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## Deformations of Metrics and Biharmonic Maps

*Aicha Benkartab, Ahmed Mohammed Cherif*

**Abstract.** We construct biharmonic non-harmonic maps between Riemannian manifolds  $(M, g)$  and  $(N, h)$  by first making the ansatz that  $\varphi: (M, g) \rightarrow (N, h)$  be a harmonic map and then deforming the metric on  $N$  by

$$\tilde{h}_\alpha = \alpha h + (1 - \alpha)df \otimes df$$

to render  $\varphi$  biharmonic, where  $f$  is a smooth function with gradient of constant norm on  $(N, h)$  and  $\alpha \in (0, 1)$ . We construct new examples of biharmonic non-harmonic maps, and we characterize the biharmonicity of some curves on Riemannian manifolds.

### 1 Introduction

Let  $(M, g)$  and  $(N, h)$  be two Riemannian manifolds. The energy functional of a map  $\varphi \in C^\infty(M, N)$  is defined by

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v^g, \tag{1}$$

where  $|d\varphi|$  is the Hilbert-Schmidt norm of the differential  $d\varphi$  and  $v^g$  is the volume element on  $(M, g)$ . A map  $\varphi \in C^\infty(M, N)$  is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1)

$$\tau(\varphi) = \text{trace } \nabla d\varphi = \nabla_{e_i}^\varphi d\varphi(e_i) - d\varphi(\nabla_{e_i}^M e_i) = 0, \tag{2}$$

where  $\{e_i\}_{i=1}^m$  is an orthonormal frame on  $(M, g)$ ,  $m = \dim M$ ,  $\nabla^M$  is the Levi-Civita connection of  $(M, g)$ , and  $\nabla^\varphi$  denote the pull-back connection on  $\varphi^{-1}TN$ .

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Affiliation:

Aicha Benkartab – Mascara University, Faculty of Science and technology, Mascara 29000, Algeria

E-mail: [benkartab.aicha@univ-mascara.dz](mailto:benkartab.aicha@univ-mascara.dz)

Ahmed Mohammed Cherif – Mascara University, Faculty of Exact Sciences, Mascara 29000, Algeria

E-mail: [a.mohammedcherif@univ-mascara.dz](mailto:a.mohammedcherif@univ-mascara.dz)

Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important role in many branches of mathematics and physics where they may serve as a model for liquid crystal (see [9]). One can refer to [6], [7], [8] for background on harmonic maps. A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map  $\varphi \in C^\infty(M, N)$  is defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v^g . \tag{3}$$

A map  $\varphi \in C^\infty(M, N)$  is called biharmonic if it is a critical point of the bi-energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (3)

$$\begin{aligned} \tau_2(\varphi) &= -\text{trace } R^N(\tau(\varphi), d\varphi)d\varphi - \text{trace}(\nabla^\varphi)^2\tau(\varphi) \\ &= -R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi \tau(\varphi) + \nabla_{\nabla_{e_i}^\varphi e_i}^\varphi \tau(\varphi) = 0, \end{aligned} \tag{4}$$

where  $R^N$  is the curvature tensor of  $(N, h)$  (see [5], [12]). Clearly, harmonic maps are biharmonic. G.Y. Jiang [12] proved that if  $M$  is compact without boundary and the sectional curvature of  $(N, h)$  is negative, then any biharmonic map  $\varphi \in C^\infty(M, N)$  is harmonic. So it is interesting to construct biharmonic non-harmonic maps. We refer the reader to [2], [5], [10], [11] for other examples and different approaches to their construction.

In this paper, we deform the codomain metric by  $\tilde{h}_\alpha = \alpha h + (1-\alpha)df \otimes df$ , where  $\alpha \in (0, 1)$  and  $f \in C^\infty(N)$ , in order to render a map biharmonic non-harmonic with respect to the new metric, we give a necessary and sufficient condition on  $f$  and  $\alpha$  such that  $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$  is biharmonic non-harmonic. So by suitable choices of  $f$ , we are able to give new examples of biharmonic non-harmonic maps.

## 2 Deformations of Metrics

Let  $M$  be a Riemannian manifold equipped with Riemannian metric  $g$ , and  $f$  a smooth function on  $M$ . We define on  $M$  a Riemannian metric, denoted  $\tilde{g}_\alpha$ , by

$$\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df ,$$

for some constant  $\alpha \in (0,1)$ . In the seminal work [4], we obtain the following results.

**Theorem 1.** *Let  $(M, g)$  be a Riemannian manifold and  $\tilde{\nabla}$  denote the Levi-Civita connection of  $(M, \tilde{g}_\alpha)$ . Then*

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{(1 - \alpha) \text{Hess}_f(X, Y)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \text{grad } f ,$$

where  $\nabla$  is the Levi-Civita connection of  $(M, g)$ ,  $\text{Hess}_f$  (resp.  $\text{grad } f$ ) is the Hessian (resp. the gradient vector) of  $f$  with respect to  $g$ , and

$$\|\text{grad } f\|^2 = g(\text{grad } f, \text{grad } f) .$$

*Proof.* Let  $X, Y, Z \in \Gamma(TM)$ . From the Koszul formula (see [13]), we have

$$\begin{aligned}
 2\tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) &= 2\alpha g(\nabla_X Y, Z) + (1 - \alpha) \left\{ X(Y(f)Z(f)) + Y(Z(f)X(f)) \right. \\
 &\quad - Z(X(f)Y(f)) + Z(f)[X, Y](f) + Y(f)[Z, X](f) \\
 &\quad \left. - X(f)[Y, Z](f) \right\}. \tag{5}
 \end{aligned}$$

Let  $\{e_i\}_{i=1}^m$  be a geodesic frame on  $(M, g)$  at  $x \in M$  (see [3]), where  $m = \dim M$ . By (5) we obtain

$$\begin{aligned}
 2\tilde{g}_\alpha(\tilde{\nabla}_X Y, e_i) &= 2\alpha g(\nabla_X Y, e_i) + (1 - \alpha) \left\{ X(Y(f)g(e_i, \text{grad } f)) \right. \\
 &\quad + Y(X(f)g(e_i, \text{grad } f)) - e_i(g(X, \text{grad } f)g(Y, \text{grad } f)) \\
 &\quad \left. + e_i(f)[X, Y](f) + Y(f)(\nabla_{e_i} X)(f) + X(f)(\nabla_{e_i} Y)(f) \right\}, \tag{6}
 \end{aligned}$$

from equation (6), and the definition of Hessian (see [13]), we get

$$\begin{aligned}
 \tilde{g}_\alpha(\tilde{\nabla}_X Y, e_i) &= \alpha g(\nabla_X Y, e_i) + (1 - \alpha)g(\nabla_X Y, \text{grad } f)g(e_i, \text{grad } f) \\
 &\quad + (1 - \alpha) \text{Hess}_f(X, Y)g(e_i, \text{grad } f), \tag{7}
 \end{aligned}$$

from equation (7), we obtain

$$\begin{aligned}
 \tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) &= \alpha g(\nabla_X Y, Z) + (1 - \alpha)g(\nabla_X Y, \text{grad } f)g(Z, \text{grad } f) \\
 &\quad + (1 - \alpha) \text{Hess}_f(X, Y)g(Z, \text{grad } f), \tag{8}
 \end{aligned}$$

by the definition of the Riemannian metric  $\tilde{g}_\alpha$ , and (8) we find that

$$\tilde{g}_\alpha(\tilde{\nabla}_X Y, Z) = \tilde{g}_\alpha(\nabla_X Y, Z) + (1 - \alpha) \text{Hess}_f(X, Y)Z(f). \tag{9}$$

Hence Theorem 1 follows from (9), with the following

$$Z(f) = \frac{1}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \tilde{g}_\alpha(Z, \text{grad } f). \quad \square$$

Now consider the curvature tensor  $\tilde{R}$  of  $(M, \tilde{g}_\alpha)$ , writing  $R$  for the curvature tensor of  $(M, g)$ . We have the following result:

**Theorem 2.** For all  $X, Y, Z \in \Gamma(TM)$ , we have

$$\begin{aligned}
 \tilde{R}(X, Y)Z &= R(X, Y)Z + \frac{(1 - \alpha)g(R(X, Y) \text{grad } f, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \text{grad } f \\
 &\quad - \frac{(1 - \alpha)^2 \text{Hess}_f(X, \text{grad } f) \text{Hess}_f(Y, Z)}{(\alpha + (1 - \alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
 &\quad + \frac{(1 - \alpha)^2 \text{Hess}_f(Y, \text{grad } f) \text{Hess}_f(X, Z)}{(\alpha + (1 - \alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
 &\quad + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \nabla_X \text{grad } f \\
 &\quad - \frac{(1 - \alpha) \text{Hess}_f(X, Z)}{\alpha + (1 - \alpha)\|\text{grad } f\|^2} \nabla_Y \text{grad } f.
 \end{aligned}$$

*Proof.* By the definition of the curvature tensor  $\tilde{R}$ ,

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z,$$

and Theorem 1 we obtain

$$\begin{aligned} \tilde{R}(X, Y)Z &= \tilde{\nabla}_X \left( \nabla_Y Z + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad - \tilde{\nabla}_Y \left( \nabla_X Z + \frac{(1 - \alpha) \text{Hess}_f(X, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad - \left( \nabla_{[X, Y]} Z + \frac{(1 - \alpha) \text{Hess}_f([X, Y], Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right), \end{aligned} \tag{10}$$

the first term of (10) is given by

$$\begin{aligned} &\tilde{\nabla}_X \left( \nabla_Y Z + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &= \nabla_X \left( \nabla_Y Z + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &\quad + \frac{(1 - \alpha) \text{Hess}_f \left( X, \nabla_Y Z + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f, \end{aligned} \tag{11}$$

by equation (11), and the definition of Hessian, we obtain

$$\begin{aligned} &\tilde{\nabla}_X \left( \nabla_Y Z + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\ &= \nabla_X \nabla_Y Z + \frac{(1 - \alpha) g(\nabla_X \nabla_Y \text{grad } f, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \\ &\quad + \frac{(1 - \alpha) \text{Hess}_f(Y, \nabla_X Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f \\ &\quad - \frac{(1 - \alpha)^2 \text{Hess}_f(X, \text{grad } f) \text{Hess}_f(Y, Z)}{(\alpha + (1 - \alpha) \|\text{grad } f\|^2)^2} \text{grad } f \\ &\quad + \frac{(1 - \alpha) \text{Hess}_f(Y, Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \nabla_X \text{grad } f \\ &\quad + \frac{(1 - \alpha) \text{Hess}_f(X, \nabla_Y Z)}{\alpha + (1 - \alpha) \|\text{grad } f\|^2} \text{grad } f. \end{aligned} \tag{12}$$

Using the similar method, the second term of (10) is given by

$$\begin{aligned}
 & -\tilde{\nabla}_Y \left( \nabla_X Z + \frac{(1-\alpha) \text{Hess}_f(X, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \right) \\
 &= -\nabla_Y \nabla_X Z - \frac{(1-\alpha)g(\nabla_Y \nabla_X \text{grad } f, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\
 &\quad - \frac{(1-\alpha) \text{Hess}_f(X, \nabla_Y Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f \\
 &\quad + \frac{(1-\alpha)^2 \text{Hess}_f(Y, \text{grad } f) \text{Hess}_f(X, Z)}{(\alpha + (1-\alpha) \|\text{grad } f\|^2)^2} \text{grad } f \\
 &\quad - \frac{(1-\alpha) \text{Hess}_f(X, Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \nabla_Y \text{grad } f \\
 &\quad - \frac{(1-\alpha) \text{Hess}_f(Y, \nabla_X Z)}{\alpha + (1-\alpha) \|\text{grad } f\|^2} \text{grad } f. \tag{13}
 \end{aligned}$$

Theorem 2 follows from equations (10), (12) and (13). □

### 3 The biharmonicity of $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$

We now consider the effects of a deformation of the codomain metric, as regards harmonic and biharmonic mappings.

**Theorem 3.** *Let  $\varphi: (M, g) \rightarrow (N, h)$  be a harmonic map between two Riemannian manifolds and let the Riemannian metric  $\tilde{h}_\alpha = \alpha h + (1-\alpha)df \otimes df$ , where  $\alpha \in (0, 1)$  and  $f \in C^\infty(N)$ . We suppose that  $\|\text{grad } f\| = 1$ . If the function  $\Delta(f \circ \varphi)$  is a non-null constant on  $M$ , then the map  $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$  is proper biharmonic if and only if the gradient vector of  $f$  is Jacobi field along  $\varphi$ , i.e.  $(\text{grad } f) \circ \varphi \in \ker J_\varphi$  where  $J_\varphi$  is a Jacobi operator corresponding to  $\varphi$ .*

*Proof.* Let  $\{e_i\}_{i=1}^m$  be a normal orthonormal frame on  $(M, g)$  at  $x$ , where  $m = \dim M$ . Then the map  $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$  is biharmonic if and only if

$$\tilde{\tau}_2(\varphi) = -\tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) - \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) = 0, \tag{14}$$

where  $\tilde{R}^N$  is the Riemannian curvature with respect to  $\tilde{h}_\alpha$ ,  $\tilde{\tau}(\varphi)$  denotes the tension field of the map  $\varphi$  with respect to  $\tilde{h}_\alpha$ , and  $\tilde{\nabla}^\varphi$  is the pull-back connection with respect to the metric  $\tilde{h}_\alpha$ . First, we compute the tension field  $\tilde{\tau}(\varphi)$ ,

$$\begin{aligned}
 \tilde{\tau}(\varphi) &= \tilde{\nabla}_{e_i}^\varphi d\varphi(e_i) = \tilde{\nabla}_{d\varphi(e_i)}^N d\varphi(e_i) \\
 &= \tau(\varphi) + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{\alpha + (1-\alpha) \|\text{grad } f\|^2 \circ \varphi} (\text{grad } f) \circ \varphi \\
 &= (1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))(\text{grad } f) \circ \varphi,
 \end{aligned}$$

since  $\Delta(f \circ \varphi) = df(\tau(\varphi)) + \text{trace Hess}_f(d\varphi, d\varphi)$  (see [3]), and  $\tau(\varphi) = 0$ , we have  $\tilde{\tau}(\varphi) = \lambda(\text{grad } f) \circ \varphi$ , with  $\lambda = (1-\alpha)\Delta(f \circ \varphi)$  is a non-null constant. Now, we

compute the first term of (14), from Theorem 2, we have

$$\begin{aligned}
 &\tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) \\
 &= \lambda \left\{ R^N(\text{grad } f, d\varphi(e_i))d\varphi(e_i) \right. \\
 &\quad + \frac{(1-\alpha)h(R^N(\text{grad } f, d\varphi(e_i)) \text{grad } f, d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \text{grad } f \\
 &\quad - \frac{(1-\alpha)^2 \text{Hess}_f(\text{grad } f, \text{grad } f) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{(\alpha + (1-\alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
 &\quad + \frac{(1-\alpha)^2 \text{Hess}_f(d\varphi(e_i), \text{grad } f) \text{Hess}_f(\text{grad } f, d\varphi(e_i))}{(\alpha + (1-\alpha)\|\text{grad } f\|^2)^2} \text{grad } f \\
 &\quad + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \nabla_{\text{grad } f}^N \text{grad } f \\
 &\quad \left. - \frac{(1-\alpha) \text{Hess}_f(\text{grad } f, d\varphi(e_i))}{\alpha + (1-\alpha)\|\text{grad } f\|^2} \nabla_{d\varphi(e_i)}^N \text{grad } f \right\} \circ \varphi, \tag{15}
 \end{aligned}$$

since  $\|\text{grad } f\| = 1$ , is constant on  $N$ , we obtain

$$\text{Hess}_f(\text{grad } f, X) = 0, \quad \nabla_{\text{grad } f}^N \text{grad } f = \frac{1}{2} \text{grad}\|\text{grad } f\|^2 = 0, \tag{16}$$

for all  $X \in \Gamma(TN)$ , the equation (15) becomes

$$\begin{aligned}
 &\tilde{R}^N(\tilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) \\
 &= \lambda \left\{ R^N(\text{grad } f, d\varphi(e_i))d\varphi(e_i) \right. \\
 &\quad \left. + (1-\alpha)h(R^N(\text{grad } f, d\varphi(e_i)) \text{grad } f, d\varphi(e_i)) \text{grad } f \right\} \circ \varphi. \tag{17}
 \end{aligned}$$

The second term of (14) is given by

$$\begin{aligned}
 \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) &= \lambda \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi (\text{grad } f) \circ \varphi \\
 &= \lambda \tilde{\nabla}_{e_i}^\varphi (\tilde{\nabla}_{d\varphi(e_i)}^N \text{grad } f) \circ \varphi \\
 &= \lambda \tilde{\nabla}_{e_i}^\varphi \left\{ (\nabla_{d\varphi(e_i)}^N \text{grad } f) \circ \varphi \right. \\
 &\quad \left. + \frac{(1-\alpha) \text{Hess}_f(d\varphi(e_i), (\text{grad } f) \circ \varphi)}{\alpha + (1-\alpha)\|\text{grad } f\|^2 \circ \varphi} (\text{grad } f) \circ \varphi \right\}, \tag{18}
 \end{aligned}$$

from equations (16) and (18), we find that

$$\begin{aligned}
 \tilde{\nabla}_{e_i}^\varphi \tilde{\nabla}_{e_i}^\varphi \tilde{\tau}(\varphi) &= \lambda \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi \\
 &\quad + (1-\alpha)\lambda \text{Hess}_f(d\varphi(e_i), \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi) (\text{grad } f) \circ \varphi, \tag{19}
 \end{aligned}$$

and note that

$$\text{Hess}_f(d\varphi(e_i), \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi) = -h((\text{grad } f) \circ \varphi, \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (\text{grad } f) \circ \varphi).$$

So, the map  $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$  is biharmonic if and only if

$$J_\varphi((\text{grad } f) \circ \varphi) - (1 - \alpha)h(J_\varphi((\text{grad } f) \circ \varphi), (\text{grad } f) \circ \varphi)(\text{grad } f) \circ \varphi = 0. \tag{20}$$

Note that, the equation (20) is equivalent to  $J_\varphi((\text{grad } f) \circ \varphi) = 0$ . □

**Example 1.** Let  $M = \mathbb{R}^2$  and  $N = \mathbb{H}^2 = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\}$ . We consider the harmonic map  $\varphi: (M, dx_1^2 + dx_2^2) \rightarrow (N, y_2^2(dy_1^2 + dy_2^2))$ ,  $(x_1, x_2) \mapsto (x_1, \sqrt{x_2^2 + 1})$ , and let the function  $f(y_1, y_2) = \frac{1}{2}y_2^2$ . A straightforward calculation shows that  $\|\text{grad } f\| = 1$ ,  $\Delta(f \circ \varphi) = 1$ ,  $(\text{grad } f) \circ \varphi = \left(0, \frac{1}{\sqrt{x_2^2 + 1}}\right)$  and  $J_\varphi((\text{grad } f) \circ \varphi) = 0$ .

Thus, with respect to metric  $\tilde{h}_\alpha = y_2^2(\alpha dy_1^2 + dy_2^2)$ , the map  $\varphi$  is biharmonic non-harmonic, with  $\tilde{\tau}(\varphi) = \left(0, \frac{1-\alpha}{\sqrt{x_2^2 + 1}}\right)$ .

**Remark 1.** • Let  $\varphi: (M, g) \rightarrow (N, h)$  be a harmonic map between two Riemannian manifolds and  $\tilde{h}_\alpha = \alpha h + (1 - \alpha)df \otimes df$ , where  $\alpha \in (0, 1)$  and  $f \in C^\infty(N)$  such that  $\|\text{grad } f\| = 1$ . Then the map  $\varphi: (M, g) \rightarrow (N, \tilde{h}_\alpha)$  is harmonic if and only if  $f \circ \varphi$  is harmonic on  $(M, g)$ .

- Let  $(M, g)$  be a Riemannian manifold, and let  $f$  be a smooth function on  $M$  such that  $\|\text{grad } f\| = 1$  and  $\Delta f = k$ , where  $k \in \mathbb{R}$ . Then, the identity map from  $(M, g)$  to  $(M, \tilde{g}_\alpha)$  is biharmonic if and only if it is harmonic. Indeed; from Theorem 3 the identity map from  $(M, g)$  to  $(M, \tilde{g}_\alpha)$  is a biharmonic map if and only if  $\text{Ricci}(\text{grad } f) = 0$ , and by Bochner-Weitzenböck formula for smooth functions (see [14])

$$\frac{1}{2}\Delta(\|\text{grad } f\|^2) = \|\text{Hess}_f\|^2 + g(\text{grad } f, \text{grad}(\Delta f)) + \text{Ric}(\text{grad } f, \text{grad } f),$$

we obtain  $\|\text{Hess}_f\| = 0$ , so that  $\Delta f = 0$ , that is the identity map from  $(M, g)$  to  $(M, \tilde{g}_\alpha)$  is harmonic map.

#### 4 The biharmonicity of the identity map $(M, \tilde{g}_\alpha) \rightarrow (M, \tilde{g}_\beta)$

Let  $(M, g)$  be a Riemannian manifold,  $f \in C^\infty(M)$ ,  $\alpha, \beta \in (0, 1)$ , and denote by

$$\begin{aligned} \tilde{I}_{\alpha, \beta}: (M, \tilde{g}_\alpha) &\rightarrow (M, \tilde{g}_\beta), \\ x &\mapsto x \end{aligned}$$

the identity map, where  $\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df$  and  $\tilde{g}_\beta = \beta g + (1 - \beta)df \otimes df$ .

**Theorem 4.** *If  $\alpha \neq \beta$ , and  $\|\text{grad } f\| = 1$ . Then the identity map  $\tilde{I}_{\alpha, \beta}$  is a proper biharmonic if and only if the function  $f$  is non-harmonic on  $M$ , and satisfying the following*

$$\begin{aligned} 2\Delta f \text{Ricci}(\text{grad } f) &= -\frac{1}{\beta}\Delta^2 f \text{grad } f - 2\nabla_{\text{grad } \Delta f} \text{grad } f - \Delta f \text{grad } \Delta f \\ &+ \frac{1 - \alpha}{\beta}\Delta f g(\text{grad } f, \text{grad } \Delta f) \text{grad } f \\ &+ \frac{1 - \alpha}{\beta}\text{Hess}_{\Delta f}(\text{grad } f, \text{grad } f) \text{grad } f, \end{aligned}$$

where  $\Delta f$  is the Laplacian of  $f$  with respect to  $g$ , and  $\Delta^2 f = \Delta(\Delta f)$ .



*Proof.* Let  $\{e_i\}_{i=1}^m$  be an orthonormal frame on  $M$  with respect to the metric  $g$ , such that  $e_1 = \text{grad } f$ , it is easy to prove that  $\{e_1, \frac{1}{\sqrt{\alpha}}e_i\}_{i=2}^m$  is a orthonormal frame on  $M$  with respect to the metric  $\tilde{g}_\alpha$ , where  $m = \dim M$ . Let  $\tilde{\nabla}^\alpha$  (resp.  $\tilde{\nabla}^\beta$ ) the Levi-Civita connection of  $(M, \tilde{g}_\alpha)$  (resp. of  $(M, \tilde{g}_\beta)$ ), then the tension field of  $\tilde{I}_{\alpha,\beta}$  is given by

$$\begin{aligned} \tau(\tilde{I}_{\alpha,\beta}) &= \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} d\tilde{I}_{\alpha,\beta}(e_1) - d\tilde{I}_{\alpha,\beta}(\tilde{\nabla}_{e_1}^\alpha e_1) + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} d\tilde{I}_{\alpha,\beta}(e_i) - d\tilde{I}_{\alpha,\beta}(\tilde{\nabla}_{e_i}^\alpha e_i) \right\} \\ &= \tilde{\nabla}_{e_1}^\beta e_1 - \tilde{\nabla}_{e_1}^\alpha e_1 + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \tilde{\nabla}_{e_i}^\beta e_i - \tilde{\nabla}_{e_i}^\alpha e_i \right\}, \end{aligned}$$

using Theorem 1, with  $\|\text{grad } f\| = 1$ , we have

$$\tau(\tilde{I}_{\alpha,\beta}) = \frac{\alpha - \beta}{\alpha} \sum_{i=2}^m \text{Hess}_f(e_i, e_i) \text{grad } f, \tag{21}$$

since  $\text{Hess}_f(e_1, e_1) = 0$ , the equation (21) becomes

$$\tau(\tilde{I}_{\alpha,\beta}) = \frac{\alpha - \beta}{\alpha} \Delta f \text{grad } f.$$

Note that  $\tilde{I}_{\alpha,\beta}$  is harmonic if and only if  $\Delta f = 0$ , i.e. the function  $f$  is harmonic on  $(M, g)$ . We compute the bitension field of the identity  $\tilde{I}_{\alpha,\beta}$ , for all  $i = 1, \dots, m$  we have

$$\tilde{R}_\beta(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i))d\tilde{I}_{\alpha,\beta}(e_i) = \frac{\alpha - \beta}{\alpha} \Delta f \tilde{R}_\beta(\text{grad } f, e_i)e_i, \tag{22}$$

where  $\tilde{R}_\beta$  is the curvature tensor of  $\tilde{\nabla}^\beta$ . From Theorem 2, and equation (22) with  $\|\text{grad } f\| = 1$ ,  $\text{Hess}_f(\text{grad } f, X) = 0$ , for all  $X \in \Gamma(TM)$ , and  $\nabla_{\text{grad } f} \text{grad } f = 0$ , we obtain the following

$$\begin{aligned} &\tilde{R}_\beta(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i))d\tilde{I}_{\alpha,\beta}(e_i) \\ &= \frac{\alpha - \beta}{\alpha} \Delta f \left\{ R(\text{grad } f, e_i)e_i + (1 - \beta)g(R(\text{grad } f, e_i) \text{grad } f, e_i) \text{grad } f \right\}, \end{aligned} \tag{23}$$

from (23) and the definition of Ricci curvature, we get

$$\begin{aligned} &\tilde{R}(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_1))d\tilde{I}_{\alpha,\beta}(e_1) + \frac{1}{\alpha} \sum_{i=2}^m \tilde{R}(\tau(\tilde{I}_{\alpha,\beta}), d\tilde{I}_{\alpha,\beta}(e_i))d\tilde{I}_{\alpha,\beta}(e_i) \\ &= \frac{\alpha - \beta}{\alpha^2} \Delta f \left\{ \text{Ricci}(\text{grad } f) \right. \\ &\quad \left. - (1 - \beta) \text{Ric}(\text{grad } f, \text{grad } f) \text{grad } f \right\}. \end{aligned} \tag{24}$$

Let  $i = 1, \dots, m$ , we compute

$$\begin{aligned}
 & \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_i}^{\alpha} e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \tilde{\nabla}_{e_i}^{\beta} \tilde{\nabla}_{e_i}^{\beta} \Delta f \operatorname{grad} f - \tilde{\nabla}_{\tilde{\nabla}_{e_i}^{\alpha} e_i}^{\beta} \Delta f \operatorname{grad} f \right\} \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \tilde{\nabla}_{e_i}^{\beta} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\tilde{\nabla}_{e_i}^{\alpha} e_i} \Delta f \operatorname{grad} f \right\} \\
 &= \frac{\alpha - \beta}{\alpha} \left\{ \nabla_{e_i} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i} e_i} \Delta f \operatorname{grad} f \right. \\
 &\quad \left. + (1 - \beta) \operatorname{Hess}_f(e_i, \nabla_{e_i} \Delta f \operatorname{grad} f) \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \alpha) \operatorname{Hess}_f(e_i, e_i) \nabla_{\operatorname{grad} f} \Delta f \operatorname{grad} f \right\}, \quad (25)
 \end{aligned}$$

a straightforward calculation shows that

$$\begin{aligned}
 & \nabla_{e_i} \nabla_{e_i} \Delta f \operatorname{grad} f - \nabla_{\nabla_{e_i} e_i} \Delta f \operatorname{grad} f \\
 &= e_i(e_i(\Delta f)) \operatorname{grad} f + 2e_i(\Delta f) \nabla_{e_i} \operatorname{grad} f \\
 &\quad + \Delta f \nabla_{e_i} \nabla_{e_i} \operatorname{grad} f - (\nabla_{e_i} e_i)(\Delta f) \operatorname{grad} f \\
 &\quad - \Delta f \nabla_{\nabla_{e_i} e_i} \operatorname{grad} f, \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 & (1 - \beta) \operatorname{Hess}_f(e_i, \nabla_{e_i} \Delta f \operatorname{grad} f) \operatorname{grad} f \\
 &= -(1 - \beta) \Delta f g(\operatorname{grad} f, \nabla_{e_i} \nabla_{e_i} \operatorname{grad} f) \operatorname{grad} f, \quad (27)
 \end{aligned}$$

and

$$\begin{aligned}
 & -(1 - \alpha) \operatorname{Hess}_f(e_i, e_i) \nabla_{\operatorname{grad} f} \Delta f \operatorname{grad} f \\
 &= -(1 - \alpha) \operatorname{Hess}_f(e_i, e_i) (\operatorname{grad} f) (\Delta f) \operatorname{grad} f, \quad (28)
 \end{aligned}$$

by equations (25)–(28), with  $\|\operatorname{grad} f\| = 1$ , we find that

$$\begin{aligned}
 & \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_1}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_1}^{\alpha} e_1}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) + \frac{1}{\alpha} \sum_{i=2}^m \left\{ \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \nabla_{e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) - \nabla_{\tilde{\nabla}_{e_i}^{\alpha} e_i}^{\tilde{I}_{\alpha,\beta}} \tau(\tilde{I}_{\alpha,\beta}) \right\} \\
 &= \frac{\alpha - \beta}{\alpha^2} \left\{ (\alpha - 1) \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f + \Delta^2 f \operatorname{grad} f \right. \\
 &\quad \left. + 2 \nabla_{\operatorname{grad} \Delta f} \operatorname{grad} f + \Delta f \operatorname{trace} \nabla^2 \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \beta) \Delta f g(\operatorname{grad} f, \operatorname{trace} \nabla^2 \operatorname{grad} f) \operatorname{grad} f \right. \\
 &\quad \left. - (1 - \alpha) \Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f) \operatorname{grad} f \right\}, \quad (29)
 \end{aligned}$$

from equations (24), (29), and the following (see [1])

$$\operatorname{trace} \nabla^2 \operatorname{grad} f = \operatorname{Ricci}(\operatorname{grad} f) + \operatorname{grad}(\Delta f),$$

the identity map  $\tilde{I}_{\alpha,\beta}$  is a proper biharmonic map if and only if

$$\begin{aligned} &2\Delta f \operatorname{Ricci}(\operatorname{grad} f) - 2(1 - \beta)\Delta f \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f \\ &\quad + \Delta^2 f \operatorname{grad} f + 2\nabla_{\operatorname{grad} \Delta f} \operatorname{grad} f + \Delta f \operatorname{grad} \Delta f \\ &\quad - (2 - \alpha - \beta)\Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f) \operatorname{grad} f \\ &\quad + (\alpha - 1) \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \operatorname{grad} f = 0, \end{aligned} \tag{30}$$

with  $\alpha \neq \beta$  and  $\Delta f \neq 0$ , taking its inner product with  $\operatorname{grad} f$ , we have

$$\begin{aligned} &- 2(1 - \beta)\Delta f \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) \\ &= \frac{1 - \beta}{\beta} \Delta^2 f - \frac{(1 - \beta)(1 - \alpha)}{\beta} \operatorname{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f) \\ &\quad - \frac{(1 - \beta)(1 - \alpha - \beta)}{\beta} \Delta f g(\operatorname{grad} f, \operatorname{grad} \Delta f). \end{aligned} \tag{31}$$

Theorem 4 follows from (30) and (31). □

**Corollary 1.** *If  $\alpha \neq \beta$ ,  $\|\operatorname{grad} f\| = 1$ ,  $\Delta f = F(f)$ , where  $F$  is a non-null function on  $I \subset \mathbb{R}$ , and  $\operatorname{Ricci}(\operatorname{grad} f) = \lambda \operatorname{grad} f$ , for some smooth function  $\lambda$  on  $M$ . Then the identity map  $\tilde{I}_{\alpha,\beta}$  is a proper biharmonic if and only if the function  $f$  satisfying the following*

$$2\beta\lambda F(f) + (\alpha + \beta)F(f)F'(f) + \alpha F''(f) = 0.$$

According to Corollary 1, we have the following example.

**Example 2.** Let  $M = (0, \infty) \times \mathbb{R}^n$  equipped with the Riemannian metric

$$g = dt^2 + \frac{dx_1^2 + \dots + dx_n^2}{t},$$

we set  $f(t, x) = t$ , for all  $(t, x) \in M$ . We have  $\operatorname{grad} f = \partial_t$ ,  $\|\operatorname{grad} f\| = 1$ ,  $\Delta f = -\frac{n}{2t}$  and  $\operatorname{Ricci}(\operatorname{grad} f) = -\frac{3n}{4t^2} \partial_t$ , so that  $F(s) = -\frac{n}{2s}$ , for all  $s \in I = (0, \infty)$  and  $\lambda(t, x) = -\frac{3n}{4t^2}$  for all  $(t, x) \in M$ . Using the Corollary 1, Then the identity map  $\tilde{I}_{\alpha,\beta}$  is proper biharmonic if and only if  $n \neq 4$  and  $\alpha = \frac{2n\beta}{n+4}$ .

### 5 Biharmonic curve in $(M, \tilde{g}_\alpha)$

Let  $\gamma: I \subset \mathbb{R} \rightarrow (M, g)$ ,  $t \mapsto \gamma(t)$  be a harmonic curve in a Riemannian manifold  $(M, g)$ , such that  $g(\dot{\gamma}, \dot{\gamma}) = 1$ , and let  $f$  be a smooth function on  $M$ . In this section we suppose that the gradient vector of  $f$  at  $\gamma(t)$  is parallel to the tangent vector  $\dot{\gamma}(t)$ . Thus,  $(\operatorname{grad} f)_{\gamma(t)} = \rho(t)\dot{\gamma}(t)$ , with  $\rho(t) = (f \circ \gamma)'(t)$ , for all  $t \in I$ . Since  $\gamma$  is harmonic we get the following formula

$$(\nabla_{\dot{\gamma}} \operatorname{grad} f)_t = \rho'(t)\dot{\gamma}(t), \quad \forall t \in I. \tag{32}$$

We set  $\tilde{g}_\alpha = \alpha g + (1 - \alpha)df \otimes df$ , where  $\alpha \in (0, 1)$ . We have the following result:

**Theorem 5.** *The curve  $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$  is biharmonic if and only if the function  $f$  satisfying the following*

$$f(\gamma(t)) = \pm \int \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}} dt,$$

where  $a, b, c \in \mathbb{R}$ , such that  $(at^2 + bt + c)^2 > \frac{\alpha}{1-\alpha}$ , for all  $t \in I$ .

*Proof.* By Theorem 1, we have

$$\tilde{\tau}(\gamma) = \tau(\gamma) + \frac{(1-\alpha) \text{Hess}_f(\dot{\gamma}, \dot{\gamma})}{\alpha + (1-\alpha)\|\text{grad } f\|^2 \circ \gamma} (\text{grad } f) \circ \gamma, \tag{33}$$

from the harmonicity condition of  $\gamma$ , and equations (32), (33), we obtain  $\tilde{\tau}(\gamma) = \lambda\dot{\gamma}$ , where

$$\lambda = \frac{(1-\alpha)\rho\rho'}{\alpha + (1-\alpha)\rho^2}. \tag{34}$$

Now, the curve  $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$  is biharmonic if and only if

$$\tilde{R}\left(\tilde{\tau}(\gamma), d\gamma\left(\frac{d}{dt}\right)\right)d\gamma\left(\frac{d}{dt}\right) + \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) = 0, \tag{35}$$

by the property of the curvature tensor, the first term on the left-hand side of (35) is

$$\tilde{R}\left(\tilde{\tau}(\gamma), d\gamma\left(\frac{d}{dt}\right)\right)d\gamma\left(\frac{d}{dt}\right) = \lambda\tilde{R}(\dot{\gamma}, \dot{\gamma})\dot{\gamma} = 0.$$

For the second term on the left-hand side of (35), we compute

$$\begin{aligned} \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) &= \tilde{\nabla}_{\frac{d}{dt}}^\gamma \lambda\dot{\gamma} \\ &= \lambda'\dot{\gamma} + \lambda\tilde{\nabla}_{\dot{\gamma}}^\gamma \dot{\gamma} \\ &= (\lambda' + \lambda^2)\dot{\gamma}, \end{aligned} \tag{36}$$

with the same method of (36), we find that

$$\begin{aligned} \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\nabla}_{\frac{d}{dt}}^\gamma \tilde{\tau}(\gamma) &= \tilde{\nabla}_{\frac{d}{dt}}^\gamma (\lambda' + \lambda^2)\dot{\gamma} \\ &= (\lambda'' + 2\lambda\lambda')\dot{\gamma} + (\lambda' + \lambda^2)\tilde{\nabla}_{\dot{\gamma}}^\gamma \dot{\gamma} \\ &= (\lambda'' + 3\lambda\lambda' + \lambda^3)\dot{\gamma}. \end{aligned}$$

So, the curve  $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$  is biharmonic if and only if  $\lambda'' + 3\lambda\lambda' + \lambda^3 = 0$ , that is the function  $\lambda$  is the form  $(2at + b)/(at^2 + bt + c)$ , where  $a, b, c \in \mathbb{R}$ , such that  $at^2 + bt + c \neq 0$ , for all  $t \in I$ . Thus, from (34) with  $(at^2 + bt + c)^2 > \frac{\alpha}{1-\alpha}$ , for all  $t \in I$ , we obtain

$$\rho(t) = \pm \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}}, \quad \forall t \in I. \tag{37}$$

Theorem 5 follows from equation (37), with  $\rho = (f \circ \gamma)'$ . □

**Remark 2.** The curve  $\gamma: I \rightarrow (M, \tilde{g}_\alpha)$  is proper biharmonic if and only if there exists  $a, b, c \in \mathbb{R}$  such that  $a^2 + b^2 > 0$ , and for all  $i = 1, \dots, m$  ( $m = \dim M$ ), and in any local coordinates  $(x_i)$  on  $M$ , such that

$$\sum_{j=1}^m g^{ij}(\gamma(t)) \frac{\partial f}{\partial x_j} \Big|_{\gamma(t)} = \pm \sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha} \frac{d\gamma^i}{dt} \Big|_t}, \quad \forall t \in I.$$

Using Theorem 5 and the previous Remark, we can construct many examples for proper biharmonic curves.

**Example 3.** Let  $M = \mathbb{R}^n$  equipped with the Riemannian metric  $g = dx_1^2 + \dots + dx_n^2$ ,

$$f(x) = \frac{2}{3} \sum_{i=1}^n (1 + x_i^2)^{\frac{3}{2}}, \quad \forall x = (x_1, \dots, x_n) \in M.$$

For  $\alpha = \frac{n}{n+1}$ , the curve

$$\gamma: I \rightarrow (M, \tilde{g}_\alpha), \quad t \mapsto \left( \frac{t}{\sqrt{n}}, \dots, \frac{t}{\sqrt{n}} \right),$$

is proper biharmonic.

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## References

- [1] P. Baird, A. Fardoun, S. Ouakkas: Conformal and semi-conformal biharmonic maps. *Annals of Global Analysis and Geometry* 34 (4) (2008) 403–414.
- [2] P. Baird, D. Kamissoko: On constructing biharmonic maps and metrics. *Annals of Global Analysis and Geometry* 23 (1) (2003) 65–75.
- [3] P. Baird, J.C. Wood: *Harmonic morphisms between Riemannian manifolds*. Oxford University Press (2003).
- [4] A. Benkartab, A.M. Cherif: New methods of construction for biharmonic maps. *Kyungpook Mathematical Journal* 59 (1) (2019) 135–147.
- [5] R. Caddeo, S. Montaldo, C. Oniciuc: Biharmonic submanifolds of  $S^3$ . *International Journal of Mathematics* 12 (08) (2001) 867–876.
- [6] J. Eells, L. Lemaire: A report on harmonic maps. *Bulletin of the London Mathematical Society* 10 (1) (1978) 1–68.
- [7] J. Eells, L. Lemaire: Another report on harmonic maps. *Bulletin of the London Mathematical Society* 20 (5) (1988) 385–524.
- [8] J. Eells, J.H. Sampson: Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics* 86 (1) (1964) 109–160.
- [9] T. Körpınar, E. Turhan: Tubular surfaces around timelike biharmonic curves in Lorentzian Heisenberg group  $\text{Heis}^3$ . *Analele Universitatii "Ovidius" Constanta – Seria Matematica* 20 (1) (2012) 431–446.

- [10] C. Oniciuc: New examples of biharmonic maps in spheres. *Colloquium Mathematicum* 97 (1) (2003) 131–139.
- [11] S. Ouakkas: Biharmonic maps, conformal deformations and the Hopf maps. *Differential Geometry and its Applications* 26 (5) (2008) 495–502.
- [12] G.Y. Jiang: 2-harmonic maps and their first and second variational formulas. *Chinese Ann. Math. Ser. A* 7 (4) (1986) 389–402.
- [13] B. O'Neill: *Semi-Riemannian geometry with applications to relativity*. Academic Press (1983).
- [14] T. Sakai: *Riemannian geometry*. Shokabo, Tokyo (1992). (in Japanese)

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