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Deformations of Metrics and Biharmonic Maps

Aicha Benkartab, Ahmed Mohammed Cherif

Abstract. We construct biharmonic non-harmonic maps between Riemannian manifolds (M, g) and (N, h) by first making the ansatz that $\varphi: (M, g) \rightarrow (N, h)$ be a harmonic map and then deforming the metric on N by

$$\tilde{h}_\alpha = \alpha h + (1 - \alpha)df \otimes df$$

to render φ biharmonic, where f is a smooth function with gradient of constant norm on (N, h) and $\alpha \in (0, 1)$. We construct new examples of biharmonic non-harmonic maps, and we characterize the biharmonicity of some curves on Riemannian manifolds.

1 Introduction

Let (M, g) and (N, h) be two Riemannian manifolds. The energy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v^g, \quad (1)$$

where $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$ and v^g is the volume element on (M, g) . A map $\varphi \in C^\infty(M, N)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1)

$$\tau(\varphi) = \text{trace } \nabla d\varphi = \nabla_{e_i}^\varphi d\varphi(e_i) - d\varphi(\nabla_{e_i}^M e_i) = 0, \quad (2)$$

where $\{e_i\}_{i=1}^m$ is an orthonormal frame on (M, g) , $m = \dim M$, ∇^M is the Levi-Civita connection of (M, g) , and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$.

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Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important role in many branches of mathematics and physics where they may serve as a model for liquid crystal (see [9]). One can refer to [6], [7], [8] for background on harmonic maps. A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v^g. \quad (3)$$

A map $\varphi \in C^\infty(M, N)$ is called biharmonic if it is a critical point of the bi-energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (3)

$$\begin{aligned} \tau_2(\varphi) &= -\text{trace } R^N(\tau(\varphi), d\varphi)d\varphi - \text{trace}(\nabla^\varphi)^2 \tau(\varphi) \\ &= -R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi \tau(\varphi) + \nabla_{\nabla_{e_i}^M e_i}^\varphi \tau(\varphi) = 0, \end{aligned} \quad (4)$$

where R^N is the curvature tensor of (N, h) (see [5], [12]). Clearly, harmonic maps are biharmonic. G.Y. Jiang [12] proved that if M is compact without boundary and the sectional curvature of (N, h) is negative, then any biharmonic map $\varphi \in C^\infty(M, N)$ is harmonic. So it is interesting to construct biharmonic non-harmonic maps. We refer the reader to [2], [5], [10], [11] for other examples and different approaches to their construction.

In this paper, we deform the codomain metric by $\tilde{h}_\alpha = \alpha h + (1-\alpha)df \otimes df$, where $\alpha \in (0, 1)$ and $f \in C^\infty(N)$, in order to render a map biharmonic non-harmonic with respect to the new metric, we give a necessary and sufficient condition on f and α such that $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$ is biharmonic non-harmonic. So by suitable choices of f , we are able to give new examples of biharmonic non-harmonic maps.

2 Deformations of Metrics

Let M be a Riemannian manifold equipped with Riemannian metric g , and f a smooth function on M . We define on M a Riemannian metric, denoted \tilde{g}_α , by

$$\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df,$$

for some constant $\alpha \in (0, 1)$. In the seminal work [4], we obtain the following results.

Theorem 1. *Let (M, g) be a Riemannian manifold and $\tilde{\nabla}$ denote the Levi-Civita connection of (M, \tilde{g}_α) . Then*

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{(1 - \alpha) \text{Hess}_f(X, Y)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f,$$

where ∇ is the Levi-Civita connection of (M, g) , Hess_f (resp. $\text{grad } f$) is the Hessian (resp. the gradient vector) of f with respect to g , and

$$\|\text{grad } f\|^2 = g(\text{grad } f, \text{grad } f).$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. From the Koszul formula (see [13]), we have

$$\begin{aligned} 2\tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) &= 2\alpha g(\nabla_X Y, Z) + (1 - \alpha) \left\{ X(Y(f)Z(f)) + Y(Z(f)X(f)) \right. \\ &\quad - Z(X(f)Y(f)) + Z(f)[X, Y](f) + Y(f)[Z, X](f) \\ &\quad \left. - X(f)[Y, Z](f) \right\}. \end{aligned} \quad (5)$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame on (M, g) at $x \in M$ (see [3]), where $m = \dim M$. By (5) we obtain

$$\begin{aligned} 2\tilde{g}_\alpha(\tilde{\nabla}_X Y, e_i) &= 2\alpha g(\nabla_X Y, e_i) + (1 - \alpha) \left\{ X(Y(f)g(e_i, \text{grad } f)) \right. \\ &\quad + Y(X(f)g(e_i, \text{grad } f)) - e_i(g(X, \text{grad } f)g(Y, \text{grad } f)) \\ &\quad \left. + e_i(f)[X, Y](f) + Y(f)(\nabla_{e_i} X)(f) + X(f)(\nabla_{e_i} Y)(f) \right\}, \end{aligned} \quad (6)$$

from equation (6), and the definition of Hessian (see [13]), we get

$$\begin{aligned} \tilde{g}_\alpha(\tilde{\nabla}_X Y, e_i) &= \alpha g(\nabla_X Y, e_i) + (1 - \alpha)g(\nabla_X Y, \text{grad } f)g(e_i, \text{grad } f) \\ &\quad + (1 - \alpha)\text{Hess}_f(X, Y)g(e_i, \text{grad } f), \end{aligned} \quad (7)$$

from equation (7), we obtain

$$\begin{aligned} \tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) &= \alpha g(\nabla_X Y, Z) + (1 - \alpha)g(\nabla_X Y, \text{grad } f)g(Z, \text{grad } f) \\ &\quad + (1 - \alpha)\text{Hess}_f(X, Y)g(Z, \text{grad } f), \end{aligned} \quad (8)$$

by the definition of the Riemannian metric \tilde{g}_α , and (8) we find that

$$\tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) = \tilde{g}_\alpha(\nabla_X Y, Z) + (1 - \alpha)\text{Hess}_f(X, Y)Z(f). \quad (9)$$

Hence Theorem 1 follows from (9), with the following

$$Z(f) = \frac{1}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \tilde{g}_\alpha(Z, \text{grad } f). \quad \square$$

Now consider the curvature tensor \tilde{R} of (M, \tilde{g}_α) , writing R for the curvature tensor of (M, g) . We have the following result:

Theorem 2. For all $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \frac{(1 - \alpha)g(R(X, Y)\text{grad } f, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \text{grad } f \\ &\quad - \frac{(1 - \alpha)^2 \text{Hess}_f(X, \text{grad } f) \text{Hess}_f(Y, Z)}{(\alpha + (1 - \alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\ &\quad + \frac{(1 - \alpha)^2 \text{Hess}_f(Y, \text{grad } f) \text{Hess}_f(X, Z)}{(\alpha + (1 - \alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\ &\quad + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \nabla_X \text{grad } f \\ &\quad - \frac{(1 - \alpha) \text{Hess}_f(X, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \nabla_Y \text{grad } f. \end{aligned}$$

Proof. By the definition of the curvature tensor \tilde{R} ,

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z,$$

and Theorem 1 we obtain

$$\begin{aligned} \tilde{R}(X, Y)Z &= \tilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad - \tilde{\nabla}_Y \left(\nabla_X Z + \frac{(1-\alpha) \text{Hess}_f(X, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad - \left(\nabla_{[X, Y]} Z + \frac{(1-\alpha) \text{Hess}_f([X, Y], Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right), \end{aligned} \quad (10)$$

the first term of (10) is given by

$$\begin{aligned} &\tilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &= \nabla_X \left(\nabla_Y Z + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad + \frac{(1-\alpha) \text{Hess}_f \left(X, \nabla_Y Z + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f, \end{aligned} \quad (11)$$

by equation (11), and the definition of Hessian, we obtain

$$\begin{aligned} &\tilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &= \nabla_X \nabla_Y Z + \frac{(1-\alpha) g(\nabla_X \nabla_Y \text{grad } f, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\ &\quad + \frac{(1-\alpha) \text{Hess}_f(Y, \nabla_X Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\ &\quad - \frac{(1-\alpha)^2 \text{Hess}_f(X, \text{grad } f) \text{Hess}_f(Y, Z)}{(\alpha + (1-\alpha) \|\text{grad } f\|^2)^2} \text{grad } f \\ &\quad + \frac{(1-\alpha) \text{Hess}_f(Y, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \nabla_X \text{grad } f \\ &\quad + \frac{(1-\alpha) \text{Hess}_f(X, \nabla_Y Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f. \end{aligned} \quad (12)$$

Using the similar method, the second term of (10) is given by

$$\begin{aligned}
& -\tilde{\nabla}_Y \left(\nabla_X Z + \frac{(1-\alpha) \text{Hess}_f(X, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\
&= -\nabla_Y \nabla_X Z - \frac{(1-\alpha) g(\nabla_Y \nabla_X \text{grad } f, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\
&\quad - \frac{(1-\alpha) \text{Hess}_f(X, \nabla_Y Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\
&\quad + \frac{(1-\alpha)^2 \text{Hess}_f(Y, \text{grad } f) \text{Hess}_f(X, Z)}{(\alpha + (1-\alpha) \|\text{grad } f\|^2)^2} \text{grad } f \\
&\quad - \frac{(1-\alpha) \text{Hess}_f(X, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \nabla_Y \text{grad } f \\
&\quad - \frac{(1-\alpha) \text{Hess}_f(Y, \nabla_X Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f.
\end{aligned} \tag{13}$$

Theorem 2 follows from equations (10), (12) and (13). \square

3 The biharmonicity of $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$

We now consider the effects of a deformation of the codomain metric, as regards harmonic and biharmonic mappings.

Theorem 3. *Let $\varphi: (M, g) \rightarrow (N, h)$ be a harmonic map between two Riemannian manifolds and let the Riemannian metric $\tilde{h}_\alpha = \alpha h + (1-\alpha)df \otimes df$, where $\alpha \in (0, 1)$ and $f \in C^\infty(N)$. We suppose that $\|\text{grad } f\| = 1$. If the function $\Delta(f \circ \varphi)$ is a non-null constant on M , then the map $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$ is proper biharmonic if and only if the gradient vector of f is Jacobi field along φ , i.e. $(\text{grad } f) \circ \varphi \in \ker J_\varphi$ where J_φ is a Jacobi operator corresponding to φ .*

Proof. Let $\{e_i\}_{i=1}^m$ be a normal orthonormal frame on (M, g) at x , where $m = \dim M$. Then the map $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$ is biharmonic if and only if

$$\tilde{\tau}_2(\varphi) = -\tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) - \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) = 0, \tag{14}$$

where \tilde{R}^N is the Riemannian curvature with respect to \tilde{h}_α , $\tilde{\tau}(\varphi)$ denotes the tension field of the map φ with respect to \tilde{h}_α , and $\tilde{\nabla}^\varphi$ is the pull-back connection with respect to the metric \tilde{h}_α . First, we compute the tension field $\tilde{\tau}(\varphi)$,

$$\begin{aligned}
\tilde{\tau}(\varphi) &= \tilde{\nabla}_{e_i}^\varphi d\varphi(e_i) = \tilde{\nabla}_{d\varphi(e_i)}^N d\varphi(e_i) \\
&= \tau(\varphi) + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{\alpha + (1-\alpha) \|\text{grad } f\|^2 \circ \varphi} (\text{grad } f) \circ \varphi \\
&= (1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i)) (\text{grad } f) \circ \varphi,
\end{aligned}$$

since $\Delta(f \circ \varphi) = df(\tau(\varphi)) + \text{trace Hess}_f(d\varphi, d\varphi)$ (see [3]), and $\tau(\varphi) = 0$, we have $\tilde{\tau}(\varphi) = \lambda(\text{grad } f) \circ \varphi$, with $\lambda = (1-\alpha)\Delta(f \circ \varphi)$ is a non-null constant. Now, we

compute the first term of (14), from Theorem 2, we have

$$\begin{aligned}
& \tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) \\
&= \lambda \left\{ R^N(\text{grad } f, d\varphi(e_i))d\varphi(e_i) \right. \\
&\quad + \frac{(1-\alpha)h(R^N(\text{grad } f, d\varphi(e_i)) \text{grad } f, d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \text{grad } f \\
&\quad - \frac{(1-\alpha)^2 \text{Hess}_f(\text{grad } f, \text{grad } f) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{(\alpha + (1-\alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
&\quad + \frac{(1-\alpha)^2 \text{Hess}_f(d\varphi(e_i), \text{grad } f) \text{Hess}_f(\text{grad } f, d\varphi(e_i))}{(\alpha + (1-\alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
&\quad + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \nabla_{\text{grad } f}^N \text{grad } f \\
&\quad \left. - \frac{(1-\alpha) \text{Hess}_f(\text{grad } f, d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \nabla_{d\varphi(e_i)}^N \text{grad } f \right\} \circ \varphi, \tag{15}
\end{aligned}$$

since $\|\text{grad } f\| = 1$, is constant on N , we obtain

$$\text{Hess}_f(\text{grad } f, X) = 0, \quad \nabla_{\text{grad } f}^N \text{grad } f = \frac{1}{2} \text{grad} \|\text{grad } f\|^2 = 0, \tag{16}$$

for all $X \in \Gamma(TN)$, the equation (15) becomes

$$\begin{aligned}
& \tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) \\
&= \lambda \left\{ R^N(\text{grad } f, d\varphi(e_i))d\varphi(e_i) \right. \\
&\quad \left. + (1-\alpha)h(R^N(\text{grad } f, d\varphi(e_i)) \text{grad } f, d\varphi(e_i)) \text{grad } f \right\} \circ \varphi. \tag{17}
\end{aligned}$$

The second term of (14) is given by

$$\begin{aligned}
\tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) &= \lambda \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi (\text{grad } f) \circ \varphi \\
&= \lambda \tilde{\nabla}_{e_i}^\varphi (\tilde{\nabla}_{d\varphi(e_i)}^N \text{grad } f) \circ \varphi \\
&= \lambda \tilde{\nabla}_{e_i}^\varphi \left\{ (\nabla_{d\varphi(e_i)}^N \text{grad } f) \circ \varphi \right. \\
&\quad \left. + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), (\text{grad } f) \circ \varphi)}{\alpha + (1-\alpha)\|\text{grad } f\|^2 \circ \varphi} (\text{grad } f) \circ \varphi \right\}, \tag{18}
\end{aligned}$$

from equations (16) and (18), we find that

$$\begin{aligned}
\tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) &= \lambda \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi \\
&\quad + (1-\alpha) \lambda \text{Hess}_f(d\varphi(e_i), \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi) (\text{grad } f) \circ \varphi, \tag{19}
\end{aligned}$$

and note that

$$\text{Hess}_f(d\varphi(e_i), \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi) = -h((\text{grad } f) \circ \varphi, \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi).$$

So, the map $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$ is biharmonic if and only if

$$J_\varphi((\text{grad } f) \circ \varphi) - (1 - \alpha)h(J_\varphi((\text{grad } f) \circ \varphi), (\text{grad } f) \circ \varphi)(\text{grad } f) \circ \varphi = 0. \quad (20)$$

Note that, the equation (20) is equivalent to $J_\varphi((\text{grad } f) \circ \varphi) = 0$. \square

Example 1. Let $M = \mathbb{R}^2$ and $N = \mathbb{H}^2 = \{(y_1, y_2) \in \mathbb{R}^2 | y_2 > 0\}$. We consider the harmonic map $\varphi: (M, dx_1^2 + dx_2^2) \rightarrow (N, y_2^2(dy_1^2 + dy_2^2))$, $(x_1, x_2) \mapsto (x_1, \sqrt{x_2^2 + 1})$, and let the function $f(y_1, y_2) = \frac{1}{2}y_2^2$. A straightforward calculation shows that $\|\text{grad } f\| = 1$, $\Delta(f \circ \varphi) = 1$, $(\text{grad } f) \circ \varphi = \left(0, \frac{1}{\sqrt{x_2^2 + 1}}\right)$ and $J_\varphi((\text{grad } f) \circ \varphi) = 0$.

Thus, with respect to metric $\tilde{h}_\alpha = y_2^2(\alpha dy_1^2 + dy_2^2)$, the map φ is biharmonic non-harmonic, with $\tilde{\tau}(\varphi) = \left(0, \frac{1-\alpha}{\sqrt{x_2^2 + 1}}\right)$.

Remark 1. • Let $\varphi: (M, g) \rightarrow (N, h)$ be a harmonic map between two Riemannian manifolds and $\tilde{h}_\alpha = \alpha h + (1 - \alpha)df \otimes df$, where $\alpha \in (0, 1)$ and $f \in C^\infty(N)$ such that $\|\text{grad } f\| = 1$. Then the map $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$ is harmonic if and only if $f \circ \varphi$ is harmonic on (M, g) .

- Let (M, g) be a Riemannian manifold, and let f be a smooth function on M such that $\|\text{grad } f\| = 1$ and $\Delta f = k$, where $k \in \mathbb{R}$. Then, the identity map from (M, g) to (M, \tilde{g}_α) is biharmonic if and only if it is harmonic. Indeed; from Theorem 3 the identity map from (M, g) to (M, \tilde{g}_α) is a biharmonic map if and only if $\text{Ricci}(\text{grad } f) = 0$, and by Bochner-Weitzenböck formula for smooth functions (see [14])

$$\frac{1}{2}\Delta(\|\text{grad } f\|^2) = \|\text{Hess}_f\|^2 + g(\text{grad } f, \text{grad}(\Delta f)) + \text{Ric}(\text{grad } f, \text{grad } f),$$

we obtain $\|\text{Hess}_f\| = 0$, so that $\Delta f = 0$, that is the identity map from (M, g) to (M, \tilde{g}_α) is harmonic map.

4 The biharmonicity of the identity map $(M, \tilde{g}_\alpha) \rightarrow (M, \tilde{g}_\beta)$

Let (M, g) be a Riemannian manifold, $f \in C^\infty(M)$, $\alpha, \beta \in (0, 1)$, and denote by

$$\begin{aligned} \tilde{I}_{\alpha, \beta}: (M, \tilde{g}_\alpha) &\rightarrow (M, \tilde{g}_\beta), \\ x &\mapsto x \end{aligned}$$

the identity map, where $\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df$ and $\tilde{g}_\beta = \beta g + (1 - \beta)df \otimes df$.

Theorem 4. *If $\alpha \neq \beta$, and $\|\text{grad } f\| = 1$. Then the identity map $\tilde{I}_{\alpha, \beta}$ is a proper biharmonic if and only if the function f is non-harmonic on M , and satisfying the following*

$$\begin{aligned} 2\Delta f \text{Ricci}(\text{grad } f) &= -\frac{1}{\beta}\Delta^2 f \text{grad } f - 2\nabla_{\text{grad } \Delta f} \text{grad } f - \Delta f \text{grad } \Delta f \\ &\quad + \frac{1 - \alpha}{\beta}\Delta f g(\text{grad } f, \text{grad } \Delta f) \text{grad } f \\ &\quad + \frac{1 - \alpha}{\beta}\text{Hess}_{\Delta f}(\text{grad } f, \text{grad } f) \text{grad } f, \end{aligned}$$

where Δf is the Laplacian of f with respect to g , and $\Delta^2 f = \Delta(\Delta f)$.

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on M with respect to the metric g , such that $e_1 = \text{grad } f$, it is easy to prove that $\{e_1, \frac{1}{\sqrt{\alpha}}e_i\}_{i=2}^m$ is a orthonormal frame on M with respect to the metric \tilde{g}_α , where $m = \dim M$. Let $\tilde{\nabla}^\alpha$ (resp. $\tilde{\nabla}^\beta$) the Levi-Civita connection of (M, \tilde{g}_α) (resp. of (M, \tilde{g}_β)), then the tension field of $\tilde{I}_{\alpha,\beta}$ is given by

$$\begin{aligned}\tau(\tilde{I}_{\alpha,\beta}) &= \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} d\tilde{I}_{\alpha,\beta}(e_1) - d\tilde{I}_{\alpha,\beta}(\tilde{\nabla}_{e_1}^\alpha e_1) + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} d\tilde{I}_{\alpha,\beta}(e_i) - d\tilde{I}_{\alpha,\beta}(\tilde{\nabla}_{e_i}^\alpha e_i) \right\} \\ &= \tilde{\nabla}_{e_1}^\beta e_1 - \tilde{\nabla}_{e_1}^\alpha e_1 + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \tilde{\nabla}_{e_i}^\beta e_i - \tilde{\nabla}_{e_i}^\alpha e_i \right\},\end{aligned}$$

using Theorem 1, with $\|\text{grad } f\| = 1$, we have

$$\tau(\tilde{I}_{\alpha,\beta}) = \frac{\alpha - \beta}{\alpha} \sum_{i=2}^m \text{Hess}_f(e_i, e_i) \text{grad } f, \quad (21)$$

since $\text{Hess}_f(e_1, e_1) = 0$, the equation (21) becomes

$$\tau(\tilde{I}_{\alpha,\beta}) = \frac{\alpha - \beta}{\alpha} \Delta f \text{grad } f.$$

Note that $\tilde{I}_{\alpha,\beta}$ is harmonic if and only if $\Delta f = 0$, i.e. the function f is harmonic on (M, g) . We compute the bitension field of the identity $\tilde{I}_{\alpha,\beta}$, for all $i = 1, \dots, m$ we have

$$\tilde{R}_\beta(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i)) d\tilde{I}_{\alpha,\beta}(e_i) = \frac{\alpha - \beta}{\alpha} \Delta f \tilde{R}_\beta(\text{grad } f, e_i) e_i, \quad (22)$$

where \tilde{R}_β is the curvature tensor of $\tilde{\nabla}^\beta$. From Theorem 2, and equation (22) with $\|\text{grad } f\| = 1$, $\text{Hess}_f(\text{grad } f, X) = 0$, for all $X \in \Gamma(TM)$, and $\nabla_{\text{grad } f} \text{grad } f = 0$, we obtain the following

$$\begin{aligned}\tilde{R}_\beta(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i)) d\tilde{I}_{\alpha,\beta}(e_i) \\ = \frac{\alpha - \beta}{\alpha} \Delta f \left\{ R(\text{grad } f, e_i) e_i + (1 - \beta) g(R(\text{grad } f, e_i) \text{grad } f, e_i) \text{grad } f \right\},\end{aligned} \quad (23)$$

from (23) and the definition of Ricci curvature, we get

$$\begin{aligned}\tilde{R}(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_1)) d\tilde{I}_{\alpha,\beta}(e_1) + \frac{1}{\alpha} \sum_{i=2}^m \tilde{R}(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i)) d\tilde{I}_{\alpha,\beta}(e_i) \\ = \frac{\alpha - \beta}{\alpha^2} \Delta f \left\{ \text{Ricci}(\text{grad } f) \right. \\ \left. - (1 - \beta) \text{Ric}(\text{grad } f, \text{grad } f) \text{grad } f \right\}.\end{aligned} \quad (24)$$

Let $i = 1, \dots, m$, we compute

$$\begin{aligned}
 & \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_i}^{\alpha}}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \tilde{\nabla}_{e_i}^{\beta} \tilde{\nabla}_{e_i}^{\beta} \Delta f \operatorname{grad} f - \tilde{\nabla}_{\tilde{\nabla}_{e_i}^{\alpha} e_i}^{\beta} \Delta f \operatorname{grad} f \right\} \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \tilde{\nabla}_{e_i}^{\beta} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\tilde{\nabla}_{e_i}^{\alpha} e_i} \Delta f \operatorname{grad} f \right\} \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \nabla_{e_i} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i} e_i} \Delta f \operatorname{grad} f \right. \\
 &\quad \left. + (1 - \beta) \operatorname{Hess}_f(e_i, \nabla_{e_i} \Delta f \operatorname{grad} f) \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \alpha) \operatorname{Hess}_f(e_i, e_i) \nabla_{\operatorname{grad} f} \Delta f \operatorname{grad} f \right\}, \quad (25)
 \end{aligned}$$

a straightforward calculation shows that

$$\begin{aligned}
 & \nabla_{e_i} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i} e_i} \Delta f \operatorname{grad} f \\
 &= e_i(e_i(\Delta f)) \operatorname{grad} f + 2e_i(\Delta f) \nabla_{e_i} \operatorname{grad} f \\
 &\quad + \Delta f \nabla_{e_i} \nabla_{e_i} \operatorname{grad} f - (\nabla_{e_i} e_i)(\Delta f) \operatorname{grad} f \\
 &\quad - \Delta f \nabla_{\nabla_{e_i} e_i} \operatorname{grad} f, \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 & (1 - \beta) \operatorname{Hess}_f(e_i, \nabla_{e_i} \Delta f \operatorname{grad} f) \operatorname{grad} f \\
 &= -(1 - \beta) \Delta f g(\operatorname{grad} f, \nabla_{e_i} \nabla_{e_i} \operatorname{grad} f) \operatorname{grad} f, \quad (27)
 \end{aligned}$$

and

$$\begin{aligned}
 & -(1 - \alpha) \operatorname{Hess}_f(e_i, e_i) \nabla_{\operatorname{grad} f} \Delta f \operatorname{grad} f \\
 &= -(1 - \alpha) \operatorname{Hess}_f(e_i, e_i) (\operatorname{grad} f) (\Delta f) \operatorname{grad} f, \quad (28)
 \end{aligned}$$

by equations (25)–(28), with $\|\operatorname{grad} f\| = 1$, we find that

$$\begin{aligned}
 & \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_1}^{\alpha} e_1}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_i}^{\alpha} e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) \right\} \\
 &= \frac{\alpha - \beta}{\alpha^2} \left\{ (\alpha - 1) \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f + \Delta^2 f \operatorname{grad} f \right. \\
 &\quad \left. + 2 \nabla_{\operatorname{grad} \Delta f} \operatorname{grad} f + \Delta f \operatorname{trace} \nabla^2 \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \beta) \Delta f g(\operatorname{grad} f, \operatorname{trace} \nabla^2 \operatorname{grad} f) \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \alpha) \Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f) \operatorname{grad} f \right\}, \quad (29)
 \end{aligned}$$

from equations (24), (29), and the following (see [1])

$$\operatorname{trace} \nabla^2 \operatorname{grad} f = \operatorname{Ricci}(\operatorname{grad} f) + \operatorname{grad}(\Delta f),$$

the identity map $\tilde{I}_{\alpha,\beta}$ is a proper biharmonic map if and only if

$$\begin{aligned} & 2\Delta f \operatorname{Ricci}(\operatorname{grad} f) - 2(1 - \beta)\Delta f \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f \\ & + \Delta^2 f \operatorname{grad} f + 2\nabla_{\operatorname{grad} \Delta f} \operatorname{grad} f + \Delta f \operatorname{grad} \Delta f \\ & - (2 - \alpha - \beta)\Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f) \operatorname{grad} f \\ & + (\alpha - 1) \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f = 0, \end{aligned} \quad (30)$$

with $\alpha \neq \beta$ and $\Delta f \neq 0$, taking its inner product with $\operatorname{grad} f$, we have

$$\begin{aligned} & -2(1 - \beta)\Delta f \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) \\ & = \frac{1 - \beta}{\beta} \Delta^2 f - \frac{(1 - \beta)(1 - \alpha)}{\beta} \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \\ & \quad - \frac{(1 - \beta)(1 - \alpha - \beta)}{\beta} \Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f). \end{aligned} \quad (31)$$

Theorem 4 follows from (30) and (31). \square

Corollary 1. *If $\alpha \neq \beta$, $\|\operatorname{grad} f\| = 1$, $\Delta f = F(f)$, where F is a non-null function on $I \subset \mathbb{R}$, and $\operatorname{Ricci}(\operatorname{grad} f) = \lambda \operatorname{grad} f$, for some smooth function λ on M . Then the identity map $\tilde{I}_{\alpha,\beta}$ is a proper biharmonic if and only if the function f satisfying the following*

$$2\beta\lambda F(f) + (\alpha + \beta)F(f)F'(f) + \alpha F''(f) = 0.$$

According to Corollary 1, we have the following example.

Example 2. Let $M = (0, \infty) \times \mathbb{R}^n$ equipped with the Riemannian metric

$$g = dt^2 + \frac{dx_1^2 + \cdots + dx_n^2}{t},$$

we set $f(t, x) = t$, for all $(t, x) \in M$. We have $\operatorname{grad} f = \partial_t$, $\|\operatorname{grad} f\| = 1$, $\Delta f = -\frac{n}{2t}$ and $\operatorname{Ricci}(\operatorname{grad} f) = -\frac{3n}{4t^2}\partial_t$, so that $F(s) = -\frac{n}{2s}$, for all $s \in I = (0, \infty)$ and $\lambda(t, x) = -\frac{3n}{4t^2}$ for all $(t, x) \in M$. Using the Corollary 1, Then the identity map $\tilde{I}_{\alpha,\beta}$ is proper biharmonic if and only if $n \neq 4$ and $\alpha = \frac{2n\beta}{n+4}$.

5 Biharmonic curve in (M, \tilde{g}_α)

Let $\gamma: I \subset \mathbb{R} \rightarrow (M, g)$, $t \mapsto \gamma(t)$ be a harmonic curve in a Riemannian manifold (M, g) , such that $g(\dot{\gamma}, \dot{\gamma}) = 1$, and let f be a smooth function on M . In this section we suppose that the gradient vector of f at $\gamma(t)$ is parallel to the tangent vector $\dot{\gamma}(t)$. Thus, $(\operatorname{grad} f)_{\gamma(t)} = \rho(t)\dot{\gamma}(t)$, with $\rho(t) = (f \circ \gamma)'(t)$, for all $t \in I$. Since γ is harmonic we get the following formula

$$(\nabla_{\dot{\gamma}} \operatorname{grad} f)_t = \rho'(t)\dot{\gamma}(t), \quad \forall t \in I. \quad (32)$$

We set $\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df$, where $\alpha \in (0, 1)$. We have the following result:

Theorem 5. *The curve $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$ is biharmonic if and only if the function f satisfying the following*

$$f(\gamma(t)) = \pm \int \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}} dt,$$

where $a, b, c \in \mathbb{R}$, such that $(at^2 + bt + c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$.

Proof. By Theorem 1, we have

$$\tilde{\tau}(\gamma) = \tau(\gamma) + \frac{(1-\alpha) \text{Hess}_f(\dot{\gamma}, \dot{\gamma})}{\alpha + (1-\alpha)\|\text{grad } f\|^2 \circ \gamma} (\text{grad } f) \circ \gamma, \quad (33)$$

from the harmonicity condition of γ , and equations (32), (33), we obtain $\tilde{\tau}(\gamma) = \lambda \dot{\gamma}$, where

$$\lambda = \frac{(1-\alpha)\rho\rho'}{\alpha + (1-\alpha)\rho^2}. \quad (34)$$

Now, the curve $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$ is biharmonic if and only if

$$\tilde{R}\left(\tilde{\tau}(\gamma), d\gamma\left(\frac{d}{dt}\right)\right)d\gamma\left(\frac{d}{dt}\right) + \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) = 0, \quad (35)$$

by the property of the curvature tensor, the first term on the left-hand side of (35) is

$$\tilde{R}\left(\tilde{\tau}(\gamma), d\gamma\left(\frac{d}{dt}\right)\right)d\gamma\left(\frac{d}{dt}\right) = \lambda \tilde{R}(\dot{\gamma}, \dot{\gamma})\dot{\gamma} = 0.$$

For the second term on the left-hand side of (35), we compute

$$\begin{aligned} \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) &= \tilde{\nabla}_{\frac{d}{dt}}^\gamma \lambda \dot{\gamma} \\ &= \lambda' \dot{\gamma} + \lambda \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} \\ &= (\lambda' + \lambda^2) \dot{\gamma}, \end{aligned} \quad (36)$$

with the same method of (36), we find that

$$\begin{aligned} \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) &= \tilde{\nabla}_{\frac{d}{dt}}^\gamma (\lambda' + \lambda^2) \dot{\gamma} \\ &= (\lambda'' + 2\lambda\lambda') \dot{\gamma} + (\lambda' + \lambda^2) \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} \\ &= (\lambda'' + 3\lambda\lambda' + \lambda^3) \dot{\gamma}. \end{aligned}$$

So, the curve $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$ is biharmonic if and only if $\lambda'' + 3\lambda\lambda' + \lambda^3 = 0$, that is the function λ is the form $(2at + b)/(at^2 + bt + c)$, where $a, b, c \in \mathbb{R}$, such that $at^2 + bt + c \neq 0$, for all $t \in I$. Thus, from (34) with $(at^2 + bt + c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$, we obtain

$$\rho(t) = \pm \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}}, \quad \forall t \in I. \quad (37)$$

Theorem 5 follows from equation (37), with $\rho = (f \circ \gamma)'$. \square

Remark 2. The curve $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$ is proper biharmonic if and only if there exists $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 > 0$, and for all $i = 1, \dots, m$ ($m = \dim M$), and in any local coordinates (x_i) on M , such that

$$\sum_{j=1}^m g^{ij}(\gamma(t)) \frac{\partial f}{\partial x_j} \Big|_{\gamma(t)} = \pm \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha} \frac{d\gamma^i}{dt} \Big|_t}, \quad \forall t \in I.$$

Using Theorem 5 and the previous Remark, we can construct many examples for proper biharmonic curves.

Example 3. Let $M = \mathbb{R}^n$ equipped with the Riemannian metric $g = dx_1^2 + \dots + dx_n^2$,

$$f(x) = \frac{2}{3} \sum_{i=1}^n (1 + x_i^2)^{\frac{3}{2}}, \quad \forall x = (x_1, \dots, x_n) \in M.$$

For $\alpha = \frac{n}{n+1}$, the curve

$$\gamma: I \rightarrow (M, \tilde{g}_\alpha), \quad t \mapsto \left(\frac{t}{\sqrt{n}}, \dots, \frac{t}{\sqrt{n}} \right),$$

is proper biharmonic.

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