submitted on 11 May 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

## $(\phi, \varphi)$ -derivations on semiprime rings and Banach algebras

Bilal Ahmad Wani

**Abstract.** Let  $\mathcal{R}$  be a semiprime ring with unity  $e$  and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . In this paper it is shown that if  $\mathcal{R}$  satisfies

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1})$$

for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ , then  $\mathcal{D}$  is an  $(\phi, \varphi)$ -derivation. Moreover, this result makes it possible to prove that if  $\mathcal{R}$  admits an additive mappings  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  satisfying the relations

$$\begin{aligned} 2\mathcal{D}(x^n) &= \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}), \\ 2\mathcal{G}(x^n) &= \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \end{aligned}$$

for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ , then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations under some torsion restriction. Finally, we apply these purely ring theoretic results to semi-simple Banach algebras.

### 1 Introduction and Results

Throughout this paper  $\mathcal{R}$  will denote an associative ring with the center  $\mathcal{Z}(\mathcal{R})$ . Recall that a ring  $\mathcal{R}$  is said to be prime if for any  $a, b \in \mathcal{R}$ ,  $a\mathcal{R}b = \{0\}$  implies  $a = 0$  or  $b = 0$ , and  $\mathcal{R}$  is semiprime if for any  $a \in \mathcal{R}$ ,  $a\mathcal{R}a = \{0\}$  implies  $a = 0$ . A ring  $\mathcal{R}$  is said to be  $n$ -torsion free, where  $n > 1$  is an integer, if  $nx = 0$  implies  $x = 0$  for all  $x \in \mathcal{R}$ . For any  $x, y \in \mathcal{R}$ , the symbol  $[x, y]$  will denote the commutator  $xy - yx$ . By a Banach algebra  $\mathfrak{B}$  we mean an algebra equipped with a norm  $\|\cdot\|$  that makes it into a Banach space and additionally satisfies the inequality  $\|uv\| \leq \|u\|\|v\|$  for all  $u, v \in \mathfrak{B}$  (see [3]). The Jacobson radical of  $\mathfrak{B}$ , denoted by  $\text{rad}(\mathfrak{B})$ , is the intersection of all the primitive ideals of  $\mathfrak{B}$ . An algebra  $\mathfrak{B}$  is called semi-simple Banach algebra if  $\text{rad}(\mathfrak{B}) = 0$ .

2020 MSC: 16N60, 46J10, 16W25

Key words: Prime ring, semiprime ring, Banach algebra, Jordan derivation,  $(\phi, \varphi)$ -derivation

Affiliation:

Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India

E-mail: bilalwanikmr@gmail.com

An additive mapping  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is said to be a derivation (resp. Jordan derivation) on  $\mathcal{R}$  if

$$\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$$

(resp.  $\mathcal{D}(x^2) = \mathcal{D}(x)x + x\mathcal{D}(x)$ ) holds for all  $x, y \in \mathcal{R}$ . A derivation  $\mathcal{D}$  is inner if there exists  $a \in \mathcal{R}$  such that  $\mathcal{D}(x) = [a, x]$  holds for all  $x \in \mathcal{R}$ . It is easy to verify that every derivation is a Jordan derivation but the converse is not true in general. A classical result of Herstein [10] states that every Jordan derivation is a derivation on a prime ring of characteristic different from two. A brief proof of Herstein's result can be found in [6]. An additive mapping  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is called a Jordan triple derivation if

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{D}(y)x + xy\mathcal{D}(x)$$

holds for all  $x, y \in \mathcal{R}$ . Obviously, every derivation is a Jordan triple derivation but not conversely. Brešar [5, Theorem 4.3], established that a Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the above result, Vukman [17] recently showed that if  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  an additive mapping on a 2-torsion free semiprime ring  $\mathcal{R}$  satisfying either

$$\mathcal{D}(xyx) = \mathcal{D}(xy)x + xy\mathcal{D}(x)$$

for all pairs  $x, y \in \mathcal{R}$  or

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{D}(yx)$$

for all pairs  $x, y \in \mathcal{R}$ , then  $\mathcal{D}$  is a derivation. In 2016 Širovnik [16] generalized the above result. In fact, he established that if  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  are two additive mappings on a 2-torsion free semiprime ring  $\mathcal{R}$  satisfying either

$$\mathcal{D}(xyx) = \mathcal{D}(xy)x + xy\mathcal{G}(x)$$

and

$$\mathcal{G}(xyx) = \mathcal{G}(xy)x + xy\mathcal{D}(x)$$

for all pairs  $x, y \in \mathcal{R}$  or

$$\mathcal{D}(xyx) = \mathcal{D}(x)yx + x\mathcal{G}(yx)$$

and

$$\mathcal{G}(xyx) = \mathcal{G}(x)yx + x\mathcal{D}(yx)$$

for all pairs  $x, y \in \mathcal{R}$ , then  $\mathcal{D}$  and  $\mathcal{G}$  are derivations and  $\mathcal{D} = \mathcal{G}$ . Following the same line, a number of results have been obtained by several authors (see [1], [2], [4], [8], [12], [13], [14], [15], [18], [19], [20]), where further references can be found.

Let  $\phi, \varphi$  be any two mappings on  $\mathcal{R}$ . An additive mapping  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is said to be an  $(\phi, \varphi)$ -derivation (resp. Jordan  $(\phi, \varphi)$ -derivation) on  $\mathcal{R}$  if

$$\mathcal{D}(xy) = \mathcal{D}(x)\phi(y) + \varphi(x)\mathcal{D}(y)$$

(resp.  $\mathcal{D}(x^2) = \mathcal{D}(x)\phi(x) + \varphi(x)\mathcal{D}(x)$ ) holds for all  $x, y \in \mathcal{R}$ . An additive mapping  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is called a Jordan triple  $(\phi, \varphi)$ -derivation if

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$$

holds for all  $x, y \in \mathcal{R}$ . Obviously, every  $(\phi, \varphi)$ -derivation is a Jordan  $(\phi, \varphi)$ -derivation and a Jordan triple  $(\phi, \varphi)$ -derivation, but not conversely. Brešar and Vukman [7] obtained that every Jordan  $(\phi, \varphi)$ -derivation is a  $(\phi, \varphi)$ -derivation on a prime ring of characteristic different from two. For these kind of results we refer the reader to ([9], [11]), where further references can be found. Liu and Shiue [11] have recently generalized the above result to 2-torsion free semiprime rings. Moreover in the same paper they showed that every Jordan triple  $(\phi, \varphi)$ -derivation is a  $(\phi, \varphi)$ -derivation on a 2-torsion free semiprime ring.

In view of the above results we begin our investigation by extending the results of Vukman [17] to  $(\phi, \varphi)$ -derivations. In fact, we have shown that an additive mapping  $\mathcal{D}$  on a semiprime ring  $\mathcal{R}$  which satisfies either of the identities

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x)$$

or

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx)$$

for all  $x, y \in \mathcal{R}$  is a  $(\phi, \varphi)$ -derivation. Further, it is also shown that if the additive mapping  $\mathcal{D}$  on  $\mathcal{R}$  satisfies

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1})$$

for all  $x \in \mathcal{R}$ , then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation. Finally, we have shown that under what conditions and additive mapping  $\mathcal{D}$  on  $\mathcal{R}$  satisfying

$$\mathcal{D}(x^n) = \sum_{j=1}^n \phi(x^{n-j})\mathcal{D}(x)\varphi(x^{j-1}) \text{ for all } x \in \mathcal{R}$$

is an  $(\phi, \varphi)$ -derivations.

## 2 Main Results

We facilitate our investigation with the following theorem:

**Theorem 1.** *Let  $\mathcal{R}$  be a 2-torsion free semiprime ring and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that either*

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) \text{ for all } x, y \in \mathcal{R}, \tag{1}$$

or

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx) \text{ for all } x, y \in \mathcal{R}. \tag{2}$$

Then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

For developing the proof of our theorem, we need the following Lemma.

**Lemma 1.** *Let  $\mathcal{R}$  be a semiprime ring and  $\phi$  be an automorphism of  $\mathcal{R}$ . Suppose  $f: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that either  $f(x)\phi(x) = 0$  holds for all  $x \in \mathcal{R}$  or  $\phi(x)f(x) = 0$  holds for all  $x \in \mathcal{R}$ , then  $f = 0$ .*

*Proof.* Since, we have

$$f(x)\phi(x) = 0 \quad \text{for all } x \in \mathcal{R}. \quad (3)$$

The linearization of the above relation gives

$$f(x)\phi(y) + f(y)\phi(x) = 0 \quad \text{for all } x, y \in \mathcal{R}. \quad (4)$$

Replace  $y$  by  $y^2$  in the above equation, we see that

$$f(x)\phi(y^2) + f(y^2)\phi(x) = 0 \quad \text{for all } x, y \in \mathcal{R}. \quad (5)$$

Right multiplication of (4) by  $\phi(y)$  gives

$$f(x)\phi(y^2) + f(y)\phi(x)\phi(y) = 0 \quad \text{for all } x, y \in \mathcal{R}. \quad (6)$$

By comparing (5) and (6), we obtain

$$f(y^2)\phi(x) - f(y)\phi(x)\phi(y) = 0 \quad \text{for all } x, y \in \mathcal{R}. \quad (7)$$

Since  $\phi$  is an automorphism, we have

$$f(y^2)z - f(y)z\phi(y) = 0 \quad \text{for all } y, z \in \mathcal{R}.$$

Replace  $z$  by  $\phi(x)f(y)$  in the above relation, and use (3), we obtain,

$$f(y^2)\phi(x)f(y) = 0 \quad \text{for all } x, y \in \mathcal{R}.$$

In view of the above relation right multiplication of (7) by  $f(y)$  yields

$$f(y)\phi(x)\phi(y)f(y) = 0$$

for all  $x, y \in \mathcal{R}$ , which leads to  $\phi(y)f(y)\phi(x)\phi(y)f(y) = 0$  for all  $x, y \in \mathcal{R}$ . Hence we have

$$\phi(y)f(y) = 0 \quad \text{for all } y \in \mathcal{R}. \quad (8)$$

Right multiplication of (4) by  $f(x)$  and using (8), we find that

$$f(x)\phi(y)f(x) = 0 \quad \text{for all } x, y \in \mathcal{R}.$$

Since  $\mathcal{R}$  is semiprime, it follows that  $f = 0$ , which completes the proof.  $\square$

*Proof.* [Proof of Theorem 1] We will restrict our attention on the relation (1), the proof in case when  $\mathcal{R}$  satisfies the relation (2) is similar and will therefore be omitted. Linearize the relation (1), we see that

$$\mathcal{D}(xyz + zyx) = \mathcal{D}(xy)\phi(z) + \mathcal{D}(zy)\phi(x) + \varphi(xy)\mathcal{D}(z) + \varphi(zy)\mathcal{D}(x),$$

for all  $x, y, z \in \mathcal{R}$ . In particular for  $z = x^2$ , the above relation gives

$$\mathcal{D}(xyx^2 + x^2yx) = \mathcal{D}(xy)\phi(x^2) + \mathcal{D}(x^2y)\phi(x) + \varphi(xy)\mathcal{D}(x^2) + \varphi(x^2y)\mathcal{D}(x), \tag{9}$$

for all  $x, y \in \mathcal{R}$ . Putting  $xy + yx$  for  $y$  in (1) and applying the relation (1), we obtain

$$\begin{aligned} \mathcal{D}(xyx^2 + x^2yx) &= \mathcal{D}(x^2y + xyx)\phi(x) + \varphi(x^2y + xyx)\mathcal{D}(x) \\ &= \mathcal{D}(x^2y)\phi(x) + \mathcal{D}(xy)\phi(x^2) + \varphi(xy)\mathcal{D}(x)\phi(x) \\ &\quad + \varphi(x^2y)\mathcal{D}(x) + \varphi(xyx)\mathcal{D}(x), \end{aligned} \tag{10}$$

for all  $x, y \in \mathcal{R}$ . By comparing (9) and (10), we have

$$\varphi(x)\varphi(y)A(x) = 0, \text{ for all } x, y \in \mathcal{R}, \tag{11}$$

where  $A(x)$  stands for  $\mathcal{D}(x^2) - \mathcal{D}(x)\phi(x) - \varphi(x)\mathcal{D}(x)$ . Since  $\varphi$  is surjective, we have

$$\varphi(x)zA(x) = 0, \text{ for all } x, z \in \mathcal{R}. \tag{12}$$

Right multiplication of (12) by  $\varphi(x)$  and left multiplication by  $A(x)$  gives,

$$A(x)\varphi(x)zA(x)\varphi(x) = 0, \text{ for all } x, z \in \mathcal{R}.$$

By the semiprimeness of  $\mathcal{R}$ , it follows that

$$A(x)\varphi(x) = 0, \text{ for all } x \in \mathcal{R}. \tag{13}$$

The substitution of  $A(x)y\varphi(x)$  for  $z$  in the relation (12), gives

$$\varphi(x)A(x)y\varphi(x)A(x) = 0$$

for all pairs  $x, y \in \mathcal{R}$ . Hence, we obtain

$$\varphi(x)A(x) = 0, \text{ for all } x \in \mathcal{R}. \tag{14}$$

The linearization of the relation (13) gives

$$B(x, y)\varphi(x) + A(x)\varphi(y) + B(x, y)\varphi(y) + A(y)\varphi(x) = 0$$

for all pairs  $x, y \in \mathcal{R}$ , where  $B(x, y)$  denotes

$$\mathcal{D}(xy + yx) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y) - \mathcal{D}(y)\phi(x) - \varphi(y)\mathcal{D}(x).$$

Putting in the above relation  $-x$  for  $x$  and comparing the relation so obtained with the above relation one obtains

$$B(x, y)\varphi(x) + A(x)\varphi(y) = 0, \text{ for all } x, y \in \mathcal{R}.$$

In view of the relation (14), right multiplication by  $A(x)$  gives,  $A(x)\varphi(y)A(x) = 0$  for all pairs  $x, y \in \mathcal{R}$ . Hence it follows that  $A(x) = 0$  for all  $x \in \mathcal{R}$ . In other words,

$\mathcal{D}$  is a Jordan  $(\phi, \varphi)$ -derivation. By [11, Corollary 1] one can conclude that  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation. It is our aim to show that Theorem 1 can be proved without using [11, Corollary 1]. From the fact that  $\mathcal{D}$  is a Jordan  $(\phi, \varphi)$ -derivation, it follows that  $\mathcal{D}$  is a Jordan triple  $(\phi, \varphi)$ -derivation. Now, comparing the relation  $\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$ , for all  $x, y \in \mathcal{R}$ , with the relation (1), we get

$$(\mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y))\phi(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

For any fixed  $y \in \mathcal{R}$ , we have an additive mapping  $x \mapsto \mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y)$  on  $\mathcal{R}$ . Thus from the above relation and Lemma 1 it follows that  $\mathcal{D}(xy) - \mathcal{D}(x)\phi(y) - \varphi(x)\mathcal{D}(y) = 0$  for all pairs  $x, y \in \mathcal{R}$ . In other words,  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation. This completes the proof.  $\square$

**Remark 1.** It is to be noted that if  $\phi$  and  $\varphi$  are the identity automorphisms on  $\mathcal{R}$ , then the above result reduces to the [17, Theorem 2].

**Theorem 2.** Let  $\mathcal{R}$  be a 2-torsion free semiprime ring and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that either

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) - \varphi(xy)\mathcal{D}(x) \text{ for all } x, y \in \mathcal{R}, \quad (15)$$

or

$$\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) - \varphi(x)\mathcal{D}(yx) \text{ for all } x, y \in \mathcal{R}. \quad (16)$$

Then  $\mathcal{D} = 0$ .

*Proof.* We will restrict our attention on the relation (15), the proof in the other case is similar. Linearization of the relation (15) gives

$$\mathcal{D}(xyz + zyx) = \mathcal{D}(xy)\phi(z) + \mathcal{D}(zy)\phi(x) - \varphi(xy)\mathcal{D}(z) - \varphi(zy)\mathcal{D}(x),$$

for all  $x, y, z \in \mathcal{R}$ . Following the same procedure as used in the above theorem we get,  $A(x) = 0$  for all pairs  $x, y \in \mathcal{R}$ , where  $A(x)$  stands for  $\mathcal{D}(x^2) - \mathcal{D}(x)\phi(x) - \varphi(x)\mathcal{D}(x)$ . Thus  $\mathcal{D}$  is a Jordan  $(\phi, \varphi)$ -derivation and hence it follows that  $\mathcal{D}$  is a Jordan triple  $(\phi, \varphi)$ -derivation. Now, comparing the relation  $\mathcal{D}(xyx) = \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(y)\phi(x) + \varphi(xy)\mathcal{D}(x)$ , for all  $x, y \in \mathcal{R}$ , with the relation (15), one obtains

$$\varphi(x)\varphi(y)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}. \quad (17)$$

Since  $\varphi$  is surjective, we have

$$\varphi(x)z\mathcal{D}(x) = 0, \text{ for all } x, z \in \mathcal{R}. \quad (18)$$

Right multiplication of (18) by  $\varphi(x)$  and left multiplication by  $\mathcal{D}(x)$  gives

$$\mathcal{D}(x)\varphi(x)z\mathcal{D}(x)\varphi(x) = 0, \text{ for all } x, z \in \mathcal{R}.$$

By the semiprimeness of  $\mathcal{R}$  it follows that

$$\mathcal{D}(x)\varphi(x) = 0, \text{ for all } x \in \mathcal{R}. \tag{19}$$

The substitution of  $\mathcal{D}(x)y\varphi(x)$  for  $z$  in the relation (18), gives

$$\varphi(x)\mathcal{D}(x)y\varphi(x)\mathcal{D}(x) = 0$$

for all pairs  $x, y \in \mathcal{R}$ . Hence, we obtain

$$\varphi(x)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}. \tag{20}$$

The linearization of the relation (19) gives

$$\mathcal{D}(x)\varphi(y) + \mathcal{D}(y)\varphi(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

In view of the relation (20), right multiplication by  $\mathcal{D}(x)$  gives,

$$\mathcal{D}(x)\varphi(y)\mathcal{D}(x) = 0, \text{ for all } x, y \in \mathcal{R}.$$

Hence it follows that  $\mathcal{D} = 0$ , which completes the proof. □

**Corollary 1.** *Let  $\mathcal{R}$  be a 2-torsion free semiprime ring and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mappings such that either*

$$\begin{aligned} \mathcal{D}(xyx) &= \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{G}(x), \\ \mathcal{G}(xyx) &= \mathcal{G}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) \end{aligned} \tag{21} \text{ for all } x, y \in \mathcal{R},$$

or

$$\begin{aligned} \mathcal{D}(xyx) &= \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{G}(yx), \\ \mathcal{G}(xyx) &= \mathcal{G}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx) \end{aligned} \tag{22} \text{ for all } x, y \in \mathcal{R}.$$

Then  $\mathcal{D}$  and  $\mathcal{G}$  are ( $\phi, \varphi$ )-derivations and  $\mathcal{D} = \mathcal{G}$ .

*Proof.* We will restrict our attention on the relations (21), the proof in case we have the relations (22) is similar and will therefore be omitted. Thus the relations are

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{G}(x), \text{ for all } x, y \in \mathcal{R}, \tag{23}$$

$$\mathcal{G}(xyx) = \mathcal{G}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x), \text{ for all } x, y \in \mathcal{R}. \tag{24}$$

Combining the relations (24) and (23), gives

$$T(xyx) = T(xy)\phi(x) - \varphi(xy)T(x), \text{ for all } x, y \in \mathcal{R}, \tag{25}$$

where  $T = \mathcal{D} - \mathcal{G}$ . By applying Theorem 2 one obtains that  $\mathcal{D} = \mathcal{G}$ . Thus relation (21) reduces to

$$\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x), \text{ for all } x, y \in \mathcal{R}.$$

Using Theorem 1, it follows that  $\mathcal{D}$  is a ( $\phi, \varphi$ )-derivation, which completes the proof. □



Disadvantage of Theorem 1 is that in identities (1) and (2) there is no symmetry. Therefore, Theorem 1, together with the desire for symmetry leads to the following conjecture.

**Conjecture 1.** Let  $\mathcal{R}$  be a 2-torsion free semiprime ring and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that

$$2\mathcal{D}(xyx) = \mathcal{D}(xy)\phi(x) + \varphi(xy)\mathcal{D}(x) + \mathcal{D}(x)\phi(yx) + \varphi(x)\mathcal{D}(yx), \quad (26)$$

holds for all pairs  $x, y \in \mathcal{R}$ . Then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

Note that in case a ring has the identity element, the proof of the above conjecture is immediate. The substitution  $y = e$  in the relation (26), where  $e$  stands for the identity element, gives that  $\mathcal{D}$  is a Jordan  $(\phi, \varphi)$ -derivation and then it follows from [11, Corollary 1] that  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

The substitution of  $y = x^{n-2}$  in the relation (26) gives

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

which leads to the following conjecture.

**Conjecture 2.** Let  $\mathcal{R}$  be a semiprime ring with a suitable torsion restriction and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

holds for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

Now we prove the above conjecture in case a ring has the identity element.

**Theorem 3.** Let  $\mathcal{R}$  be a  $(n-1)!$ -torsion free semiprime ring with identity  $e$  and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

*Proof.* We have the relation

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \quad (27)$$

holds for all  $x \in \mathcal{R}$ . The substitution of  $x = e$  in the relation (27) gives  $\mathcal{D}(e) = 0$ . Let  $y$  be any element of the center  $\mathcal{Z}(\mathcal{R})$ . Putting  $x + y$  for  $x$  in the above relation,

we obtain

$$\begin{aligned}
 2 \sum_{i=0}^n \binom{n}{i} \mathcal{D}(x^{n-i}y^i) &= \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \mathcal{D}(x^{n-1-i}y^i) \right) \phi(x+y) \\
 &+ \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \varphi(x^{n-1-i}y^i) \right) \mathcal{D}(x+y) \\
 &+ \mathcal{D}(x+y) \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \phi(x^{n-1-i}y^i) \right) \\
 &+ \varphi(x+y) \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \mathcal{D}(x^{n-1-i}y^i) \right).
 \end{aligned}$$

Using (27) in the above relation and rearranging it in sense of collecting together terms involving equal number of factors of  $y$ , we obtain

$$\sum_{i=1}^{n-1} f_i(x, y) = 0, \tag{28}$$

where  $f_i(x, y)$  stands for the expression of terms involving  $i$  factors of  $y$ . Replace  $x$  by  $x + 2y, x + 3y, \dots, x + (n - 1)y$  in the relation (27) and expressing the resulting system of  $(n - 2)$  homogeneous equations of variables  $f_i(x, y)$  for  $i = 1, 2, \dots, n - 1$  together with (28), we see that the coefficient matrix of the system of  $(n - 1)$  homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix}
 1 & 1 & \dots & 1 \\
 2 & 2^2 & \dots & 2^{n-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 n-1 & (n-1)^2 & \dots & (n-1)^{n-1}
 \end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular, if  $y$  is replaced with the identity element  $e$ , we obtain

$$\begin{aligned}
 f_{n-2}(x, e) &= 2 \binom{n}{n-2} \mathcal{D}(x^2) - \binom{n-1}{n-2} \mathcal{D}(x)\phi(x) - \binom{n-1}{n-3} \mathcal{D}(x^2) \\
 &- \binom{n-1}{n-2} \varphi(x)\mathcal{D}(x) - \binom{n-1}{n-3} \varphi(x^2)\mathcal{D}(e) - \binom{n-1}{n-2} \mathcal{D}(x)\phi(x) \\
 &- \binom{n-1}{n-3} \mathcal{D}(e)\phi(x^2) - \binom{n-1}{n-3} \mathcal{D}(x^2) - \binom{n-1}{n-2} \varphi(x)\mathcal{D}(x).
 \end{aligned}$$

After few calculations and considering the relation  $\mathcal{D}(e) = 0$ , we obtain

$$(n(n-1) - (n-1)(n-2))\mathcal{D}(x^2) = 2(n-1)(\mathcal{D}(x)\phi(x) + \varphi(x)\mathcal{D}(x)).$$

Since  $\mathcal{R}$  is  $(n - 1)!$ -torsion free, it follows from the above relation that

$$\mathcal{D}(x^2) = \mathcal{D}(x)\phi(x) + \varphi(x)\mathcal{D}(x) \text{ for all } x \in \mathcal{R}.$$

Hence  $\mathcal{D}$  is a Jordan  $(\phi, \varphi)$ -derivation. By [11, Corollary 1],  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation, which completes the proof.  $\square$

**Theorem 4.** *Let  $\mathcal{R}$  be a  $(n - 1)!$ -torsion free semiprime ring with identity  $e$  and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose there exist additive mappings  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  satisfying the relations*

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}),$$

$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}),$$

for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations.

*Proof.* We have

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}), \tag{29}$$

$$2\mathcal{G}(x^n) = \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \tag{30}$$

for all  $x \in \mathcal{R}$ , where  $n \geq 2$  is a fixed integer. Subtracting the two relations of equation, we obtain

$$2T(x^n) = T(x^{n-1})\phi(x) - \varphi(x^{n-1})T(x) - T(x)\phi(x^{n-1}) - \varphi(x)T(x^{n-1}), \tag{31}$$

where  $T = \mathcal{D} - \mathcal{G}$ . We denote the identity element of the ring  $\mathcal{R}$  by  $e$ . Putting  $e$  for  $x$  in the above relation gives

$$T(e) = 0. \tag{32}$$

Let  $y$  be any element of the center  $\mathcal{Z}(\mathcal{R})$ . Putting  $x + y$  for  $x$  in the relation 31 and follow the same procedure as used in Theorem 3, we arrive at

$$\begin{aligned} f_{n-1}(x, e) &= 2\binom{n}{n-1}T(x) - \binom{n-1}{n-1}\left(T(e)\varphi(x) + eT(x) + T(x)e + \phi(x)T(e)\right) \\ &\quad - \binom{n-1}{n-2}\left(T(x)e + \phi(x)T(e) + T(e)\varphi(x) + eT(x)\right) \\ &= 0. \end{aligned}$$

Using 32 in the above identity, we obtain

$$2nT(x) = 2T(x) - 2(n - 1)T(x)$$

Since  $\mathcal{R}$  is  $(n - 1)!$ -torsion free, it follows from the above relation that  $T(x) = 0$  for all  $x \in \mathcal{R}$ . Therefore, we get  $\mathcal{D} = \mathcal{G}$ . Thus equations 29 and 30 reduces into one relation, which is

$$2\mathcal{D}(x^n) = \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}).$$

Using Theorem 3, we conclude that  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations. This completes the proof.  $\square$

Following are the immediate consequences of above theorems.

Since every semi-simple Banach algebra  $\mathcal{B}$  is a semiprime ring (see [3] for details), we have the following results.

**Corollary 2.** *Let  $\mathcal{B}$  be a semi-simple Banach algebra and  $\phi, \varphi$  be automorphisms of  $\mathcal{B}$ . Suppose  $\mathcal{D}, \mathcal{G}: \mathcal{B} \rightarrow \mathcal{B}$  are linear mappings such that either*

$$\begin{aligned} \mathcal{D}(uvu) &= \mathcal{D}(uv)\phi(u) + \varphi(uv)\mathcal{G}(u), \\ \mathcal{G}(uvu) &= \mathcal{G}(uv)\phi(u) + \varphi(uv)\mathcal{D}(u) \quad \text{for all } u, v \in \mathcal{B}, \end{aligned}$$

or

$$\begin{aligned} \mathcal{D}(uvu) &= \mathcal{D}(u)\phi(vu) + \varphi(u)\mathcal{G}(vu), \\ \mathcal{G}(uvu) &= \mathcal{G}(u)\phi(vu) + \varphi(u)\mathcal{D}(vu) \quad \text{for all } u, v \in \mathcal{B}. \end{aligned}$$

Then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations and  $\mathcal{D} = \mathcal{G}$ .

**Corollary 3.** *Let  $\mathcal{B}$  be a semi-simple Banach algebra with identity  $e$  and  $\phi, \varphi$  be automorphisms of  $\mathcal{B}$ . Suppose  $\mathcal{D}, \mathcal{G}: \mathcal{B} \rightarrow \mathcal{B}$  are additive mappings such that*

$$\begin{aligned} 2\mathcal{D}(u^n) &= \mathcal{D}(u^{n-1})\phi(u) + \varphi(u^{n-1})\mathcal{G}(u) + \mathcal{G}(u)\phi(u^{n-1}) + \varphi(u)\mathcal{G}(u^{n-1}), \\ 2\mathcal{G}(u^n) &= \mathcal{G}(u^{n-1})\phi(u) + \varphi(u^{n-1})\mathcal{D}(u) + \mathcal{D}(u)\phi(u^{n-1}) + \varphi(u)\mathcal{D}(u^{n-1}), \end{aligned}$$

holds for all  $u \in \mathcal{B}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations.

Theorem 4 and Corollary 1 leads to the following conjectures. So, we conclude our paper by giving the following conjectures:

**Conjecture 3.** *Let  $\mathcal{R}$  be a semiprime ring with a suitable torsion restriction and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  are additive mappings satisfying the relations*

$$\begin{aligned} 2\mathcal{D}(x^n) &= \mathcal{D}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{G}(x) + \mathcal{G}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{G}(x^{n-1}), \\ 2\mathcal{G}(x^n) &= \mathcal{G}(x^{n-1})\phi(x) + \varphi(x^{n-1})\mathcal{D}(x) + \mathcal{D}(x)\phi(x^{n-1}) + \varphi(x)\mathcal{D}(x^{n-1}), \end{aligned}$$

for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations.

**Conjecture 4.** *Let  $\mathcal{R}$  be a semiprime ring with a suitable torsion restriction and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}, \mathcal{G}: \mathcal{R} \rightarrow \mathcal{R}$  are additive mappings such that either*

$$\begin{aligned} \mathcal{D}(x^3) &= \mathcal{D}(x^2)\phi(x) + \varphi(x^2)\mathcal{G}(x), \\ \mathcal{G}(x^3) &= \mathcal{G}(x^2)\phi(x) + \varphi(x^2)\mathcal{D}(x) \quad \text{for all } x, y \in \mathcal{R}, \end{aligned} \tag{33}$$

or

$$\begin{aligned} \mathcal{D}(x^3) &= \mathcal{D}(x)\phi(x^2) + \varphi(x)\mathcal{G}(x^2), \\ \mathcal{G}(x^3) &= \mathcal{G}(x)\phi(x^2) + \varphi(x)\mathcal{D}(x^2) \quad \text{for all } x, y \in \mathcal{R}. \end{aligned} \tag{34}$$

Then  $\mathcal{D}$  and  $\mathcal{G}$  are  $(\phi, \varphi)$ -derivations and  $\mathcal{D} = \mathcal{G}$ .

**Conjecture 5.** Let  $\mathcal{R}$  be a semiprime ring with a suitable torsion restriction and  $\phi, \varphi$  be automorphisms of  $\mathcal{R}$ . Suppose  $\mathcal{D}: \mathcal{R} \rightarrow \mathcal{R}$  is an additive mapping such that

$$\mathcal{D}(x^n) = \sum_{j=1}^n \phi(x^{n-j})\mathcal{D}(x)\varphi(x^{j-1}),$$

holds for all  $x \in \mathcal{R}$  and some fixed integer  $n \geq 2$ . Then  $\mathcal{D}$  is a  $(\phi, \varphi)$ -derivation.

## References

- [1] M. Ashraf, N. Rehman, S. Ali: On Lie ideals and Jordan generalized derivations of prime rings. *Indian Journal of Pure & Applied Mathematics* 34 (2) (2003) 291–294.
- [2] M. Ashraf, N. Rehman: On Jordan ideals and Jordan derivations of a prime rings. *Demonstratio Mathematica* 37 (2) (2004) 275–284.
- [3] F.F. Bonsall, J. Duncan: *Complete Normed Algebras*. Springer-Verlag, New York (1973).
- [4] M. Brešar: Jordan derivations on semiprime rings. *Proceedings of the American Mathematical Society* 104 (4) (1988) 1003–1006.
- [5] M. Brešar: Jordan mappings of semiprime rings. *Journal of Algebra* 127 (1) (1989) 218–228.
- [6] M. Brešar, J. Vukman: Jordan derivations on prime rings. *Bulletin of the Australian Mathematical Society* 37 (3) (1988) 321–322.
- [7] M. Brešar, J. Vukman: Jordan  $(\theta, \phi)$ -derivations. *Glasnik Matematički* 16 (1991) 13–17.
- [8] J.M. Cusack: Jordan derivations on rings. *Proceedings of the American Mathematical Society* 53 (2) (1975) 321–324.
- [9] A. Fošner, J. Vukman: On certain functional equations related to Jordan triple  $(\theta, \phi)$ -derivations on semiprime rings. *Monatshefte für Mathematik* 162 (2) (2011) 157–165.
- [10] I.N. Herstein: Jordan derivations of prime rings. *Proceedings of the American Mathematical Society* 8 (6) (1957) 1104–1110.
- [11] C. K. Liu, W. K. Shiue: Generalized Jordan triple  $(\theta, \phi)$ -derivations on semiprime rings. *Taiwanese Journal of Mathematics* 11 (5) (2007) 1397–1406.
- [12] N. Rehman, N. Širovnik, T. Bano: On certain functional equations on standard operator algebras. *Mediterranean Journal of Mathematics* 14 (1) (2017) 1–10.
- [13] N. Rehman, T. Bano: A result on functional equations in semiprime rings and standard operator algebras. *Acta Mathematica Universitatis Comenianae* 85 (1) (2016) 21–28.
- [14] N. Širovnik: On certain functional equation in semiprime rings and standard operator algebras. *Cubo (Temuco)* 16 (1) (2014) 73–80.
- [15] N. Širovnik, J. Vukman: On certain functional equation in semiprime rings. In: *Algebra Colloquium*. World Scientific (2016) 65–70.
- [16] N. Širovnik: On functional equations related to derivations in semiprime rings and standard operator algebras. *Glasnik Matematički* 47 (1) (2012) 95–104.
- [17] J. Vukman: Some remarks on derivations in semiprime rings and standard operator algebras. *Glasnik Matematički* 46 (1) (2011) 43–48.
- [18] J. Vukman: Identities with derivations and automorphisms on semiprime rings. *International Journal of Mathematics and Mathematical Sciences* 2005 (7) (2005) 1031–1038.

- [19] J. Vukman: Identities related to derivations and centralizers on standard operator algebras. *Taiwanese Journal of Mathematics* 11 (1) (2007) 255–265.
- [20] J. Vukman, I. Kosi-Ulbl: A note on derivations in semiprime rings. *International Journal of Mathematics and Mathematical Sciences* 2005 (20) (2005) 3347–3350.

*Received:* 23 February 2019

*Accepted for publication:* 4 December 2019

*Communicated by:* Eric Swartz