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# A Deformed Quon Algebra

Hery Randriamaro

**Abstract.** The quon algebra is an approach to particle statistics in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated by a small amount. The quons are particles whose annihilation and creation operators obey the quon algebra which interpolates between fermions and bosons. In this paper we generalize these models by introducing a deformation of the quon algebra generated by a collection of operators  $a_{i,k}$ ,  $(i, k) \in \mathbb{N}^* \times [m]$ , on an infinite dimensional vector space satisfying the deformed  $q$ -mutator relations  $a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j}$ . We prove the realizability of our model by showing that, for suitable values of  $q$ , the vector space generated by the particle states obtained by applying combinations of  $a_{i,k}$ 's and  $a_{i,k}^\dagger$ 's to a vacuum state  $|0\rangle$  is a Hilbert space. The proof particularly needs the investigation of the new statistic *cinv* and representations of the colored permutation group.

## 1 Introduction

Let  $\mathbb{R}(q)$  be the fraction field of the real polynomials with variable  $q$ . By a deformed quon algebra  $\mathbf{A}$ , we mean the free algebra

$$\mathbb{R}(q)[a_{i,k} \mid (i, k) \in \mathbb{N}^* \times [m]]$$

subject to the anti-involution  $\dagger$  exchanging  $a_{i,k}$  with  $a_{i,k}^\dagger$ , and to the commutation relation

$$a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j},$$

where  $\delta_{i,j}$  is the Kronecker delta and

$$\beta_{k,l} = \begin{cases} 0 & \text{if } l - k \equiv m \pmod{m} \\ 1 & \text{otherwise} \end{cases}.$$

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This algebra is a generalization of the quon algebra introduced by Greenberg [2], subject to the commutation relation  $a_j a_i^\dagger = q a_i^\dagger a_j + \delta_{i,j}$  obeyed by the annihilation and creation operators of the quon particles, and generating a model of infinite statistics. Moreover, the quon algebra is a generalization of the classical Bose and Fermi algebras corresponding to the restrictions  $q = 1$  and  $q = -1$  respectively, as well as of the intermediate case  $q = 0$  suggested by Hegstrom and investigated by Greenberg [1].

In a Fock-like representation, the generators of  $\mathbf{A}$  are the linear operators  $a_{i,k}, a_{i,k}^\dagger: \mathbf{V} \rightarrow \mathbf{V}$  on an infinite dimensional real vector space  $\mathbf{V}$  satisfying the commutation relations

$$a_{j,l} a_{i,k}^\dagger - q a_{i,k}^\dagger a_{j,l} = q^{\beta_{k,l}} \delta_{i,j},$$

and the relations

$$a_{i,k} |0\rangle = 0,$$

where  $a_{i,k}^\dagger$  is the adjoint of  $a_{i,k}$ , and  $|0\rangle$  is a nonzero distinguished vector of  $\mathbf{V}$ . The  $a_{i,k}$ 's are the annihilation operators and the  $a_{i,k}^\dagger$ 's the creation operators.

Let  $\mathbf{H}$  be the vector subspace of  $\mathbf{V}$  generated by the particle states obtained by applying combinations of  $a_{i,k}$ 's and  $a_{i,k}^\dagger$ 's to  $|0\rangle$ , or

$$\mathbf{H} := \{a|0\rangle \mid a \in \mathbf{A}\}.$$

The aim of this article is to prove the realizability of this model through the following theorem.

**Theorem 1.**  $\mathbf{H}$  is a Hilbert space for the bilinear form  $(\cdot, \cdot): \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}(q)$  defined by

$$(a|0\rangle, b|0\rangle) := \langle 0|a^\dagger b|0\rangle \quad \text{with} \quad \langle 0|0\rangle = 1,$$

and for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1-m} < q < 1 \text{ if } m > 1.$$

Theorem 1 is a generalization of the realizability of the quon algebra model in infinite statistics proved by Zagier [3, Theorem 1].

To prove Theorem 1, we begin by showing in Section 3 that

$$\mathcal{B} := \{a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle \mid (i_u, k_u) \in \mathbb{N}^* \times [m], n \in \mathbb{N}\}$$

is a basis of  $\mathbf{H}$ , so that we can assume that

$$\mathbf{H} = \left\{ \sum_{i=1}^n \lambda_i b_i \mid n \in \mathbb{N}^*, \lambda_i \in \mathbb{R}(q), b_i \in \mathcal{B} \right\}.$$

Denote by  $\mathbb{U}_m$  the group of all  $m^{\text{th}}$  roots of unity, and  $\mathfrak{S}_n$  the permutation group on  $[n]$ . We represent an element  $\pi$  of the colored permutation group of  $m$  colors  $\mathbb{U}_m \wr \mathfrak{S}_n$  by

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ (\sigma(1), k_1) & (\sigma(2), k_2) & \dots & (\sigma(n), k_n) \end{pmatrix},$$

where  $k_1, \dots, k_n \in [m]$ , and  $\sigma$  is a permutation of  $[n]$ . But we also adopt the notation  $\pi = (\sigma, \alpha)$  meaning that  $\sigma \in \mathfrak{S}_n$  and  $\alpha: [n] \rightarrow [m]$  such that

$$\forall i \in [n], \pi(i) = (\sigma(i), \alpha(i)).$$

More generally, let  $I$  be a multiset of  $n$  elements in  $\mathbb{N}^*$ , and  $\mathfrak{S}_I$  its permutation set. An element  $\theta$  of the colored permutation set  $\mathbb{U}_m \wr \mathfrak{S}_I$  is defined by  $\theta := (\varphi, \epsilon)$  meaning that  $\varphi \in \mathfrak{S}_I$  and  $\epsilon: [n] \rightarrow [m]$  such that

$$\forall i \in [n], \theta(i) = (\varphi(i), \epsilon(i)).$$

Denote the infinite matrix associated to the bilinear form in Theorem 1 by

$$\mathbf{M} := ((f, g))_{f, g \in \mathcal{B}}.$$

Let  $\left[ \begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]$  be the set of multisets of  $n$  elements in  $\mathbb{N}^*$ . We also prove in Section 3 that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[ \begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]} \mathbf{M}_I \quad \text{with} \quad \mathbf{M}_I = \left( \langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger | 0 \rangle \right)_{\vartheta, \theta \in \mathbb{U}_m \wr \mathfrak{S}_I}.$$

For  $m = 3$  for example, we have

$$\mathbf{M}_{[2]} = \begin{pmatrix} 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 \\ q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 \\ q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 \\ q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 \\ q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 \\ q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 \\ q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 \\ q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 \\ q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q \\ q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 \\ q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 \\ q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q \\ q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 \\ q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 \\ q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q \\ q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q \\ q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q \\ q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 \end{pmatrix}.$$

We need to introduce the statistic  $\text{cinv}: \mathbb{U}_m \wr \mathfrak{S}_n \rightarrow \mathbb{N}$  defined by

$$\text{cinv}(\sigma, \alpha) := \#\{(i, j) \in [n]^2 \mid i < j, \sigma(i) > \sigma(j)\} + \#\{i \in [n] \mid \alpha(i) \neq m\}.$$

Still in Section 3, we prove that  $\mathbf{M}_I$  is the representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \tag{1}$$

on the  $\mathbb{U}_m \wr \mathfrak{S}_n$ -module  $\mathbb{R}[\mathbb{U}_m \wr \mathfrak{S}_I]$ . Hence if the regular representation of (1), which is  $\mathbf{M}_{[n]}$ , is positive definite, then  $\mathbf{M}_I$  is positive definite.

We prove in Section 4 that

$$\det \mathbf{M}_{[n]} = \left( (1 + (m - 1)q)(1 - q)^{m-1} \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\binom{n-i}{i^2+i}} \right)^{m^n n!}.$$

We particularly can infer that  $\mathbf{M}_{[n]}$  is nonsingular for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1 - m} < q < 1 \text{ if } m > 1.$$

Since  $\mathbf{M}_{[n]}$  is the identity matrix of order  $m^n n!$  if  $q = 0$ , we deduce by continuity that  $\mathbf{M}_{[n]}$  is positive definite for the values of  $q$  mentioned above. For these suitable values of  $q$ ,  $\mathbf{M}$  is then a symmetric positive definite matrix or, in other terms, the bilinear form of Theorem 1 is an inner product on  $\mathbf{H}$ .

But before investigating the deformed quon algebra, it is necessary to recall some notions in representation theory and do some computations in Section 2.

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## 2 Representation Theory

We recall the useful notions on representation theory of group and do some calculations for the cyclic groups.

Take a group  $G$  and a finite-dimensional vector space  $V$  over a field  $\mathbb{K}$ . Let  $g, h \in G$ ,  $a, b \in \mathbb{K}$ , and  $u, v \in V$ . Then  $V$  is a  $G$ -module if there is a multiplication  $\cdot$  of elements of  $V$  by elements of  $G$  such that

- $u \cdot g \in V$ .
- $(au + bv) \cdot g = a(u \cdot g) + b(v \cdot g)$ ,
- $u \cdot (gh) = (u \cdot g) \cdot h$ ,
- $u \cdot 1 = u$  where 1 is the neutral element of  $G$ .

Take an element  $x$  in the group algebra  $\mathbb{K}[G]$ . Suppose that  $\{v_1, \dots, v_n\}$  is a basis of  $V$ , and that  $v_j \cdot x = \sum_{i \in [n]} \mu_{i,j} v_i$ . Then the representation of  $x$  on the  $G$ -module  $V$  is the matrix

$$R_V(x) := (\mu_{i,j})_{i,j \in [n]}.$$

In particular if  $x = \sum_{g \in G} \lambda_g g \in \mathbb{K}[G]$  with  $\lambda_g \in \mathbb{R}$ , then the regular representation of  $x$  is

$$R_{\mathbb{K}[G]}(x) := (\lambda_{h^{-1}g})_{g,h \in G}.$$

**Lemma 1.** *Let  $G$  be a finite group,  $H \leq G$ , and  $x \in \mathbb{K}[H]$ . Then,*

$$\det R_{\mathbb{K}[G]}(x) = (\det R_{\mathbb{K}[H]}(x))^{|G:H|}.$$

*Proof.* Let  $H = \{h_1, \dots, h_r\}$ , and  $\{g_1, \dots, g_k\}$  be a left coset representative set of  $H$ . On the ordered basis  $(g_1h_1, \dots, g_1h_r, g_2h_1, \dots, g_2h_r, \dots, g_kh_1, \dots, g_kh_r)$  of  $\mathbb{K}[G]$ , we have

$$R_{\mathbb{K}[G]}(x) = R_{\mathbb{K}[H]}(x) \otimes I_{|G:H|},$$

where  $I_{|G:H|}$  is the unit matrix of size  $|G : H|$ . □

Now consider the cyclic group  $Z_m$  of order  $m$  generated by  $\gamma$ , and take a variable  $z$ . We need the following equalities on the group algebra  $\mathbb{R}(z)[Z_m]$ .

**Lemma 2.** *We have*

$$\det R_{\mathbb{R}(z)[Z_m]}(1 + z \sum_{k \in [m-1]} \gamma^k) = (1 + (m-1)z)(1-z)^{m-1}.$$

*Proof.* The regular representation of  $1 + z \sum_{k \in [m-1]} \gamma^k$  is the  $m \times m$  circulant matrix with associated polynomial  $f(x) = 1 + z \sum_{j \in [m-1]} x^j$ . The determinant of this circulant matrix is  $\prod_{i \in [m]} f(\zeta^i)$ . If  $i \in [m-1]$ , then

$$\sum_{j \in [m-1]} \zeta^{ij} = \frac{1 - \zeta^i}{1 - \zeta^i} \sum_{j \in [m-1]} \zeta^{ij} = \frac{\zeta^i - 1}{1 - \zeta^i} = -1.$$

Thus  $f(1) = 1 + (m-1)z$ , and  $f(\zeta^i) = 1 - z$  for  $i \in [m-1]$ . □

**Lemma 3.** *We have*

$$\left(1 + z \sum_{k \in [m-1]} \gamma^k\right)^{-1} = \frac{1}{(1 + (m-1)z)(1-z)} \left(1 + (m-2)z - z \sum_{k \in [m-1]} \gamma^k\right).$$

*Proof.* The form of  $1 + z \sum_{k \in [m-1]} \gamma^k$  gives us the intuition that its inverse has the form  $x + y \sum_{k \in [m-1]} \gamma^k$ . The calculation

$$\begin{aligned} \left(1 + z \sum_{k \in [m-1]} \gamma^k\right) \cdot \left(x + y \sum_{k \in [m-1]} \gamma^k\right) \\ = x + (m-1)zy + \left(zx + (1 + (m-2)z)y\right) \sum_{k \in [m-1]} \gamma^k \end{aligned}$$

confirms the intuition since it leads us to solve the equation system

$$\begin{cases} x + (m-1)zy = 1 \\ zx + (1 + (m-2)z)y = 0 \end{cases}$$

to get the inverse of  $1 + z \sum_{k \in [m-1]} \gamma^k$ . We obtain

$$x = \frac{1 + (m-2)z}{(1 + (m-1)z)(1-z)} \quad \text{and} \quad y = -\frac{z}{(1 + (m-1)z)(1-z)}. \quad \square$$

**Lemma 4.** *We have*

$$(1 - z\gamma)^{-1} = \frac{1}{1 - z^m} \sum_{i=0}^{m-1} z^i \gamma^i.$$

*Proof.* It comes from  $(1 - z\gamma)(1 + z\gamma + \dots + z^{m-1}\gamma^{m-1}) = 1 - z^m$ .  $\square$

### 3 The Bilinear Form $(\cdot, \cdot)$

We first show that  $\mathbf{H}$  is linearly generated by the particle states obtained by applying combinations of  $a_{i,k}^\dagger$ 's to  $|0\rangle$ . Then we prove that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[ \begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]} \mathbf{M}_I,$$

where  $\mathbf{M}_I$  is a representation of  $\sum_{\pi \in \mathbb{U}_{m,l} \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$ .

**Lemma 5.** *The vector space generated by our particle states is*

$$\mathbf{H} = \left\{ \sum_{i=1}^n \lambda_i b_i \mid n \in \mathbb{N}^*, \lambda_i \in \mathbb{R}(q), b_i \in \mathcal{B} \right\}.$$

*Proof.* Let  $(j, l) \in \mathbb{N}^* \times [m]$ . We have,

$$\begin{aligned} a_{j,l} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger &= q^r a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger a_{j,l} \\ &+ \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta - k_u, l} a_{i_1, k_1}^\dagger \dots \widehat{a_{i_u, k_u}^\dagger} \dots a_{i_r, k_r}^\dagger, \end{aligned}$$

where the hat over the  $u^{\text{th}}$  term of the product indicates that this term is omitted. So

$$a_{j,l} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger |0\rangle = \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta - k_u, l} a_{i_1, k_1}^\dagger \dots \widehat{a_{i_u, k_u}^\dagger} \dots a_{i_r, k_r}^\dagger |0\rangle.$$

Thus one can recursively remove every annihilation operator  $a_{j,l}$  of an element  $a|0\rangle$  of  $\mathbf{H}$ .  $\square$

**Lemma 6.** *Let  $((j_1, l_1), \dots, (j_s, l_s)) \in (\mathbb{N}^* \times [m])^s$  and  $((i_1, k_1), \dots, (i_r, k_r)) \in (\mathbb{N}^* \times [m])^r$ . If, as multisets,  $\{j_1, \dots, j_s\} \neq \{i_1, \dots, i_r\}$ , then*

$$\langle 0 | a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger | 0 \rangle = 0.$$

*Proof.* Suppose that  $v$  is the smallest integer in  $[s]$  such that

$$j_v \notin \{i_1, \dots, i_r\} \setminus \{j_1, \dots, j_{v-1}\}.$$

Then

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = P a_{j_v, l_v} \dots a_{j_1, l_1} + Q a_{j_v, l_v} \quad \text{with } P, Q \in \mathbf{A}.$$

We deduce that

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger |0\rangle = Pa_{j_v, l_v} \dots a_{j_1, l_1} |0\rangle + Qa_{j_v, l_v} |0\rangle = 0.$$

In the same way, suppose that  $u$  is the smallest integer in  $[r]$  such that  $i_u$  does not belong to the multiset  $\{j_1, \dots, j_s\} \setminus \{i_1, \dots, i_{u-1}\}$ . Then

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = a_{i_1, k_1}^\dagger \dots a_{i_u, k_u}^\dagger P' + a_{i_u, k_u}^\dagger Q' \text{ with } P', Q' \in \mathbf{A}.$$

And  $\langle 0 | a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = \langle 0 | a_{i_1, k_1}^\dagger \dots a_{i_u, k_u}^\dagger P' + \langle 0 | a_{i_u, k_u}^\dagger Q' = 0. \quad \square$

We just then need to investigate the product  $\langle 0 | a_{j_n, l_n} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle$ , where  $(j_1, \dots, j_n)$  is a permutation of  $(i_1, \dots, i_n)$ . Consider a multiset  $I$  of  $n$  elements in  $\mathbb{N}^*$ .

**Lemma 7.** *Let  $\theta, \vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I$ . Then,*

$$\langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger |0\rangle = \sum_{\substack{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n \\ \vartheta = \theta\pi}} q^{\text{cinv } \pi}.$$

*Proof.* Let  $(j_1, \dots, j_n)$  be a permutation of  $(i_1, \dots, i_n)$ . Then,

$$\begin{aligned} & a_{j_n, l_n} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} \prod_{s \in [n]} q^{u_s - 1 - \#\{r \in [s-1] \mid u_r < u_s\}} q^{\beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} \prod_{s \in [n]} q^{\#\{r \in [s-1] \mid u_r > u_s\}} q^{\beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} q^{\#\{(r, s) \in [n]^2 \mid r < s, u_r > u_s\} + \sum_{s \in [n]} \beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \forall s \in [n], j_s = i_{\sigma(s)}}} q^{\#\{(r, s) \in [n]^2 \mid r < s, \sigma(r) > \sigma(s)\} + \sum_{s \in [n]} \beta_{k_{\sigma(s)}, l_s}} |0\rangle \\ &= \sum_{\substack{\pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \forall s \in [n], \\ j_s = i_{\sigma(s)}, l_s \equiv k_{\sigma(s)} + \alpha(s) \pmod{m}}} q^{\text{cinv } \pi} |0\rangle. \end{aligned}$$

We obtain the result by replacing  $a_{j_n, l_n} \dots a_{j_1, l_1}$  and  $a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger$  by  $a_{\vartheta(n)} \dots a_{\vartheta(1)}$  and  $a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger$  respectively.  $\square$

For example, take  $m = 4$ ,

$$\vartheta = \begin{pmatrix} 1 & 2 & 3 \\ (2, 4) & (5, 1) & (2, 4) \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & 2 & 3 \\ (5, 2) & (2, 3) & (2, 1) \end{pmatrix}.$$



Then

$$\begin{aligned} \langle 0 | a_{2,4} a_{5,1} a_{2,4}^\dagger a_{5,2}^\dagger a_{2,3}^\dagger a_{2,1}^\dagger | 0 \rangle \\ &= q^{\text{cinv} \left( \begin{smallmatrix} 1 \\ (2,1) \end{smallmatrix} \begin{smallmatrix} 2 \\ (1,3) \end{smallmatrix} \begin{smallmatrix} 3 \\ (3,3) \end{smallmatrix} \right)} + q^{\text{cinv} \left( \begin{smallmatrix} 1 \\ (3,3) \end{smallmatrix} \begin{smallmatrix} 2 \\ (1,3) \end{smallmatrix} \begin{smallmatrix} 3 \\ (2,1) \end{smallmatrix} \right)} \\ &= q^4 + q^5 \end{aligned}$$

Define the multiplication of an element  $\theta = (\varphi, \epsilon)$  of  $\mathbb{U}_m \wr \mathfrak{S}_I$  by an element  $\pi = (\sigma, \alpha)$  of  $\mathbb{U}_m \wr \mathfrak{S}_n$  by

$$\theta \cdot \pi = (\psi, \eta) \in \mathbb{U}_m \wr \mathfrak{S}_I \quad \text{with} \quad \forall i \in [n], \psi(i) = \varphi\sigma(i), \eta(i) \equiv \epsilon\sigma(i) + \alpha(i) \pmod{m}.$$

Consider the vector space of linear combinations of colored permutations

$$\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I] := \left\{ \sum_{\theta \in \mathbb{U}_m \wr \mathfrak{S}_I} z_\theta \theta \mid z_\theta \in \mathbb{R}(q) \right\}.$$

One can easily check that, relatively to the multiplication  $\cdot$ ,  $\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]$  is a  $\mathbb{U}_m \wr \mathfrak{S}_n$ -module.

**Proposition 1.** *We have*

$$\mathbf{M}_I = R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]} \left( \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} \right).$$

*Proof.* Using Lemma 7, we obtain for  $\theta \in \mathbb{U}_m \wr \mathfrak{S}_I$

$$\begin{aligned} \theta \cdot \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \left( \sum_{\substack{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n \\ \vartheta = \theta\pi}} q^{\text{cinv} \pi} \right) \vartheta \\ &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger | 0 \rangle \vartheta. \quad \square \end{aligned}$$

#### 4 The Determinant of $\mathbf{M}_{[n]}$

We compute the determinant and the inverse of the regular representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} \pi.$$

Consider the subgroup  $\mathfrak{C}_n$  of  $\mathbb{U}_m \wr \mathfrak{S}_n$  defined by

$$\mathfrak{C}_n := \{ \pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \mid \forall i \in [n], \sigma(i) = i \}.$$

For  $i \in [n]$ , let  $\xi_i$  be the colored permutation

$$\left( \begin{array}{cccccc} 1 & 2 & \dots & i & \dots & n \\ (1, m) & (2, m) & \dots & (i, 1) & \dots & (n, m) \end{array} \right)$$

in  $\mathfrak{C}_n$ . We need the following lemma.

**Lemma 8.** *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left( \sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right) = \left( (1 + (m-1)q)(1-q)^{m-1} \right)^{m^n n!}.$$

*Proof.* Remark that

$$\sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi = \prod_{i \in [n]} \left( 1 + q \sum_{k \in [m-1]} \xi_i^k \right).$$

Then, using Lemma 1 and Lemma 2, we obtain

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left( 1 + q \sum_{k \in [m]} \xi_i^k \right) = \left( (1 + (m-1)q)(1-q)^{m-1} \right)^{m^{n-1} n!}. \quad \square$$

Now we can compute the determinant of  $\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$ .

**Theorem 2.** *We have*

$$\begin{aligned} \det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left( \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \right) \\ = \left( (1 + (m-1)q)(1-q)^{m-1} \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}} \right)^{m^n n!}. \end{aligned}$$

*Proof.* Every  $\pi \in \mathbb{U}_m \wr \mathfrak{S}_n$  has a decomposition  $\pi = \sigma \xi$  such that

$$\sigma \in \mathfrak{S}_n, \xi \in \mathfrak{C}_n, \text{ and } \text{cinv } \pi = \text{cinv } \sigma + \text{cinv } \xi.$$

Then,

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi = \left( \sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right) \left( \sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right).$$

It is known that [3, Theorem 2]

$$\det R_{\mathbb{R}(q)[\mathfrak{S}_n]} \left( \sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right) = \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\frac{(n-i)n!}{(i^2+i)}}.$$

We finally obtain the result by using Lemma 1 and Lemma 8. □

For  $k \in [n]$ , denote by  $t_{k,n}$  the permutation  $(n \ n-1 \ \dots \ k)$  in cycle notation. Let

$$\gamma_n = \prod_{k \in [n-1]}^{\rightarrow} (1 - q^{n-k} t_{k,n}) \quad \text{and} \quad \varepsilon_n = \prod_{k \in [n]}^{\leftarrow} \frac{\sum_{i=0}^{n-k} q^{(n-k+2)i} t_{k,n}^i}{1 - q^{(n-k+1)(n-k+2)}}.$$

Furthermore, let

$$\rho_k = \frac{1 + (m-2)q - q \sum_{i \in [m-1]} \xi_k^i}{(1 + (m-1)q)(1-q)}.$$

We finish with the inverse of  $\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$ .

**Proposition 2.** *We have*

$$\left( \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \right)^{-1} = \prod_{i \in [n]} \rho_i \cdot \prod_{i \in [n-1]}^{\leftarrow} \gamma_{i+1} \varepsilon_i.$$

*Proof.* We obtain

$$\left( \sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right)^{-1} = \prod_{i \in [n]} \rho_i$$

by means of Lemma 3. Then [3, Proposition 2] and Lemma 4 permit us to write

$$\left( \sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right)^{-1} = \prod_{i \in [n-1]}^{\leftarrow} \gamma_{i+1} \varepsilon_i. \quad \square$$

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