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► **To cite this version:**

Florian Luca, Sylvester Manganye. On the gaps between  $q$ -binomial coefficients. Communications in Mathematics, 2021, Volume 29 (2021), Issue 3 (3), pp.431 - 442. 10.2478/cm-2020-0010 . hal-03664983

**HAL Id: hal-03664983**

**<https://hal.science/hal-03664983>**

Submitted on 11 May 2022

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# On the gaps between $q$ -binomial coefficients

Florian Luca, Sylvester Manganye

**Abstract.** In this note, we estimate the distance between two  $q$ -nomial coefficients  $\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right|$ , where  $(n, k) \neq (n', k')$  and  $q \geq 2$  is an integer.

## 1 Introduction

In this paper,  $q \geq 2$  is an integer and for  $n > k \geq 1$ ,

$$\binom{n}{k}_q := \frac{(q^{n-k+1} - 1)(q^{n-k+2} - 1) \cdots (q^n - 1)}{(q - 1)(q^2 - 1) \cdots (q^k - 1)}$$

is the  $q$ -binomial coefficient. We are interested in the distinct values of  $\binom{n}{k}_q$ . Since  $\binom{n}{k}_q = \binom{n-k}{k}_q$ , we assume that  $n \geq 2k$ . It was shown in [1] that under these conditions

$$\binom{n}{k}_q \neq \binom{n'}{k'}_q \quad \text{for } (n, k) \neq (n', k'), \quad n \geq 2k, \quad n' \geq 2k'.$$

The proof is an easy application of the primitive divisor theorem for members of Lucas sequences. Thus, taking

$$\mathcal{B}_q := \left\{ \binom{n}{k}_q : n \geq 2k \geq 2 \right\},$$

2020 MSC: 11B65, 11B39

Key words:  $q$ -binomial coefficients

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the elements from  $\mathcal{B}_q$  are distinct. Assume  $\mathcal{B}_q = \{b_1, b_2, \dots\}$ , where the elements  $b_i$  are listed increasingly. We are interested in a lower bound for  $b_{i+1} - b_i$ . We have the following theorem:

**Theorem 1.** *The inequality*

$$b_{N+1} - b_N \geq \exp\left((\log b_N)^{1/3}\right)$$

holds for all  $q \geq 2$  and all  $N \geq 163,000$ .

**Corollary 1.** *The inequality  $b_{N+1} - b_N > 100$  always holds except when  $N \leq 8$  for  $q = 2$  or  $N \leq 4$  for  $q \in \{3, 4, 5, 6, 7, 8, 9, 10\}$ .*

## 2 Some auxiliary results

We put  $m := k(n - k)$ .

**Lemma 1.** *We have*

$$\frac{q^m}{4} < \binom{n}{k}_q < 4q^m$$

for all  $q \geq 2$  and  $n \geq 2k$ .

*Proof.* We have

$$\binom{n}{k}_q = \frac{q^{n-(k-1)+n-(k-2)+\dots+n}}{q^{k+k-1+\dots+1}} \left( \prod_{1 \leq j \leq k} \left(1 - \frac{1}{q^{n-j+1}}\right) \right) \left( \prod_{j=1}^k \left(1 - \frac{1}{q^j}\right) \right)^{-1}.$$

The first factor in the right-hand side above is  $q^m$ . As for the others, the inequality

$$\frac{1}{4} < 0.288 < \prod_{j \geq 1} \left(1 - \frac{1}{2^j}\right) \leq \prod_{a \leq j \leq b} \left(1 - \frac{1}{q^j}\right) < 1$$

holds for all positive integers  $a < b$  and  $q \geq 2$ . Taking  $(a, b) = (n - k + 1, k)$ , or  $(a, b) = (1, k)$ , respectively, we get that

$$\frac{1}{4} < \left( \prod_{j=1}^k \left(1 - \frac{1}{q^{n-j+1}}\right) \right) \left( \prod_{j=1}^k \left(1 - \frac{1}{q^j}\right) \right)^{-1} < 4,$$

which finishes the proof. □

From now on,  $(n, k) \neq (n', k')$  are such that  $n \geq 2k$ ,  $n' \geq 2k'$ . For a positive integer  $\ell$  we write

$$\Phi_\ell(X) = \prod_{\substack{1 \leq j \leq \ell \\ \gcd(j, \ell) = 1}} (X - e^{2\pi i j / \ell}) \in \mathbb{Z}[X]$$

for the  $\ell$ th cyclotomic polynomial.

**Lemma 2.** Assume that  $[n - k + 1, n] \cap [n' - k' + 1, n'] \neq \emptyset$ . Then

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \Phi_\ell(q), \quad \text{where } \ell \in [n - k + 1, n] \cap [n' - k' + 1, n'].$$

*Proof.* Since  $q^\ell - 1 = \prod_{d|\ell} \Phi_d(q)$ , it follows that

$$\binom{n}{k}_q = \prod_{d \in \mathcal{D}(n, k)} \Phi_d(q)^{\alpha(d, n, k)},$$

where

$$\mathcal{D}(n, k) = \bigcup_{j \in [1, k]} \{d \geq 1 : d \mid n - j + 1 \text{ or } d \mid j\},$$

and  $\alpha(d, h, k)$  are some integers. Since  $\binom{n}{k}_q$  is a rational function in  $q$  which is an integer for all  $q \geq 2$ , it follows that  $\alpha(d, n, k) \geq 0$  for all  $d \in \mathcal{D}(n, k)$ . Further, it is easy to see that  $d = n - j + 1$  has  $\alpha(d, n, k) \geq 1$  for all  $j \in [1, k]$ , since  $\Phi_{n-j+1}(q) \mid q^{n-j+1} - 1$  and  $\Phi_{n-j+1}(q)$  is not a factor of  $\prod_{i=1}^k (q^i - 1)$  because  $n - j + 1 \geq n - k + 1 > k$ . Thus, if  $\ell \in [n - k + 1, n] \cap [n' - k' + 1, n']$ , then  $\Phi_\ell(q)$  is a factor of both  $\binom{n}{k}_q$  and  $\binom{n'}{k'}_q$ . Thus, their difference is nonzero and a multiple of  $\Phi_\ell(q)$ , which finishes the proof of the lemma.  $\square$

**Lemma 3.** Assume that  $[n - k + 1, n] \cap [n' - k' + 1, n'] = \emptyset$ . Put again  $m := k(n - k)$ ,  $m' := k'(n - k')$ . Then:

(i) If  $m' < m$ , then

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \frac{1}{7} \binom{n'}{k'}_q.$$

(ii) If  $m' = m$  and  $k' < k$ , then

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \frac{2}{q^{n+1}} \binom{n'}{k'}_q.$$

*Proof.* From the arguments from the proof of Lemma 1, we have

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| = \left| q^m \left( \frac{\prod_{j=1}^k (1 - 1/q^{n-j+1})}{\prod_{j=1}^k (1 - 1/q^j)} \right) - q^{m'} \left( \frac{\prod_{j=1}^{k'} (1 - 1/q^{n'-j+1})}{\prod_{j=1}^{k'} (1 - 1/q^j)} \right) \right|.$$

We analyze the two cases.

(i) In this case,

$$\begin{aligned} & \left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \\ &= \binom{n'}{k'}_q \left| q^{m-m'} \left( \frac{\prod_{j=1}^k (1 - 1/q^{n-j+1})}{\prod_{j=1}^k (1 - 1/q^j)} \right) \left( \frac{\prod_{j=1}^{k'} (1 - 1/q^{n'-j+1})}{\prod_{j=1}^{k'} (1 - 1/q^j)} \right)^{-1} - 1 \right|. \end{aligned}$$

In the right, the coefficient of  $q^{m-m'}$  is  $(P/Q)(Q'/P')$ , where

$$P = \prod_{j=1}^k (1 - 1/q^{n-j+1}) \quad Q = \prod_{j=1}^k (1 - 1/q^j),$$

and  $P', Q'$  are obtained from  $P, Q$  by changing  $(k, n)$  to  $(k', n')$ , respectively. All of  $P, Q, P', Q'$  are smaller than 1. We have the following lemma:

**Lemma 4.** *The inequality*

$$\prod_{j=a}^b (1 - 1/q^j) \geq q^{-1/3} \tag{1}$$

holds for all  $q \geq 2$  and  $a \geq 1$  and any  $b \geq a$  except for possibly

$$(a, q) = (1, 2), (1, 3), (2, 2), (3, 2).$$

*Proof.* Taking logarithms, the desired inequality becomes

$$\sum_{j=a}^b \log \left( 1 - \frac{1}{q^j} \right) > -\frac{\log q}{3}.$$

The inequality  $\log(1 - x) > -2x$  holds for all  $x \in (0, 1/2)$ . So, using this with  $x = 1/q^j$  for  $j \in [a, b]$ , it suffices to show that

$$-\sum_{j=a}^b \frac{2}{q^j} > -\frac{\log q}{3},$$

which is equivalent to

$$\sum_{j=a}^b \frac{1}{q^j} < \frac{\log q}{6}.$$

Taking the sum on the left to infinity, it is a geometrical progression whose sum is  $1/(q^{a-1}(q - 1))$ . Thus, it suffices that

$$q^{a-1}(q - 1) \geq \frac{6}{\log q}.$$

The above inequality holds for all  $a \geq 1$  and  $q \geq 5$ . It also holds for  $a \geq 5$  and any  $q \geq 2$ . So, it remains to check the given inequality for  $(a, q)$  with  $a \in [1, 4]$  and  $q \in [2, 4]$ , and we get the list of exceptions.  $\square$

To apply the above lemma, notice that  $(P/Q)(P'/Q')^{-1} = PQ'(QP')^{-1}$ , and  $(QP')^{-1} > 1$ . Furthermore,  $P$  is a product as the one appearing in (1) with  $a = n - k + 1 \geq k + 1 \geq 2$ , while  $Q'$  is a product like the one appearing in (1) but with  $a = 1$ . Thus, by Lemma 4, we have that the inequality

$$\min\{P, Q'\} \geq q^{-1/3}$$

holds unless  $q \in \{2, 3\}$ . So, unless  $q \in \{2, 3\}$ , we have that

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |q^{m-m'-2/3} - 1| \geq |q^{1/3} - 1| \geq |2^{1/3} - 1| > 1/4.$$

Assume next that  $q = 2, 3$ . If  $q = 3$ , then

$$\min\{P, Q'\} \geq \prod_{j=1}^{\infty} (1 - 1/3^j) > 0.56, \quad \max\{P, Q'\} \geq \prod_{j \geq 2} (1 - 1/3^j) > 0.84,$$

so

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |3 \times 0.56 \times 0.84 - 1| > 0.4 > 1/4.$$

It remains to treat the case  $q = 2$ . If  $k' \leq k$ , then  $P/Q(P'/Q')^{-1} = P(Q/Q')^{-1}P'^{-1}$  and both  $Q/Q' \leq 1$ ,  $P' < 1$ . Furthermore,  $P$  is a product like in (1) starting at  $n - k + 1$ . Thus, if  $n - k + 1 \geq 4$ , then

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |2^{m-m'-1/3} - 1| \geq |2^{2/3} - 1| > 1/2.$$

If  $m - m' \geq 2$ , then since

$$P \geq \prod_{j \geq 1} (1 - 1/2^j) > 0.288,$$

we get

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |2^2 \times 0.288 - 1| > 1/7.$$

Thus, we only need to analyze the situation  $n - k + 1 \leq 3$  and  $m' = m - 1$ . Since  $n - k \geq k$ , this gives  $k \leq 2$  and then  $n \leq k + 2 \leq 4$ . Thus,  $(n, k) = (2, 1), (3, 1), (4, 1), (4, 2)$ . Further,  $m = nk - k^2 = k(n - k) \leq 4$ . Since  $m' < m$ , we get  $m' = k'(n' - k') < 4$ , so  $(n', k') = (2, 1), (3, 1), (4, 1)$ . Now we compute

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right|$$

over all such possibilities  $(n, k, n', k')$  and  $q = 2$ , and conclude that the desired inequality holds in these cases as well.

This was if  $k' \leq k$ . Assume next that  $k' > k$ . Then

$$(P/Q)(P'/Q')^{-1} = P(Q'/Q)P'^{-1}$$

and  $Q'/Q$  is a product as in (1) starting at  $a = k' + 1 \geq 3$ . Thus, if  $\min\{n - k + 1, k' + 1\} \geq 4$ , then (1) holds and so

$$|q^{m-m'}P(Q'/Q)P'^{-1} - 1| \geq |2^{1/3} - 1| > 1/4.$$

Thus, we treat the case  $\min\{n - k + 1, k' + 1\} \leq 3$ . Since  $n - k + 1 \geq k + 1$  and  $k' > k$ , it follows that

$$k + 1 = \min\{k + 1, k' + 1\} \leq \min\{n - k + 1, k' + 1\} \leq 3,$$

so  $k \in \{1, 2\}$ . Thus,

$$\min\{n - 1, k' + 1\} \leq \min\{n - k + 1, k' + 1\} \leq 3,$$

so either  $n \leq 4$  or  $(k', k) = (2, 1)$ . If  $m - m' \geq 2$ , then since

$$\prod_{j \geq 2} (1 - 1/2^j) \geq 0.57,$$

it follows that

$$|q^{m-m'} P(Q'/P)P'^{-1} - 1| \geq |4 \times (0.57)^2 - 1| > 1/4.$$

Thus, it remains to treat the case  $m' = m - 1$ . If  $n \leq 4$ , then

$$k'^2 \leq k'(n' - k') = m' = m - 1 = k(n - k) - 1 \leq 3,$$

so  $k' = 1$ , contradicting the fact that  $k' > k$ . Thus,  $(k', k) = (2, 1)$  so  $Q'/Q$  is a product like in (1) starting at  $k' + 1 = 3$ . If also  $n - k + 1 \geq 3$ , then since

$$\prod_{j \geq 3} (1 - 1/2^j) > 0.77,$$

it follows that

$$|q^{m-m'} P(Q'/Q)P'^{-1} - 1| \geq |2 \times (0.77)^2 - 1| > 1/6.$$

Hence, it remains to treat the case when  $n - k + 1 = 2$ , so  $(n, k) = (2, 1)$ , so  $m = 1$  and then  $m' = m - 1 = 0$ , a contradiction. This takes care of (i).

(ii). In this case, since  $k(n - k) = k'(n' - k')$  and  $k' < k$ , it follows that  $n' - k' > n - k$  and since  $[n - k + 1, n]$  and  $[n' - k' + 1, n']$  are disjoint, it follows that  $n' - k' \geq n$ . With the notations from part (i), we have

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| = q^m |(P/Q) - (P'/Q')| = \binom{n'}{k'}_q |(P/(Q/Q')P'^{-1} - 1|.$$

Now

$$P/(Q/Q')P'^{-1} = \prod_{j=1}^{k'} \left( \frac{1 - 1/q^{n-k+j}}{1 - 1/q^{n'-k'+j}} \right) \prod_{j=k'}^{k-1} \left( \frac{1 - 1/q^{n-(k-j)+1}}{1 - 1/q^{j+1}} \right). \tag{2}$$

Let us notice the following order

$$k' + 1 \leq \dots \leq k < n - k + 1 \leq \dots \leq n < n' - k' + 1 < \dots < n'.$$

Using the inequalities

$$1 - 1/q^\ell > \exp\left(-\frac{2}{q^\ell}\right) \quad \text{and} \quad 1 - 1/q^\ell < \exp\left(-\frac{1}{q^\ell}\right),$$

for  $\ell$  an index participating in the numerator, respectively, denominator of the right-hand side of (2), we get to get that

$$P/(Q/Q')P'^{-1} > \exp\left(\frac{1}{q^{k'+1}} + \dots + \frac{1}{q^k} - \frac{2}{q^{n-k+1}} - \dots - \frac{2}{q^n} + \frac{1}{q^{n'-k'+1}} + \dots + \frac{1}{q^{n'}}\right).$$

Now

$$\frac{2}{q^{n-k+1}} + \dots + \frac{2}{q^n} < 2\left(\sum_{j \geq n-k+1} \frac{1}{q^j}\right) - \frac{2}{q^{n+1}} = \frac{2}{q^{n-k}(q-1)} - \frac{2}{q^{n+1}}.$$

Hence,

$$P/(Q/Q')P'^{-1} > \exp\left(\frac{1}{q^{k'+1}} + \dots + \frac{1}{q^k} - \frac{2}{q^{n-k}(q-1)} + \frac{2}{q^{n+1}} + \frac{1}{q^{n'-k'+1}} + \dots + \frac{1}{q^{n'}}\right). \tag{3}$$

If  $q \geq 3$ , then since  $n - k \geq k$ , it follows that

$$\frac{1}{q^{k'+1}} + \dots + \frac{1}{q^k} - \frac{2}{q^{n-k}(q-1)} \geq \frac{1}{q^k} - \frac{1}{q^{n-k}} \geq 0,$$

so the amount under the exponential in the right-hand side of (3) is at least  $2/q^{n+1}$ . Since  $e^x - 1 > x$  for positive  $x$ , it follows that in these cases

$$|P/(Q/Q')P'^{-1} - 1| > \frac{2}{q^{n+1}}.$$

The same conclusion holds if  $q = 2$  and either  $k < n - k$ , or  $k' < k - 1$ . But if  $q = 2$ ,  $k = n - k$  and  $k' = k - 1$ , then

$$m = k(n - k) = k^2 = m' = (k - 1)(n' - (k - 1)).$$

Thus,  $k - 1$  divides  $k^2$ , which is possible only for  $k = 2$ . Hence,  $(k, n) = (2, 4)$ , and then  $k' = 1$  and

$$4 = m = m' = n' - k' = n - 1,$$

so  $n' = 5$ . In this case,

$$|P/(Q/Q')P'^{-1} - 1| = \left| \frac{(1 - 1/2^3)(1 - 1/2^4)}{(1 - 1/2^2)(1 - 1/2^5)} - 1 \right| > 0.12 > \frac{2}{q^{n+1}}.$$

Hence,

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| > \frac{2}{q^{n+1}} \binom{n'}{k'}_q,$$

holds in all cases, which completes the proof of this lemma. □



### 3 The proof of Theorem 1

We are now ready to do some estimates. We distinguish several cases.

#### 3.1 The case of Lemma 3 (i)

In this case, putting  $b_{N'} = \binom{n'}{m'}_q$ , we need to decide when the inequality

$$\frac{1}{7}b_{N'} \geq \exp((\log b_{N'})^{1/3})$$

holds. This is equivalent to

$$\log b_{N'} \geq \log 7 + (\log b_{N'})^{1/3}.$$

Using also Lemma 1, it is enough to show that

$$m' \log q - \log 4 \geq \log 7 + (m' \log q + \log 4)^{1/3}.$$

Dividing by  $\log q$  and using the fact that  $q \geq 2$ , it is enough that

$$m' \geq \frac{\log 28}{\log 2} + \left( \frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3} \right)^{1/3},$$

an inequality which holds for all  $m' \geq 8$ .

#### 3.2 The case of Lemma 3 (ii)

In this case,  $m' = m$  and we want that

$$\log b_{N'} + \log 2 - (n+1) \log q \geq (\log b_{N'})^{1/3}.$$

Using again Lemma 1, it suffices that

$$m' \log q - \log 4 + \log 2 \geq (n+1) \log q + (m' \log q + \log 4)^{1/3}.$$

We have  $k(n-k) = m'$ , so  $n+1 = m'/k + k + 1$  and  $k > k'$ . Thus,  $k \in [2, \sqrt{m'}]$ . The function  $x \mapsto m'/x + x + 1$  in the interval  $[2, \sqrt{m'}]$  since its derivative is  $-m'/x^2 + 1 \leq 0$ . Thus,  $n+1 \leq m'/2 + 3$ . Hence, it suffices that the inequality

$$m' \log q \geq \log 2 + (m'/2 + 3) \log q + (m' \log q + \log 4)^{1/3}$$

holds. Dividing by  $\log q$  and using the fact that  $q \geq 2$ , it suffices that

$$\frac{m'}{2} - 3 \geq 1 + \left( \frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3} \right)^{1/3},$$

an inequality which holds for all  $m' \geq 15$ .

### 3.3 The case of Lemma 2 and $q \geq 3$

We assume  $\ell \geq 3$ . In this case,

$$\Phi_\ell(q) = \prod_{\substack{1 \leq j \leq \ell \\ \gcd(j, \ell) = 1}} |q - e^{2\pi i j / \ell}| \geq (q - 1)^{\phi(\ell)} = \exp(\phi(\ell) \log(q - 1)). \tag{4}$$

So, we need to show that

$$\phi(\ell) \log(q - 1) \geq (\log b_{N'})^{1/3},$$

or, using again Lemma 1, that

$$\phi(\ell) \log(q - 1) \geq (m' \log q + \log 4)^{1/3}.$$

By Theorem 15 in [2], we have

$$\phi(\ell) > \frac{\ell}{1.8 \log \log \ell + 2.6 / \log \ell} \quad \text{for all } \ell \geq 3.$$

Thus, dividing also by  $\log(q - 1)$ , it suffices to show that

$$\frac{\ell}{1.8 \log \log \ell + 2.6 / \log \ell} \geq \left( \frac{m' \log q}{(\log(q - 1))^3} + \frac{\log 4}{(\log(q - 1))^3} \right)^{1/3}.$$

The functions

$$x \mapsto \frac{x}{1.8 \log \log x + 2.6 / \log x} \quad \text{and} \quad x \mapsto \frac{\log x}{(\log(x - 1))^3}$$

have the property that the first one is increasing and the second one is decreasing for  $x \geq 3$ , as it can be confirmed by computing their derivatives. Since  $\ell \geq n' - k' + 1 \geq \sqrt{m'} + 1$ , it suffices that

$$\frac{\sqrt{m'} + 1}{1.8 \log \log(\sqrt{m'} + 1) + 2.6 / \log(\sqrt{m'} + 1)} \geq \left( \frac{m' \log 3}{(\log 2)^3} + \frac{\log 4}{(\log 2)^3} \right)^{1/3},$$

an inequality which holds for  $m' > 15,300$ .

### 3.4 The case of Lemma 2 and $q = 2$

Here,  $q - 1 = 1$ , so inequality (4) is useless. Instead we use the formula

$$\Phi_\ell(2) = \prod_{d|n} (2^{n/d} - 1)^{\mu(d)},$$

where  $\mu$  is the Möbius function. Factoring out the “main” terms, we get

$$\Phi_\ell(2) \geq 2^{\sum_{a|\ell} \mu(d)\ell/d} \prod_{j \geq 1} (1 - 1/2^j) > 2^{\phi(\ell) - 2}.$$

Thus, we get that

$$\Phi_\ell(2) \geq \exp((\phi(\ell) - 2) \log 2).$$

Thus, in order to prove the desired inequality it suffices, again via Lemma 1, to show that

$$\phi(\ell) - 2 \geq \left( \frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3} \right)$$

for some  $\ell \in [n' - k' + 1, n']$ . The argument from Subsection 3.3 shows that this inequality holds provided that

$$\frac{\sqrt{m'} + 1}{1.8 \log \log(\sqrt{m'} + 1) + 2.6/\log(\sqrt{m'} + 1)} - 2 \geq \left( \frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3} \right)^{1/3},$$

an inequality which holds for  $m' > 8100$ .

To summarize, we proved:

**Lemma 5.** *If  $m \geq m' \geq 15,300$ , then*

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \exp \left( \left( \log \binom{n'}{k'}_q \right)^{1/3} \right).$$

Thus, the inequality in the theorem may fail only if  $b_{N'} = \binom{n'}{k'}_q$  for some  $m' \leq M' := 15,300$ . Since  $m' = k'(n' - k')$ , it follows that for a fixed  $m'$ , the number of pairs  $(n', k')$  with  $m' = k'(n' - k')$  is at most  $\tau(m')$ , where  $\tau(s)$  is the number of divisors of  $s$  (in fact, it is smaller than that since  $k' \leq n' - k'$ , but we will not get into such details). Thus, those  $N'$  can be at most the first

$$\begin{aligned} \sum_{m' \leq M'} \tau(m') &= \sum_{m' \leq M'} \sum_{d' | m'} 1 \leq \sum_{d' \leq M'} \sum_{\substack{m' \leq M' \\ m' \equiv 0 \pmod{d'}}} 1 \\ &= \sum_{d' \leq M'} \left\lfloor \frac{M'}{d'} \right\rfloor \leq M' \sum_{d' \leq M'} \frac{1}{d'} \\ &\leq M' \left( 1 + \int_1^{M'} \frac{dt}{t} \right) \leq M'(1 + \log M') < 163,000, \end{aligned}$$

which finishes the proof.

### 4 The proof of the Corollary 1

We follow the previous steps of the proof of Theorem 1. For the situation treated in Subsection 3.1, we need  $b_{N'} > 700$ . By Lemma 1, this gives  $q^{m'} > 700$ , which is satisfied for  $m' \geq 9$ . Since  $m' = k'(n' - k') \geq n'/2$ , it follows that the last inequality is satisfied for  $n' \geq 18$ . Thus, it remains to study the case  $n' < 17$ . In this case,  $m' \leq (n'/2)^2$ , so  $m' \leq 72$ . If  $m \geq 83$ , then  $m - m' \geq 11$ , so by Lemma 1, we have

$$\binom{n}{k}_q \geq \frac{q^m}{4} \geq 2^7(4q^{m'}) > 2^7 \binom{n'}{k'}_q,$$

so

$$\binom{n}{k}_q - \binom{n'}{k'}_q \geq (2^7 - 1) \binom{n'}{k'}_q > 100.$$

Thus, it suffices to consider the case  $m \leq 82$ , leading to  $n/2 \leq k(n - k) \leq m$ , so  $n \leq 164$ . Thus, for Subsection 3.1, it suffices to check in the range  $\max\{n, n'\} \leq 164$ . A similar argument works for the situation treated in Subsection 3.2. Namely, here we need that  $2b_{N'}/q^{n+1} > 100$ . Together with Lemma 1, this is satisfied for  $q^{m'-n-1} > 200$ , which in turn holds if  $m' - n \geq 9$ . Now  $m' = k(n - k)$ , where  $k > k'$  so either  $k = 2$ , or  $k \geq 3$ . When  $k = 2$ , we have

$$9 \leq m' - n = 2(n - 2) - n = n - 4$$

so the desired inequality is satisfied for  $n \geq 13$ . When  $k \geq 3$ , we have that

$$m' - n = k(n - k) - n \geq 3n/2 - n = n/2$$

and so the desired inequality holds for  $n \geq 18$ . Thus, it suffices to assume that  $n \leq 17$ , leading to  $m \leq (17/2)^2$ , so  $m \leq 72$ . Since in this case we have  $m = m'$ , we get that  $n'/2 \leq m' = m \leq 72$ , so  $n' \leq 144$ . Thus, in this case it suffices to check in the range  $\max\{n, n'\} \leq 144$ . For Subsection 3.3, all we need is that  $2^{\phi(\ell)} \geq 100$ , so  $\phi(\ell) > 6$  for some  $\ell \in [n - k + 1, n] \cap [n' - k' + 1, n']$ . Now  $\phi(\ell) > 6$  for  $\ell > 18$ , so the desired inequality is satisfied provided that  $n - k + 1 \geq 19$ . Since  $n - k \geq n/2$ , the last inequality holds for  $n \geq 36$ . Thus, it suffices to check it for  $n \leq 35$  and since  $[n' - k' + 1, n']$  intersects nontrivially  $[n - k + 1, n]$ , we get that  $n' - k' + 1 \leq n \leq 35$ . Thus,  $n'/2 \leq n' - k' \leq 34$ , so  $n' \leq 68$ . Thus, in this case it suffices check the range  $\max\{n, n'\} \leq 68$ . Finally, for Subsection 3.4, we want  $\Phi_n(2) > 100$  and we checked that this is so for all  $n \geq 19$ . To do so, we use a consequence of the Primitive Divisor Theorem to the effect that  $\Phi_n(2)$  is divisible by a prime  $p \equiv 1 \pmod{n}$  for all  $n > 6$  (this is a primitive prime factor of  $2^n - 1$ ). In particular,  $\Phi_n(2) > 100$  if  $n > 100$ , so we only needed to check the values of  $\Phi_n(2)$  for  $n \leq 100$  and got that the largest  $n$  with  $\Phi_n(2) \leq 100$  is  $n = 18$ . Thus, it suffices to consider the case  $n \leq 18$ , and since  $n' - k' + 1 \leq n \leq 18$ , we get that  $n' \leq 34$ . Thus, in all cases  $\max\{n, n'\} \leq 200$ . Putting everything together, we conclude that  $b_{N+1} - b_N > 100$  unless both  $b_N, b_{N+1}$  correspond to  $q$ -nomial coefficients  $\binom{n}{k}_q$  or  $\binom{n'}{k'}_q$  with  $\max\{n, n'\} \leq 200$ . Further, unless  $q = 2$ , we are in the cases from Subsections 3.1, 3.2, 3.3, respectively, and in these there cases, invoking Lemma 1, the lower bounds on  $b_{N+1} - b_N$  are  $q^m/28, q^{m-(n+1)}/2, (q - 1)^{\phi(\ell)}$ , respectively. In the first case we have the exponent  $m \geq 2$ , while in the other two cases the exponents are  $m - (n + 1) \geq 1, \phi(\ell) \geq 1$ . Thus, the inequality  $b_{N+1} - b_N > 100$  is satisfied if  $q > 201$  independently on  $(n, k, n', k')$ . Hence, we only need to check the situations  $q \leq 201$  and  $\max\{n, n'\} \leq 200$ . A computation in this range finishes the job.

### Acknowledgments

The first author supported in part by grant CPRR160325161141 from the NRF of South Africa and the Focus Area Number Theory grant RTNUM19 from CoEMaSS Wits.

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*Received:* 7 July 2019

*Accepted for publication:* 27 July 2019

*Communicated by:* Yuri Bilu