



HAL
open science

Approach of q -Derivative Operators to Terminating q -Series Formulae

Xiaoyuan Wang, Wenchang Chu

► **To cite this version:**

Xiaoyuan Wang, Wenchang Chu. Approach of q -Derivative Operators to Terminating q -Series Formulae. *Communications in Mathematics*, 2018, Volume 26 (2018), Issue 2 (2), pp.99 - 111. 10.2478/cm-2018-0007. hal-03664958

HAL Id: hal-03664958

<https://hal.science/hal-03664958v1>

Submitted on 11 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Approach of q -Derivative Operators to Terminating q -Series Formulae

Xiaoyuan Wang, Wenchang Chu

Abstract. The q -derivative operator approach is illustrated by reviewing several typical summation formulae of terminating basic hypergeometric series.

1 Introduction and Motivation

The q -derivative operator is a useful tool for proving q -series identities (cf. Carlitz [10], Chu [16], [18] and Liu [28]). It is defined by

$$\mathcal{D}_x f(x) := \frac{f(x) - f(qx)}{x} \quad \text{and} \quad \mathcal{D}^n f = \mathcal{D}(\mathcal{D}^{n-1})f \quad \text{for } n = 2, 3, \dots$$

with the convention that $\mathcal{D}_x^0 f(x) = f(x)$ for the identity operator. One can show, by means of the induction principle, the following explicit formula

$$\mathcal{D}_x^n f(x) = x^{-n} \sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} f(q^k x), \quad (1)$$

where the q -shifted factorial of x is given by $(x; q)_0 \equiv 1$ and

$$(x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \quad \text{for } n = 1, 2, \dots,$$

2010 MSC: Primary 33C20, Secondary 05A30

Key words: Terminating q -series, the q -derivative operator, well-poised series, balanced series, Pfaff-Saalschütz summation theorem, Gasper's q -Karlsson-Minton formula

Affiliation:

X. Wang – School of Science, Dalian Jiaotong University, Dalian 116028, P. R. China
E-mail: xiaoyuanwang27@yahoo.com

W. Chu – School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou (Henan), P. R. China and
 Dipartimento di Matematica e Fisica, Università del Salento, Arnesano,
 P. O. Box 193, 73100 Lecce, Italy
E-mail: chu.wenchang@unisalento.it

with product and quotient forms being abbreviated, respectively, to

$$[a, b, \dots, c; q]_n = (a; q)_n (b; q)_n \cdots (c; q)_n,$$

$$\left[\begin{matrix} a, b, \dots, c \\ A, B, \dots, C \end{matrix} \middle| q \right]_n = \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(A; q)_n (B; q)_n \cdots (C; q)_n}.$$

According to (1), we can verify, by means of the q -binomial theorem and the q -Chu-Vandermonde formula, that for a monic polynomial $P_m(x)$ of degree $m \leq n$, the following evaluation formulae for higher q -derivatives hold:

$$\mathcal{D}_x^n P_m(x) = \chi(m = n)(q; q)_n, \quad (2)$$

$$\mathcal{D}^n \frac{P_m(x)}{1 - \lambda x} = \frac{\lambda^n (q; q)_n}{(\lambda; q)_{n+1}} P_m(1/\lambda), \quad (3)$$

where χ denotes the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$ otherwise.

The objective of this paper is to review several typical summation formulae of terminating basic hypergeometric series by means of the q -derivative operator. The approach will consist of the following three steps:

- First, for a given a q -series identity, identifying a parameter x as a variable and expressing the q -sum in terms of the higher q -derivatives displayed in (1).
- Then, evaluating the q -sum for particular values of x with the help of q -derivative properties (2) and/or (3).
- Finally, confirming the q -series identity via the fundamental theorem of algebra, i.e., “two polynomials of degrees $\leq n$ are identical if they agree at $n + 1$ distinct points”.

Throughout the paper, the basic hypergeometric series (cf. Bailey [5] and Gasper-Rahman [23]) is defined by

$${}_{1+\ell}\phi_\ell \left[\begin{matrix} a_0, a_1, \dots, a_\ell \\ b_1, \dots, b_\ell \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} z^n \left[\begin{matrix} a_0, a_1, \dots, a_\ell \\ q, b_1, \dots, b_\ell \end{matrix} \middle| q \right]_n,$$

which becomes terminating if one of the numerator parameters $\{a_i\}_{0 \leq i \leq \ell}$ results in q^{-m} with m being a nonnegative integer.

2 The q -Pfaff-Saalschütz theorem

As a warm-up, we start with the following fundamental formula; see [5, Chapter 8] and [31, §3.3].

Theorem 1 (The q -Pfaff-Saalschütz theorem).

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix} \middle| q; q \right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix} \middle| q \right]_n = a^n \left[\begin{matrix} c/a, q^{1-n}b/c \\ c, q^{1-n}ab/c \end{matrix} \middle| q \right]_n. \quad (4)$$

Proof. Multiplying across this equation by $(q^{1-n}ab/c; q)_n$, we see that both sides are polynomials of degree n in b . To prove the identity, it suffices to show that the equality holds for $n + 1$ distinct values of b . First it is trivial to see that both sides are equal to 1 for $b = 1$. Then for $b = q^{m-1}c$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be written as the q -binomial sum

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, a, q^{m-1}c \\ c, q^{m-n}a \end{matrix} \middle| q; q \right] = \sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(q^{m-n+k}a; q)_{n-m} (q^k c; q)_{m-1}}{(q^{m-n}a; q)_{n-m} (c; q)_{m-1}}.$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it results in a multiple of the n th q -derivative of the polynomial $(cx; q)_{m-1} (q^{m-n}ax; q)_{n-m}$ of degree $n - 1$ in x . Therefore we have validated the equality for $n + 1$ distinct values of b and completed proof of (4). \square

3 The q -Watson Formula

In this section we give a proof of an identity that was first found by Andrews [2, Theorem 1] in 1976 as a terminating q -series analogue of the Watson formula; see also Gasper-Rahman [23, II-17].

Theorem 2. *Let $b = q^{-\delta-2n}$ with $\delta = 0$ or 1 and $n \in \mathbb{N}_0$. Then the following terminating series identity holds:*

$${}_4\phi_3 \left[\begin{matrix} a, b, \sqrt{c}, -\sqrt{c} \\ c, \sqrt{qab}, -\sqrt{qab} \end{matrix} \middle| q; q \right] = (1 - \delta) \left[\begin{matrix} q, qc/a \\ q/a, qc \end{matrix} \middle| q^2 \right]_n. \tag{5}$$

For different proofs, see Chu [19, Corollary 7] and Verma-Jain [32, Eq. 1.1].

Proof. Multiplying across the equality (5) by $(qc; q^2)_n$, we can rewrite the resulting equation equivalently as

$$\sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(a; q)_k (c; q^2)_k (qc; q^2)_n}{(c; q)_k (q^{1-\delta-2n}a; q^2)_k} q^k = (1 - \delta) \frac{(q; q^2)_n (qc/a; q^2)_n}{(q/a; q^2)_n}. \tag{6}$$

According to the relation

$$\frac{(c; q^2)_k (qc; q^2)_n}{(c; q)_k} = \begin{cases} \frac{(qc; q^2)_n (c; q)_{2k}}{(qc; q^2)_k (c; q)_k}, & k \leq n; \\ \frac{(c; q^2)_k (c; q)_{2n}}{(c; q^2)_n (c; q)_k}, & k > n; \end{cases}$$

both sides of (6) are polynomials of degree $\leq n$ in c . In order to prove (5), it suffices to show that the equality holds for $n + 1$ distinct values of c .

Let $\mathcal{S}(c)$ be the ${}_4\phi_3$ -series displayed in (5). Then for $c = q^{1-\delta-2m}a$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be restated as the following sum:

$$\begin{aligned} \mathcal{S}(q^{1-\delta-2m}a) &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(a; q)_k (q^{1-\delta-2m}a; q^2)_k}{(q^{1-\delta-2m}a; q)_k (q^{1-\delta-2n}a; q^2)_k} q^k \\ &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(q^{1-\delta-2m+k}a; q)_{\delta-1+2m} (q^{1-\delta-2n+2k}a; q^2)_{n-m}}{(q^{1-\delta-2m}a; q)_{\delta-1+2m} (q^{1-\delta-2n}a; q^2)_{n-m}} q^k. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 - \delta \leq m \leq n$ because it results in a multiple of the $(\delta + 2n)$ th q -derivatives of the polynomial

$$(q^{1-\delta-2m}ax; q)_{\delta-1+2m}(q^{1-\delta-2n}ax^2; q^2)_{n-m}$$

of degree $\delta - 1 + 2n$ in x . Therefore, (5) is confirmed when $\delta = 1$ because it holds for $n + 1$ distinct values of $c \in \{q^{-2m}a\}_{0 \leq m \leq n}$.

When $\delta = 0$ and $m = 0$, reformulating the last sum and then evaluating it by (3),

$$\begin{aligned} \sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(q^{1-2n+2k}a; q^2)_n}{(q^{1-2n}a; q^2)_n} \frac{1-a}{1-q^ka} &= a^{2n} \frac{(q^{1-2n}/a; q^2)_n}{(q^{1-2n}a; q^2)_n} \frac{(q; q)_{2n}}{(qa; q)_{2n}} \\ &= \left[\begin{matrix} q, & q^2 \\ q/a, & q^2a \end{matrix} \middle| q^2 \right]_n \end{aligned}$$

which coincides with the right member of (5) under the conditions $\delta = 0$ and $c = qa$. Hence, we have validated (5) also when $\delta = 0$ for $n + 1$ distinct values of $c \in \{q^{1-2m}a\}_{0 \leq m \leq n}$. This completes the proof of the theorem. \square

4 Two Balanced ${}_4\phi_3$ -Series

Unlike the q -Pfaff-Saalschütz theorem, there exist two summation formulae of balanced ${}_4\phi_3$ -series with one less free parameter. They are, in fact, the q -analogues of the particular case $b = 2$ of the following well-known Hagen-Rothe convolution identity (cf. Chu [20] and Gould [25]):

$$\sum_{k=0}^n \frac{a}{a+bk} \binom{a+bk}{k} \binom{c-bk}{n-k} = \binom{a+c}{n}.$$

The first one is due to Al-Salam and Verma [1]; see also Andrews [4, Eq. 7.6], Chu [11, Eq. 5.3b] and [30, Eq. 17.7.12].

Theorem 3.

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, qa, q^{2n}b^2 \\ b, qb, q^2a^2 \end{matrix} \middle| q^2; q^2 \right] = a^n \left[\begin{matrix} -q, b/a \\ -qa, b \end{matrix} \middle| q \right]_n. \quad (7)$$

Proof. Multiplying both sides of (7) by $(-qa; q)_n$ and observing that

$$\frac{(a; q)_{2k}(-qa; q)_n}{(q^2a^2; q^2)_k} = \frac{(a; q)_{2k}(-q^{k+1}a; q)_{n-k}}{(qa; q)_k},$$

we infer that the resulting equation is a polynomial identity of degree n in a . In order to prove (7), we need only to validate it for $n + 1$ distinct values of a . First of all, (7) is obviously valid for $a = 1$. Then denote by $\mathcal{S}(a)$ the ${}_4\phi_3$ -series in (7). For $a = q^{m-1}b$ with $1 \leq m \leq n$, the right member of (7) is equal to zero. The

corresponding left member can be reformulated as

$$\begin{aligned} S(q^{m-1}b) &= \sum_{k=0}^n \frac{(q^{-2n}; q^2)_k}{(q^2; q^2)_k} \frac{(q^{m-1}b; q)_{2k}}{(b; q)_{2k}} \frac{(q^{2n}b^2; q^2)_k}{(q^{2m}b^2; q^2)_k} q^{2k} \\ &= \sum_{k=0}^n \frac{(q^{-2n}; q^2)_k}{(q^2; q^2)_k} \frac{(q^{2k}b; q)_{m-1}}{(b; q)_{m-1}} \frac{(q^{2m+2k}b^2; q^2)_{n-m}}{(q^{2m}b^2; q^2)_{n-m}} q^{2k}, \end{aligned}$$

which vanishes because in base q^2 , it results in the n th q -derivative of the polynomial $(bx; q)_{m-1}(q^{2m}b^2x; q^2)_{n-m}$ of degree $n - 1$. This confirms (7). \square

The second balanced ${}_4\phi_3$ -series identity is due to Andrews [3, Eq. 4.3] & [4, Eq. 7.7]; see also Gessel-Stanton [24, Eq. 4.22], Chu [12, Eq. 4.3d] and [30, Eq. 17.7.13].

Theorem 4 (Variant of Theorem 3).

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, qa, q^{2n-2}b^2 \\ b, qb, a^2 \end{matrix} \middle| q^2; q^2 \right] = a^n \left[\begin{matrix} -q, b/a \\ -a, b \end{matrix} \middle| q \right]_n \frac{1 - q^{n-1}b}{1 - q^{2n-1}b}.$$

In exactly the same manner, this identity can be proved after the equation having been multiplied by $(-a; q)_n$ across.

5 Four Terminating Well-Poised Series

This section will be devoted to four terminating well-poised series identities of Dixon’s type. The first one is essentially due to Jackson [26, Eq. 2]. See Bailey [6, Eq. 2], Bressoud [8, Eq. 2]), Carlitz [9, Eq. 1.2], Chu [21], Verma-Joshi [33, Eq. 3.10] for different proofs, and Bailey [7, Eqs. 2.2 and 1.2], Chu [17], Verma-Jain [32, Eq. 5.5] for the nonterminating form.

Theorem 5. For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:

$$\begin{aligned} {}_3\phi_2 \left[\begin{matrix} q^{-2n}, b, d \\ q^{1-2n}/b, q^{1-2n}/d \end{matrix} \middle| q; q^{1+\delta-n}/bd \right] \\ = q^{n(\delta-1)} \left[\begin{matrix} b, d \\ b, bd \end{matrix} \middle| q \right]_n \left[\begin{matrix} q, bd \\ b, d \end{matrix} \middle| q \right]_{2n}. \end{aligned} \tag{8}$$

Proof. Multiplying across (8) by $(q^n b; q)_n$, we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(b; q)_k (q^n b; q)_n (d; q)_k}{(q^{1-2n}/b; q)_k (q^{1-2n}/d; q)_k} \left(\frac{q^{1+\delta-n}}{bd} \right)^k = \frac{(q^{n+1}; q)_n (q^n bd; q)_n}{q^{n(1-\delta)} (q^n d; q)_n}. \tag{9}$$

Observing the relation

$$\begin{aligned} \frac{(b; q)_k (q^n b; q)_n}{b^k (q^{1-2n}/b; q)_k} &= (-1)^k q^{2nk - \binom{k+1}{2}} \frac{(b; q)_k (q^n b; q)_n}{(q^{2n-k} b; q)_k} \\ &= (-1)^k q^{2nk - \binom{k+1}{2}} \frac{(b; q)_k (b; q)_{2n-k}}{(b; q)_n}, \end{aligned}$$

we assert that (9) is a polynomial identity of degree n in b . In order to prove (8), it suffices to show that the equality holds for $n + 1$ distinct values of b .

Let $\mathcal{S}(b)$ be the ${}_3\phi_2$ -series displayed in (8). Then for $b = q^{m-2n}/d$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{m-2n}/d) &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} d, & q^{m-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_k q^{k(1+\delta-m+n)} \\ &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} q^{1+k-m}d, & q^{1+k-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_{m-1} q^{k(1+\delta-m+n)}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it results in a multiple of the $(2n)$ th q -derivative of the polynomial

$$(q^{1-m}xd; q)_{m-1}(q^{1-2n}x/d; q)_{m-1}x^{\delta-m+n}$$

of degree $\delta - 2 + m + n < 2n$ in x .

When $m = 0$, the last sum can be reformulated by partial fractions and then evaluated by (3) as

$$\begin{aligned} &\sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(1-d)(1-q^{-2n}/d)}{(1-q^k d)(1-q^{k-2n}/d)} q^{k(\delta+n)} \\ &= \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \left\{ \frac{q^{k(\delta+n)}}{1-q^{k-2n}/d} - \frac{q^{k(\delta+n)+2n}d^2}{1-q^k d} \right\} \\ &= \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \left\{ \frac{(q; q)_{2n}(q^{2n}d)^{\delta-n}}{(q^{-2n}/d; q)_{2n+1}} - \frac{(q; q)_{2n}q^{2n}d^{n-\delta+2}}{(d; q)_{2n+1}} \right\}. \end{aligned}$$

The above expression can be easily simplified to

$$(q^{\delta-1}d)^n \frac{(q; q)_{2n}}{(d; q)_{2n}} \frac{1-d}{1-q^n d},$$

which agrees with the right member of (8) specified by $b = q^{-2n}/d$. Therefore for $n + 1$ distinct values of $b \in \{q^{m-2n}/d\}_{0 \leq m \leq n}$, we have validated (8), which completes the proof. \square

The next formula serves as a counterpart of (8) whose nonterminating version was found by Bailey [7, Eq. 2.3]. For different proofs, the reader may refer to Carlitz [9, Eq. 2.12], Chu [17], Chu-Wang [21] and Verma-Joshi [33, Eq. 3.13].

Theorem 6. For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:

$$\begin{aligned} &{}_3\phi_2 \left[\begin{matrix} q^{-1-2n}, & b, & d \\ q^{-2n}/b, & q^{-2n}/d \end{matrix} \middle| q; q^{2\delta-n}/bd \right] \\ &= \left\{ 1 - q^{(2n+1)(2\delta-1)} \right\} \left[\begin{matrix} qb, & qd \\ q, & qbd \end{matrix} \middle| q \right]_n \left[\begin{matrix} q, & qbd \\ qb, & qd \end{matrix} \middle| q \right]_{2n}. \quad (10) \end{aligned}$$

Proof. Multiplying across (10) by $(q^{n+1}b; q)_n$, we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{(b; q)_k (q^{n+1}b; q)_n (d; q)_k}{(q^{-2n}/b; q)_k (q^{-2n}/d; q)_k} \left(\frac{q^{2\delta-n}}{bd}\right)^k = \left(1 - q^{(2n+1)(2\delta-1)}\right) \frac{(q^{n+1}; q)_n (q^{n+1}bd; q)_n}{(q^{n+1}d; q)_n}. \tag{11}$$

Observing that

$$\begin{aligned} \frac{(b; q)_k (q^{n+1}b; q)_n}{b^k (q^{-2n}/b; q)_k} &= (-1)^k q^{2nk - \binom{k}{2}} \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{1+2n-k}b; q)_k} \\ &= (-1)^k q^{2nk - \binom{k}{2}} \frac{(b; q)_k (b; q)_{1+2n-k}}{(b; q)_{n+1}}, \end{aligned}$$

we assert that both sides of (11) are polynomials of degree n in b . In order to prove (10), it suffices to show that the equality holds for $n + 1$ distinct values of b .

Let $\mathcal{S}(b)$ be the ${}_3\phi_2$ -series displayed in (10). Then for $b = q^{-m-n}/d$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{-m-n}/d) &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} d, & q^{-m-n}/d \\ q^{m-n}d, & q^{-2n}/d \end{matrix} \middle| q \right]_k q^{k(2\delta+m)} \\ &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} q^{k+m-n}d, & q^{k-2n}/d \\ q^{m-n}d, & q^{-2n}/d \end{matrix} \middle| q \right]_{n-m} q^{k(2\delta+m)}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it is a multiple of the $(2n + 1)$ th q -derivative of the polynomial

$$(q^{m-n}xd; q)_{n-m} (q^{-2n}x/d; q)_{n-m} x^{2\delta-1+m}$$

of degree $2n + 2\delta - m - 1 < 2n + 1$ in x .

When $m = n + 1$, we can rewrite the last sum by partial fractions and then evaluate it by (3) as

$$\begin{aligned} \sum_{k=0}^{2n+1} q^k \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{(1-d)(1-q^{-1-2n}/d)}{(1-q^k d)(1-q^{k-1-2n}/d)} q^{k(2\delta+n)} &= \frac{(1-d)(1-q^{-2n-1}/d)}{1-q^{1+2n}d^2} \\ &\times \sum_{k=0}^{2n+1} q^k \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left\{ \frac{q^{k(2\delta+n)}}{1-q^{k-1-2n}/d} - \frac{q^{k(2\delta+n)+1+2n}d^2}{1-q^k d} \right\} \\ &= \frac{(1-d)(1-q^{-1-2n}/d)}{1-q^{1+2n}d^2} \\ &\times \left\{ \frac{(q; q)_{2n+1} (q^{1+2n}d)^{2\delta-1-n}}{(q^{-1-2n}/d; q)_{2n+2}} - \frac{(q; q)_{2n+1} q^{1+2n} d^{3+n-2\delta}}{(d; q)_{2n+2}} \right\}. \end{aligned}$$

The above expression can be further simplified to

$$d^n \frac{(q; q)_{2n}}{(d; q)_{2n}} \frac{1-d}{1-q^{1+2n}d} \left\{ 1 - q^{(2n+1)(2\delta-1)} \right\},$$

which agrees with the right member of (10) specified by $b = q^{-1-2n}/d$. Therefore for $n+1$ distinct values of $b \in \{q^{m-2n}/d\}_{1 \leq m \leq n+1}$, we have validated (10), which completes the proof. \square

There is also the following more general well-poised ${}_5\phi_4$ -series identity discovered by Jackson [26, Eq. 1], that different proofs can be found in Bailey [7, Eq. 3.1], Bressoud [8, Eq. 1], Chu [14, §2] and Verma-Joshi [33, Eq. 3.8].

Theorem 7. For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:

$$\begin{aligned} {}_5\phi_4 \left[\begin{matrix} q^{-2n}, & b, & c, & d, & q^{1-3n}/bcd \\ q^{1-2n}/b, & q^{1-2n}/c, & q^{1-2n}/d, & q^n bcd \end{matrix} \middle| q; q^{1+\delta} \right] \\ = q^{n(\delta-1)} \left[\begin{matrix} b, & c, & d, & bcd \\ q, & bc, & bd, & cd \end{matrix} \middle| q \right]_n \left[\begin{matrix} q, & bc, & bd, & cd \\ b, & c, & d, & bcd \end{matrix} \middle| q \right]_{2n}. \end{aligned} \quad (12)$$

Proof. Multiplying across (12) by $(q^n b; q)_n (q^n bcd; q)_n$, we may rewrite the resulting equation equivalently as

$$\begin{aligned} \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{[b, c, d, q^{1-3n}/bcd; q]_k (q^n b; q)_n (q^n bcd; q)_n}{[q^{1-2n}/b, q^{1-2n}/c, q^{1-2n}/d, q^n bcd; q]_k} q^{k(1+\delta)} \\ = q^{n(\delta-1)} \frac{(q^{n+1}; q)_n (q^n bc; q)_n (q^n bd; q)_n (q^n cd; q)_n}{(q^n c; q)_n (q^n d; q)_n}. \end{aligned} \quad (13)$$

According to the relation

$$\begin{aligned} \frac{(b; q)_k (q^n b; q)_n}{(q^{1-2n}/b; q)_k} \frac{(q^{1-3n}/bcd; q)_k (q^n bcd; q)_n}{(q^n bcd; q)_k} \\ = \left(\frac{q^{-n}}{cd} \right)^k \frac{(b; q)_k (q^n b; q)_n}{(q^{2n-k} b; q)_k} \frac{(q^{3n-k} bcd; q)_k (q^n bcd; q)_n}{(q^n bcd; q)_k} \\ = \left(\frac{q^{-n}}{cd} \right)^k \frac{(b; q)_k (b; q)_{2n-k}}{(b; q)_n} \frac{(bcd; q)_{2n} (bcd; q)_{3n}}{(bcd; q)_{n+k} (bcd; q)_{3n-k}}, \end{aligned}$$

both sides of (13) become polynomials of degree $2n$ in b . In order to prove (12), it suffices to show that the equality holds for $2n+1$ distinct values of b .

First, for $b = 1$ in (12), the ${}_5\phi_4$ -series becomes $1 + q^{n(2\delta-1)}$ because only the two extreme terms survive. This coincides with the corresponding right member.

Let $\mathcal{S}(b)$ be the ${}_5\phi_4$ -series displayed in (12). Then for $b = q^{m-2n}/c$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be

written as the expression

$$\begin{aligned} \mathcal{S}(q^{m-2n}/c) &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} c, & d, & q^{m-2n}/c, & q^{1-m-n}/d \\ q^{1-m}c, & q^{m-n}d, & q^{1-2n}/c, & q^{1-2n}/d \end{matrix} \middle| q \right]_k q^{k(1+\delta)} \\ &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} q^{1+k-m}c, & q^{1+k-2n}/c \\ q^{1-m}c, & q^{1-2n}/c \end{matrix} \middle| q \right]_{m-1} \\ &\quad \times \left[\begin{matrix} q^{k+m-n}d, & q^{1+k-2n}/d \\ q^{m-n}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_{n-m} q^{k+k\delta}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it results in a multiple of the $(2n)$ th q -derivative of the following polynomial

$$x^\delta [q^{1-m}xc, q^{1-2n}x/c; q]_{m-1} [q^{m-n}xd, q^{1-2n}x/d; q]_{n-m}$$

of degree $\delta - 2 + 2n < 2n$ in x .

Because (12) is symmetric with respect to c and d , it holds also for $b = q^{m-2n}/d$ with $1 \leq m \leq n$. Therefore for $2n + 1$ distinct values of

$$b \in \{1\} \cup \{q^{m-2n}/c\}_{1 \leq m \leq n} \cup \{q^{m-2n}/d\}_{1 \leq m \leq n},$$

we have validated (12), which completes the proof. □

Finally, we record the following counterpart of (12) due to Bailey [7, Eq. 3.2] for which the reader can find different proofs in Carlitz [9, Eq. 3.4], Chu [14, §2] and Verma-Joshi [33, Eq. 3.12].

Theorem 8. For $\delta = 0$ or 1 and $n \in \mathbb{N}_0$, there holds the terminating series identity:

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} q^{-1-2n}, & b, & c, & d, & q^{-1-3n}/bcd \\ q^{-2n}/b, & q^{-2n}/c, & q^{-2n}/d, & q^{1+n}bcd \end{matrix} \middle| q; q^{1+2\delta} \right] \\ &= (-q^{1+2n})^{\delta-1} (q; q)_{2n+1} \left[\begin{matrix} qb, & qc, & qd, & qbcd \\ q, & qbc, & qbd, & qcd \end{matrix} \middle| q \right]_n \left[\begin{matrix} qbc, & qbd, & qcd \\ qb, & qc, & qd, & qbcd \end{matrix} \middle| q \right]_{2n}. \end{aligned} \quad (14)$$

Proof. Multiplying across (14) by $(q^{n+1}b; q)_n (q^{n+1}bcd; q)_n$, we may rewrite the resulting equation equivalently as

$$\begin{aligned} & \sum_{k=0}^{2n} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{[b, c, d, q^{-1-3n}/bcd; q]_k (q^{n+1}b; q)_n (q^{n+1}bcd; q)_n}{[q^{-2n}/b, q^{-2n}/c, q^{-2n}/d, q^{1+n}bcd; q]_k} q^{k(1+2\delta)} \\ &= (-q^{1+2n})^{\delta-1} \frac{(q^{n+1}; q)_{n+1} (q^{n+1}bc; q)_n (q^{n+1}bd; q)_n (q^{n+1}cd; q)_n}{(q^{n+1}c; q)_n (q^{n+1}d; q)_n}. \end{aligned} \quad (15)$$

Observing that

$$\begin{aligned} & \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{-2n}/b; q)_k} \frac{(q^{-1-3n}/bcd; q)_k (q^{n+1}bcd; q)_n}{(q^{n+1}bcd; q)_k} \\ &= \left(\frac{q^{-n-1}}{cd} \right)^k \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{2n+1-k}b; q)_k} \frac{(q^{2+3n-k}bcd; q)_k (q^{n+1}bcd; q)_n}{(q^{n+1}bcd; q)_k} \\ &= \left(\frac{q^{-n-1}}{cd} \right)^k \frac{(b; q)_k (b; q)_{1+2n-k}}{(b; q)_{n+1}} \frac{(bcd; q)_{2n+1} (bcd; q)_{3n+2}}{(bcd; q)_{n+k+1} (bcd; q)_{3n+2-k}} \end{aligned}$$

we assert that both sides of (15) are polynomials of degree $2n$ in b . In order to prove (14), it suffices to show that the equality holds for $2n + 1$ distinct values of b .

First, for $b = 1$ in (14), the corresponding ${}_5\phi_4$ -series reduces to $1 - q^{(2n+1)(2\delta-1)}$ because only the two extreme terms survive. It is trivial to check that the right member has the same value in this case.

Let $\mathcal{S}(b)$ be the ${}_5\phi_4$ -series displayed in (14). Then for $b = q^{-m-n}/c$ with $1 \leq m \leq n$, the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{-m-n}/c) &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} c, & d, & q^{-m-n}/c, & q^{m-2n-1}/d \\ q^{m-n}c, & q^{1-m}d, & q^{-2n}/c, & q^{-2n}/d \end{matrix} \middle| q \right]_k q^{k+2k\delta} \\ &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[\begin{matrix} q^{k+m-n}c, & q^{k-2n}/c \\ q^{m-n}c, & q^{-2n}/c \end{matrix} \middle| q \right]_{n-m} \\ &\quad \times \left[\begin{matrix} q^{1+k-m}d, & q^{k-2n}/d \\ q^{1-m}d, & q^{-2n}/d \end{matrix} \middle| q \right]_{m-1} q^{k+2k\delta}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for $1 \leq m \leq n$ because it results in a multiple of the $(2n + 1)$ th q -derivative of the polynomial

$$x^{2\delta} [q^{m-n}xc, q^{-2n}x/c; q]_{n-m} [q^{1-m}xd, q^{-2n}x/d; q]_{m-1}$$

of degree $2\delta - 2 + 2n < 2n + 1$ in x .

Analogously, (14) is valid also for $b = q^{-m-n}/d$ with $1 \leq m \leq n$ for its symmetry with respect to c and d . In conclusion, we have shown (14) for $2n + 1$ distinct values of $b \in \{1\} \cup \{q^{-m-n}/c\}_{1 \leq m \leq n} \cup \{q^{-m-n}/d\}_{1 \leq m \leq n}$, which completes the proof. \square

6 Gasper's q -Karlsson-Minton Formula

Finally, we examine the q -analogue of Gasper [22] (see also Chu [13]) for a classical hypergeometric sum due to Minton [29] and subsequently extended by Karlsson [27].

Theorem 9. For nonnegative integers m_i and n with $n \geq \sum_{i=1}^{\ell} m_i$, we have

$${}_{\ell+2}\phi_{\ell+1} \left[\begin{matrix} q^{-n}, & \lambda, & \{q^{m_i}a_i\}_{i=1}^{\ell} \\ q\lambda, & \{a_i\}_{i=1}^{\ell} \end{matrix} \middle| q; q \right] = \lambda^n \frac{(q; q)_n}{(q\lambda; q)_n} \prod_{i=1}^{\ell} \frac{(a_i/\lambda; q)_{m_i}}{(a_i; q)_{m_i}}. \quad (16)$$

Its nonterminating form and extensions can be found in Gasper [22] and Chu [13], [15]. However, we believe that the proof given here is the simplest.

Proof. According to the relation

$$\frac{(q^{m_i}a_i; q)_k}{(a_i; q)_k} = \frac{(q^k a_i; q)_{m_i}}{(a_i; q)_{m_i}}$$

we may express (16) equivalently as the equality

$$\sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{\prod_{i=1}^{\ell} (q^k a_i; q)_{m_i}}{1 - q^k \lambda} = \lambda^n \frac{(q; q)_n}{(\lambda; q)_{n+1}} \prod_{i=1}^{\ell} (a_i/\lambda; q)_{m_i}. \tag{17}$$

Writing the last sum in terms of q -derivatives (1) and then evaluating it, by (3), as

$$\mathcal{D}^n \frac{\prod_{i=1}^{\ell} (a_i x; q)_{m_i}}{1 - \lambda x} \Big|_{x=1} = \lambda^n \frac{(q; q)_n}{(\lambda; q)_{n+1}} \prod_{i=1}^{\ell} (a_i/\lambda; q)_{m_i},$$

we confirm (17) and so Gasper’s summation formula (16). □

7 Concluding comments

It should be pointed out that the approach presented in this paper works only for some q -series identities. For example, we have failed to verify the following q -Whipple formula due to Andrews [2, Theorem 2] (cf. Chu [19, Corollary 10] and Verma-Jain [32, Eq. 1.2]):

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{1+n}, \sqrt{c}, -\sqrt{c} \\ -q, e, qc/e \end{matrix} \middle| q; q \right] = q^{\binom{n+1}{2}} \frac{(q^{-n}e; q^2)_n (q^{1-n}c/e; q^2)_n}{(e; q)_n (qc/e; q)_n},$$

even though this will evidently become a polynomial identity of degree n in c if multiplying it by the factorial $(qc/e; q)_n$.

References

- [1] W.A. Al-Salam, A. Verma: On quadratic transformations of basic series. *SIAM J. Math.* 15 (1984) 414–420.
- [2] G.E. Andrews: On q -analogues of the Watson and Whipple summations. *SIAM J. Math.* 7 (3) (1976) 332–336.
- [3] G.E. Andrews: Connection coefficient problems and partitions. *Proc. Sympos. Pure Math.* 34 (1979) 1–24.
- [4] G.E. Andrews: Pfaff’s method (II): diverse applications. *J. Comp. and Appl. Math.* 68 (1–2) (1996) 15–23.
- [5] W.N. Bailey: *Generalized Hypergeometric Series*. Cambridge University Press, Cambridge (1935).
- [6] W.N. Bailey: A note on certain q -identities. *Quart. J. Math. (Oxford)* 12 (1941) 173–175.
- [7] W.N. Bailey: On the analogue of Dixon’s theorem for bilateral basic hypergeometric series. *Quart. J. Math. (Oxford)* 1 (1950) 318–320.
- [8] D.M. Bressoud: Almost poised basic hypergeometric series. *Proc. Indian Acad. Sci. Math. Sci.* 97 (1–3) (1987) 61–66.
- [9] L. Carlitz: Some formulas of F.H. Jackson. *Monatsh. Math.* 73 (1969) 193–198.
- [10] L. Carlitz: Some q -expansion formulas. *Glasnik Mat.* 8 (28) (1973) 205–213.
- [11] W. Chu: Inversion techniques and combinatorial identities: Basic hypergeometric identities. *Publ. Math. Debrecen* 44 (3–4) (1994) 301–320.

- [12] W. Chu: Inversion techniques and combinatorial identities: Strange evaluations of basic hypergeometric series. *Compos. Math.* 91 (2) (1994) 121–144.
- [13] W. Chu: Partial fractions and bilateral summations. *J. Math. Phys.* 35 (4) (1994) 2036–2042.
- [14] W. Chu: *Basic Almost-Poised Hypergeometric Series*. Mem. Amer. Math. Soc. (1998).
- [15] W. Chu: Partial-fraction expansions and well-poised bilateral series. *Acta Sci. Mat.* 64 (1998) 495–513.
- [16] W. Chu: q -Derivative operators and basic hypergeometric series. *Results Math.* 49 (1-2) (2006) 25–44.
- [17] W. Chu: Abel's Method on summation by parts and Bilateral Well-Poised ${}_3\psi_3$ -Series Identities. In: S. Elaydi et al.: *Difference Equations, Special Functions and Orthogonal Polynomials Proceedings of the International Conference (München, Germany, July 2005)*. World Scientific Publishers (2007) 752–761.
- [18] W. Chu: Divided differences and generalized Taylor series. *Forum Math.* 20 (6) (2008) 1097–1108.
- [19] W. Chu: Abel's method on summation by parts and balanced q -series identities. *Appl. Anal. Discrete Math.* 4 (2010) 54–65.
- [20] W. Chu: Elementary proofs for convolution identities of Abel and Hagen-Rothe. *Electron. J. Combin.* 17 (1) (2010) 24.
- [21] W. Chu, C. Wang: Bilateral inversions and terminating basic hypergeometric series identities. *Discrete Math.* 309 (12) (2009) 3888–3904.
- [22] G. Gasper: Summation formulas for basic hypergeometric series. *SIAM J. Math. Anal.* 12 (2) (1981) 196–200.
- [23] G. Gasper, M. Rahman: *Basic Hypergeometric Series*. Cambridge University Press (2004). 2nd Ed.
- [24] I.M. Gessel, D. Stanton: Applications of q -Lagrange inversion to basic hypergeometric series. *Trans. Amer. Math. Soc.* 277 (1983) 173–201.
- [25] H.W. Gould: Some generalizations of Vandermonde's convolution. *Amer. Math. Monthly* 63 (1) (1956) 84–91.
- [26] F.H. Jackson: Certain q -identities. *Quart. J. Math.* 12 (1941) 167–172.
- [27] P.W. Karlsson: Hypergeometric formulas with integral parameter differences. *J. Math. Phys.* 12 (2) (1971) 270–271.
- [28] Z. Liu: An expansion formula for q -series and applications. *Ramanujan J.* 6 (4) (2002) 429–447.
- [29] B.M. Minton: Generalized hypergeometric function of unit argument. *J. Math. Phys.* 11 (4) (1970) 1375–1376.
- [30] NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/17.7>
- [31] I.J. Slater: *Generalized Hypergeometric Functions*. Cambridge University Press, Cambridge (1966).
- [32] A. Verma, V.K. Jain: Some summation formulae for nonterminating basic hypergeometric series. *SIAM J. Math. Anal.* 16 (3) (1985) 647–655.

- [33] A. Verma, C.M. Joshi: Some remarks on summation of basic hypergeometric series.
Houston J. Math. 5 (2) (1979) 277–294.

Received: 25 February, 2017

Accepted for publication: 14 June, 2018

Communicated by: Karl Dilcher