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Symmetric identity for polynomial sequences satisfying $A'_{n+1}(x) = (n + 1)A_n(x)$

Farid Bencherif, Rachid Boumahdi, Tarek Garici

Abstract. Using umbral calculus, we establish a symmetric identity for any sequence of polynomials satisfying $A'_{n+1}(x) = (n + 1)A_n(x)$ with $A_0(x)$ a constant polynomial. This identity allows us to obtain in a simple way some known relations involving Apostol-Bernoulli polynomials, Apostol-Euler polynomials and generalized Bernoulli polynomials attached to a primitive Dirichlet character.

1 Introduction and preliminaries

With each formal power series $S(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!} \in \mathbb{C}[[z]]$ we can associate a polynomial sequence $(A_k(x))_{k \geq 0}$ defined by

$$\sum_{k=0}^{\infty} A_k(x) \frac{z^k}{k!} = S(z)e^{xz}, \tag{1}$$

or equivalently by $A_k(x) = \sum_{i=0}^k \binom{k}{i} a_i x^{k-i}$. Clearly

$$A'_k(x) = kA_{k-1}(x), \quad k \geq 1, \quad A'_0(x) = 0. \tag{2}$$

Reciprocally, it is not difficult to prove that if a polynomial sequence $(A_k(x))_{k \geq 0}$ satisfies (2) then (1) holds for

$$S(z) = \sum_{k=0}^{\infty} A_k(0) \frac{z^k}{k!}.$$

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Affiliation:

Farid Bencherif – Faculté de Mathématiques, USTHB, Alger, Algérie

E-mail: fbencherif@usthb.dz

Rachid Boumahdi – Ecole Nationale Supérieure d’Informatique, Alger, Algérie

E-mail: r_boumehdi@esi.dz

Tarek Garici – Faculté de Mathématiques, USTHB, Alger, Algérie

E-mail: tgarici@usthb.dz

Apostol-Bernoulli polynomials $(\mathcal{B}_k(x; \lambda))_{k \geq 0}$ and Apostol-Euler polynomials $(\mathcal{E}_k(x; \lambda))_{k \geq 0}$ in the variable x defined for $\lambda \in \mathbb{C}$ by

$$\frac{z}{\lambda e^z - 1} e^{xz} = \sum_{k=0}^{\infty} \mathcal{B}_k(x; \lambda) \frac{z^k}{k!}$$

and for $\lambda \in \mathbb{C} \setminus \{-1\}$ by

$$\frac{2}{\lambda e^z + 1} e^{xz} = \sum_{k=0}^{\infty} \mathcal{E}_k(x; \lambda) \frac{z^k}{k!}$$

provide an example of such polynomials. If χ is a primitive Dirichlet character with conductor $f = f_\chi$, then $B_{n,\chi}(x)$, $n = 0, 1, 2, \dots$, the generalized Bernoulli polynomials attached to χ are defined by the generating function [1]

$$F_\chi(t, x) = \frac{t}{e^{ft} - 1} \sum_{a=1}^f \chi(a) e^{at} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{f}.$$

The polynomial sequence $(B_{n,\chi}(x))_{n \geq 0}$ is another example of polynomial sequence satisfying (1). Furthermore, if $a_0 \neq 0$, the polynomial sequence $(A_k(x))_{k \geq 0}$ is called an Appell sequence [2]. Which is the case for the classical Bernoulli and Euler polynomials $B_k(x)$ and $E_k(x)$ defined respectively by

$$\frac{z}{e^z - 1} e^{xz} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}$$

and

$$\frac{2}{e^z + 1} e^{xz} = \sum_{n=0}^{\infty} E_k(x) \frac{z^k}{k!}.$$

In 2003, motivated by the work of Kaneko [8], Momiyama [10] and Wu et al. [17], Sun [14] derived a general combinatorial identity in terms of polynomials with dual sequences of coefficients. More precisely, he considered the two following polynomial sequences

$$R_k(x) = \sum_{i=0}^k \binom{k}{i} (-1)^i u_i x^{k-i} \quad \text{and} \quad R_k^*(x) = \sum_{i=0}^k \binom{k}{i} (-1)^i u_i^* x^{k-i}, \quad (3)$$

where $(u_n)_{n \geq 0}$ is any complex sequence and $(u_n^*)_{n \geq 0}$ is its dual sequence defined for $k \geq 0$ by $u_k^* = \sum_{i=0}^k \binom{k}{i} (-1)^i u_i$, and proved [14, Theorem 1.1], for any integers $n, m \geq 0$ and $x + y + z = 1$, the following identities

$$\begin{aligned} (-1)^{n-1} \sum_{k=0}^m \binom{m}{k} x^{m-k} \frac{R_{n+k+1}(y)}{n+k+1} \\ + (-1)^{m+1} \sum_{k=0}^n \binom{n}{k} x^{n-k} \frac{R_{m+k+1}^*(z)}{m+k+1} = \frac{m!n!x^{m+n+1}u_0}{(m+n+1)!}, \quad (4) \end{aligned}$$

$$(-1)^{n-1} \sum_{k=0}^m \binom{m}{k} x^{m-k} R_{n+k}(y) + (-1)^m \sum_{k=0}^n \binom{n}{k} x^{n-k} R_{m+k}^*(z) = 0, \quad (5)$$

$$\begin{aligned} & (-1)^n \left\{ \sum_{k=0}^m \binom{m+1}{k} x^{m+1-k} (n+1+k) R_{n+k}(y) \right. \\ & \qquad \qquad \qquad \left. + (n+m+2) R_{n+m+1}(y) \right\} \\ & + (-1)^m \left\{ \sum_{k=0}^n \binom{n+1}{k} x^{n+1-k} (m+1+k) R_{m+k}^*(z) \right. \\ & \qquad \qquad \qquad \left. + (n+m+2) R_{m+n+1}^*(z) \right\} = 0. \quad (6) \end{aligned}$$

Which allowed him to derive various known identities involving Bernoulli numbers.

Chen and Sun [3], by applying the extended Zeilberger’s algorithm, established several recurrence relations for Bernoulli numbers and polynomials which generalize the relations of Momiyama [10], Gessel [6] and Gelfand [5]. Prévost [11], by using the Padé approximation of the exponential function, extended Chen and Sun’s results and obtained numerous recurrence relations involving Apostol-Bernoulli polynomials $\mathcal{B}_k(x; \lambda)$ or Apostol-Euler polynomials $\mathcal{E}_k(x; \lambda)$. Recently, Agoh [1] found some shortened recurrence relations for generalized Bernoulli polynomials attached to a primitive Dirichlet character χ which allowed him to reprove and generalize several identities on classical Bernoulli numbers and polynomials such as Saalschütz-Gelfand [5], von Ettiinghausen-Seidel-Stern-Kaneko’s [15], [16], [13], [8] and Chen and Sun [3] formulas. In particular, he showed [1, Theorem 2.1] that for any integers $n, m \geq 0$ and $s \geq 1$, we have

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \frac{(sf)^{m-k} B_{n+1+k, \chi}(x)}{n+1+k} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(sf)^{n-k} B_{m+1+k, \chi}(x)}{m+1+k} \\ & = \sum_{a=1}^f \chi(a) \sum_{r=0}^{s-1} (a+x+rf)^m (a+x+(r-s)f)^n \\ & \qquad \qquad \qquad + \frac{(-1)^{n+1}}{m+n+1} \binom{n+m}{n}^{-1} (sf)^{n+m+1} B_{0, \chi}(x). \quad (7) \end{aligned}$$

With the use of the well-known relation [1, Eq. (2.4)]

$$B_{n, \chi}(x+sf) = B_{n, \chi}(x) + \sum_{k=0}^{s-1} \sum_{a=1}^f \chi(a) n(a+x+kf)^{n-1}, \quad (8)$$

where n and s are non-negative integers, identity (7) can be deduced from the

following relation, due to He and Zhang [7, Theorem 1.2]

$$\sum_{k=0}^m \binom{m}{k} \frac{\alpha^{m-k} A_{n+k+q}(x)}{(n+q+k)^{\underline{q}}} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{\alpha^{n-k} A_{m+q+k}(x+\alpha)}{(m+q+k)^{\underline{q}}} = \frac{(-1)^{n+1} \alpha^{n+m+1}}{(q-1)!} \int_0^1 (1-t)^m t^n A_{q-1}(x+\alpha t) dt, \tag{9}$$

where $(A_n(x))_{n \geq 0}$ is any sequence of polynomials defined by relation (1), α a complex number, and $n, m \geq 0, q \geq 1$ any integers. Recall that the falling factorial $z^{\underline{q}}$ is a polynomial in z defined by $z^{\underline{0}} = 1$ and

$$z^{\underline{q}} = \prod_{j=0}^{q-1} (z-j),$$

for $q \geq 1$.

It is the main aim of this paper to establish in Theorem 1 a Saalschütz-Gelfand type identity for the polynomial sequence $(A_n(x))_{n \geq 0}$ satisfying (2). This identity allows us to reprove some known identities for generalized Bernoulli polynomials attached to a primitive Dirichlet character due to Agoh [1] and Apostol-Bernoulli polynomials due to Prévost [11, Theorem 2] in a simple way.

2 Main result

Before giving our main result it is convenient to have the following lemma.

Lemma 1. *For any non-negative integers n, m and q , we have the following identity*

$$\sum_{k=0}^m \binom{m}{k} \frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(1+z)^{m+k+q}}{(m+k+q)^{\underline{q}}} = \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{m!(n+q-k-1)!}{(m+n+q-k)!} z^k. \tag{10}$$

Proof. Consider the polynomial $P(z) = z^n(1+z)^m$ and let $Q(z)$ denote the q -fold primitive of $P(z)$ defined by

$$Q(z) = \int_0^z \int_0^{z_1} \cdots \int_0^{z_{q-1}} P(t) dt dz_{q-1} \cdots dz_1. \tag{11}$$

On the one hand, by the Cauchy well-known formula [4, p. 115] for repeated integration, we have

$$Q(z) = \frac{1}{(q-1)!} \int_0^z (z-t)^{q-1} P(t) dt. \tag{12}$$

Changing variable t into $u - 1$ in the right-hand side of (12) gives

$$\begin{aligned} Q(z) &= \frac{1}{(q-1)!} \int_0^{z+1} ((z+1) - u)^{q-1} u^m (u-1)^n du \\ &\quad - \frac{1}{(q-1)!} \int_0^1 (z + (1-u))^{q-1} u^m (u-1)^n du \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{1}{(q-1)!} \int_0^{z+1} ((z+1) - u)^{q-1} u^{m+k} du \\ &\quad - \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} (-1)^n z^k \int_0^1 u^m (1-u)^{n+q-k-1} du. \end{aligned}$$

Recall that for any non-negative integers i and j ,

$$\int_0^1 u^i (1-u)^j du = \frac{i!j!}{(i+j+1)!}$$

and for any non-negative integer k , from the Cauchy formula (12) we have

$$\frac{1}{(q-1)!} \int_0^z (z-u)^{q-1} u^k du = \frac{z^{k+q}}{(k+q)^{\underline{q}}}. \tag{13}$$

From this, we can deduce that

$$\begin{aligned} Q(z) &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(z+1)^{m+k+q}}{(m+k+q)^{\underline{q}}} \\ &\quad - \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} (-1)^n \frac{m!(n+q-k-1)!}{(m+n+q-k)!} z^k. \end{aligned} \tag{14}$$

On the other hand, expanding $P(t)$ in (11) and using the well-known identity

$$\int_0^z \int_0^{z_1} \dots \int_0^{z_{q-1}} t^{n+k} dt dz_{q-1} \dots dz_1 = \frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}},$$

where k is any non-negative integer, give

$$Q(z) = \sum_{k=0}^m \binom{m}{k} \frac{z^{n+k+q}}{(n+k+q)^{\underline{q}}}. \tag{15}$$

Finally, by equalling both expressions (14) and (15) of $Q(z)$ we obtain (10). \square

Theorem 1. Let $S(z) \in \mathbb{C}[[z]]$ be a formal power series and $(A_k(x))_{k \geq 0}$ be a sequence of polynomials given by

$$\sum_{k=0}^{+\infty} A_k(x) \frac{z^k}{k!} = S(z)e^{xz}. \tag{16}$$

For any complex number α and any integers $n, m \geq 0$ and $q \geq 1$, we have

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} \frac{\alpha^{m-k} A_{n+q+k}(x)}{(n+q+k)^{\underline{q}}} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{\alpha^{n-k} A_{m+q+k}(x+\alpha)}{(m+q+k)^{\underline{q}}} \\ = \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \alpha^{n+m+q-k} \frac{m!(n+q-k-1)!}{(m+n+q-k)!} A_k(x). \end{aligned} \tag{17}$$

Proof. Let

$$S(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}.$$

Then, from (16) we have, for $r \geq 0$,

$$A_r(x) = \sum_{k=0}^r \binom{r}{k} a_{r-k} x^k.$$

Inspired by the work of Rota [12], let us consider the linear map L from $\mathbb{C}[x, y]$ to $\mathbb{C}[x]$ defined for any non-negative integers r and s by

$$L(x^r y^s) = a_s x^r. \tag{18}$$

For $\alpha = 0$, it is obvious that equation (17) holds. Assume that $\alpha \neq 0$. As for any $r \geq 0$ we have

$$\begin{aligned} L((x+y)^r) &= \sum_{i=0}^r \binom{r}{i} a_{r-i} x^i = A_r(x), \\ L((x+\alpha+y)^r) &= \sum_{i=0}^r \binom{r}{i} a_{r-i} (x+\alpha)^i = A_r(x+\alpha), \end{aligned}$$

replacing z with $(\frac{x+y}{\alpha})$ in (10) and applying L we get the desired relation (17). \square

Remark 1. If I and J denote respectively the right-hand sides of relations (9) and (17), then it is not difficult to show that $I = J$. In fact, for each fixed x and α , the Taylor expansion of the polynomial $A_{q-1}(x + \alpha t)$ in power of t is given by

$$A_{q-1}(x + \alpha t) = \sum_{k=0}^{q-1} \binom{q-1}{k} t^{q-1-k} \alpha^{q-1-k} A_k(x).$$

Thus,

$$J = \frac{\alpha^{n+m+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \alpha^{q-1-k} A_k(x) \int_0^1 t^{n+q-1-k} (1-t)^m dt = I.$$

Which leads to another proof of Theorem 1.2 of He and Zhang [7].

3 Corollaries

Using Theorem 1, we can give an alternative proof of He and Zhang’s theorem [7, Theorem 1.1 and Corollary 1.2] as follows.

Corollary 1. *Let $S(z) \in \mathbb{C}[[z]]$ be a formal power series and $(A_k(x))_{k \geq 0}$ be a sequence of polynomials given by*

$$\sum_{k=0}^{+\infty} A_k(x) \frac{z^k}{k!} = S(z)e^{xz}.$$

For any complex number α and any non-negative integers n, m and q , we have

$$\sum_{k=0}^m \binom{m}{k} (n+k)^q \alpha^{m-k} A_{n-q+k}(x) - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)^q \alpha^{n-k} A_{m-q+k}(x+\alpha) = 0. \quad (19)$$

Proof. From (2) we have for each $k \geq 1$, $A'_k(x) = kA_{k-1}(x)$ and $A'_0(x) = 0$. Taking $q = 1$ in relation (17) and then differentiating the obtained relation $q + 1$ times with respect to x gives (19). \square

Remark 2. Relation (19) is clearly satisfied for $\alpha = 0$. Note that in the case $\alpha \neq 0$ relation (19) can also be obtained by using the same method as in the proof of Theorem 1. Indeed, by differentiating the following identity

$$\sum_{k=0}^m \binom{m}{k} z^{n+k} = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (1+z)^{m+k}$$

q times with respect to z , we obtain

$$\sum_{k=0}^m \binom{m}{k} (n+k)^q z^{n-q+k} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)^q (1+z)^{m-q+k} = 0.$$

Replacing z with $(\frac{x+y}{\alpha})$ and applying the linear map L defined by (18) we get (19).

Next applying Theorem 1 and Corollary 1 for $\alpha = sf$ to the sequence

$$(B_{n,\chi}(x))_{n \geq 0},$$

with the help of relation (8) we get the following two corollaries.

Corollary 2. *For integers $q, s \geq 1$, and $n, m \geq 0$, we have*

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \frac{(sf)^{m-k} B_{n+q+k,\chi}(x)}{(n+q+k)^q} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(sf)^{n-k} B_{m+q+k,\chi}(x)}{(m+q+k)^q} \\ &= \sum_{i=0}^{s-1} \sum_{a=1}^f \chi(a) \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(sf)^{n-k} (a+x+if)^{m+q-1+k}}{(m+q-1+k)^{q-1}} \\ &+ \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{(sf)^{n+m+q-k} m!(n+q-k-1)!}{(m+n+q-k)!} B_{k,\chi}(x). \quad (20) \end{aligned}$$

With the help of relation (13), the right hand-side of (20) can also be written, for $q \geq 2$, as

$$\begin{aligned} & \frac{1}{(q-2)!} \sum_{i=0}^{s-1} \sum_{a=1}^f \chi(a) \int_0^{a+x+if} (a+x+if-u)^{q-2} u^m (u-sf)^n du \\ & + \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{(sf)^{n+m+q-k} m!(n+q-k-1)!}{(m+n+q-k)!} B_{k,\chi}(x). \end{aligned}$$

Taking $q = 1$ in (20) gives immediately Theorem 2.1 of Agoh [1].

Corollary 3. *For integers $s \geq 1$, and $n, m, q \geq 0$, we have*

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (n+k)^{\underline{q}} (sf)^{m-k} B_{n-q+k,\chi}(x) \\ & - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k)^{\underline{q}} (sf)^{n-k} B_{m-q+k,\chi}(x) \\ = & (q+1)! \sum_{a=1}^f \chi(a) \sum_{r=0}^{s-1} \sum_{j=0}^{q+1} \binom{m}{q+1-j} \binom{n}{j} (a+x+rf)^{m-q-1+j} (a+x+(r-s)f)^{n-j}. \end{aligned}$$

One observes that the left hand side of the previous identity can be expressed as

$$\frac{d^{q+1}}{dx^{q+1}} \sum_{a=1}^f \chi(a) \sum_{r=0}^{s-1} (a+x+rf)^m (a+x+(r-s)f)^n.$$

By taking respectively $q = 0$ and $q = 1$ in Corollary 3, we get Theorems 2.3 and 2.4 of Agoh [1].

The following corollary enables us to obtain Sun’s result by giving different values to q .

Corollary 4. *Let $(R_k(x))_{n \geq 0}$ and $(R_k^*(x))_{n \geq 0}$ be the polynomial sequences defined by (3), then for any non-negative integers n, m and q and for any complex number α , we have*

$$\begin{aligned} & (-1)^{n-1} \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} \frac{R_{n+k+q}(x)}{(n+k+q)^{\underline{q}}} \\ & + (-1)^{m+q} \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \frac{R_{m+k+q}^*(1-x-\alpha)}{(m+k+q)^{\underline{q}}} \\ = & \frac{1}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{m!(n+q-k-1)!}{(m+n+q-k)!} \alpha^{m+n+q-k} R_k(x), \quad (21) \end{aligned}$$

$$(-1)^{n-1} \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} (n+k)^q R_{n-q+k}(x) + (-1)^{m-q} \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} (m+k)^q R_{m-q+k}^*(1-x-\alpha) = 0. \quad (22)$$

Proof. If

$$F(z) = \sum_{k=0}^{\infty} u_k \frac{z^k}{k!}$$

denotes the exponential generating function for the sequence u , then it is easy to see that the polynomial sequences $(R_k(x))_{k \geq 0}$ and $(R_k^*(x))_{k \geq 0}$ satisfy

$$\sum_{k=0}^{+\infty} R_k(x) \frac{z^k}{k!} = F(-z)e^{zx} \quad \text{and} \quad \sum_{k=0}^{+\infty} R_k^*(x) \frac{z^k}{k!} = F(z)e^{z(x-1)}.$$

Thus, it comes immediately that

$$R_k(x) = (-1)^k R_k^*(1-x).$$

With this, a direct application of Theorem 1 and Corollary 1 for the polynomial sequence $(R_k(x))_{k \geq 0}$ gives (21) and (22). \square

For $q = 1$ in (21) and $q \in \{0, 1\}$ in (22) we obtain respectively Sun's results (4), (5), and (6).

To conclude this section, we deal with Apostol-Bernoulli polynomials $(\mathcal{B}_k(x; \lambda))_{k \geq 0}$ defined [9] for any complex number λ , by means of the following generating function

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{k=0}^{\infty} \mathcal{B}_k(x; \lambda) \frac{z^k}{k!},$$

with $|z + \ln \lambda| < 2\pi$. Let ℓ be a positive integer. Expanding the both sides of the functional relation

$$\frac{\lambda^\ell z e^{(\ell+x)z}}{\lambda e^z - 1} = \frac{z e^{xz}}{\lambda e^z - 1} + \sum_{i=0}^{\ell-1} \lambda^i z e^{iz} e^{xz}$$

into power series of z and then equating the coefficients of $\frac{z^k}{k!}$ ($k \geq 0$), yields

$$\lambda^\ell \mathcal{B}_0(x + \ell; \lambda) = \mathcal{B}_0(x; \lambda)$$

and

$$\lambda^\ell \mathcal{B}_k(x + \ell; \lambda) = \mathcal{B}_k(x; \lambda) + \sum_{i=0}^{\ell-1} \lambda^i k(x+i)^{k-1}$$

for $k \geq 1$. One can see that the previous equality also holds true even when $k = 0$. Therefore, for any positive integer ℓ and any non-negative integer k

$$\lambda^\ell \mathcal{B}_k(x + \ell; \lambda) = \mathcal{B}_k(x; \lambda) + \sum_{i=0}^{\ell-1} \lambda^i k(x+i)^{k-1}. \quad (23)$$

Theorem 1 enables us to obtain simply the following corollary due to Prévost [11, Theorem 2].

Corollary 5. For any integers $n, m, p \geq 0$ and $\ell \geq 1$, if $p \geq n + m + 1$ then

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \lambda^\ell \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} - \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} \\ &= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(n+m-k)! (x+i)^{p-k-1}}{(p-k-1)! \ell^{p-k}} \\ & \quad - \lambda^\ell \frac{(-1)^n n! m!}{p!} \sum_{k=0}^{p-m-n-1} \binom{p}{k} \binom{p-m-k-1}{n} \frac{\mathcal{B}_k(x; \lambda)}{\ell^k}, \end{aligned} \tag{24}$$

and if $p \leq n + m$ then

$$\begin{aligned} & \sum_{k=0}^{\min\{m,p\}} \binom{m}{k} \lambda^\ell \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} \\ & \quad - \sum_{k=0}^{\min\{n,p\}} \binom{n}{k} (-1)^k \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} \\ &= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^{\min(p-1,n)} \binom{n}{k} (-1)^k \frac{(n+m-k)! (x+i)^{p-k-1}}{(p-k-1)! \ell^{p-k}}. \end{aligned} \tag{25}$$

Proof. Applying Theorem 1 and Corollary 1 for $S(z) = \frac{z}{\lambda e^z - 1}$ and $\alpha = \ell$, multiplying the whole by λ^ℓ and using identity (23) give

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \frac{\lambda^\ell \mathcal{B}_{n+q+k}(x; \lambda)}{(n+q+k) \underline{q} \ell^{n+q+k}} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{\mathcal{B}_{m+q+k}(x; \lambda)}{(m+q+k) \underline{q} \ell^{m+q+k}} \\ &= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(m+q+k)(x+i)^{m+q+k-1}}{(m+q+k) \underline{q} \ell^{m+q+k}} \\ & \quad + \lambda^\ell \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{m!(n+q-k-1)! \mathcal{B}_k(x; \lambda)}{(n+m+q-k)! \ell^k}, \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (n+k) \underline{q} \frac{\lambda^\ell \mathcal{B}_{n-q+k}(x; \lambda)}{\ell^{n-q+k}} - \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k) \underline{q} \frac{\mathcal{B}_{m-q+k}(x; \lambda)}{\ell^{m-q+k}} \\ &= \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (m+k) \underline{q} \frac{(m-q+k)(x+i)^{m-q+k-1}}{\ell^{m-q+k}}. \end{aligned}$$

Reverse the order of summations in the two previous relations to obtain

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \frac{\lambda^\ell \mathcal{B}_{n+m+q-k}(x; \lambda)}{(n+m+q-k)^\underline{q} \ell^{n+m+q-k}} \\ & \quad - \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{\mathcal{B}_{n+m+q-k}(x; \lambda)}{(n+m+q-k)^\underline{q} \ell^{m+n+q-k}} \\ & = \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(n+m+q-k)(x+i)^{n+m+q-k-1}}{(n+m+q-k)^\underline{q} \ell^{n+m+q-k}} \\ & \quad + \lambda^\ell \frac{(-1)^{n+1}}{(q-1)!} \sum_{k=0}^{q-1} \binom{q-1}{k} \frac{m!(n+q-k-1)! \mathcal{B}_k(x; \lambda)}{(n+m+q-k)! \ell^k}, \quad (26) \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (n+m-k)^\underline{q} \frac{\lambda^\ell \mathcal{B}_{n+m-q-k}(x; \lambda)}{\ell^{n+m-q-k}} \\ & \quad - \sum_{k=0}^n \binom{n}{k} (-1)^k (n+m-k)^\underline{q} \frac{\mathcal{B}_{n+m-q-k}(x; \lambda)}{\ell^{n+m-q-k}} \\ & = \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^k (n+m-k)^\underline{q} \frac{(n+m-q-k)(x+i)^{n+m-q-k-1}}{\ell^{n+m-q+k}}. \quad (27) \end{aligned}$$

Suppose first that $p \geq n+m+1$. Substituting q by $p-n-m$ in (26) gives

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \lambda^\ell \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} - \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(n+m-k)! \mathcal{B}_{p-k}(x; \lambda)}{(p-k)! \ell^{p-k}} \\ & = \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{(n+m-k)!(x+i)^{p-k-1}}{(p-k-1)! \ell^{p-k}} \\ & \quad - \lambda^\ell \frac{(-1)^n n! m!}{p!} \sum_{k=0}^{p-n-m-1} \binom{p}{k} \binom{p-m-k-1}{n} \frac{\mathcal{B}_k(x; \lambda)}{\ell^k}. \end{aligned}$$

Thus, we obtain relation (24). Suppose now that $p \leq n+m$. Taking $q = n+m-p$ in (27) gives

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} (n+m-k)^{n+m-p} \frac{\lambda^\ell \mathcal{B}_{p-k}(x; \lambda)}{\ell^{p-k}} \\ & \quad - \sum_{k=0}^n \binom{n}{k} (-1)^k (n+m-k)^{n+m-p} \frac{\mathcal{B}_{p-k}(x; \lambda)}{\ell^{p-k}} \\ & = \sum_{i=0}^{\ell-1} \lambda^i \sum_{k=0}^n \binom{n}{k} (-1)^k (n+m-k)^{n+m-p} \frac{(p-k)(x+i)^{p-k-1}}{\ell^{p+k}}. \quad (28) \end{aligned}$$

Noting that the factor $(n+m-k)^{\frac{n+m-p}{p-k}}$ in the left hand side vanishes when $k > p$, and the factor $(n+m-k)^{\frac{n+m-p}{p-k}}$ in the right hand side vanishes when $k > p-1$, relation (28) leads to relation (25). \square

Remark 3. In a similar way, by making use of

$$\lambda^\ell \mathcal{E}_k(x+\ell; \lambda) = (-1)^\ell \mathcal{E}_k(x; \lambda) + 2(-1)^{\ell-1} \sum_{i=0}^{\ell-1} (-1)^i \lambda^i (x+i)^k,$$

we obtain an identity due to Prévost [11, Theorem 2], for Apostol-Euler polynomials $(\mathcal{E}_k(x; \lambda))_{k \geq 0}$ defined [9] for $\lambda \in \mathbb{C} \setminus \{-1\}$ by means of the following generating function

$$\frac{2}{\lambda e^z + 1} e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda) \frac{z^n}{n!} \quad (|z + \ln \lambda| < \pi).$$

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