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Lightlike hypersurfaces of an indefinite Kaehler manifold of a quasi-constant curvature

Dae Ho Jin, Jae Won Lee

Abstract. We study lightlike hypersurfaces M of an indefinite Kaehler manifold \bar{M} of quasi-constant curvature subject to the condition that the characteristic vector field ζ of \bar{M} is tangent to M . First, we provide a new result for such a lightlike hypersurface. Next, we investigate such a lightlike hypersurface M of \bar{M} such that

- (1) the screen distribution $S(TM)$ is totally umbilical or
- (2) M is screen conformal.

1 Introduction

In the classical theory of Riemannian geometry, Chen-Yano [2] introduced the notion of a *Riemannian manifold of a quasi-constant curvature* as a Riemannian manifold (\bar{M}, \bar{g}) endowed with a curvature tensor \bar{R} satisfying

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} &= f_1 \{ \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} \} \\ &+ f_2 \{ \theta(\bar{Y})\theta(\bar{Z})\bar{X} - \theta(\bar{X})\theta(\bar{Z})\bar{Y} + \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta - \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta \}, \end{aligned} \quad (1)$$

where f_1 and f_2 are smooth functions which are called the *curvature functions*, ζ is a vector field which is called the *characteristic vector field* of \bar{M} , and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. In the followings, we denote by \bar{X}, \bar{Y} and \bar{Z} the smooth vector fields on \bar{M} . If $f_2 = 0$, then \bar{M} is reduced to a space of constant curvature.

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In this paper, we study lightlike hypersurfaces M of an indefinite Kaehler manifold \bar{M} of quasi-constant curvature subject such that ζ is tangent to M . After then, under the condition that ζ is tangent to M , we investigate lightlike hypersurfaces M of \bar{M} such that

- (1) the screen distribution $S(TM)$ of M is totally umbilical in M or
- (2) M is screen conformal.

2 Preliminaries

Let (M, g) be a lightlike hypersurface, with a screen distribution $S(TM)$, of a semi-Riemannian manifold \bar{M} . Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E . Also denote by $(8)_i$ the i -th equation of (8). We use same notations for any others. We follow Duggal-Bejancu [3] for notations and structure equations used in this article. It is well known that

$$TM = TM^\perp \oplus_{\text{orth}} S(TM),$$

where \oplus_{orth} denotes the orthogonal direct sum. For any null section ξ of TM^\perp on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $\text{tr}(TM)$ of rank 1 in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ in \bar{M} satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Then the tangent bundle $T\bar{M}$ of \bar{M} is decomposed as follow

$$T\bar{M} = TM \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM).$$

We call $\text{tr}(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$, respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of \bar{M} and P the projection morphism of TM on $S(TM)$. In the sequel, denote by X, Y, Z and W the smooth vector fields on M , unless otherwise specified. The local Gauss and Weingartan formulae for M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2)$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad (3)$$

$$\nabla_X P Y = \nabla_X^* P Y + C(X, P Y)\xi, \quad (4)$$

$$\nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \quad (5)$$

where ∇ and ∇^* are the liner connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A_ξ^* are the shape operators and τ is a 1-form on TM .

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. As $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, B is independent of the choice of $S(TM)$ and

$$B(X, \xi) = 0. \quad (6)$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \quad (7)$$

where η is a 1-form such that $\eta(X) = \bar{g}(X, N)$. But ∇^* is metric. The above local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A_\xi^* X, Y), \quad \bar{g}(A_\xi^* X, N) = 0, \quad (8)$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \quad (9)$$

From (8), A_ξ^* is $S(TM)$ -valued and self-adjoint on TM such that

$$A_\xi^* \xi = 0. \quad (10)$$

Denote by \bar{R} , R and R^* the curvature tensors of the connections $\bar{\nabla}$, ∇ and ∇^* , respectively. Using (2)–(5), we obtain the Gauss-Codazzi equations:

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N, \end{aligned} \quad (11)$$

$$\begin{aligned} \bar{R}(X, Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X, Y] \\ &+ \tau(X)A_N Y - \tau(Y)A_N X + \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X, Y)\}N, \end{aligned} \quad (12)$$

$$\begin{aligned} R(X, Y)PZ &= R^*(X, Y)PZ + C(X, PZ)A_\xi^* Y - C(Y, PZ)A_\xi^* X \\ &+ \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ)\}\xi, \end{aligned} \quad (13)$$

$$\begin{aligned} R(X, Y)\xi &= -\nabla_X^*(A_\xi^* Y) + \nabla_Y^*(A_\xi^* X) + A_\xi^*[X, Y] - \tau(X)A_\xi^* Y + \tau(Y)A_\xi^* X \\ &+ \{C(Y, A_\xi^* X) - C(X, A_\xi^* Y) - 2d\tau(X, Y)\}\xi. \end{aligned} \quad (14)$$

In the case $R = 0$, we say that M is *flat*.

The *Ricci tensor*, denoted by $\bar{\text{Ric}}$, of \bar{M} is defined by

$$\bar{\text{Ric}}(\bar{X}, \bar{Y}) = \text{trace}\{\bar{Z} \rightarrow \bar{R}(\bar{X}, \bar{Z})\bar{Y}\}.$$

Let $\dim \bar{M} = n + 2$. Locally, $\bar{\text{Ric}}$ is given by

$$\bar{\text{Ric}}(\bar{X}, \bar{Y}) = \sum_{i=1}^{n+2} \epsilon_i \bar{g}(\bar{R}(E_i, \bar{X})\bar{Y}, E_i),$$

where $\{E_1, \dots, E_{n+2}\}$ is an orthonormal basis of $T\bar{M}$.

Let $R^{(0,2)}$ denote the induced tensor of type $(0, 2)$ on M given by

$$R^{(0,2)}(X, Y) = \text{trace}\{Z \rightarrow R(X, Z)Y\}. \quad (15)$$

Due to [4], using (8), (9) and the Gauss equation (11), we get

$$R^{(0,2)}(X, Y) = \overline{\text{Ric}}(X, Y) + B(X, Y) \text{tr} A_N - g(A_N X, A_\xi^* Y) - \bar{g}(\bar{R}(\xi, Y)X, N). \quad (16)$$

Using the transversal part of (12) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y).$$

This shows that $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ of M , given by (15), is called the *induced Ricci tensor*, denoted by Ric , of M if it is symmetric. In this case, M is said to be *Ricci flat* if $\text{Ric} = 0$. M is called an *Einstein manifold* if there exist a smooth function κ such that

$$\text{Ric} = \kappa g. \quad (17)$$

Let $\nabla_X^\perp N = \pi_1(\bar{\nabla}_X N)$, where π_1 is the projection morphism of $T\bar{M}$ on $\text{tr}(TM)$. Then ∇^\perp is a linear connection on the transversal vector bundle $\text{tr}(TM)$ of M . We say that ∇^\perp is the *transversal connection* of M . We define the curvature tensor R^\perp on $\text{tr}(TM)$ by

$$R^\perp(X, Y)N = \nabla_X^\perp \nabla_Y^\perp N - \nabla_Y^\perp \nabla_X^\perp N - \nabla_{[X, Y]}^\perp N.$$

The transversal connection ∇^\perp of M is said to be *flat* [5] if $R^\perp = 0$.

We quote the following result due to Jin [5].

Theorem 1. *Let M be a lightlike hypersurface of a semi-Riemannian manifold \bar{M} . The following assertions are equivalent:*

- (1) *The transversal connection of M is flat, i.e., $R^\perp = 0$.*
- (2) *The 1-form τ is closed, i.e., $d\tau = 0$, on any neighborhood $\mathcal{U} \subset M$.*
- (3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M .*

Remark 1. Due to [3, Section 4.2–4.3], we shown the following results:

- (1) $d\tau$ is independent to the choice of the section $\xi \in \Gamma(TM^\perp)$, that is, suppose τ and $\bar{\tau}$ are 1-forms with respect to the sections ξ and $\bar{\xi}$, respectively, then $d\tau = d\bar{\tau}$.
- (2) If $d\tau = 0$, then we can take a 1-form τ such that $\tau = 0$.

3 Quasi-constant curvature

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$ -dimensional indefinite Kähler manifold, where \bar{g} is a semi-Riemannian metric of index $q = 2v$, $0 < v < m$, and J is an almost complex metric structure on \bar{M} satisfying

$$J^2 = -I, \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \quad (\bar{\nabla}_{\bar{X}} J)\bar{Y} = 0. \quad (18)$$

Let (M, g) be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} , where g is a degenerate metric on M induced by \bar{g} . Due to [3, Section 6.2], we show that $J(TM^\perp) \oplus J(\text{tr}(TM))$ is a subbundle of $S(TM)$ of rank 2. There exist two non-degenerate almost complex distributions D_o and D on M with respect to J , i.e., $J(D_o) = D_o$ and $J(D) = D$, such that

$$\begin{aligned} S(TM) &= \left\{ J(TM^\perp) \oplus J(\text{tr}(TM)) \right\} \oplus_{\text{orth}} D_o, \\ D &= \left\{ TM^\perp \oplus_{\text{orth}} J(TM^\perp) \right\} \oplus_{\text{orth}} D_o. \end{aligned}$$

In this case, TM is decomposed as follow

$$TM = D \oplus J(\text{tr}(TM)). \quad (19)$$

Consider lightlike vector fields U and V , and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \quad (20)$$

Denote by S the projection morphism of TM on D with respect to (19). Then, for any vector field X on M , JX is expressed as follow

$$JX = FX + u(X)N, \quad (21)$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$. Applying $\bar{\nabla}_X$ to (20)_{1,2} and using (2)–(5) and (18)–(21), we have

$$B(X, U) = C(X, V), \quad (22)$$

$$\nabla_X U = F(A_N X) + \tau(X)U, \quad (23)$$

$$\nabla_X V = F(A_\xi^* X) - \tau(X)V. \quad (24)$$

From now and in the sequel, let \bar{M} be an indefinite Kaehler manifold of a quasi-constant curvature. We shall assume that the characteristic vector field ζ of \bar{M} is tangent to M and let $\alpha = \theta(N)$.

Theorem 2. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of a quasi-constant curvature such that ζ is tangent to M . Then the curvature functions f_1 and f_2 , given by (1), are satisfied*

$$f_1 = 0, \quad f_2\theta(V) = 0, \quad \alpha f_2 = 0.$$

Proof. Comparing the tangent and transversal components of the two forms (1) and (11) of the curvature tensor \bar{R} of \bar{M} , we get

$$\begin{aligned} R(X, Y)Z &= B(Y, Z)A_N X - B(X, Z)A_N Y + f_1 \{ \bar{g}(Y, Z)X - \bar{g}(X, Z)Y \} \\ &\quad + f_2 \left\{ [\theta(Y)X - \theta(X)Y]\theta(Z) + [g(Y, Z)\theta(X) - g(X, Z)\theta(Y)]\zeta \right\}, \end{aligned} \quad (25)$$

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) = 0. \quad (26)$$

Taking the product with N to (11) and using (9)₂ and (13), we get

$$\begin{aligned} & (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ) \\ &= f_1\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ) \\ & \quad + \alpha f_2\{\theta(X)g(Y, PZ) - \theta(Y)g(X, PZ)\}. \end{aligned} \quad (27)$$

Applying ∇_Y to (22) and using (8), (9) and (22)–(24), we have

$$\begin{aligned} (\nabla_X B)(Y, U) &= (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) \\ & \quad - g(A_\xi^* X, F(A_N Y)) - g(A_\xi^* Y, F(A_N X)). \end{aligned}$$

Substituting this equation into (26) with $Z = U$, we get

$$(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) + \tau(Y)C(X, V) = 0.$$

Comparing this equation and (27) such that $PZ = V$, we obtain

$$\begin{aligned} & f_1\{\eta(X)u(Y) - \eta(Y)u(X)\} + f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(V) \\ & \quad + f_2\alpha\{\theta(X)u(Y) - \theta(Y)u(X)\} = 0. \end{aligned} \quad (28)$$

Replacing Y by ξ to this equation and using the fact that $\theta(\xi) = 0$, we have

$$f_1 u(X) + f_2 \theta(X) \theta(V) = 0.$$

Taking $X = V$ and $X = U$ to this equation by turns, we get

$$f_2 \theta(V) = 0, \quad f_1 + f_2 \theta(U) \theta(V) = 0.$$

From these two equations, we get $f_1 = 0$. Taking $Y = \zeta$ to (28) and using $f_1 = 0$ and $f_2 \theta(V) = 0$, we have $\alpha f_2 u(X) = 0$. It follows that $\alpha f_2 = 0$. \square

4 Totally umbilical screen distribution

Definition 1. A screen distribution $S(TM)$ is said to be *totally umbilical* [3], [6] in M if there exists a smooth function γ such that $A_N X = \gamma P X$, i.e.,

$$C(X, P Y) = \gamma g(X, Y). \quad (29)$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M .

Theorem 3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of a quasi-constant curvature such that ζ is tangent to M . If $S(TM)$ is totally umbilical, then*

- (1) $S(TM)$ is totally geodesic and parallel distribution,
- (2) $f_1 = f_2 = 0$, i.e., \bar{M} is flat, and M is also flat,
- (3) the transversal connection of M is flat, and

(4) M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where \mathcal{C}_ξ is a null geodesic tangent to TM^\perp , and M^* is a semi-Euclidean leaf of $S(TM)$.

Proof. Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (7), we have

$$(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this and (29) into (27) such that $f_1 = f_2\alpha = 0$, we obtain

$$\begin{aligned} & \{X\gamma - \gamma\tau(X)\}g(Y, PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X, PZ) \\ & \quad + \gamma\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\} \\ & \quad = f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ). \end{aligned}$$

Replacing Y by ξ to this and using (6) and the fact that $\theta(\xi) = 0$, we get

$$\gamma B(X, Y) = \{\xi\gamma - \gamma\tau(\xi)\}g(X, Y) - f_2\theta(X)\theta(Y). \quad (30)$$

Taking $Y = U$ to this equation and using (20), (22) and (29), we have

$$\gamma^2 u(X) = \{\xi\gamma - \gamma\tau(\xi)\}v(X) - f_2\theta(X)\theta(U).$$

Replacing X by V to this and using the fact that $f_2\theta(V) = 0$, we obtain

$$\xi\gamma - \gamma\tau(\xi) = 0, \quad \gamma^2 u(X) = -f_2\theta(X)\theta(U). \quad (31)$$

Assume that $f_2 \neq 0$. Taking $X = \zeta$ to (31)₂, we have

$$\gamma^2\theta(V) = -f_2\theta(U).$$

Taking the product with f_2 to this and using $f_2\theta(V) = 0$, we get $f_2\theta(U) = 0$. Using this, from (31)₂, we see that $\gamma = 0$. Taking $X = Y = \zeta$ to (30), we have $f_2 = 0$. It is a contradiction. Thus $f_2 = 0$. We obtain $\gamma = 0$ by (31)₂.

- (1) As $\gamma = 0$, $S(TM)$ is totally geodesic. Therefore, $S(TM)$ is a parallel distribution by (4) and the fact that $C = 0$.
- (2) As $f_1 = f_2 = 0$, \bar{M} is flat. As $f_1 = f_2 = A_N = 0$, from (27), we see that $R = 0$. Thus M is also flat.
- (3) As $R = 0$, from (15), M is Ricci flat and $d\tau = 0$ by Theorem 2.1. Thus the transversal connection of M is flat.
- (4) From (5) and (10), we see that TM^\perp is an auto-parallel distribution. As $S(TM)$ is a parallel distribution and $TM = TM^\perp \oplus S(TM)$, by the decomposition theorem [7], M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where \mathcal{C}_ξ is a null geodesic tangent to TM^\perp and M^* is a leaf of $S(TM)$. As $R = 0$ and $C = 0$, from (13) we see that $R^* = 0$. Thus M^* is semi-Euclidean. \square

Denote by $\mathcal{G} = J(TM^\perp) \oplus_{\text{orth}} D_o$. Then \mathcal{G} is a complementary vector subbundle to $J(\text{tr}(TM))$ in $S(TM)$ and we have the decomposition:

$$S(TM) = J(\text{tr}(TM)) \oplus \mathcal{G}.$$

Theorem 4. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of quasi-constant curvature such that ζ is tangent to M . If $S(TM)$ is totally umbilical, then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where \mathcal{C}_ξ and \mathcal{C}_U are null geodesics tangent to TM^\perp and $J(\text{tr}(TM))$ respectively and M^\sharp is a semi-Euclidean leaf of \mathcal{G} .*

Proof. By Theorem 4.1, we show that $d\tau = 0$ and $A_N = C = 0$. As $d\tau = 0$, we can take $\tau = 0$ by Remark 2.2, without loss generality. As $C = 0$, from (22) we see that $B(X, U) = 0$. Also, since $A_N = 0$, from (23) we have

$$\nabla_X U = 0. \quad (32)$$

Thus $J(\text{tr}(TM))$ is a parallel distribution on M . From (5) and (10), TM^\perp is also a parallel distribution on M . Using (32), we derive

$$g(\nabla_X Y, U) = 0, \quad g(\nabla_X V, U) = 0, \quad \forall X \in \Gamma(\mathcal{G}), \forall Y \in \Gamma(D_o).$$

Thus \mathcal{G} is also a parallel distribution. By the decomposition theorem [7], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where \mathcal{C}_ξ and \mathcal{C}_U are null geodesics tangent to TM^\perp and $J(\text{tr}(TM))$ respectively and M^\sharp is a leaf of \mathcal{G} . Let π_2 be the projection morphism of $S(TM)$ on \mathcal{G} . Then $\pi_2 \circ R^*$ is the curvature tensor of \mathcal{G} . As $R = 0$ and $C = 0$, we have $R^* = 0$. Therefore, $\pi_2 \circ R^* = 0$ and M^\sharp is a semi-Euclidean space. \square

5 Screen conformal lightlike hypersurfaces

Definition 2. A lightlike hypersurface M is called *screen conformal* [1], [4] if there exists a non-vanishing smooth function φ such that $A_N = \varphi A_\xi^*$, i.e.,

$$C(X, PY) = \varphi B(X, Y).$$

If φ is a non-zero constant, then we say that M is *screen homothetic*.

Remark 2. If M is screen conformal, then, using (1) and the fact $f_1 = 0$,

$$\bar{g}(R(\xi, X)Y, N) = f_2 \theta(X) \theta(Y)$$

and

$$\bar{\text{Ric}}(X, Y) = f_2 \{g(X, Y) + n\theta(X)\theta(Y)\}.$$

Thus the form (16) of the Ricci type tensor $R^{(0,2)}$ is reduced to

$$R^{(0,2)}(X, Y) = f_2 \{g(X, Y) + (n-1)\theta(X)\theta(Y)\} + B(X, Y) \text{tr} A_N - \varphi g(A_\xi^* X, A_\xi^* Y). \quad (33)$$

Thus $R^{(0,2)}$ is symmetric. Thus $d\tau = 0$ and the transversal connection is flat by Theorem 2.1. As $d\tau = 0$, we can take $\tau = 0$ by Remark 2.2.

Proposition 1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of a quasi-constant curvature such that ζ is tangent to M . If M is screen conformal, then the curvature function f_2 is satisfied $f_2\theta(U) = 0$.*

Proof. Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (26) and using (25), we obtain

$$(X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ) = f_2\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(PZ). \quad (34)$$

Taking $Y = \xi$ to (34) and using (6) and the fact that $\theta(\xi) = 0$, we get

$$(\xi\varphi)B(X, Y) = f_2\theta(X)\theta(Y). \quad (35)$$

Replacing Y by V to (35) and using the fact that $f_2\theta(V) = 0$, we have

$$(\xi\varphi)B(X, V) = 0.$$

Taking $Y = U$ to (35) and using the fact $B(X, U) = C(X, V) = \varphi B(X, V)$, we obtain $f_2\theta(X)\theta(U) = 0$. Replacing X by ζ , we have $f_2\theta(U) = 0$. \square

Corollary 1. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of a quasi-constant curvature such that ζ is tangent to M . If M is screen homothetic, then $f_1 = f_2 = 0$, i.e., \bar{M} is flat.*

Proof. As M is screen homothetic, we get $\xi\varphi = 0$. Taking $X = Y = \zeta$ to (35) such that $\xi\varphi = 0$, we obtain $f_2 = 0$. As $f_1 = f_2 = 0$, \bar{M} is flat. \square

As $\{U, V\}$ is a null basis of $J(TM^\perp) \oplus J(\text{tr}(TM))$, let

$$\mu = U - \varphi V, \quad \nu = U + \varphi V,$$

then $\{\mu, \nu\}$ is an orthogonal basis of $J(TM^\perp) \oplus J(\text{tr}(TM))$ and satisfies

$$B(X, \mu) = 0, \quad A_\xi^* \mu = 0, \quad (36)$$

due to (22). Thus μ is an eigenvector field of A_ξ^* on $S(TM)$ corresponding to the eigenvalue 0. As $f_2\theta(V) = 0$ and $f_2\theta(U) = 0$, we also have

$$f_2\theta(\mu) = 0, \quad f_2\theta(\nu) = 0. \quad (37)$$

Let $\mathcal{H}' = \text{Span}\{\mu\}$. Then $\mathcal{H} = D_o \oplus_{\text{orth}} \text{Span}\{\nu\}$ is a complementary vector subbundle to \mathcal{H}' in $S(TM)$ and we have the following decomposition

$$S(TM) = \mathcal{H}' \oplus_{\text{orth}} \mathcal{H}. \quad (38)$$

Theorem 5. *Let M be a screen homothetic lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of quasi-constant curvature such that ζ is tangent to M . Then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where \mathcal{C}_ξ and \mathcal{C}_μ are null and non-null geodesics tangent to TM^\perp and \mathcal{H}' , respectively and M^\natural is a leaf of a non-degenerate distribution \mathcal{H} .*

Proof. In general, from (23), (24) and the fact that F is linear, we have

$$\nabla_X \mu = -(X\varphi)V.$$

Therefore, if M is screen homothetic, then we have

$$\nabla_X \mu = 0. \quad (39)$$

This implies that \mathcal{H}' is a parallel distribution on M . From (5) and (10), TM^\perp is also a parallel distribution on M . Using (39), we derive

$$\begin{aligned} g(\nabla_X Y, \mu) &= g(\bar{\nabla}_X Y, \mu) = -g(Y, \nabla_X \mu) = 0, \\ g(\nabla_X \nu, \mu) &= -g(\nu, \nabla_X \mu) = X\varphi = 0, \end{aligned}$$

for $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(D_o)$. Thus \mathcal{H} is also a parallel distribution. By the decomposition theorem of de Rham [7], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\sharp$, where \mathcal{C}_ξ and \mathcal{C}_μ are null and non-null geodesics tangent to TM^\perp and \mathcal{H}' respectively and M^\sharp is a leaf of \mathcal{H} . □

Theorem 6. *Let M be an Einstein lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of a quasi-constant curvature such that ζ is tangent to M . If M is screen conformal, then the function κ , given by (17), satisfies $\kappa = f_2$. If M is screen homothetic, then it is Ricci flat, i.e., $\kappa = 0$.*

Proof. Since M is Einstein manifold, (33) is reduced to

$$g(A_\xi^* X, A_\xi^* Y) - \ell g(A_\xi^* X, Y) - \varphi^{-1} \{ (\kappa - f_2)g(X, Y) - f_2(n-1)\theta(X)\theta(Y) \} = 0, \quad (40)$$

where $\ell = \text{tr } A_\xi^*$ is the trace of A_ξ^* . Put $X = Y = \mu$ in (40) and using (36)₂ and (37)₁, we have $\kappa = f_2$. If M is screen homothetic, then M is Ricci flat as $f_2 = 0$ by Corollary 5.3. □

Theorem 7. *Let M be a screen homothetic Einstein lightlike hypersurface of an indefinite Kaehler manifold \bar{M} of quasi-constant curvature such that $q = 2$ and ζ is tangent to M . Then M is locally a product manifold*

$$M = \mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\sharp \quad \text{or} \quad M = \mathcal{C}_\xi \times \mathcal{C}_\mu \times \mathcal{C}_\ell \times M^\sharp,$$

where \mathcal{C}_ξ , \mathcal{C}_μ and \mathcal{C}_ℓ are null geodesic, timelike geodesic and spacelike geodesic respectively, and M^\sharp and M^\sharp are Euclidean spaces.

Proof. In this proof, we set $\mu = \frac{1}{\sqrt{2\epsilon\varphi}}\{U - \varphi V\}$ where $\epsilon = \text{sgn } \varphi$. Then μ is a unit timelike eigenvector of A_ξ^* corresponding to the eigenvalue 0 by (36) and \mathcal{H} is a parallel Riemannian distribution by Theorem 5.4 due to $q = 2$. Since $g(A_\xi^* X, N) = 0$ and $g(A_\xi^* X, \mu) = 0$, A_ξ^* is \mathcal{H} -valued real self-adjoint operator. Thus A_ξ^* have $(n-1)$ real orthonormal eigenvectors in \mathcal{H} and is diagonalizable. Consider a frame field of eigenvectors $\{\mu, e_1, \dots, e_{n-1}\}$ of A_ξ^* on $S(TM)$ such that $\{e_1, \dots, e_{n-1}\}$ is an orthonormal frame field of \mathcal{H} . Then $A_\xi^* e_i = \lambda_i e_i$ ($1 \leq i \leq n-1$).

Put $X = Y = e_i$ in (40) such that $\kappa = f_2 = 0$, we show that each eigenvalue λ_i of A_ξ^* is a solution of

$$x(x - \ell) = 0. \quad (41)$$

The equation (41) has at most two distinct real solutions 0 and ℓ on \mathcal{U} . Assume that there exists $p \in \{1, \dots, n-1\}$ such that $\lambda_1 = \dots = \lambda_p = 0$ and $\lambda_{p+1} = \dots = \lambda_{n-1} = \ell$, by renumbering if necessary. Then we have

$$\ell = \text{tr } A_\xi^* = (n - p - 1)\ell.$$

If $\ell = 0$, then $A_\xi^* = 0$ and also $A_N = 0$. Thus M and $S(TM)$ are totally geodesic. From (11) and (13), we have $R^*(X, Y)Z = \bar{R}(X, Y)Z = 0$ for all $X, Y, Z \in \Gamma(S(TM))$. Thus M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where \mathcal{C}_ξ and \mathcal{C}_μ are null and timelike geodesic tangent to TM^\perp and \mathcal{H}' respectively and M^\natural is a leaf of \mathcal{H} , where the leaf $M^*(= \mathcal{C}_\mu \times M^\natural)$ of $S(TM)$ is a Minkowski space. Since $\nabla_X \mu = 0$ and

$$g(\nabla_X^* Y, \mu) = -g(Y, \nabla_X^* \mu) = -g(Y, \nabla_X \mu) = 0,$$

for all $X, Y, Z \in \Gamma(S(TM))$, we have $\nabla_X^* Y \in \Gamma(\mathcal{H})$ and $R^*(X, Y)Z \in \Gamma(\mathcal{H})$. This imply $\nabla_X^* Y = Q(\nabla_X^* Y)$, i.e., M^\natural is totally geodesic and $Q(R^*(X, Y)Z) = R^*(X, Y)Z = 0$, where Q is a projection morphism of $S(TM)$ on \mathcal{H} with respect to (38). Thus M^\natural is a Euclidean space.

If $\ell \neq 0$, then $p = n - 2$. Consider the following two distributions on \mathcal{H} ;

$$\begin{aligned} \Gamma(E_0) &= \{X \in \Gamma(\mathcal{H}) | A_\xi^* X = 0\}, \\ \Gamma(E_\ell) &= \{X \in \Gamma(\mathcal{H}) | A_\xi^* X = \ell X\}. \end{aligned}$$

Then we know that the distributions E_0 and E_ℓ are mutually orthogonal non-degenerate subbundle of \mathcal{H} , of rank $(n-2)$ and 1 respectively, satisfy $\mathcal{H} = E_0 \oplus_{\text{orth}} E_\ell$. From (40), we get $A_\xi^*(A_\xi^* - \ell Q) = 0$. Using this equation, we have

$$\text{Im } A_\xi^* \subset \Gamma(E_\ell) \quad \text{and} \quad \text{Im}(A_\xi^* - \ell Q) \subset \Gamma(E_0).$$

For any $X, Y \in \Gamma(E_0)$ and $Z \in \Gamma(\mathcal{H})$, we get

$$(\nabla_X B)(Y, Z) = -g(A_\xi^* \nabla_X Y, Z).$$

Using this and the fact that

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z),$$

we have $g(A_\xi^*[X, Y], Z) = 0$. If we take $Z \in \Gamma(E_\ell)$, since $\text{Im } A_\xi^* \subset \Gamma(E_\ell)$ and E_ℓ is non-degenerate, we have $A_\xi^*[X, Y] = 0$. Thus $[X, Y] \in \Gamma(E_0)$ and E_0 is integrable. From (11) and (13), we have

$$R^*(X, Y)Z = \bar{R}(X, Y)Z = 0$$

for all $X, Y, Z \in \Gamma(E_0)$.

Since $g(\nabla_X^* Y, \mu) = 0$ and $g(\nabla_X^* Y, e_{n-1}) = -g(Y, \nabla_X e_{n-1}) = 0$ for all $X, Y \in \Gamma(E_0)$ because $\nabla_X W \in \Gamma(E_\ell)$ for $X \in \Gamma(E_0)$ and $W \in \Gamma(E_\ell)$. In fact, from (26) such that $\tau = 0$, we get

$$g\left(\{(A_\xi^* - \ell Q)\nabla_X W - A_\xi^* \nabla_W X\}, Z\right) = 0,$$

for all $X \in \Gamma(E_0), W \in \Gamma(E_\ell)$ and $Z \in \Gamma(\mathcal{H})$. Using the fact that \mathcal{H} is non-degenerate distribution, we have

$$(A_\xi^* - \ell Q)\nabla_X W = A_\xi^* \nabla_W X.$$

Since the left term of this equation is in $\Gamma(E_0)$ and the right term is in $\Gamma(E_\ell)$ and $E_0 \cap E_\ell = \{0\}$, we have

$$(A_\xi^* - \ell Q)\nabla_X W = 0 \quad \text{and} \quad A_\xi^* \nabla_W X = 0.$$

These imply that $\nabla_X W \in \Gamma(E_\ell)$. Thus $\nabla_X^* Y = \pi_3 \nabla_X^* Y$ for all $X, Y \in \Gamma(E_0)$, where π_3 is the projection morphism of $S(TM)$ on E_0 and $\pi_3 \nabla^*$ is the induced connection on E_0 . These imply that the leaf M^\sharp of E_0 is totally geodesic. Thus E_0 is a parallel distribution and M is locally a product manifold $\mathcal{C}_\xi \times M^* (= \mathcal{C}_\mu \times \mathcal{C}_\ell \times M^\sharp)$, where \mathcal{C}_ℓ is a spacelike curve and M^\sharp is an $(n-2)$ -dimensional Riemannian manifold satisfies $A_\xi^* = 0$. As

$$g(R^*(X, Y)Z, \mu) = 0 \quad \text{and} \quad g(R^*(X, Y)Z, e_{n-1}) = 0$$

for all $X, Y, Z \in \Gamma(E_0)$, we have

$$R^*(X, Y)Z = \pi_3 R^*(X, Y)Z \in \Gamma(E_0)$$

and the curvature tensor $\pi_3 R^*$ of E_0 is flat. Thus M^\sharp is a Euclidean space. \square

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