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## A note on the volume of $\nabla$ -Einstein manifolds with skew-torsion

*Ioannis Chrysikos*

**Abstract.** We study the volume of compact Riemannian manifolds which are Einstein with respect to a metric connection with (parallel) skew-torsion. We provide a result for the sign of the first variation of the volume in terms of the corresponding scalar curvature. This generalizes a result of M. Ville [15] related with the first variation of the volume on a compact Einstein manifold.

### Introduction

Consider a compact Einstein manifold  $(M^n, g)$  and let  $\mathcal{M}$  be the space of Riemannian metrics on  $M$ . A smooth variation of the metric  $g$  in the direction of some arbitrary but fixed smooth symmetric  $(2, 0)$ -tensor  $h \in \Gamma(S^2T^*M) =: \mathcal{S}^2(M)$  on  $M$ , is a smooth curve  $g(t) : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$  with  $g(0) = g$  and  $\dot{g}(0) = \frac{d}{dt}\big|_{t=0}g(t) = h$ . We view the scalar curvature related with the Levi-Civita connection  $\nabla^g$  as a functional  $\mathcal{M} \rightarrow C^\infty(M)$ , defined by  $g \mapsto \text{Scal}_g$ , and in a line with the notation of [5], we denote its differential  $\text{Scal}'_g$  at  $g$  in the direction of  $h$  (or the first variation in the direction of  $h$ ) by

$$\text{Scal}'_g(h) := \frac{d}{dt}\bigg|_{t=0} \text{Scal}_{g+th} = \frac{d}{dt}\bigg|_{t=0} \text{Scal}_{g(t)} .$$

By Ville [15] it is known that the first variation of the volume  $\text{vol}_g(M)$  of  $(M^n, g)$  enjoys some nice properties which can be expressed in terms of the scalar curvature. In particular,

$$\int_M (\text{Scal}'_g h) dV_g = -\frac{2}{n} \text{Scal}_g \text{vol}_g(M)' h ,$$

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where  $dV_g$  is the volume element, which can be also written as

$$\int_M \left. \frac{d}{dt} \right|_{t=0} \text{Scal}_{g(t)} dV_g = -\frac{2}{n} \text{Scal}_g \left. \frac{d}{dt} \right|_{t=0} \text{vol}_{g(t)}(M).$$

Here,  $\text{Scal}_{g(t)}$  and  $\text{vol}_{g(t)}(M)$  denote the scalar curvature and volume of  $g(t)$ , respectively. Therefore, if  $\text{Scal}_g = 0$ , or if  $\text{vol}_g(M)'h = 0$ , then the differential  $\text{Scal}'_g$  cannot have constant sign, unless it is identically zero (see also [5, Prop. 1.188]).

Our purpose in this short note is to extend Ville’s results on Riemannian manifolds which are “Einstein” with respect to a metric connection  $\nabla$  with skew-torsion  $T \in \Omega^3(M)$ , i.e.

$$\text{Ric}_S^\nabla = \frac{\text{Scal}^\nabla}{n} g,$$

where  $\text{Ric}_S^\nabla$  denotes the symmetric part of the Ricci tensor  $\text{Ric}^\nabla$  associated to  $\nabla$  (see below). Such geometric structures are called  $\nabla$ -Einstein manifolds with skew-torsion and play a key role in the theory of non-integrable geometries, due to the so-called *characteristic connection* (cf. [12]). This is a metric connection  $\nabla$  with skew-torsion as above, i.e.

$$\nabla = \nabla^g + \frac{1}{2} T$$

which preserves the underlying non-integrable geometry and hence it is a natural replacement of the Levi-Civita connection. A first systematic study of  $\nabla$ -Einstein manifolds with skew-torsion, in a variety of different dimensions, was given in [1]. It is remarkable that this work also provides the existence of  $\nabla$ -Einstein manifolds with skew-torsion, which are *not* Einstein with respect to the corresponding metric, e.g. the Allof-Wallach spaces. Further such examples were constructed on Berger spheres in [11], while similar existence results are known even for non-Einstein Lorentzian metrics. Notice also that there are  $\nabla$ -Einstein structures with skew-torsion which provide examples of manifolds satisfying important spinorial equations, a fact which yields a strong interplay with string theory (see e.g. [3], [7], [8], [12]). Hence, the last decades the geometry of  $\nabla$ -Einstein manifolds with skew-torsion has attracted more attention. For instance, the recent works [6], [9], [10], [11], present classification results of such structures for particular families of homogeneous spaces (in terms of invariant connections and representation theory).

At this point we need to emphasize an important difference in comparison with the classical notion of Einstein manifolds, namely:  $\nabla$ -Einstein manifolds with skew-torsion  $(M^n, g, T)$  may have non-constant scalar curvature  $\text{Scal}^\nabla$  ([1]). This comes true since for example in this case the corresponding *Einstein tensor*

$$G^\nabla := -\text{Ric}_S^\nabla + \frac{1}{2} \text{Scal}^\nabla g$$

is not necessarily divergence free. In this short note by assuming that the scalar curvature  $\text{Scal}^\nabla$  is constant, we provide an extension of Ville’s result for compact  $\nabla$ -Einstein manifolds with skew-torsion. Examples of  $\nabla$ -Einstein manifolds which verify our unique assumption are those whose torsion form  $T$  is  $\nabla$ -parallel. Consequently, one can present a wealth of examples for which our result makes sense,

e.g. 6-dimensional nearly Kähler manifolds, 7-dimensional weak  $G_2$ -manifolds, 7-dimensional 3-Sasakian manifolds and naturally reductive spaces are few of them (see for example [1], [3], [7], [8] and the references therein).

### 1 $\nabla$ -Einstein manifolds with skew-torsion

Let  $(M^n, g)$  be a connected oriented Riemannian manifold. We fix once and for all a  $g$ -orthonormal basis  $\{e_1, \dots, e_n\}$  of  $T_x M$  at some point  $x \in M$ . Recall that a linear connection

$$\nabla: \Gamma(TM) \mapsto \Gamma(T^*M \otimes TM)$$

is said to be metric if  $\nabla g = 0$ . The torsion  $T: \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$  of  $\nabla$  is the vector-valued 2-form defined by  $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ , for any vector field  $X, Y \in \Gamma(TM)$ .  $\nabla$  is said to be with totally anti-symmetric torsion if the induced tensor  $T(X, Y, Z) := g(T(X, Y), Z)$  is a 3-form on  $M$  and then the following identity holds

$$g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{2}T(X, Y, Z).$$

Thus, in such a case the condition  $T \in \Omega^3(M)$  induces the dimensional restriction  $n = \dim_{\mathbb{R}} M \geq 3$ .

Let us denote by

$$d_{\nabla}: \Omega^p(M) \rightarrow \Omega^{p+1}(M), \quad d_{\nabla}^*: \Omega^p(M) \rightarrow \Omega^{p-1}(M),$$

the differential and co-differential induced by  $\nabla$ , which are the differential operators defined by  $d_{\nabla} \omega := \sum_i e_i \wedge \nabla_{e_i} \omega$  and  $d_{\nabla}^* \omega := -\sum_i e_i \lrcorner \nabla_{e_i} \omega$ , respectively. In a line with the Riemannian case,  $d_{\nabla}$  and  $d_{\nabla}^*$  are formally adjoint each other, but in general one has  $(d_{\nabla})^2 \neq 0$ . The difference between  $d_{\nabla} T$  and  $dT$ , where we set  $d \equiv d_g$ , is a 4-form which (up to a factor) will be denoted by  $\sigma_T$ , namely  $d_{\nabla} T - dT = -2\sigma_T$ . This 4-form  $\sigma_T$  does not play some explicit role in this note and we refer to [12] for alternative definitions. Notice however that the co-differential of the torsion form  $T$  satisfies  $d_{\nabla}^* T = d^* T$  and under the assumption  $\nabla T = 0$  the following hold (see [2], [12])

$$d_{\nabla}^* T = 0 = d^* T, \quad dT = 2\sigma_T.$$

For the curvature tensor  $R^{\nabla}$  associated to  $\nabla$ , we adopt the convention

$$R^{\nabla}(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

and set  $R^{\nabla}(X, Y, Z, W) := g(R^{\nabla}(X, Y)Z, W)$ . In terms of the  $g$ -orthonormal frame  $\{e_1, \dots, e_n\}$ , the Ricci tensor associated to  $\nabla$  is given by

$$\text{Ric}^{\nabla}(X, Y) = \sum_i g(R^{\nabla}(X, e_i)e_i, Y) = \sum_i R^{\nabla}(X, e_i, e_i, Y).$$

Moreover, the two Ricci tensors are related by (see e.g. [12])

$$\text{Ric}^{\nabla}(X, Y) = \text{Ric}^g(X, Y) - \frac{1}{4}S(X, Y) - \frac{1}{2}(d^* T)(X, Y),$$

where  $S$  is the symmetric tensor defined by

$$S(X, Y) := \sum_{i=1}^n g(T(e_i, X), T(e_i, Y)) = \sum_{i,j=1}^n T(e_i, X, e_j) \cdot T(e_i, Y, e_j). \quad (1)$$

Thus, in contrast to the Riemannian Ricci tensor  $\text{Ric}^g$ , the Ricci tensor of  $\nabla$  is not in general symmetric; it decomposes into a symmetric and anti-symmetric part  $\text{Ric}^\nabla = \text{Ric}_S^\nabla + \text{Ric}_A^\nabla$ , given by

$$\text{Ric}_S^\nabla(X, Y) := \text{Ric}^g(X, Y) - \frac{1}{4}S(X, Y), \quad \text{Ric}_A^\nabla(X, Y) := -\frac{1}{2}(d^*T)(X, Y),$$

respectively. As we said above, when  $T$  is  $\nabla$ -parallel, then  $d^*T = 0$  and hence  $\text{Ric}^\nabla = \text{Ric}_S^\nabla$ . Finally, the scalar curvature  $\text{Scal}^\nabla = \text{tr Ric}^\nabla$  of  $(M, g, T)$  with respect to  $\nabla$  satisfies the identity  $\text{Scal}^\nabla = \text{Scal}^g - \frac{3}{2}\|T\|_g^2$ .

**Definition 1.** A triple  $(M^n, g, T)$  ( $n \geq 3$ ) is called a  $\nabla$ -Einstein manifold with skew-torsion  $0 \neq T \in \Omega^3(TM)$ , or in short, a  $\nabla$ -Einstein manifold, if the symmetric part  $\text{Ric}_S^\nabla$  of the Ricci tensor associated to the metric connection  $\nabla = \nabla^g + \frac{1}{2}T$  satisfies the equation

$$\text{Ric}_S^\nabla = \frac{\text{Scal}^\nabla}{n}g, \quad (2)$$

where  $\text{Scal}^\nabla$  is the scalar curvature associated to  $\nabla$  and  $n = \dim_{\mathbb{R}} M$ . If  $\nabla T = 0$ , then  $(M, g, T)$  is called a  $\nabla$ -Einstein manifold with parallel skew-torsion.

In a line with the case of Einstein manifolds, in [1] it was shown that  $\nabla$ -Einstein manifolds  $(M, g, T)$  attain a variational approach which reads in terms of a generalized scalar curvature functional  $\mathcal{L}$ , given by

$$(g, T) \mapsto \int_M \left( \text{Scal}^\nabla - 2\Lambda \right) dV_g = \int_M \left( \text{Scal}^g - \frac{3}{2}\|T\|_g^2 - 2\Lambda \right) dV_g.$$

In particular, if  $\Lambda = 0$ , then the critical points of  $\mathcal{L}$  are triples  $(M, g, T)$  satisfying  $\text{Ric}_S^\nabla = 0$  identically, and conversely. If  $\Lambda \neq 0$  and  $\nabla T = 0$ , then the critical points are  $\text{Ric}^\nabla$ -flat manifolds. For  $\Lambda \neq 0$ , critical points of  $\mathcal{L}$  are  $\nabla$ -Einstein manifolds  $(M, g, T)$  with  $\text{Scal}^\nabla \neq 0$ , i.e.  $\text{Ric}_S = \frac{\text{Scal}^\nabla}{n}g$  (so-called strictly  $\nabla$ -Einstein metrics), and conversely.

## 2 An extension of a result of Ville

Let us now prove the analogue of the infinitesimal result of Ville [15, Prop. 3] about the sign of the first variation of the volume on compact Einstein manifolds (see also Proposition 1.188 in Besse’s book [5]). The volume of a triple  $(M^n, g, T)$  will be denoted by  $\text{Vol}_g(M) := \int_M dV_g$ . Also, given  $h_1, h_2 \in \mathcal{S}^2(M)$ , we shall denote their inner product with respect to  $g$  by

$$(h_1, h_2)_g := \sum_{i,j} h_1(e_i, e_j)h_2(e_i, e_j).$$

On the other hand, for the square norm of the 3-form  $T$  with respect to  $g$  we fix the normalization

$$\|T\|_g^2 := \frac{1}{3!} \sum_{i,j} g(T(e_i, e_j), T(e_i, e_j)). \tag{3}$$

**Theorem 1.** *Let  $(M^n, g, T)$  be a compact  $\nabla$ -Einstein manifold with skew-torsion, i.e.  $\text{Ric}_S^\nabla = \frac{1}{n} \text{Scal}^\nabla g$ . Assume that  $\text{Scal}^\nabla$  is constant. Then, the first variation of the volume  $\text{Vol}_{g(t)}(M)$  with respect to the curve  $g(t) \in \mathcal{M}$  in the direction of  $h \in \mathcal{S}^2(M)$ , satisfies the relation*

$$-\frac{2 \text{Scal}^\nabla}{n} \frac{d}{dt} \Big|_{t=0} \text{Vol}_{g(t)}(M) = \int_M \frac{d}{dt} \Big|_{t=0} \text{Scal}_{g(t)}^\nabla dV_g, \tag{4}$$

where  $\text{Scal}_{g(t)}^\nabla := \text{Scal}^{g(t)} - \frac{3}{2} \|T\|_{g(t)}^2$  such that  $\text{Scal}_{g(0)}^\nabla \equiv \text{Scal}_g^\nabla =: \text{Scal}^\nabla$ .

*Proof.* Set  $\text{Vol}_{g(t)}(M) = \int_M dV_{g(t)}$ . The first variation of the volume element  $dV_g$  is given by (see for example [4], [14])

$$\frac{d}{dt} \Big|_{t=0} dV_{g(t)} = \frac{1}{2} (g, h)_g dV_g = \frac{1}{2} (\text{tr}_g h) dV_g,$$

thus

$$\frac{d}{dt} \Big|_{t=0} \text{Vol}_{g(t)}(M) = \frac{1}{2} \int_M \text{tr}_g h dV_g. \tag{5}$$

Since  $(M, g, T)$  is  $\nabla$ -Einstein, it follows that

$$(\text{Ric}_S^\nabla, h)_g = \frac{\text{Scal}^\nabla}{n} \text{tr}_g h$$

which is equivalent with the relation

$$(\text{Ric}^g, h)_g = \frac{\text{Scal}^\nabla}{n} \text{tr}_g h + \frac{1}{4} (S, h)_g. \tag{6}$$

Now we need the first variation of the square norm of the torsion form in the direction of some  $h \in \mathcal{S}^2(M)$ . This is given by (see [1] and see also our Appendix for an alternative proof)

$$\frac{d}{dt} \Big|_{t=0} \|T\|_{g(t)}^2 = -\frac{1}{6} (S, h)_g. \tag{7}$$

Consequently,

$$\int_M (S, h)_g dV_g = -6 \int_M \frac{d}{dt} \Big|_{t=0} \|T\|_{g(t)}^2 dV_g. \tag{8}$$

Finally, let us recall the first variation of  $\text{Scal}^g$  (see [4], [14])

$$\frac{d}{dt} \Big|_{t=0} \text{Scal}^{g(t)} = \Delta_g(\text{tr}_g h) + \text{div}_g(\text{div}_g h) - (\text{Ric}^g, h)_g,$$

where we write  $\operatorname{div}_g : \mathcal{S}^p(M) \rightarrow \mathcal{S}^{p-1}(M)$  for the divergence of a symmetric tensor field  $F$  on  $M$ , i.e.

$$\operatorname{div}_g(F)(X_1, \dots, X_{p-1}) := - \sum_{i=1}^n (\nabla_{e_i}^g F)(e_i, X_1, \dots, X_{p-1}).$$

Thus, and since  $M$  is compact, by integrating one computes

$$\begin{aligned} & \int_M \frac{d}{dt} \Big|_{t=0} \operatorname{Scal}^{g(t)} dV_g \\ &= - \int_M (\operatorname{Ric}_g, h)_g dV_g \\ &\stackrel{(6)}{=} - \int_M \frac{\operatorname{Scal}^\nabla}{n} \operatorname{tr}_g h dV_g - \int_M \frac{1}{4} (S, h)_g dV_g \\ &\stackrel{(5),(8)}{=} - \frac{2 \operatorname{Scal}^\nabla}{n} \frac{d}{dt} \Big|_{t=0} \operatorname{Vol}_{g(t)}(M) + \frac{3}{2} \int_M \frac{d}{dt} \Big|_{t=0} \|T\|_{g(t)}^2 dV_g, \end{aligned}$$

or equivalently

$$\int_M \frac{d}{dt} \Big|_{t=0} \left( \operatorname{Scal}^{g(t)} - \frac{3}{2} \|T\|_{g(t)}^2 \right) dV_g = - \frac{2 \operatorname{Scal}^\nabla}{n} \frac{d}{dt} \Big|_{t=0} \operatorname{Vol}_{g(t)}(M).$$

This proves our assertion. □

**Remark 1.** It should be clear that the right hand side of (4) does not coincide with the variation of the generalized total scalar curvature functional  $\mathcal{L}$  (in the direction of  $h$ ). This only happens when the first variation of the volume element is zero,  $\frac{d}{dt} \Big|_{t=0} dV_{g(t)} = 0$ .

As an immediate corollary of Theorem 1 we get that

**Corollary 1.** *Let  $(M^n, g, T)$  be a compact  $\nabla$ -Einstein manifold with a constant  $\operatorname{Scal}^\nabla$ . If*

$$\frac{d}{dt} \Big|_{t=0} \operatorname{Vol}_{g(t)}(M) = 0,$$

*or if  $\operatorname{Scal}^\nabla = 0$ , then the differential  $(\operatorname{Scal}_g^\nabla)'$  cannot have a constant sign (unless it is identically zero).*

Theorem 1 shows that [5, Prop. 1.188] holds more general for any compact Riemannian manifold which is  $\nabla$ -Einstein with respect to a metric connection with *parallel skew-torsion*. Indeed, when  $\nabla T = 0$ , then we can show that the associated Einstein tensor

$$G^\nabla := - \operatorname{Ric}_S^\nabla + \frac{1}{2} \operatorname{Scal}^\nabla g = G^g + \frac{1}{4} S - \frac{3}{4} \|T\|_g^2 g$$

is divergence free, and as in the classical case for  $n \geq 3$  this yields the constancy of  $\operatorname{Scal}^\nabla$  (see also [1]). Hence we deduce

**Corollary 2.** *Theorem 1 applies on any compact  $\nabla$ -Einstein manifold  $(M^n, g, T)$  ( $n \geq 3$ ) with  $\nabla T = 0$ .*

### 3 Appendix

Here we present a proof of the formula (7), in a slightly different way than the one discussed in [1].

Recall that given any two Riemannian metrics  $a, b \in \mathcal{M}$ , there is a unique self-adjoint positive definite endomorphism  $B_{a,b}$  on the tangent space  $T_x M$  ( $x \in M$ ), such that

$$b(u, v) = a(B_{a,b}u, v),$$

for all  $u, v \in T_x M$ . Hence  $d_b^a := B_{a,b}^{-1/2}$  is an isometry on  $T_x M$  and this extends to an isometry on whole tangent bundle  $TM$ , which we denote by the same letter,  $d_b^a : T^a M \equiv TM \rightarrow T^b M \equiv TM$ ,  $a(X, Y) = b(d_b^a X, d_b^a Y)$  for any  $X, Y \in \Gamma(TM)$ . Consider the variation  $g(t) \equiv g_t = g + th$  of  $g$  in the direction of some symmetric  $(2, 0)$ -tensor  $h$  on  $M$ . Since there exists an open neighborhood  $I = (-\epsilon, \epsilon) \subset \mathbb{R}$  of  $0 \in \mathbb{R}$  such that  $g_t \in \mathcal{M}$  for any  $t \in I$ , one has  $g(G_{g,g_t}^t u, w) = g_t(u, w)$ , for any  $u, w \in T_x M$ , where  $G_{g,g_t}^t \in \text{End}(T_x M)$  is the self-adjoint positive definite endomorphism corresponding to  $g_t$ . Then, by the definition of  $g_t$  it follows that

$$\begin{aligned} g(G_{g,g_t}^t u, w) &= g_t(u, w) \\ &= g(u, w) + th(u, w) \\ &= g(u, w) + tg(H_{g,h}u, w) \\ &= g((\text{Id} + tH_{g,h})u, w), \end{aligned}$$

i.e.  $G_{g,g_t}^t = \text{Id} + tH_{g,h}$  and hence<sup>1</sup>

$$d_{g_t}^g = (G_{g,g_t}^t)^{-1/2} = (\text{Id} + tH_{g,h})^{-1/2}.$$

Consequently

$$\left. \frac{d}{dt} \right|_{t=0} d_{g_t}^g = -\frac{1}{2}(\text{Id} + tH_{g,h})^{-3/2} H_{g,h} \Big|_{t=0} = -\frac{1}{2} H_{g,h}. \tag{9}$$

Consider now some  $g$ -orthonormal frame  $\{e_i\}$  of  $M$ . Then, the set

$$\{e_i(t) := d_{g_t}^g(e_i) : 1 \leq i \leq n\}$$

forms a  $g(t)$ -orthonormal frame such that  $e_i(0) = d_{g_0}^g(e_i) = \text{Id}(e_i) = e_i$ , for any  $i$ . Since  $h(e_i, e_j) = g(H_{g,h}e_i, e_j)$  we obtain

$$H_{g,h}e_i = \sum_j g(H_{g,h}e_i, e_j)e_j = \sum_j h(e_i, e_j)e_j$$

and hence (9) yields the formula

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} d_{g_t}^g(e_i) &= \left. \frac{d}{dt} \right|_{t=0} e_i(t) := \dot{e}_i(t) \Big|_{t=0} \\ &= \dot{e}_i(0) = -\frac{1}{2} H_{g,h}e_i = -\frac{1}{2} \sum_j h(e_i, e_j)e_j. \end{aligned} \tag{10}$$

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<sup>1</sup>Note that any arbitrary element  $h \in \mathcal{S}^2(M)$  induces some endomorphism  $H_{g,h} \in \text{End}(T_x M)$  defined by  $h(u, v) = g(H_{g,h}u, v)$ , see for example [13, p. 134].



Now, by (3) we see that

$$\begin{aligned}
 & \frac{d}{dt} \|T\|_{g(t)}^2 \\
 &= \frac{1}{6} \sum_{i,j} \frac{d}{dt} \left[ (g + th) \left( T(e_i(t), e_j(t)), T(e_i(t), e_j(t)) \right) \right] \\
 &= \frac{1}{6} \sum_{i,j} \dot{g}(t) \left( T(e_i(t), e_j(t)), T(e_i(t), e_j(t)) \right) \\
 &\quad + \frac{1}{3} \sum_{i,j} g(t) \left( \frac{d}{dt} T(e_i(t), e_j(t)), T(e_i(t), e_j(t)) \right) \\
 &= \frac{1}{6} \sum_{i,j} \dot{g}(t) \left( \sum_k g(T(e_i(t), e_j(t)), e_k(t)) e_k(t), \sum_\ell g(T(e_i(t), e_j(t)), e_\ell(t)) e_\ell(t) \right) \\
 &\quad + \frac{1}{3} \sum_{i,j} g(t) \left( T(\dot{e}_i(t), e_j(t)) + T(e_i(t), \dot{e}_j(t)), T(e_i(t), e_j(t)) \right) \\
 &= \frac{1}{6} \sum_{i,j,k,\ell} T(e_i(t), e_j(t), e_k(t)) \cdot T(e_i(t), e_j(t), e_\ell(t)) \cdot \dot{g}(t) (e_k(t), e_\ell(t)) \\
 &\quad + \frac{1}{3} \sum_{i,j} g(t) \left( T(\dot{e}_i(t), e_j(t)) + T(e_i(t), \dot{e}_j(t)), T(e_i(t), e_j(t)) \right).
 \end{aligned}$$

Therefore, due to (10) and the tensor  $S$  defined in (1), the evaluation of the previous relation at  $t = 0$  gives the result:

$$\begin{aligned}
 \frac{d}{dt} \Big|_{t=0} \|T\|_{g(t)}^2 &= \frac{1}{6} \sum_{k,\ell} S(e_k, e_\ell) h(e_k, e_\ell) - \frac{1}{6} \sum_{i,j} g(T(H_{g,h} e_i, e_j), T(e_i, e_j)) \\
 &\quad - \frac{1}{6} \sum_{i,j} g(T(e_i, H_{g,h} e_j), T(e_i, e_j)) \\
 &= \frac{1}{6} (S, h)_g - \frac{1}{6} \sum_i S(H_{g,h} e_i, e_i) - \frac{1}{6} \sum_j S(H_{g,h} e_j, e_j) \\
 &= \frac{1}{6} (S, h)_g - \frac{1}{3} \sum_i S(H_{g,h} e_i, e_i) \\
 &= \frac{1}{6} (S, h)_g - \frac{1}{3} \sum_i S\left(\sum_j h(e_i, e_j) e_j, e_i\right) \\
 &= \frac{1}{6} (S, h)_g - \frac{1}{3} \sum_{i,j} h(e_i, e_j) S(e_i, e_j) \\
 &= \frac{1}{6} (S, h)_g - \frac{1}{3} (S, h)_g \\
 &= -\frac{1}{6} (S, h)_g.
 \end{aligned}$$

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