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A Deformed Quon Algebra

Hery Randriamaro

Abstract. The quon algebra is an approach to particle statistics in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated by a small amount. The quons are particles whose annihilation and creation operators obey the quon algebra which interpolates between fermions and bosons. In this paper we generalize these models by introducing a deformation of the quon algebra generated by a collection of operators $a_{i,k}$, $(i, k) \in \mathbb{N}^* \times [m]$, on an infinite dimensional vector space satisfying the deformed q -mutator relations $a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j}$. We prove the realizability of our model by showing that, for suitable values of q , the vector space generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to a vacuum state $|0\rangle$ is a Hilbert space. The proof particularly needs the investigation of the new statistic *cinv* and representations of the colored permutation group.

1 Introduction

Let $\mathbb{R}(q)$ be the fraction field of the real polynomials with variable q . By a deformed quon algebra \mathbf{A} , we mean the free algebra

$$\mathbb{R}(q)[a_{i,k} \mid (i, k) \in \mathbb{N}^* \times [m]]$$

subject to the anti-involution \dagger exchanging $a_{i,k}$ with $a_{i,k}^\dagger$, and to the commutation relation

$$a_{j,l}a_{i,k}^\dagger = qa_{i,k}^\dagger a_{j,l} + q^{\beta_{k,l}}\delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta and

$$\beta_{k,l} = \begin{cases} 0 & \text{if } l - k \equiv m \pmod{m} \\ 1 & \text{otherwise} \end{cases}.$$

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This algebra is a generalization of the quon algebra introduced by Greenberg [2], subject to the commutation relation $a_j a_i^\dagger = q a_i^\dagger a_j + \delta_{i,j}$ obeyed by the annihilation and creation operators of the quon particles, and generating a model of infinite statistics. Moreover, the quon algebra is a generalization of the classical Bose and Fermi algebras corresponding to the restrictions $q = 1$ and $q = -1$ respectively, as well as of the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [1].

In a Fock-like representation, the generators of \mathbf{A} are the linear operators $a_{i,k}, a_{i,k}^\dagger: \mathbf{V} \rightarrow \mathbf{V}$ on an infinite dimensional real vector space \mathbf{V} satisfying the commutation relations

$$a_{j,l} a_{i,k}^\dagger - q a_{i,k}^\dagger a_{j,l} = q^{\beta_{k,l}} \delta_{i,j},$$

and the relations

$$a_{i,k} |0\rangle = 0,$$

where $a_{i,k}^\dagger$ is the adjoint of $a_{i,k}$, and $|0\rangle$ is a nonzero distinguished vector of \mathbf{V} . The $a_{i,k}$'s are the annihilation operators and the $a_{i,k}^\dagger$'s the creation operators.

Let \mathbf{H} be the vector subspace of \mathbf{V} generated by the particle states obtained by applying combinations of $a_{i,k}$'s and $a_{i,k}^\dagger$'s to $|0\rangle$, or

$$\mathbf{H} := \{a|0\rangle \mid a \in \mathbf{A}\}.$$

The aim of this article is to prove the realizability of this model through the following theorem.

Theorem 1. \mathbf{H} is a Hilbert space for the bilinear form $(\cdot, \cdot): \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}(q)$ defined by

$$(a|0\rangle, b|0\rangle) := \langle 0|a^\dagger b|0\rangle \quad \text{with} \quad \langle 0|0\rangle = 1,$$

and for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1-m} < q < 1 \text{ if } m > 1.$$

Theorem 1 is a generalization of the realizability of the quon algebra model in infinite statistics proved by Zagier [3, Theorem 1].

To prove Theorem 1, we begin by showing in Section 3 that

$$\mathcal{B} := \{a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle \mid (i_u, k_u) \in \mathbb{N}^* \times [m], n \in \mathbb{N}\}$$

is a basis of \mathbf{H} , so that we can assume that

$$\mathbf{H} = \left\{ \sum_{i=1}^n \lambda_i b_i \mid n \in \mathbb{N}^*, \lambda_i \in \mathbb{R}(q), b_i \in \mathcal{B} \right\}.$$

Denote by \mathbb{U}_m the group of all m^{th} roots of unity, and \mathfrak{S}_n the permutation group on $[n]$. We represent an element π of the colored permutation group of m colors $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ (\sigma(1), k_1) & (\sigma(2), k_2) & \dots & (\sigma(n), k_n) \end{pmatrix},$$

where $k_1, \dots, k_n \in [m]$, and σ is a permutation of $[n]$. But we also adopt the notation $\pi = (\sigma, \alpha)$ meaning that $\sigma \in \mathfrak{S}_n$ and $\alpha: [n] \rightarrow [m]$ such that

$$\forall i \in [n], \pi(i) = (\sigma(i), \alpha(i)).$$

More generally, let I be a multiset of n elements in \mathbb{N}^* , and \mathfrak{S}_I its permutation set. An element θ of the colored permutation set $\mathbb{U}_m \wr \mathfrak{S}_I$ is defined by $\theta := (\varphi, \epsilon)$ meaning that $\varphi \in \mathfrak{S}_I$ and $\epsilon: [n] \rightarrow [m]$ such that

$$\forall i \in [n], \theta(i) = (\varphi(i), \epsilon(i)).$$

Denote the infinite matrix associated to the bilinear form in Theorem 1 by

$$\mathbf{M} := ((f, g))_{f, g \in \mathcal{B}}.$$

Let $\left[\begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]$ be the set of multisets of n elements in \mathbb{N}^* . We also prove in Section 3 that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[\begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]} \mathbf{M}_I \quad \text{with} \quad \mathbf{M}_I = \left(\langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger | 0 \rangle \right)_{\vartheta, \theta \in \mathbb{U}_m \wr \mathfrak{S}_I}.$$

For $m = 3$ for example, we have

$$\mathbf{M}_{[2]} = \begin{pmatrix} 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 \\ q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 \\ q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 \\ q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 \\ q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 \\ q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 \\ q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 \\ q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 \\ q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q \\ q & q^2 & q^2 & q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & 1 & q & q & q & q^2 & q^2 & q & q^2 & q^2 \\ q^2 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q & 1 & q & q^2 & q & q^2 & q^2 & q & q^2 \\ q^2 & q^2 & q & q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q & q & 1 & q^2 & q^2 & q & q^2 & q^2 & q \\ q^2 & q^3 & q^3 & q & q^2 & q^2 & q^2 & q^3 & q^3 & q & q^2 & q^2 & 1 & q & q & q & q^2 & q^2 \\ q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 \\ q^3 & q^3 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^2 & q^2 & q^2 & q & q & q & 1 & q^2 & q^2 & q \\ q^2 & q^3 & q^3 & q^2 & q^3 & q^3 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & 1 & q & q \\ q^3 & q^2 & q^3 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q & 1 & q \\ q^3 & q^3 & q^2 & q^3 & q^3 & q^2 & q^2 & q^2 & q & q^2 & q^2 & q & q^2 & q^2 & q & q & q & 1 \end{pmatrix}.$$

We need to introduce the statistic $\text{cinv}: \mathbb{U}_m \wr \mathfrak{S}_n \rightarrow \mathbb{N}$ defined by

$$\text{cinv}(\sigma, \alpha) := \#\{(i, j) \in [n]^2 \mid i < j, \sigma(i) > \sigma(j)\} + \#\{i \in [n] \mid \alpha(i) \neq m\}.$$

Still in Section 3, we prove that \mathbf{M}_I is the representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \tag{1}$$

on the $\mathbb{U}_m \wr \mathfrak{S}_n$ -module $\mathbb{R}[\mathbb{U}_m \wr \mathfrak{S}_I]$. Hence if the regular representation of (1), which is $\mathbf{M}_{[n]}$, is positive definite, then \mathbf{M}_I is positive definite.

We prove in Section 4 that

$$\det \mathbf{M}_{[n]} = \left((1 + (m - 1)q)(1 - q)^{m-1} \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\binom{n-i}{i^2+i}} \right)^{m^n n!}.$$

We particularly can infer that $\mathbf{M}_{[n]}$ is nonsingular for

$$-1 < q < 1 \text{ if } m = 1 \quad \text{and} \quad \frac{1}{1 - m} < q < 1 \text{ if } m > 1.$$

Since $\mathbf{M}_{[n]}$ is the identity matrix of order $m^n n!$ if $q = 0$, we deduce by continuity that $\mathbf{M}_{[n]}$ is positive definite for the values of q mentioned above. For these suitable values of q , \mathbf{M} is then a symmetric positive definite matrix or, in other terms, the bilinear form of Theorem 1 is an inner product on \mathbf{H} .

But before investigating the deformed quon algebra, it is necessary to recall some notions in representation theory and do some computations in Section 2.

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2 Representation Theory

We recall the useful notions on representation theory of group and do some calculations for the cyclic groups.

Take a group G and a finite-dimensional vector space V over a field \mathbb{K} . Let $g, h \in G$, $a, b \in \mathbb{K}$, and $u, v \in V$. Then V is a G -module if there is a multiplication \cdot of elements of V by elements of G such that

- $u \cdot g \in V$.
- $(au + bv) \cdot g = a(u \cdot g) + b(v \cdot g)$,
- $u \cdot (gh) = (u \cdot g) \cdot h$,
- $u \cdot 1 = u$ where 1 is the neutral element of G .

Take an element x in the group algebra $\mathbb{K}[G]$. Suppose that $\{v_1, \dots, v_n\}$ is a basis of V , and that $v_j \cdot x = \sum_{i \in [n]} \mu_{i,j} v_i$. Then the representation of x on the G -module V is the matrix

$$R_V(x) := (\mu_{i,j})_{i,j \in [n]}.$$

In particular if $x = \sum_{g \in G} \lambda_g g \in \mathbb{K}[G]$ with $\lambda_g \in \mathbb{R}$, then the regular representation of x is

$$R_{\mathbb{K}[G]}(x) := (\lambda_{h^{-1}g})_{g,h \in G}.$$

Lemma 1. *Let G be a finite group, $H \leq G$, and $x \in \mathbb{K}[H]$. Then,*

$$\det R_{\mathbb{K}[G]}(x) = (\det R_{\mathbb{K}[H]}(x))^{|G:H|}.$$

Proof. Let $H = \{h_1, \dots, h_r\}$, and $\{g_1, \dots, g_k\}$ be a left coset representative set of H . On the ordered basis $(g_1h_1, \dots, g_1h_r, g_2h_1, \dots, g_2h_r, \dots, g_kh_1, \dots, g_kh_r)$ of $\mathbb{K}[G]$, we have

$$R_{\mathbb{K}[G]}(x) = R_{\mathbb{K}[H]}(x) \otimes I_{|G:H|},$$

where $I_{|G:H|}$ is the unit matrix of size $|G : H|$. □

Now consider the cyclic group Z_m of order m generated by γ , and take a variable z . We need the following equalities on the group algebra $\mathbb{R}(z)[Z_m]$.

Lemma 2. *We have*

$$\det R_{\mathbb{R}(z)[Z_m]}(1 + z \sum_{k \in [m-1]} \gamma^k) = (1 + (m-1)z)(1-z)^{m-1}.$$

Proof. The regular representation of $1 + z \sum_{k \in [m-1]} \gamma^k$ is the $m \times m$ circulant matrix with associated polynomial $f(x) = 1 + z \sum_{j \in [m-1]} x^j$. The determinant of this circulant matrix is $\prod_{i \in [m]} f(\zeta^i)$. If $i \in [m-1]$, then

$$\sum_{j \in [m-1]} \zeta^{ij} = \frac{1 - \zeta^i}{1 - \zeta^i} \sum_{j \in [m-1]} \zeta^{ij} = \frac{\zeta^i - 1}{1 - \zeta^i} = -1.$$

Thus $f(1) = 1 + (m-1)z$, and $f(\zeta^i) = 1 - z$ for $i \in [m-1]$. □

Lemma 3. *We have*

$$\left(1 + z \sum_{k \in [m-1]} \gamma^k\right)^{-1} = \frac{1}{(1 + (m-1)z)(1-z)} \left(1 + (m-2)z - z \sum_{k \in [m-1]} \gamma^k\right).$$

Proof. The form of $1 + z \sum_{k \in [m-1]} \gamma^k$ gives us the intuition that its inverse has the form $x + y \sum_{k \in [m-1]} \gamma^k$. The calculation

$$\begin{aligned} \left(1 + z \sum_{k \in [m-1]} \gamma^k\right) \cdot \left(x + y \sum_{k \in [m-1]} \gamma^k\right) \\ = x + (m-1)zy + \left(zx + (1 + (m-2)z)y\right) \sum_{k \in [m-1]} \gamma^k \end{aligned}$$

confirms the intuition since it leads us to solve the equation system

$$\begin{cases} x + (m-1)zy = 1 \\ zx + (1 + (m-2)z)y = 0 \end{cases}$$

to get the inverse of $1 + z \sum_{k \in [m-1]} \gamma^k$. We obtain

$$x = \frac{1 + (m-2)z}{(1 + (m-1)z)(1-z)} \quad \text{and} \quad y = -\frac{z}{(1 + (m-1)z)(1-z)}. \quad \square$$

Lemma 4. *We have*

$$(1 - z\gamma)^{-1} = \frac{1}{1 - z^m} \sum_{i=0}^{m-1} z^i \gamma^i.$$

Proof. It comes from $(1 - z\gamma)(1 + z\gamma + \dots + z^{m-1}\gamma^{m-1}) = 1 - z^m$. \square

3 The Bilinear Form (\cdot, \cdot)

We first show that \mathbf{H} is linearly generated by the particle states obtained by applying combinations of $a_{i,k}^\dagger$'s to $|0\rangle$. Then we prove that

$$\mathbf{M} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{I \in \left[\begin{smallmatrix} \mathbb{N}^* \\ n \end{smallmatrix} \right]} \mathbf{M}_I,$$

where \mathbf{M}_I is a representation of $\sum_{\pi \in \mathbb{U}_{m,l} \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$.

Lemma 5. *The vector space generated by our particle states is*

$$\mathbf{H} = \left\{ \sum_{i=1}^n \lambda_i b_i \mid n \in \mathbb{N}^*, \lambda_i \in \mathbb{R}(q), b_i \in \mathcal{B} \right\}.$$

Proof. Let $(j, l) \in \mathbb{N}^* \times [m]$. We have,

$$\begin{aligned} a_{j,l} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger &= q^r a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger a_{j,l} \\ &+ \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta - k_u, l} a_{i_1, k_1}^\dagger \dots \widehat{a_{i_u, k_u}^\dagger} \dots a_{i_r, k_r}^\dagger, \end{aligned}$$

where the hat over the u^{th} term of the product indicates that this term is omitted. So

$$a_{j,l} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger |0\rangle = \sum_{\substack{u \in [r] \\ i_u = j}} q^{u-1} q^{\beta - k_u, l} a_{i_1, k_1}^\dagger \dots \widehat{a_{i_u, k_u}^\dagger} \dots a_{i_r, k_r}^\dagger |0\rangle.$$

Thus one can recursively remove every annihilation operator $a_{j,l}$ of an element $a|0\rangle$ of \mathbf{H} . \square

Lemma 6. *Let $((j_1, l_1), \dots, (j_s, l_s)) \in (\mathbb{N}^* \times [m])^s$ and $((i_1, k_1), \dots, (i_r, k_r)) \in (\mathbb{N}^* \times [m])^r$. If, as multisets, $\{j_1, \dots, j_s\} \neq \{i_1, \dots, i_s\}$, then*

$$\langle 0 | a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger | 0 \rangle = 0.$$

Proof. Suppose that v is the smallest integer in $[s]$ such that

$$j_v \notin \{i_1, \dots, i_r\} \setminus \{j_1, \dots, j_{v-1}\}.$$

Then

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = P a_{j_v, l_v} \dots a_{j_1, l_1} + Q a_{j_v, l_v} \quad \text{with } P, Q \in \mathbf{A}.$$

We deduce that

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger |0\rangle = P a_{j_v, l_v} \dots a_{j_1, l_1} |0\rangle + Q a_{j_v, l_v} |0\rangle = 0.$$

In the same way, suppose that u is the smallest integer in $[r]$ such that i_u does not belong to the multiset $\{j_1, \dots, j_s\} \setminus \{i_1, \dots, i_{u-1}\}$. Then

$$a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = a_{i_1, k_1}^\dagger \dots a_{i_u, k_u}^\dagger P' + a_{i_u, k_u}^\dagger Q' \text{ with } P', Q' \in \mathbf{A}.$$

And $\langle 0 | a_{j_s, l_s} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_r, k_r}^\dagger = \langle 0 | a_{i_1, k_1}^\dagger \dots a_{i_u, k_u}^\dagger P' + \langle 0 | a_{i_u, k_u}^\dagger Q' = 0. \quad \square$

We just then need to investigate the product $\langle 0 | a_{j_n, l_n} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle$, where (j_1, \dots, j_n) is a permutation of (i_1, \dots, i_n) . Consider a multiset I of n elements in \mathbb{N}^* .

Lemma 7. *Let $\theta, \vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I$. Then,*

$$\langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger |0\rangle = \sum_{\substack{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n \\ \vartheta = \theta \pi}} q^{\text{cinv } \pi}.$$

Proof. Let (j_1, \dots, j_n) be a permutation of (i_1, \dots, i_n) . Then,

$$\begin{aligned} & a_{j_n, l_n} \dots a_{j_1, l_1} a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} \prod_{s \in [n]} q^{u_s - 1 - \#\{r \in [s-1] \mid u_r < u_s\}} q^{\beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} \prod_{s \in [n]} q^{\#\{r \in [s-1] \mid u_r > u_s\}} q^{\beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{(u_1, \dots, u_n) \in [n]^n \\ i_{u_1} = j_1, \dots, i_{u_n} = j_n}} q^{\#\{(r, s) \in [n]^2 \mid r < s, u_r > u_s\} + \sum_{s \in [n]} \beta_{k_{u_s}, l_s}} |0\rangle \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \forall s \in [n], j_s = i_{\sigma(s)}}} q^{\#\{(r, s) \in [n]^2 \mid r < s, \sigma(r) > \sigma(s)\} + \sum_{s \in [n]} \beta_{k_{\sigma(s)}, l_s}} |0\rangle \\ &= \sum_{\substack{\pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \forall s \in [n], \\ j_s = i_{\sigma(s)}, l_s \equiv k_{\sigma(s)} + \alpha(s) \pmod{m}}} q^{\text{cinv } \pi} |0\rangle. \end{aligned}$$

We obtain the result by replacing $a_{j_n, l_n} \dots a_{j_1, l_1}$ and $a_{i_1, k_1}^\dagger \dots a_{i_n, k_n}^\dagger$ by $a_{\vartheta(n)} \dots a_{\vartheta(1)}$ and $a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger$ respectively. \square

For example, take $m = 4$,

$$\vartheta = \begin{pmatrix} 1 & 2 & 3 \\ (2, 4) & (5, 1) & (2, 4) \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} 1 & 2 & 3 \\ (5, 2) & (2, 3) & (2, 1) \end{pmatrix}.$$

Then

$$\begin{aligned} \langle 0 | a_{2,4} a_{5,1} a_{2,4}^\dagger a_{5,2}^\dagger a_{2,3}^\dagger a_{2,1}^\dagger | 0 \rangle \\ &= q^{\text{cinv} \left(\begin{smallmatrix} 1 \\ (2,1) \end{smallmatrix} \begin{smallmatrix} 2 \\ (1,3) \end{smallmatrix} \begin{smallmatrix} 3 \\ (3,3) \end{smallmatrix} \right)} + q^{\text{cinv} \left(\begin{smallmatrix} 1 \\ (3,3) \end{smallmatrix} \begin{smallmatrix} 2 \\ (1,3) \end{smallmatrix} \begin{smallmatrix} 3 \\ (2,1) \end{smallmatrix} \right)} \\ &= q^4 + q^5 \end{aligned}$$

Define the multiplication of an element $\theta = (\varphi, \epsilon)$ of $\mathbb{U}_m \wr \mathfrak{S}_I$ by an element $\pi = (\sigma, \alpha)$ of $\mathbb{U}_m \wr \mathfrak{S}_n$ by

$$\theta \cdot \pi = (\psi, \eta) \in \mathbb{U}_m \wr \mathfrak{S}_I \quad \text{with} \quad \forall i \in [n], \psi(i) = \varphi\sigma(i), \eta(i) \equiv \epsilon\sigma(i) + \alpha(i) \pmod{m}.$$

Consider the vector space of linear combinations of colored permutations

$$\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I] := \left\{ \sum_{\theta \in \mathbb{U}_m \wr \mathfrak{S}_I} z_\theta \theta \mid z_\theta \in \mathbb{R}(q) \right\}.$$

One can easily check that, relatively to the multiplication \cdot , $\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]$ is a $\mathbb{U}_m \wr \mathfrak{S}_n$ -module.

Proposition 1. *We have*

$$\mathbf{M}_I = R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_I]} \left(\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} \right).$$

Proof. Using Lemma 7, we obtain for $\theta \in \mathbb{U}_m \wr \mathfrak{S}_I$

$$\begin{aligned} \theta \cdot \sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \left(\sum_{\substack{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n \\ \vartheta = \theta\pi}} q^{\text{cinv} \pi} \right) \vartheta \\ &= \sum_{\vartheta \in \mathbb{U}_m \wr \mathfrak{S}_I} \langle 0 | a_{\vartheta(n)} \dots a_{\vartheta(1)} a_{\theta(1)}^\dagger \dots a_{\theta(n)}^\dagger | 0 \rangle \vartheta. \quad \square \end{aligned}$$

4 The Determinant of $\mathbf{M}_{[n]}$

We compute the determinant and the inverse of the regular representation of

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv} \pi} \pi.$$

Consider the subgroup \mathfrak{C}_n of $\mathbb{U}_m \wr \mathfrak{S}_n$ defined by

$$\mathfrak{C}_n := \{ \pi = (\sigma, \alpha) \in \mathbb{U}_m \wr \mathfrak{S}_n \mid \forall i \in [n], \sigma(i) = i \}.$$

For $i \in [n]$, let ξ_i be the colored permutation

$$\left(\begin{array}{cccccc} 1 & 2 & \dots & i & \dots & n \\ (1, m) & (2, m) & \dots & (i, 1) & \dots & (n, m) \end{array} \right)$$

in \mathfrak{C}_n . We need the following lemma.

Lemma 8. *We have*

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left(\sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right) = \left((1 + (m-1)q)(1-q)^{m-1} \right)^{m^n n!}.$$

Proof. Remark that

$$\sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi = \prod_{i \in [n]} \left(1 + q \sum_{k \in [m-1]} \xi_i^k \right).$$

Then, using Lemma 1 and Lemma 2, we obtain

$$\det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left(1 + q \sum_{k \in [m]} \xi_i^k \right) = \left((1 + (m-1)q)(1-q)^{m-1} \right)^{m^{n-1} n!}. \quad \square$$

Now we can compute the determinant of $\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$.

Theorem 2. *We have*

$$\begin{aligned} \det R_{\mathbb{R}(q)[\mathbb{U}_m \wr \mathfrak{S}_n]} \left(\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \right) \\ = \left((1 + (m-1)q)(1-q)^{m-1} \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\frac{(n-i)}{(i^2+i)}} \right)^{m^n n!}. \end{aligned}$$

Proof. Every $\pi \in \mathbb{U}_m \wr \mathfrak{S}_n$ has a decomposition $\pi = \sigma \xi$ such that

$$\sigma \in \mathfrak{S}_n, \xi \in \mathfrak{C}_n, \text{ and } \text{cinv } \pi = \text{cinv } \sigma + \text{cinv } \xi.$$

Then,

$$\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi = \left(\sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right) \left(\sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right).$$

It is known that [3, Theorem 2]

$$\det R_{\mathbb{R}(q)[\mathfrak{S}_n]} \left(\sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right) = \prod_{i=1}^{n-1} (1 - q^{i^2+i})^{\frac{(n-i)n!}{(i^2+i)}}.$$

We finally obtain the result by using Lemma 1 and Lemma 8. □

For $k \in [n]$, denote by $t_{k,n}$ the permutation $(n \ n-1 \ \dots \ k)$ in cycle notation. Let

$$\gamma_n = \prod_{k \in [n-1]}^{\rightarrow} (1 - q^{n-k} t_{k,n}) \quad \text{and} \quad \varepsilon_n = \prod_{k \in [n]}^{\leftarrow} \frac{\sum_{i=0}^{n-k} q^{(n-k+2)i} t_{k,n}^i}{1 - q^{(n-k+1)(n-k+2)}}.$$

Furthermore, let

$$\rho_k = \frac{1 + (m-2)q - q \sum_{i \in [m-1]} \xi_k^i}{(1 + (m-1)q)(1-q)}.$$

We finish with the inverse of $\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi$.

Proposition 2. *We have*

$$\left(\sum_{\pi \in \mathbb{U}_m \wr \mathfrak{S}_n} q^{\text{cinv } \pi} \pi \right)^{-1} = \prod_{i \in [n]} \rho_i \cdot \prod_{i \in [n-1]}^{\leftarrow} \gamma_{i+1} \varepsilon_i.$$

Proof. We obtain

$$\left(\sum_{\xi \in \mathfrak{C}_n} q^{\text{cinv } \xi} \xi \right)^{-1} = \prod_{i \in [n]} \rho_i$$

by means of Lemma 3. Then [3, Proposition 2] and Lemma 4 permit us to write

$$\left(\sum_{\sigma \in \mathfrak{S}_n} q^{\text{cinv } \sigma} \sigma \right)^{-1} = \prod_{i \in [n-1]}^{\leftarrow} \gamma_{i+1} \varepsilon_i. \quad \square$$

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