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## Finite groups with prime graphs of diameter 5

Ilya B. Gorshkov, Andrey V. Kukharev

**Abstract.** In this paper we consider a prime graph of finite groups. In particular, we expect finite groups with prime graphs of maximal diameter.

### 1 Introduction

In this paper, all groups are finite. The prime graph  $\Gamma(G)$  of a group  $G$  is defined as follows. The vertex set is  $\pi(G)$  the set of prime divisors of the order of  $G$ , and two distinct primes  $r, s \in \pi(G)$  considered as vertices of the graph are adjacent if and only if there exists an element  $g \in G$  such that  $|g| = rs$ .

We say that the graph  $\Gamma$  is isomorphic to the prime graph of the group  $G$  if the vertices of the graph  $\Gamma$  can be marked of primes so that the resulting graph  $\Gamma'$  is equal to the graph  $\Gamma(G)$ .

The structure of a finite group  $G$  such that  $\Gamma(G)$  is disconnected has been determined by Williams [6]. All simple groups  $G$  such that  $\Gamma(G)$  is disconnected have been described in [2] and [6].

The complement of the prime graph of a solvable group does not contain triangles that have been proved by Lucido [3]. In [1] were completely classified of a prime graphs of solvable groups. In particular, the graph  $\Gamma$  is isomorphic to  $\Gamma(G)$ , where  $G$  is a solvable group if and only if the complement of  $\Gamma$  is without triangles and 3-colorable. However, the general question remains open.

**Question 1.** *When a graph  $\Gamma$  is isomorphic to a prime graph of some group?*

In this paper we consider the diameter of  $\Gamma(G)$ . We write  $d(p, q)$  to denote the distance between elements  $p$  and  $q$  if they are in the same connected component of  $\Gamma(G)$ . We define the diameter of  $\Gamma(G)$  as follows:

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$$\text{diam}(\Gamma(G)) = \max\{d(p, q) \mid p, q \text{ in the same connected component of } \Gamma(G)\}.$$

Lucido [3] proved that  $\text{diam}(\Gamma(G)) \leq 5$ . Moreover, she classifies all almost simple groups with property that the diameter of its prime graph is equal to 5.

**Theorem 1.** [3, Proposition 12] *Let  $G$  be an almost simple group such that  $\text{diam}(\Gamma(G)) = 5$ . Then  $G$  has the following structure:*

$G \simeq E_8(q) \cdot H$ , where  $q = p^n$ ,  $q \equiv 0, 1 \text{ or } 4 \pmod{5}$ ,  $n = 3^s 5^t$  where  $s, t > 0$  and  $H = \langle \gamma \rangle$ , where  $\gamma$  is a field automorphism of order  $n$ .

The main goal of this paper is to describe groups with a prime graph of diameter 5.

A chain graph is a graph without any cycles. Denote by  $s(G)$  the number of connected components of  $\Gamma(G)$  and by  $\pi_i(G), i = 1, \dots, s(G)$ , its  $i$ th connected component. If  $G$  has even order, then put  $2 \in \pi_1(G)$ .

**Theorem 2.** *Let  $G$  be a group such that  $\text{diam}(\Gamma(G)) = 5$  and  $a, b_1, b_2, b_3, b_4, c$  is a chain in  $\Gamma(G)$ . Then  $G = K.S.A$  where  $K$  is a nilpotent subgroup,  $S$  is a simple group and  $A \leq \text{Out}(S)$ ,  $\{a, b_1, b_4, c\} \subseteq \pi(S) \setminus (\pi(K) \cup \pi(A))$ . Furthermore one of the following statements holds:*

1.  $\text{diam}(\Gamma(G) \setminus \Gamma(K)) = 5$ . In particular,  $G/K$  is an almost simple group satisfying Lucido theorem.
2. If  $b_2 \in \pi(K)$ , then  $b_3 \in \pi(K) \cup \pi(A)$ . The graph  $\Gamma(S)$  contains more than 2 connected components and  $\{a, b_1, b_4, c\} \cap \pi_1(S) = \emptyset$ .

**Remark 1.** Note that, when  $G$  satisfies the statement 1 of Theorem 2, then  $G/K$  satisfies the Lucido theorem. Hence,  $G/K \simeq E_8(q).H$ , where  $q = p^n$ ,  $q \equiv 0, 1 \text{ or } 4 \pmod{5}$ ,  $n = 3^s 5^t$  where  $s, t > 0$  and  $H = \langle \gamma \rangle$ , where  $\gamma$  is a field automorphism of order  $n$ . The existence of groups satisfying the statement 2 of Theorem 2 depends on the validity of the following conjecture.

**Conjecture 1.** *Let  $S$  be a finite simple group with disconnected prime graph,  $a, b \in \pi_2(S)$  be different primes and  $x, y \in S$  be elements of order  $a$  and  $b$  respectively. If there exists an irreducible  $S$ -module  $M$  such that  $C_M(x) > 1$ , then  $C_M(y) > 1$ .*

It is easy to see that if the conjecture is true, then statement 2 of Theorem 2 is impossible.

Williams [6] and Kondratev [2] described simple groups  $G$  such that  $\Gamma(G)$  is disconnected. In particular, from this description follows that  $S$  is isomorphic  $A_1(q)$ , where  $3 < q \equiv \pm 1(4)$ , or simple group of exceptional Lie type.

## 2 Definitions and preliminary results

**Lemma 1.** [1, Theorem 2] *An unlabeled graph  $F$  is isomorphic to the prime graph of some solvable group if and only if its complement  $\overline{F}$  is 3-colorable and triangle-free.*

**Lemma 2.** [3, Proposition 1] *Let  $G$  be a finite solvable group. If  $p, q, r$  are three distinct primes of  $\pi(G)$ , then  $G$  contains an element of order the product of two of these primes.*

**Lemma 3.** [3, Lemma 5] *Let  $G$  be a finite simple group. If  $p \in \pi_1(G)$ , then  $d(2, p) \leq 2$ .*

For graph  $\Gamma$  we will denote  $\overline{\Gamma}$  the complement of  $\Gamma$ . Denote by  $t(G)$  the maximal number of primes in  $\pi(G)$  pairwise nonadjacent in  $\Gamma(G)$ . In graph theory this number is called the independence number of a graph. By analogy we denote by  $t(2, G)$  the maximal number of vertices in the independent sets of  $\Gamma(G)$  containing 2. Let  $\omega(G)$  be the spectrum of  $G$ ; i.e., the set of element orders of  $G$ . Denote by  $\omega_i(G)$  the set of  $n \in \omega(G)$  such that every prime divisor of  $n$  belongs to  $\pi_i(G)$ .

**Lemma 4.** [4] *Let  $G$  be a finite group satisfying the two conditions:*

- (a) *there exist three primes in  $\pi(G)$  pairwise nonadjacent in  $\Gamma(G)$ ;*
- (b) *there exists an odd prime in  $\pi(G)$  nonadjacent in  $\Gamma(G)$  to the prime 2.*

*Then there exists a non-abelian simple group  $S$  such that  $S \leq G = G/K \leq \text{Aut}(S)$  for the maximal normal solvable subgroup  $K$  of  $G$ . Furthermore,  $t(S) \geq t(G) - 1$ , and one of the following statements holds:*

(1)  *$S \simeq A_7$  or  $L_2(q)$  for some odd  $q$ , and  $t(S) = t(2, S) = 3$ .*

(2) *For every prime  $p \in \pi(G)$  nonadjacent to 2 in  $\Gamma(G)$  a Sylow  $p$ -subgroup of  $G$  is isomorphic to a Sylow  $p$ -subgroup of  $S$ . In particular,  $t(2, S) \geq t(2, G)$ .*

**Lemma 5.** [5, Lemma 3.6] *Let  $s$  and  $p$  be distinct primes, a group  $H$  be a semidirect product of a normal  $p$ -subgroup  $T$  and a cyclic subgroup  $C = \langle g \rangle$  of order  $s$ , and let  $[T, g] \neq 1$ . Suppose that  $H$  acts faithfully on a vector space  $V$  of positive characteristic  $t$  not equal to  $p$ . If the minimal polynomial of  $g$  on  $V$  does not equal  $x^s - 1$ , then*

- (i)  *$C_T(g) \neq 1$ ;*
- (ii)  *$T$  is non-abelian;*
- (iii)  *$p = 2$  and  $s = 2^{2^\delta} + 1$  is a Fermat prime.*

**Lemma 6.** [7, Lemma 2.12] *Let  $S$  be a finite simple group of Lie type. If  $r, s, t \in \pi(S)$  and  $rt, st \in \omega(S)$ , but  $rs \notin \omega(S)$ , then a Sylow  $t$ -subgroup of  $S$  is not cyclic.*

## 3 Proof of Theorem 2

Let  $G$  be such that  $\text{diam}(\Gamma(G)) = 5$ ,  $a, b_1, b_2, b_3, b_4, c \in V(\Gamma(G))$  be a path of length 5. The numbers  $a, b_2, c$  is a triangle in  $\Gamma(G)$ . Lemma 2 implies that  $G$  is non-solvable. Therefore,  $2 \in \pi(G)$ . Moreover, since  $\text{diam}(\Gamma(G)) > 2$  it follows that  $t(2, G) \geq 3$ . Using Lemma 4, we get  $G \simeq K.S.A$ , where  $K$  is a solvable radical of  $G$ ,  $S$  is a non-abelian simple group and  $A \leq \text{Out}(S)$ .

**Lemma 7.**  $S \not\cong L_2(q)$  for odd  $q$ .

*Proof.* Notice, that  $\Gamma(L_2(q))$  is a graph with three connected components every of which is complete. In particular,  $\Gamma(L_2(q))$  does not contains chains.

The proof is by contradiction. Let  $S \simeq L_2(q)$  for some  $q = p^n$  where  $p$  is odd prime. Assume that  $a \in \pi(K)$ . Lemma 4 implies that  $b_2, b_3, b_4, c \in \pi(S) \setminus \pi(|K||A|)$ . Therefore,  $\Gamma(S)$  contains a chain of length 4; a contradiction with the fact that  $\Gamma(S)$  does not contain chain. By analogy, we can show that  $\pi(K)$  does not contain  $b_1, b_4, c$ .

Assume that  $b_2 \in \pi(K)$ . Therefore,  $a, b_1 \in \pi((q - 1)/2)$  or  $c, b_4 \in \pi((q - 1)/2)$ . Let  $r \in \{a, c\} \cap \pi((q - 1)/2)$ . We have that  $L_2(q)$  includes a Frobenius group  $F$  with kernel  $Q$  of order  $q$  and complement  $K$  of order  $(q - 1)/2$ . Let  $\overline{G} = T \triangleleft G$  such that  $T < K$  and  $K/T$  includes a normal Sylow  $b_2$ -subgroup  $H$ . Let  $P$  be a Sylow  $p$ -subgroup of  $\overline{G}$ . Assume that  $P < C_{\overline{G}}(H)$ . Since  $H$  is a normal subgroup of  $\overline{G}$  we have that  $X = \langle P^G \rangle < C_{\overline{G}}(H)$ . We have that  $X\overline{K}/\overline{K}$  is a normal subgroup of  $S$ . Therefore,  $X\overline{K}/\overline{K} = S$  and  $r \in \pi(C_{\overline{G}}(H))$ ; a contradiction. Therefore,  $P$  acts no trivial on  $H$ . From Frattini's argument it follows that  $N_{\overline{G}}(P)$  includes a  $r$ -subgroup  $T$  such that  $T\overline{K}/\overline{K}$  is a Sylow  $r$ -subgroup of  $S$ . From Lemma 5 it follows that  $C_{\overline{G}}(H)$  contains an element of order  $b_2r$ ; a contradiction. By analogy, we can show that  $\pi(K)$  does not contain  $b_3$ . Thus  $a, b_1, b_2, b_3, b_4, c \in \pi(S.A)$ ; a contradiction with Theorem 1. □

**Lemma 8.**  $S \not\cong Alt_7$ .

*Proof.* Assume that  $S \simeq Alt_7$ . Then a Sylow 3-subgroup of  $S$  is an elementary abelian group of order 9. Therefore, for every  $p \in \pi(K)$  we have  $3p \in \omega(G)$ . Using this fact it is easy show that  $\text{diam}(G) \leq 3$ . □

**Lemma 9.** If  $t \in \pi(K)$ , then  $2t \in \omega(G)$ .

*Proof.* Assume that there exists  $t \in \pi(K)$  such that  $2t \notin \omega(G)$ . Let  $R \triangleleft G$  be such that  $\overline{K} = K/R$  includes a normal  $t$ -subgroup  $T$ . Let  $P \in Syl_2(K.S/R)$ . We have that  $P$  is cyclic or generalized quaternion group. Since  $PR/R$  is a Sylow subgroup of  $S$ , we see that  $PR/R$  is a dihedral group. A finite simple group has a dihedral Sylow 2-subgroup only in the case isomorphic to  $Alt_7$  or  $L_2(q)$ . From Lemmas 7 and 8 we have that  $S$  does not isomorphic  $L_2(q)$  and  $Alt_7$ . □

**Lemma 10.**  $b_1, b_4, c, a \notin \pi(K)$ .

*Proof.* Assume that  $c \in \pi(K)$ . Lemma 4 implies that  $a, b_1, b_2, b_3 \in \pi(S) \setminus (\pi(A) \cap \pi(K))$ . Since  $\pi_n(S)$ , where  $n > 1$ , is a full component of  $\Gamma(S)$ , we see that  $a, b_1, b_2, b_3 \in \pi_1(S)$ . From Lemma 9 it follows that  $2c \in \omega(G)$ . Therefore,  $\omega(G)$  does not contain an element of order  $2t$  for  $t \in \{a, b_1, b_2\}$ . Since  $a, b_1, b_2, b_3 \in \pi_1(S)$ , we obtain  $d_S(2, a) > 3$ . It is a contradiction with Lemma 3.

Assume that  $b_1 \in \pi(K)$ . From Lemma 4 it follows that  $b_3, c \in \pi(S)$ . If  $b_4 \in \pi(S)$ , then similar as above, we can get a contradiction. Let  $b_4 \in \pi(K)$ . From the Frattini argument we have that automorphisms groups of Sylow  $b_1$ - and

$b_4$ -subgroups are non-solvable. Therefore, Sylow  $b_1$ - and  $b_2$ -subgroups are not cyclic, in particular  $b_1b_4 \in \omega(K)$ ; a contradiction. Assume that  $b_4 \in \pi(A)$ . Hence,  $2b_4 \in \omega(S.A)$ . Since  $S \not\cong Alt_7$  and  $S \not\cong L_2(q)$ , we see that  $2b_1 \in \omega(G)$ . Thus  $a, b_1, 2, b_4, c$  is a chain in  $\Gamma(G)$ ; a contradiction. Therefore,  $b_1 \in \pi(K)$ . Similarly, it can be shown that  $a, b_4 \notin \pi(K)$ .  $\square$

**Lemma 11.**  $\{b_1, b_4, c, a\} \cap \pi(A) = \emptyset$ .

*Proof.* Similar to Lemma 10.  $\square$

**Lemma 12.**  $K$  is nilpotent.

*Proof.* Assume that  $\pi(K)$  includes  $p, q$  such that a Hall  $\{p, q\}$ -subgroup  $H$  of  $K$  does not nilpotent. From the Frattini argument it follows that  $N_G(H)$  includes a subgroup  $T$  such that  $T/N_K(H) \simeq S$ . Let  $X \triangleleft N_G(H)$  be such that a Hall  $\{p, q\}$ -subgroup of  $N_K(H)/X$  is not nilpotent but for every  $X < Y \triangleleft N_G(H)$  a Hall  $\{p, q\}$ -subgroup of  $N_K(H)/Y$  is nilpotent. Denote by  $\tilde{\cdot} : N_G(H) \rightarrow N_G(H)/X$  the natural homomorphism. Let  $N = N_G(H)$ . From definition follows that  $\tilde{N}$  includes a normal  $t$ -subgroup  $T$ , where  $t \in \{p, q\}$ . Let  $r \in \{p, q\} \setminus \{t\}$ ,  $R \in Syl_r(\tilde{N})$ . From definition of  $\tilde{N}$  follows that  $R$  acts no trivial on  $T$ . The Frattini argument implies that  $N_{\tilde{G}}(R)/N_{\tilde{K}(R)}$  includes a subgroup isomorphic  $S$ . Let  $x, y \in N_{\tilde{G}}(R)$  be elements of order  $a$  and  $c$  respectively. Assume that  $Z(R) < C_{\tilde{G}}(x)$ . Therefore,  $\langle x^{N_{\tilde{G}}(R)} \rangle < C_{\tilde{G}}(Z(R))$ . We know that  $\langle x^{N_{\tilde{G}}(R)} \rangle / \langle x^{N_{\tilde{G}}(R)} \rangle \cap N_{\tilde{K}}(R) \simeq S$ . Therefore,  $y \in C_{\tilde{G}}(Z(R))$  and  $d(a, c) = 2$ ; a contradiction. We have  $x \notin C_{\tilde{G}}(Z(R))$ . Similar we can show that  $x \notin C_{\tilde{G}}(Z(R))$ . Applying Lemma 5 for groups  $T.Z(R).\langle x \rangle$  and  $T.Z(R).\langle y \rangle$  we obtain  $\tilde{G}$  contains elements of order  $ta$  and  $tc$ , in particular,  $d(a, c) = 2$ ; a contradiction.

We show that for every  $p, q \in \pi(K)$  a Hall  $\{p, q\}$ -subgroup of  $K$  is nilpotent. Therefore, Sylow  $p$ - and  $q$ -subgroups are commute. Hence,  $K$  is nilpotent.  $\square$

**Lemma 13.** If  $b_2 \in \pi(K) \setminus \pi(G/K)$ , then  $b_3 \in \pi(K) \cup \pi(A)$ . The graph  $\Gamma(S)$  contains more than 2 connected components and  $a, b_1, b_4, c \notin \pi_1(S)$ .

*Proof.* From Lemmas 10 and 11 follow that  $a, b_1, b_4, c \in \pi(S)$ . Assume that  $c, b_4 \in \pi_1(S)$ . Since  $d(2, c) = 2$  there exists  $h \in \pi(S)$  such that  $2h, ch \in \omega(S)$ . From Lemma 6 follows that a Sylow  $h$ -subgroup of  $S$  is not cyclic. Consequently  $hb_2 \in \omega(G)$ . Thus  $a, b_1, b_2, h, c$  is a chain in  $\Gamma(G)$ , in particular  $d(a, c) < 5$ ; a contradiction. We have  $c, b_4 \in \pi_2(S)$ . Since  $\pi_2(G)$  is a full component of  $\Gamma(S)$ , we see that  $b_3 \in \pi(K) \cup \pi(A)$ . Assume that  $a, b_1 \in \pi_1(S)$ . If  $b_3 \in \pi(K)$ , then similar as above we can show that  $a, c \notin \pi_1(S)$ ; a contradiction. Therefore,  $b_3 \in \pi(A)$ . In this case we know that  $2b_3 \in \omega(S.A)$ . Hence,  $a, b_1, 2, b_3, b_4, c$  is a chain of  $\Gamma(A.S)$ . From Lucido Theorem 1 follows that  $a, b_1 \notin \pi_1(S)$ ; a contradiction.  $\square$

The main assertion of the theorem follows from Lemmas 10, 11 and 12. The statements 1) and 2) of Theorem follows from Lemmas 1 and 13.

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