



HAL
open science

On a question of Schmidt and Summerer concerning 3-systems

Johannes Schleisnitz

► **To cite this version:**

Johannes Schleisnitz. On a question of Schmidt and Summerer concerning 3-systems. Communications in Mathematics, 2021, Volume 28 (2020), Issue 3 (3), pp.253 - 262. 10.2478/cm-2020-0007 . hal-03664987

HAL Id: hal-03664987

<https://hal.science/hal-03664987>

Submitted on 11 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0
International License

On a question of Schmidt and Summerer concerning 3-systems

Johannes Schleisnitz

Abstract. Following a suggestion of W.M. Schmidt and L. Summerer, we construct a proper 3-system (P_1, P_2, P_3) with the property $\bar{\varphi}_3 = 1$. In fact, our method generalizes to provide n -systems with $\bar{\varphi}_n = 1$, for arbitrary $n \geq 3$. We visualize our constructions with graphics. We further present explicit examples of numbers ξ_1, \dots, ξ_{n-1} that induce the n -systems in question.

1 Parametric Diophantine approximation in dimension three

Let ξ_1, ξ_2 be real numbers so that the set $\{1, \xi_1, \xi_2\}$ is linearly independent over \mathbb{Q} . For $q > 0$ a parameter, let $K(q)$ be the box of points $(z_0, z_1, z_2) \in \mathbb{R}^3$ that satisfy

$$|z_0| \leq e^{2q}, \quad |z_1| \leq e^{-q}, \quad |z_2| \leq e^{-q}.$$

Further let Λ be the lattice consisting of the points

$$\{(x, \xi_1 x - y_1, \xi_2 x - y_2) : x, y_1, y_2 \in \mathbb{Z}\}.$$

The successive minima $\lambda_1(q), \lambda_2(q), \lambda_3(q)$ of $K(q)$ with respect to Λ as functions of q contain the essential information on the simultaneous rational approximation to ξ_1, ξ_2 . It is convenient to study the logarithms of the functions $\lambda_j(q)$, denoted by $L_j(q) = \log \lambda_j(q)$ for $j = 1, 2, 3$. These functions have the nice property that their slopes are among $\{-2, 1\}$, and their sum is absolutely bounded uniformly in the parameter q . These properties motivated Schmidt and Summerer [7] to define so called 3-systems. A 3-system $P = (P_1, P_2, P_3)$ is a triple of functions $P_j : [0, \infty) \rightarrow \mathbb{R}$ with slopes among $\{-2, 1\}$ with the properties that

$$P_1(0) = P_2(0) = P_3(0) = 0, \\ P_1(q) \leq P_2(q) \leq P_3(q)$$

2020 MSC: 11J13, 11H06

Key words: parametric geometry of numbers, simultaneous approximation

Affiliation:

Middle East Technical University, Northern Cyprus Campus, Kalkanli, Güzelyurt

E-mail: johannes.schleisnitz@univie.ac.at

and

$$P_1(q) + P_2(q) + P_3(q) = 0 \quad \text{for every } q \geq 0.$$

Hence, locally in a neighborhood of any $q > 0$, precisely one of the three functions decays while the other two rise, unless q is a switch point where some P_j are not differentiable (change slope). Moreover, for P to be a 3-system, it is additionally required that if at a switch point q some P_i changes from falling to rising and some other P_j from rising to falling, then $i < j$ unless $P_i(q) = P_j(q)$. It has been shown in [7] that every function triple (L_1, L_2, L_3) as above, associated to some (ξ_1, ξ_2) , corresponds to a 3-system P up to a bounded amount, and conversely by Roy [2] that for any 3-system P there exist ξ_1, ξ_2 satisfying the \mathbb{Q} -linear independence condition above and so that $\sup_{q>0} \max_{j=1,2,3} |P_j(q) - L_j(q)| \ll 1$. Roy’s result employs a minor technical condition on the mesh of the system P , we do not rephrase it here. Both results [2], [7] are established in more generality.

For given ξ_1, ξ_2 with induced functions $L_j(q)$, let $\varphi_j(q) = L_j(q)/q$ and put

$$\underline{\varphi}_j = \liminf_{q \rightarrow \infty} \varphi_j(q), \quad \overline{\varphi}_j = \limsup_{q \rightarrow \infty} \varphi_j(q),$$

for $j = 1, 2, 3$. Since L_j have slopes -2 and 1 only, it is clear that

$$-2 \leq \underline{\varphi}_j \leq \overline{\varphi}_j \leq 1, \quad j = 1, 2, 3. \tag{1}$$

By virtue of the results from [2], [7] quoted above, in the sequel we will identify the values $\underline{\varphi}_j, \overline{\varphi}_j$ with quantities derived from an associated 3-system P via

$$\underline{\varphi}_j \longleftrightarrow \liminf_{q \rightarrow \infty} \frac{P_j(q)}{q}, \quad \overline{\varphi}_j \longleftrightarrow \limsup_{q \rightarrow \infty} \frac{P_j(q)}{q}, \quad j = 1, 2, 3, \tag{2}$$

and vice versa. M. Laurent [1] provided estimates for classical exponents of approximation related to any pair (ξ_1, ξ_2) that is \mathbb{Q} -linearly independent with $\{1\}$. As pointed out in [9] they translate into the language of the functions φ_j as

$$0 \leq \underline{\varphi}_3 \leq \overline{\varphi}_3 \leq 1, \tag{3}$$

$$\underline{\varphi}_3 + \underline{\varphi}_3 \overline{\varphi}_1 + \overline{\varphi}_1 = 0, \tag{3}$$

$$2\underline{\varphi}_1 + \overline{\varphi}_3 \leq -\underline{\varphi}_3(3 + \underline{\varphi}_1 + 2\overline{\varphi}_3), \tag{4}$$

$$2\overline{\varphi}_3 + \underline{\varphi}_1 \geq -\overline{\varphi}_1(3 + \overline{\varphi}_3 + 2\underline{\varphi}_1). \tag{5}$$

Schmidt and Summerer [9] recently provided additional information by including the second successive minimum in the picture.

Theorem 1 (Schmidt/Summerer, 2017). *For any ξ_1, ξ_2 with $\{1, \xi_1, \xi_2\}$ linearly independent over \mathbb{Q} , if $0 \leq \underline{\varphi}_3 < 1$, additionally to the above relations we have*

$$\overline{\varphi}_2 \leq \overline{\Omega} := \frac{\overline{\varphi}_1 - \underline{\varphi}_1}{2 - \overline{\varphi}_1 - \overline{\varphi}_1 \underline{\varphi}_1}, \tag{6}$$

and

$$\varphi_2 \geq \underline{\Omega} := \frac{\varphi_3 - \bar{\varphi}_3}{2 - \varphi_3 - \bar{\varphi}_3 \varphi_3}. \tag{7}$$

Moreover, these estimates are best possible in the sense that for given numbers $\varphi_1, \bar{\varphi}_1, \varphi_3, \bar{\varphi}_3$ with $0 \leq \varphi_3 < 1$ and (3), (4), (5) there are ξ_1, ξ_2 with $\{1, \xi_1, \xi_2\}$ linearly independent over \mathbb{Q} for whose approximation constants we have $\varphi_2 = \underline{\Omega}$ and $\bar{\varphi}_2 = \bar{\Omega}$.

Schmidt and Summerer enclose a remark to Theorem 1 pointing out that in the case $\varphi_3 = 1$ excluded in its claim, we have $\bar{\varphi}_2 = 1$ and $\bar{\varphi}_1 = \varphi_2 = -1/2$ (by mistake they denoted $-1/3$ instead in [9]). However, there is a gap in Theorem 1 concerning the existence of graphs with the property $\varphi_3 = 1$, and related real numbers ξ_1, ξ_2 . In [9] they state “But one really should prove that $\underline{\xi} = (\xi_1, \xi_2)$ with $(1, \xi_1, \xi_2)$ linearly independent over \mathbb{Q} with $\bar{\varphi}_3 = 1$ exist. We invite the reader to construct a proper 3-system P with this property.” The main purpose of this paper is to provide the desired construction. Before we turn to constructing the 3-system, we point out that explicit examples of \mathbb{Q} -linearly independent $\{1, \xi_1, \xi_2\}$ inducing $\varphi_3 = \bar{\varphi}_3 = 1$ can be derived from previous results of the author. Concretely [4, Corollary 2.11], upon putting $k = n - 1 = 2$ and $C = \infty$, yields the following example.

Theorem 2. *Let*

$$\xi_1 = \sum_{k=1}^{\infty} 10^{-(2k-1)!}, \quad \xi_2 = \sum_{k=1}^{\infty} 10^{-(2k)!}. \tag{8}$$

Then

$$\varphi_1 = -2, \quad \varphi_2 = -\frac{1}{2}, \quad \varphi_3 = 1, \tag{9}$$

$$\bar{\varphi}_1 = -\frac{1}{2}, \quad \bar{\varphi}_2 = 1, \quad \bar{\varphi}_3 = 1. \tag{10}$$

While the results in [4] are originally formulated in the language of another type of exponents, the two types of exponents determine each other via the identities of [6, Theorem 1.4], and we derive Theorem 2. We note that for the sole purpose of $\bar{\varphi}_3 = 1$, as desired in [9] and rephrased above, in fact any numbers ξ_1, ξ_2 which are simultaneously approximable to any order by rational numbers can be chosen. In particular, one may choose the pair (ξ, ξ^2) with ξ any Liouville number, see [5, Theorem 3.1]. However, then we always have $\varphi_3 = 0$. For Liouville’s constant given as $\xi = 10^{-11} + 10^{-21} + 10^{-31} + \dots$, by [5, Theorem 3.2] in place of (9), (10) we have

$$\varphi_1 = -2, \quad \varphi_2 = -\frac{1}{2}, \quad \varphi_3 = 0, \tag{11}$$

$$\bar{\varphi}_1 = 0, \quad \bar{\varphi}_2 = 1, \quad \bar{\varphi}_3 = 1. \tag{12}$$

Alternatively to the above examples, the pure existence of pairs (ξ_1, ξ_2) inducing $\bar{\varphi}_3 = 1$ (or $\varphi_3 = 1$) also follows from Roy’s results [2] and [3, Theorem 11.5]

(the latter result, already quoted in [9], provides an explicit description of the spectrum of sextuples $\varphi_1, \varphi_2, \varphi_3, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3$ by a system of complicated inequalities). The main concern of the question of Schmidt and Summerer appears to be the construction of a suitable 3-system, carried out in Section 2.1 below.

2 Construction of a 3-system with $\bar{\varphi}_3 = 1$

We want to present an effective construction of a 3-system with (9), (10), in particular $\bar{\varphi}_3 = 1$. It resembles the combined graph (L_1, L_2, L_3) with respect to the pair (ξ_1, ξ_2) in (8), in an idealized form. In fact the resulting 3-system can be interpreted as the idealized extremal case of the regular graph defined in [8], for the parameter $\rho = \infty$. In Section 2.3 we will briefly sketch how to modify the method to obtain a graph with (11), (12) instead, and give generalizations to n -systems.

2.1 The construction

We construct the graphs piecewise as follows. Let

$$0 < l_0 < l_1 < l_2 < l_3 < \dots ,$$

be a fast increasing lacunary sequence of real numbers with the property

$$\lim_{i \rightarrow \infty} \frac{l_{i+1}}{l_i} = \infty . \tag{13}$$

Let $r_0 = 0$. In the interval $[r_0, l_0] = [0, l_0]$ let P_1 decay with slope -2 and P_2, P_3 rise with slope 1 , so that $P_1(l_0) = -2l_0$ and $P_2(l_0) = P_3(l_0) = l_0$. Let $w_0 = l_0$ for consistency with later notation. Let l_0 be the first switch point where P_1 starts to rise and P_2 starts to decay. Then the graph of P_1 will meet the graph of P_2 at some point $(r_1, P_1(r_1))$ with $r_1 > l_0$. We may assume $l_1 > r_1$. In the interval $[r_1, l_1]$ we define P_1 as decaying with slope -2 again and the other two functions rising with slope 1 . Note that $P_3(l_1) = l_1$ since it has not changed slope yet. Assume this construction of the graphs in $[0, l_1]$ was step 0 of our construction. Now we carry out how to complete the process with identical steps $1, 2, 3, \dots$ where in step i we define the graphs of P_1, P_2, P_3 in the interval $[l_i, l_{i+1}]$. At position $q = l_1$ we let P_1 and P_3 change slopes so that P_1 rises with slope 1 and P_3 decays with slope -2 . The function P_2 still rises with slope 1 . We keep these slopes until P_2 meets P_3 at position $q = w_1$. Then we let P_2 decay with slope -2 and the other functions rise with slope 1 until P_2 meets P_1 at some point $(r_2, P_1(r_2))$. We may assume $l_2 > r_2$. Then we let P_1 decay with slope -2 up to $q = l_2$, and the other two functions rise with slope 1 in this interval. This completes step 1. At $q = l_2$ we let P_1 again switch from decaying to rising and conversely for P_3 , and so on. When we repeat the whole process ad infinitum, we claim that P_1, P_2, P_3 represent the combined graph of a 3-system with the properties (9), (10). A sketch of such a 3-system in an initial interval is shown in Figure 1 below. For size reasons we used the slopes $-1, 1/2$ instead of $-2, 1$, thereby sketching $P_j(q)/2$ for $j = 1, 2, 3$.

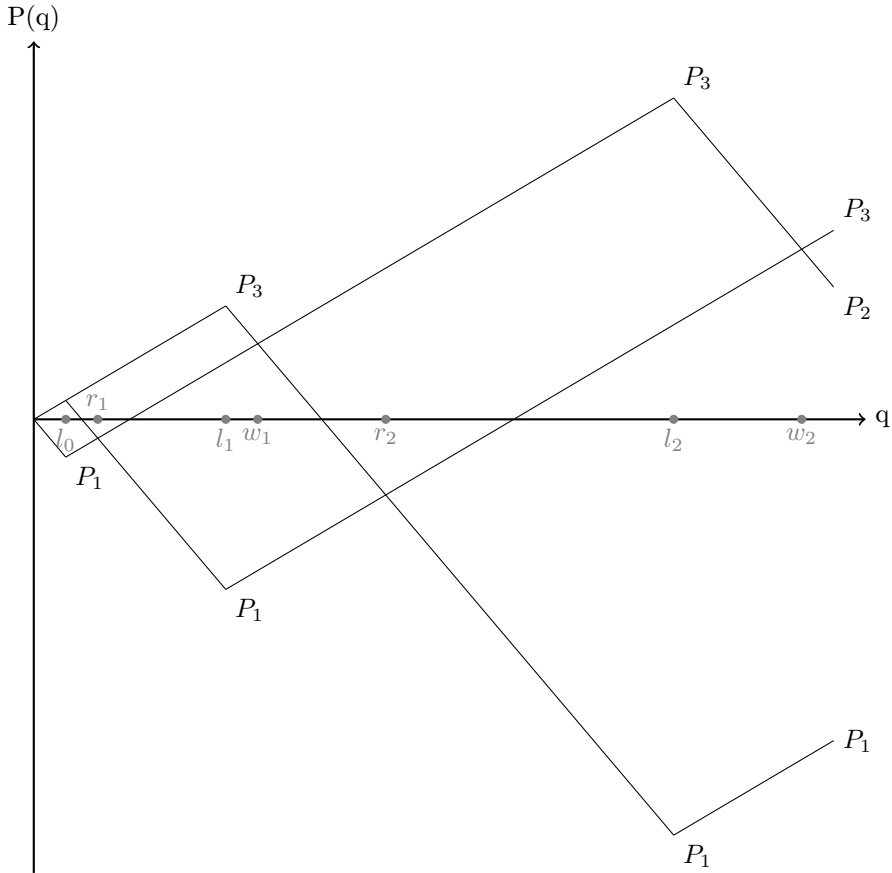


Figure 1: Visualization of case $\bar{\varphi}_3 = 1$, slopes scaled by factor $1/2$.

2.2 The proof

Keep in mind for the following that the switching positions in our construction are ordered

$$0 = r_0 < l_0 = w_0 < r_1 < l_1 < w_1 < r_2 < l_2 < w_2 < \dots,$$

and also the identification (2). First it is clear that the process yields the combined graph of a 3-system P . Indeed, by construction there is always precisely one P_j decaying, there are infinitely many positions where $P_1 = P_2$ and $P_2 = P_3$ respectively hold, and the switches occur in a way that respects the additional 3-system condition on a local maximum having higher index than a local minimum at switch points mentioned in the introduction. To obtain (9), (10), we first look at positions $q = l_i$ and claim that

$$\lim_{i \rightarrow \infty} \frac{P_1(l_i)}{l_i} = -2, \quad \lim_{i \rightarrow \infty} \frac{P_2(l_i)}{l_i} = \lim_{i \rightarrow \infty} \frac{P_3(l_i)}{l_i} = 1. \tag{14}$$

By the identification (2) and by (1) this implies $\varphi_1 = -2$ and $\bar{\varphi}_2 = \bar{\varphi}_3 = 1$. By construction P_1 decays with slope -2 in intervals of the form $I_t := [r_t, l_t]$ for $t \geq 0$ and rises in intervals $J_t := [l_{t-1}, r_t]$ for $t \geq 1$. We next check that

$$r_t < 2l_{t-1}, \quad t \geq 1. \tag{15}$$

We trivially have $P_3(l_{t-1}) - P_1(l_{t-1}) \leq l_{t-1} - (-2l_{t-1}) = 3l_{t-1}$. On the other hand, since P_1 decays in J_t with slope -2 whereas P_3 rises with slope 1 , the function $P_3 - P_1$ has slope 3 in J_t so that they must meet within distance $3l_{t-1}/3 = l_{t-1}$ in the first coordinate on the right from l_{t-1} . This intersection point has first coordinate r_t , and we deduce (15).

The estimate (15) and the assumption (13) clearly imply that the sums of the lengths of the intervals I_t over $t = 1, 2, \dots, i$ exceeds the according sums of the intervals J_t by any given factor $\rho > 0$ for large enough i , i.e.

$$\sum_{t=0}^i |I_t| > \rho \sum_{t=1}^i |J_t|, \quad i \geq i_0(\rho).$$

Thus since

$$l_i = \sum_{t=0}^i |I_t| + \sum_{t=1}^i |J_t|$$

and

$$P_1(l_i) = -2 \sum_{t=0}^i |I_t| + \sum_{t=1}^i |J_t|,$$

indeed for sufficiently large i we have

$$\frac{P_1(l_i)}{l_i} < -\frac{2 + \rho^{-1}}{1 + \rho^{-1}}.$$

As we can choose ρ arbitrarily large indeed $\lim_{i \rightarrow \infty} P_1(l_i)/l_i = -2$, hence $\varphi_1 = -2$ by (1), (2). Since P_2 and P_3 rise with slope 1 in any I_t we infer the remaining claims of (14) by a very similar argument, or directly by using the bounded sum property at $q = l_i$.

Next we show

$$\lim_{q \rightarrow \infty} \frac{P_3(q)}{q} = 1. \tag{16}$$

By construction P_3 has local minima precisely at positions w_i and it rises with slope 1 everywhere outside of the intervals $[l_i, w_i]$, in which it decays with slope -2 . In view of (1) it suffices to check that

$$\liminf_{i \rightarrow \infty} \frac{P_3(w_i)}{w_i} \geq 1. \tag{17}$$

By construction

$$P_3(w_i) = l_0 - 2 \sum_{j=0}^i (w_j - l_j) + \sum_{j=0}^{i-1} (l_{j+1} - w_j).$$

Hence, in view of (13), to verify (17) it suffices to check

$$\lim_{i \rightarrow \infty} \frac{w_i}{l_i} = 1. \tag{18}$$

Now by construction in the interval $[l_i, w_i]$ the function P_2 rises with slope 1 whereas P_3 decays with slope -2 , hence $w_i = l_i + u_i$ with u_i defined implicitly by the identity $P_3(l_i) - 2u_i = P_2(l_i) + u_i$, that is $w_i = l_i + (P_3(l_i) - P_2(l_i))/3$. On the other hand, by (14) we have $P_2(l_i) = l_i(1 + o(1))$ and $P_3(l_i) = l_i(1 + o(1))$, hence inserting we derive $w_i = l_i(1 + o(1))$ as $i \rightarrow \infty$, as desired. Thus (16) is shown.

Finally we show that

$$\lim_{i \rightarrow \infty} \frac{P_1(r_i)}{r_i} = \lim_{i \rightarrow \infty} \frac{P_2(r_i)}{r_i} = -\frac{1}{2}. \tag{19}$$

Since by construction the local maxima of P_1 and the local minima of P_2 both are attained precisely at the positions r_i , the remaining identities from (9) and (10) are implied. Let $K_t = [w_{t-1}, r_t]$, so that $K_t \subseteq J_t$ and by (13), (18) the complement $J_t \setminus K_t$ is small compared to J_t . In K_t , the function P_1 rises with slope 1 whereas P_2 decays with slope -2 . Moreover, by (14) and (18) and since the slopes are bounded

$$\lim_{i \rightarrow \infty} \frac{P_1(w_i)}{w_i} = -2, \quad \lim_{i \rightarrow \infty} \frac{P_2(w_i)}{w_i} = 1.$$

Combining these two facts and by definition of r_i , for large i we readily conclude $r_i = w_{i-1}(2 - o(1))$ and thus the asymptotic value at r_i is

$$P_1(r_i) = P_1(w_{i-1}) + r_i - w_{i-1} = w_{i-1}(-1 + o(1)),$$

hence indeed $P_1(r_i)/r_i = -1/2 + o(1)$ for large i . Thus (19) holds and the proof is finished.

2.3 Generalizations and variations

A similar construction as in Section 2.1 can be done in arbitrary dimension n , where the slopes of the P_j are among $\{-n + 1, 1\}$. Instead of one sequence $(w_i)_{i \geq 0}$ with $l_i < w_i < r_{i+1}$, we obtain $n - 2$ sequences $(w_i^h)_{i \geq 0}, 1 \leq h \leq n - 2$, induced by positions where P_{n-h+1} meets P_{n-h} , ordered

$$l_i < w_i^1 < w_i^2 < \dots < w_i^{n-2} < r_{i+1}.$$

We derive n -systems $P = (P_1, \dots, P_n)$ whose approximation constants (via identification (2)) satisfy

$$\varphi_1 = -n + 1, \quad \varphi_2 = \frac{2 - n}{2}, \quad \varphi_j = 1, \quad 3 \leq j \leq n,$$

and

$$\bar{\varphi}_1 = \frac{2 - n}{2}, \quad \bar{\varphi}_j = 1, \quad 2 \leq j \leq n.$$

Again this resembles the special case $\rho = \infty$ of the regular graph [8] in dimension n , and suitable numbers $(\xi_1, \dots, \xi_{n-1})$ inducing these approximation constants arise from [4, Corollary 2.11] upon taking $k = n - 1$, $C = \infty$, a particular choice is

$$\xi_j = \sum_{k=0}^{\infty} 10^{-(k(n-1)+j)!}, \quad 1 \leq j \leq n - 1.$$

Finally, we sketch the construction of a 3-system P with the properties (11), (12) in place of (9), (10). We have to alternate between the construction of Section 2.1 and another type of intermediate construction. Take i a large integer and follow the construction from Section 2.1 up to $q = q_0 =: l_i$. Recall $P_1(l_i) \approx -2l_i$ and $P_j(l_i) \approx l_i$ for $j = 2, 3$ by (14). Then we make the first intermediate construction. Starting from q_0 , let P_1 rise with slope 1 and P_2, P_3 decay with slope roughly $-1/2$ in not too short intervals. The latter can be easily realized by changing the slopes of P_2, P_3 rapidly so that there are many positions q with equality $P_2(q) = P_3(q)$. One may take these equality positions an arithmetic sequence $b_0, b_1 = b_0 + D, b_2 = b_0 + 2D, \dots, b_h = b_0 + hD$ with some $b_0 \geq q_0, h \geq 0$ and some small increment $D > 0$, in the following way. Fix $D > 0$ small. Let $(b_0, P_2(b_0))$ be the intersection point of the line passing through $(q_0, P_2(q_0))$ with slope 1 (graph of P_2) and the line passing through $(q_0, P_3(q_0))$ with slope -2 (graph of P_3), corresponding to w_i in Section 2.1. In $[b_0, b_0 + D/2]$, let P_2 decay with slope -2 and P_3 rise with slope 1. Then at $q_0 + D/2$ interchange the slopes, such that at $b_1 = b_0 + D$ we have $P_2(b_1) = P_3(b_1) = P_2(b_0) - D/2$. We repeat this procedure and stop at the largest index h so that the resulting graphs of P_2, P_3 remain positive on $[0, b_h]$. For simplicity let $\tilde{q} := b_h$. Notice that $P_j(b_l) - P_j(b_0) = -(b_l - b_0)/2 = -lD/2$ for $l = 0, 1, \dots, h$. Therefore, by (14) and since D is small, it is easy to see that $|P_j(\tilde{q})|$ are all small for $j = 1, 2, 3$. Now starting at \tilde{q} , let P_1, P_3 rise with slope 1 and P_2 decay with slope -2 until the graphs of P_1 and P_2 meet at some position q_1 . Since $|P_j(\tilde{q})|$ are all small, the expressions $q_1 - \tilde{q}$ and $|P_j(q_1)|$ for $j = 1, 2, 3$, are small (like $o(q_1)$) as well. This ends the first intermediate construction, illustrated in Figure 2 below (again slopes are scaled with factor $1/2$). Now we essentially apply the initial construction (step 0) from Section 2.1 from the interval $[0, l_1]$ again, starting from $q = q_1$ instead of $q = 0$. Let us denote by q_2 the right endpoint in this construction, that is the value corresponding to l_1 from Section 2.1. Notice that P_1 has a local minimum inside the interval $[q_1, q_2]$, corresponding to l_0 from Section 2.1, and another one at the right endpoint q_2 . Since $|P_j(q_1)|$ are small for $j = 1, 2, 3$, the P_j indeed behave in $[q_1, q_2]$ essentially like they do in the construction of Section 2.1 in the interval $[0, l_1]$ (see Figure 1). In particular, as for $q = q_0$, at $q = q_2$ again we have $P_1(q_2) \approx -2q_2$ and $P_j(q_2) \approx q_2$ for $j = 2, 3$. Hence at this point we again switch to the intermediate construction to define the P_j in some interval $[q_2, q_3]$. We repeat this iterative process of constructing P in $[q_{2k}, q_{2k+1}]$ and then in $[q_{2k+1}, q_{2k+2}]$, for all $k \geq 1$. It can be checked that the resulting combined graph satisfies (11), (12). Notice hereby that the condition $\varphi_2 = -1/2$ forced us to copy the behavior of the P_j on $[0, l_1]$, and not only on $[0, l_0]$, in intervals $[q_{2k+1}, q_{2k+2}]$. The procedure can again be generalized to dimension n to provide n -systems with

the properties

$$\varphi_1 = -n + 1, \quad \varphi_2 = \frac{2 - n}{2}, \quad \varphi_j = 0, \quad 3 \leq j \leq n,$$

and

$$\bar{\varphi}_1 = 0, \quad \bar{\varphi}_j = 1, \quad 2 \leq j \leq n.$$

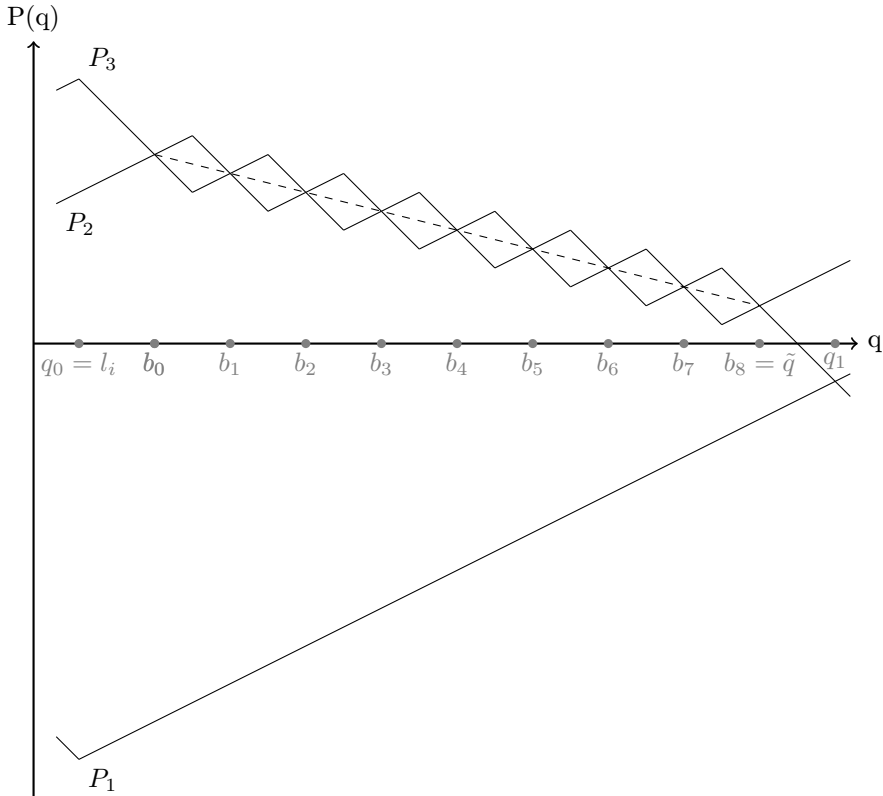


Figure 2: Intermediate construction in $[q_0, q_1]$, slopes scaled by factor $1/2$.

Acknowledgement

The author thanks the referee for the careful reading and for pointing out some inaccuracies!

References

- [1] M. Laurent: Exponents of Diophantine approximation in dimension two. *Canadian Journal of Mathematics* 61 (1) (2009) 165–189.
- [2] D. Roy: On Schmidt and Summerer parametric geometry of numbers. *Annals of Mathematics* (2015) 739–786.

- [3] D. Roy: On the topology of Diophantine approximation spectra. *Compositio Mathematica* 153 (7) (2017) 1512–1546.
- [4] J. Schleisnitz: Diophantine approximation and special Liouville numbers. *Communications in Mathematics* 21 (1) (2013) 39–76.
- [5] J. Schleisnitz: On approximation constants for Liouville numbers. *Glasnik matematički* 50 (2) (2015) 349–361.
- [6] W.M. Schmidt, L. Summerer: Parametric geometry of numbers and applications. *Acta Arithmetica* 140 (2009) 67–91.
- [7] W.M. Schmidt, L. Summerer: Diophantine approximation and parametric geometry of numbers. *Monatshefte für Mathematik* 169 (1) (2013) 51–104.
- [8] W.M. Schmidt, L. Summerer: Simultaneous approximation to three numbers. *Moscow Journal of Combinatorics and Number Theory* 3 (1) (2013) 84–107.
- [9] W.M. Schmidt, L. Summerer: Simultaneous approximation to two reals: bounds for the second successive minimum. *Mathematika* 63 (3) (2017) 1136–1151.

Received: 10 December 2018

Accepted for publication: 29 May 2019

Communicated by: Attila Bérczes