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# Approach of $q$ -Derivative Operators to Terminating $q$ -Series Formulae

Xiaoyuan Wang, Wenchang Chu

**Abstract.** The  $q$ -derivative operator approach is illustrated by reviewing several typical summation formulae of terminating basic hypergeometric series.

## 1 Introduction and Motivation

The  $q$ -derivative operator is a useful tool for proving  $q$ -series identities (cf. Carlitz [10], Chu [16], [18] and Liu [28]). It is defined by

$$\mathcal{D}_x f(x) := \frac{f(x) - f(qx)}{x} \quad \text{and} \quad \mathcal{D}^n f = \mathcal{D}(\mathcal{D}^{n-1})f \quad \text{for } n = 2, 3, \dots$$

with the convention that  $\mathcal{D}_x^0 f(x) = f(x)$  for the identity operator. One can show, by means of the induction principle, the following explicit formula

$$\mathcal{D}_x^n f(x) = x^{-n} \sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} f(q^k x), \quad (1)$$

where the  $q$ -shifted factorial of  $x$  is given by  $(x; q)_0 \equiv 1$  and

$$(x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \quad \text{for } n = 1, 2, \dots,$$

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with product and quotient forms being abbreviated, respectively, to

$$[a, b, \dots, c; q]_n = (a; q)_n (b; q)_n \cdots (c; q)_n,$$

$$\left[ \begin{matrix} a, b, \dots, c \\ A, B, \dots, C \end{matrix} \middle| q \right]_n = \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(A; q)_n (B; q)_n \cdots (C; q)_n}.$$

According to (1), we can verify, by means of the  $q$ -binomial theorem and the  $q$ -Chu-Vandermonde formula, that for a monic polynomial  $P_m(x)$  of degree  $m \leq n$ , the following evaluation formulae for higher  $q$ -derivatives hold:

$$\mathcal{D}_x^n P_m(x) = \chi(m = n)(q; q)_n, \quad (2)$$

$$\mathcal{D}^n \frac{P_m(x)}{1 - \lambda x} = \frac{\lambda^n (q; q)_n}{(\lambda; q)_{n+1}} P_m(1/\lambda), \quad (3)$$

where  $\chi$  denotes the logical function with  $\chi(\text{true}) = 1$  and  $\chi(\text{false}) = 0$  otherwise.

The objective of this paper is to review several typical summation formulae of terminating basic hypergeometric series by means of the  $q$ -derivative operator. The approach will consist of the following three steps:

- First, for a given a  $q$ -series identity, identifying a parameter  $x$  as a variable and expressing the  $q$ -sum in terms of the higher  $q$ -derivatives displayed in (1).
- Then, evaluating the  $q$ -sum for particular values of  $x$  with the help of  $q$ -derivative properties (2) and/or (3).
- Finally, confirming the  $q$ -series identity via the fundamental theorem of algebra, i.e., “two polynomials of degrees  $\leq n$  are identical if they agree at  $n + 1$  distinct points”.

Throughout the paper, the basic hypergeometric series (cf. Bailey [5] and Gasper-Rahman [23]) is defined by

$${}_{1+\ell}\phi_{\ell} \left[ \begin{matrix} a_0, a_1, \dots, a_{\ell} \\ b_1, \dots, b_{\ell} \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} z^n \left[ \begin{matrix} a_0, a_1, \dots, a_{\ell} \\ q, b_1, \dots, b_{\ell} \end{matrix} \middle| q \right]_n,$$

which becomes terminating if one of the numerator parameters  $\{a_i\}_{0 \leq i \leq \ell}$  results in  $q^{-m}$  with  $m$  being a nonnegative integer.

## 2 The $q$ -Pfaff-Saalschütz theorem

As a warm-up, we start with the following fundamental formula; see [5, Chapter 8] and [31, §3.3].

**Theorem 1 (The  $q$ -Pfaff-Saalschütz theorem).**

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix} \middle| q; q \right] = \left[ \begin{matrix} c/a, c/b \\ c, c/ab \end{matrix} \middle| q \right]_n = a^n \left[ \begin{matrix} c/a, q^{1-n}b/c \\ c, q^{1-n}ab/c \end{matrix} \middle| q \right]_n. \quad (4)$$

*Proof.* Multiplying across this equation by  $(q^{1-n}ab/c; q)_n$ , we see that both sides are polynomials of degree  $n$  in  $b$ . To prove the identity, it suffices to show that the equality holds for  $n + 1$  distinct values of  $b$ . First it is trivial to see that both sides are equal to 1 for  $b = 1$ . Then for  $b = q^{m-1}c$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be written as the  $q$ -binomial sum

$${}_3\phi_2 \left[ \begin{matrix} q^{-n}, a, q^{m-1}c \\ c, q^{m-n}a \end{matrix} \middle| q; q \right] = \sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(q^{m-n+k}a; q)_{n-m} (q^k c; q)_{m-1}}{(q^{m-n}a; q)_{n-m} (c; q)_{m-1}}.$$

In view of (1) and (2), the last sum vanishes for  $1 \leq m \leq n$  because it results in a multiple of the  $n$ th  $q$ -derivative of the polynomial  $(cx; q)_{m-1} (q^{m-n}ax; q)_{n-m}$  of degree  $n - 1$  in  $x$ . Therefore we have validated the equality for  $n + 1$  distinct values of  $b$  and completed proof of (4).  $\square$

### 3 The $q$ -Watson Formula

In this section we give a proof of an identity that was first found by Andrews [2, Theorem 1] in 1976 as a terminating  $q$ -series analogue of the Watson formula; see also Gasper-Rahman [23, II-17].

**Theorem 2.** *Let  $b = q^{-\delta-2n}$  with  $\delta = 0$  or  $1$  and  $n \in \mathbb{N}_0$ . Then the following terminating series identity holds:*

$${}_4\phi_3 \left[ \begin{matrix} a, b, \sqrt{c}, -\sqrt{c} \\ c, \sqrt{qab}, -\sqrt{qab} \end{matrix} \middle| q; q \right] = (1 - \delta) \left[ \begin{matrix} q, qc/a \\ q/a, qc \end{matrix} \middle| q^2 \right]_n. \tag{5}$$

For different proofs, see Chu [19, Corollary 7] and Verma-Jain [32, Eq. 1.1].

*Proof.* Multiplying across the equality (5) by  $(qc; q^2)_n$ , we can rewrite the resulting equation equivalently as

$$\sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(a; q)_k (c; q^2)_k (qc; q^2)_n}{(c; q)_k (q^{1-\delta-2n}a; q^2)_k} q^k = (1 - \delta) \frac{(q; q^2)_n (qc/a; q^2)_n}{(q/a; q^2)_n}. \tag{6}$$

According to the relation

$$\frac{(c; q^2)_k (qc; q^2)_n}{(c; q)_k} = \begin{cases} \frac{(qc; q^2)_n (c; q)_{2k}}{(qc; q^2)_k (c; q)_k}, & k \leq n; \\ \frac{(c; q^2)_k (c; q)_{2n}}{(c; q^2)_n (c; q)_k}, & k > n; \end{cases}$$

both sides of (6) are polynomials of degree  $\leq n$  in  $c$ . In order to prove (5), it suffices to show that the equality holds for  $n + 1$  distinct values of  $c$ .

Let  $\mathcal{S}(c)$  be the  ${}_4\phi_3$ -series displayed in (5). Then for  $c = q^{1-\delta-2m}a$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be restated as the following sum:

$$\begin{aligned} \mathcal{S}(q^{1-\delta-2m}a) &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(a; q)_k (q^{1-\delta-2m}a; q^2)_k}{(q^{1-\delta-2m}a; q)_k (q^{1-\delta-2n}a; q^2)_k} q^k \\ &= \sum_{k=0}^{\delta+2n} \frac{(q^{-\delta-2n}; q)_k}{(q; q)_k} \frac{(q^{1-\delta-2m+k}a; q)_{\delta-1+2m} (q^{1-\delta-2n+2k}a; q^2)_{n-m}}{(q^{1-\delta-2m}a; q)_{\delta-1+2m} (q^{1-\delta-2n}a; q^2)_{n-m}} q^k. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for  $1 - \delta \leq m \leq n$  because it results in a multiple of the  $(\delta + 2n)$ th  $q$ -derivatives of the polynomial

$$(q^{1-\delta-2m}ax; q)_{\delta-1+2m}(q^{1-\delta-2n}ax^2; q^2)_{n-m}$$

of degree  $\delta - 1 + 2n$  in  $x$ . Therefore, (5) is confirmed when  $\delta = 1$  because it holds for  $n + 1$  distinct values of  $c \in \{q^{-2m}a\}_{0 \leq m \leq n}$ .

When  $\delta = 0$  and  $m = 0$ , reformulating the last sum and then evaluating it by (3),

$$\begin{aligned} \sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(q^{1-2n+2k}a; q^2)_n}{(q^{1-2n}a; q^2)_n} \frac{1-a}{1-q^ka} &= a^{2n} \frac{(q^{1-2n}/a; q^2)_n}{(q^{1-2n}a; q^2)_n} \frac{(q; q)_{2n}}{(qa; q)_{2n}} \\ &= \left[ \begin{matrix} q, & q^2 \\ q/a, & q^2a \end{matrix} \middle| q^2 \right]_n \end{aligned}$$

which coincides with the right member of (5) under the conditions  $\delta = 0$  and  $c = qa$ . Hence, we have validated (5) also when  $\delta = 0$  for  $n + 1$  distinct values of  $c \in \{q^{1-2m}a\}_{0 \leq m \leq n}$ . This completes the proof of the theorem.  $\square$

## 4 Two Balanced ${}_4\phi_3$ -Series

Unlike the  $q$ -Pfaff-Saalschütz theorem, there exist two summation formulae of balanced  ${}_4\phi_3$ -series with one less free parameter. They are, in fact, the  $q$ -analogues of the particular case  $b = 2$  of the following well-known Hagen-Rothe convolution identity (cf. Chu [20] and Gould [25]):

$$\sum_{k=0}^n \frac{a}{a+bk} \binom{a+bk}{k} \binom{c-bk}{n-k} = \binom{a+c}{n}.$$

The first one is due to Al-Salam and Verma [1]; see also Andrews [4, Eq. 7.6], Chu [11, Eq. 5.3b] and [30, Eq. 17.7.12].

### Theorem 3.

$${}_4\phi_3 \left[ \begin{matrix} q^{-2n}, a, qa, q^{2n}b^2 \\ b, qb, q^2a^2 \end{matrix} \middle| q^2; q^2 \right] = a^n \left[ \begin{matrix} -q, b/a \\ -qa, b \end{matrix} \middle| q \right]_n. \quad (7)$$

*Proof.* Multiplying both sides of (7) by  $(-qa; q)_n$  and observing that

$$\frac{(a; q)_{2k}(-qa; q)_n}{(q^2a^2; q^2)_k} = \frac{(a; q)_{2k}(-q^{k+1}a; q)_{n-k}}{(qa; q)_k},$$

we infer that the resulting equation is a polynomial identity of degree  $n$  in  $a$ . In order to prove (7), we need only to validate it for  $n + 1$  distinct values of  $a$ . First of all, (7) is obviously valid for  $a = 1$ . Then denote by  $\mathcal{S}(a)$  the  ${}_4\phi_3$ -series in (7). For  $a = q^{m-1}b$  with  $1 \leq m \leq n$ , the right member of (7) is equal to zero. The

corresponding left member can be reformulated as

$$\begin{aligned} S(q^{m-1}b) &= \sum_{k=0}^n \frac{(q^{-2n}; q^2)_k}{(q^2; q^2)_k} \frac{(q^{m-1}b; q)_{2k}}{(b; q)_{2k}} \frac{(q^{2n}b^2; q^2)_k}{(q^{2m}b^2; q^2)_k} q^{2k} \\ &= \sum_{k=0}^n \frac{(q^{-2n}; q^2)_k}{(q^2; q^2)_k} \frac{(q^{2k}b; q)_{m-1}}{(b; q)_{m-1}} \frac{(q^{2m+2k}b^2; q^2)_{n-m}}{(q^{2m}b^2; q^2)_{n-m}} q^{2k}, \end{aligned}$$

which vanishes because in base  $q^2$ , it results in the  $n$ th  $q$ -derivative of the polynomial  $(bx; q)_{m-1}(q^{2m}b^2x; q^2)_{n-m}$  of degree  $n - 1$ . This confirms (7).  $\square$

The second balanced  ${}_4\phi_3$ -series identity is due to Andrews [3, Eq. 4.3] & [4, Eq. 7.7]; see also Gessel-Stanton [24, Eq. 4.22], Chu [12, Eq. 4.3d] and [30, Eq. 17.7.13].

**Theorem 4 (Variant of Theorem 3).**

$${}_4\phi_3 \left[ \begin{matrix} q^{-2n}, a, qa, q^{2n-2}b^2 \\ b, qb, a^2 \end{matrix} \middle| q^2; q^2 \right] = a^n \left[ \begin{matrix} -q, b/a \\ -a, b \end{matrix} \middle| q \right]_n \frac{1 - q^{n-1}b}{1 - q^{2n-1}b}.$$

In exactly the same manner, this identity can be proved after the equation having been multiplied by  $(-a; q)_n$  across.

**5 Four Terminating Well-Poised Series**

This section will be devoted to four terminating well-poised series identities of Dixon’s type. The first one is essentially due to Jackson [26, Eq. 2]. See Bailey [6, Eq. 2], Bressoud [8, Eq. 2]), Carlitz [9, Eq. 1.2], Chu [21], Verma-Joshi [33, Eq. 3.10] for different proofs, and Bailey [7, Eqs. 2.2 and 1.2], Chu [17], Verma-Jain [32, Eq. 5.5] for the nonterminating form.

**Theorem 5.** For  $\delta = 0$  or  $1$  and  $n \in \mathbb{N}_0$ , there holds the terminating series identity:

$$\begin{aligned} {}_3\phi_2 \left[ \begin{matrix} q^{-2n}, b, d \\ q^{1-2n}/b, q^{1-2n}/d \end{matrix} \middle| q; q^{1+\delta-n}/bd \right] \\ = q^{n(\delta-1)} \left[ \begin{matrix} b, d \\ q, bd \end{matrix} \middle| q \right]_n \left[ \begin{matrix} q, bd \\ b, d \end{matrix} \middle| q \right]_{2n}. \end{aligned} \tag{8}$$

*Proof.* Multiplying across (8) by  $(q^n b; q)_n$ , we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(b; q)_k (q^n b; q)_n (d; q)_k}{(q^{1-2n}/b; q)_k (q^{1-2n}/d; q)_k} \left( \frac{q^{1+\delta-n}}{bd} \right)^k = \frac{(q^{n+1}; q)_n (q^n bd; q)_n}{q^{n(1-\delta)} (q^n d; q)_n}. \tag{9}$$

Observing the relation

$$\begin{aligned} \frac{(b; q)_k (q^n b; q)_n}{b^k (q^{1-2n}/b; q)_k} &= (-1)^k q^{2nk - \binom{k+1}{2}} \frac{(b; q)_k (q^n b; q)_n}{(q^{2n-k} b; q)_k} \\ &= (-1)^k q^{2nk - \binom{k+1}{2}} \frac{(b; q)_k (b; q)_{2n-k}}{(b; q)_n}, \end{aligned}$$

we assert that (9) is a polynomial identity of degree  $n$  in  $b$ . In order to prove (8), it suffices to show that the equality holds for  $n + 1$  distinct values of  $b$ .

Let  $\mathcal{S}(b)$  be the  ${}_3\phi_2$ -series displayed in (8). Then for  $b = q^{m-2n}/d$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{m-2n}/d) &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} d, & q^{m-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_k q^{k(1+\delta-m+n)} \\ &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} q^{1+k-m}d, & q^{1+k-2n}/d \\ q^{1-m}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_{m-1} q^{k(1+\delta-m+n)}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for  $1 \leq m \leq n$  because it results in a multiple of the  $(2n)$ th  $q$ -derivative of the polynomial

$$(q^{1-m}xd; q)_{m-1}(q^{1-2n}x/d; q)_{m-1}x^{\delta-m+n}$$

of degree  $\delta - 2 + m + n < 2n$  in  $x$ .

When  $m = 0$ , the last sum can be reformulated by partial fractions and then evaluated by (3) as

$$\begin{aligned} &\sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{(1-d)(1-q^{-2n}/d)}{(1-q^k d)(1-q^{k-2n}/d)} q^{k(\delta+n)} \\ &= \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \sum_{k=0}^{2n} q^k \frac{(q^{-2n}; q)_k}{(q; q)_k} \left\{ \frac{q^{k(\delta+n)}}{1-q^{k-2n}/d} - \frac{q^{k(\delta+n)+2n}d^2}{1-q^k d} \right\} \\ &= \frac{(1-d)(1-q^{-2n}/d)}{1-q^{2n}d^2} \left\{ \frac{(q; q)_{2n}(q^{2n}d)^{\delta-n}}{(q^{-2n}/d; q)_{2n+1}} - \frac{(q; q)_{2n}q^{2n}d^{n-\delta+2}}{(d; q)_{2n+1}} \right\}. \end{aligned}$$

The above expression can be easily simplified to

$$(q^{\delta-1}d)^n \frac{(q; q)_{2n}}{(d; q)_{2n}} \frac{1-d}{1-q^n d},$$

which agrees with the right member of (8) specified by  $b = q^{-2n}/d$ . Therefore for  $n + 1$  distinct values of  $b \in \{q^{m-2n}/d\}_{0 \leq m \leq n}$ , we have validated (8), which completes the proof.  $\square$

The next formula serves as a counterpart of (8) whose nonterminating version was found by Bailey [7, Eq. 2.3]. For different proofs, the reader may refer to Carlitz [9, Eq. 2.12], Chu [17], Chu-Wang [21] and Verma-Joshi [33, Eq. 3.13].

**Theorem 6.** For  $\delta = 0$  or 1 and  $n \in \mathbb{N}_0$ , there holds the terminating series identity:

$$\begin{aligned} &{}_3\phi_2 \left[ \begin{matrix} q^{-1-2n}, & b, & d \\ q^{-2n}/b, & q^{-2n}/d \end{matrix} \middle| q; q^{2\delta-n}/bd \right] \\ &= \left\{ 1 - q^{(2n+1)(2\delta-1)} \right\} \left[ \begin{matrix} qb, & qd \\ q, & qbd \end{matrix} \middle| q \right]_n \left[ \begin{matrix} q, & qbd \\ qb, & qd \end{matrix} \middle| q \right]_{2n}. \quad (10) \end{aligned}$$

*Proof.* Multiplying across (10) by  $(q^{n+1}b; q)_n$ , we may rewrite the resulting equation equivalently as

$$\sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{(b; q)_k (q^{n+1}b; q)_n (d; q)_k}{(q^{-2n}/b; q)_k (q^{-2n}/d; q)_k} \left( \frac{q^{2\delta-n}}{bd} \right)^k = \left( 1 - q^{(2n+1)(2\delta-1)} \right) \frac{(q^{n+1}; q)_n (q^{n+1}bd; q)_n}{(q^{n+1}d; q)_n}. \tag{11}$$

Observing that

$$\begin{aligned} \frac{(b; q)_k (q^{n+1}b; q)_n}{b^k (q^{-2n}/b; q)_k} &= (-1)^k q^{2nk - \binom{k}{2}} \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{1+2n-k}b; q)_k} \\ &= (-1)^k q^{2nk - \binom{k}{2}} \frac{(b; q)_k (b; q)_{1+2n-k}}{(b; q)_{n+1}}, \end{aligned}$$

we assert that both sides of (11) are polynomials of degree  $n$  in  $b$ . In order to prove (10), it suffices to show that the equality holds for  $n + 1$  distinct values of  $b$ .

Let  $\mathcal{S}(b)$  be the  ${}_3\phi_2$ -series displayed in (10). Then for  $b = q^{-m-n}/d$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{-m-n}/d) &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} d, & q^{-m-n}/d \\ q^{m-n}d, & q^{-2n}/d \end{matrix} \middle| q \right]_k q^{k(2\delta+m)} \\ &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} q^{k+m-n}d, & q^{k-2n}/d \\ q^{m-n}d, & q^{-2n}/d \end{matrix} \middle| q \right]_{n-m} q^{k(2\delta+m)}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for  $1 \leq m \leq n$  because it is a multiple of the  $(2n + 1)$ th  $q$ -derivative of the polynomial

$$(q^{m-n}xd; q)_{n-m} (q^{-2n}x/d; q)_{n-m} x^{2\delta-1+m}$$

of degree  $2n + 2\delta - m - 1 < 2n + 1$  in  $x$ .

When  $m = n + 1$ , we can rewrite the last sum by partial fractions and then evaluate it by (3) as

$$\begin{aligned} &\sum_{k=0}^{2n+1} q^k \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{(1-d)(1-q^{-1-2n}/d)}{(1-q^k d)(1-q^{k-1-2n}/d)} q^{k(2\delta+n)} \\ &= \frac{(1-d)(1-q^{-2n-1}/d)}{1-q^{1+2n}d^2} \\ &\quad \times \sum_{k=0}^{2n+1} q^k \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left\{ \frac{q^{k(2\delta+n)}}{1-q^{k-1-2n}/d} - \frac{q^{k(2\delta+n)+1+2n}d^2}{1-q^k d} \right\} \\ &= \frac{(1-d)(1-q^{-1-2n}/d)}{1-q^{1+2n}d^2} \\ &\quad \times \left\{ \frac{(q; q)_{2n+1} (q^{1+2n}d)^{2\delta-1-n}}{(q^{-1-2n}/d; q)_{2n+2}} - \frac{(q; q)_{2n+1} q^{1+2n} d^{3+n-2\delta}}{(d; q)_{2n+2}} \right\}. \end{aligned}$$



The above expression can be further simplified to

$$d^n \frac{(q; q)_{2n}}{(d; q)_{2n}} \frac{1-d}{1-q^{1+2n}d} \left\{ 1 - q^{(2n+1)(2\delta-1)} \right\},$$

which agrees with the right member of (10) specified by  $b = q^{-1-2n}/d$ . Therefore for  $n+1$  distinct values of  $b \in \{q^{m-2n}/d\}_{1 \leq m \leq n+1}$ , we have validated (10), which completes the proof.  $\square$

There is also the following more general well-poised  ${}_5\phi_4$ -series identity discovered by Jackson [26, Eq. 1], that different proofs can be found in Bailey [7, Eq. 3.1], Bressoud [8, Eq. 1], Chu [14, §2] and Verma-Joshi [33, Eq. 3.8].

**Theorem 7.** For  $\delta = 0$  or 1 and  $n \in \mathbb{N}_0$ , there holds the terminating series identity:

$$\begin{aligned} {}_5\phi_4 \left[ \begin{matrix} q^{-2n}, & b, & c, & d, & q^{1-3n}/bcd \\ q^{1-2n}/b, & q^{1-2n}/c, & q^{1-2n}/d, & q^n bcd \end{matrix} \middle| q; q^{1+\delta} \right] \\ = q^{n(\delta-1)} \left[ \begin{matrix} b, & c, & d, & bcd \\ q, & bc, & bd, & cd \end{matrix} \middle| q \right]_n \left[ \begin{matrix} q, & bc, & bd, & cd \\ b, & c, & d, & bcd \end{matrix} \middle| q \right]_{2n}. \end{aligned} \quad (12)$$

*Proof.* Multiplying across (12) by  $(q^n b; q)_n (q^n bcd; q)_n$ , we may rewrite the resulting equation equivalently as

$$\begin{aligned} \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \frac{[b, c, d, q^{1-3n}/bcd; q]_k (q^n b; q)_n (q^n bcd; q)_n}{[q^{1-2n}/b, q^{1-2n}/c, q^{1-2n}/d, q^n bcd; q]_k} q^{k(1+\delta)} \\ = q^{n(\delta-1)} \frac{(q^{n+1}; q)_n (q^n bc; q)_n (q^n bd; q)_n (q^n cd; q)_n}{(q^n c; q)_n (q^n d; q)_n}. \end{aligned} \quad (13)$$

According to the relation

$$\begin{aligned} \frac{(b; q)_k (q^n b; q)_n}{(q^{1-2n}/b; q)_k} \frac{(q^{1-3n}/bcd; q)_k (q^n bcd; q)_n}{(q^n bcd; q)_k} \\ = \left( \frac{q^{-n}}{cd} \right)^k \frac{(b; q)_k (q^n b; q)_n}{(q^{2n-k} b; q)_k} \frac{(q^{3n-k} bcd; q)_k (q^n bcd; q)_n}{(q^n bcd; q)_k} \\ = \left( \frac{q^{-n}}{cd} \right)^k \frac{(b; q)_k (b; q)_{2n-k}}{(b; q)_n} \frac{(bcd; q)_{2n} (bcd; q)_{3n}}{(bcd; q)_{n+k} (bcd; q)_{3n-k}}, \end{aligned}$$

both sides of (13) become polynomials of degree  $2n$  in  $b$ . In order to prove (12), it suffices to show that the equality holds for  $2n+1$  distinct values of  $b$ .

First, for  $b = 1$  in (12), the  ${}_5\phi_4$ -series becomes  $1 + q^{n(2\delta-1)}$  because only the two extreme terms survive. This coincides with the corresponding right member.

Let  $\mathcal{S}(b)$  be the  ${}_5\phi_4$ -series displayed in (12). Then for  $b = q^{m-2n}/c$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be

written as the expression

$$\begin{aligned} \mathcal{S}(q^{m-2n}/c) &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} c, & d, & q^{m-2n}/c, & q^{1-m-n}/d \\ q^{1-m}c, & q^{m-n}d, & q^{1-2n}/c, & q^{1-2n}/d \end{matrix} \middle| q \right]_k q^{k(1+\delta)} \\ &= \sum_{k=0}^{2n} \frac{(q^{-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} q^{1+k-m}c, & q^{1+k-2n}/c \\ q^{1-m}c, & q^{1-2n}/c \end{matrix} \middle| q \right]_{m-1} \\ &\quad \times \left[ \begin{matrix} q^{k+m-n}d, & q^{1+k-2n}/d \\ q^{m-n}d, & q^{1-2n}/d \end{matrix} \middle| q \right]_{n-m} q^{k+k\delta}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for  $1 \leq m \leq n$  because it results in a multiple of the  $(2n)$ th  $q$ -derivative of the following polynomial

$$x^\delta [q^{1-m}xc, q^{1-2n}x/c; q]_{m-1} [q^{m-n}xd, q^{1-2n}x/d; q]_{n-m}$$

of degree  $\delta - 2 + 2n < 2n$  in  $x$ .

Because (12) is symmetric with respect to  $c$  and  $d$ , it holds also for  $b = q^{m-2n}/d$  with  $1 \leq m \leq n$ . Therefore for  $2n + 1$  distinct values of

$$b \in \{1\} \cup \{q^{m-2n}/c\}_{1 \leq m \leq n} \cup \{q^{m-2n}/d\}_{1 \leq m \leq n},$$

we have validated (12), which completes the proof. □

Finally, we record the following counterpart of (12) due to Bailey [7, Eq. 3.2] for which the reader can find different proofs in Carlitz [9, Eq. 3.4], Chu [14, §2] and Verma-Joshi [33, Eq. 3.12].

**Theorem 8.** For  $\delta = 0$  or 1 and  $n \in \mathbb{N}_0$ , there holds the terminating series identity:

$$\begin{aligned} & {}_5\phi_4 \left[ \begin{matrix} q^{-1-2n}, & b, & c, & d, & q^{-1-3n}/bcd \\ q^{-2n}/b, & q^{-2n}/c, & q^{-2n}/d, & q^{1+n}bcd \end{matrix} \middle| q; q^{1+2\delta} \right] \\ &= (-q^{1+2n})^{\delta-1} (q; q)_{2n+1} \left[ \begin{matrix} qb, & qc, & qd, & qbcd \\ q, & qbc, & qbd, & qcd \end{matrix} \middle| q \right]_n \left[ \begin{matrix} qbc, & qbd, & qcd \\ qb, & qc, & qd, & qbcd \end{matrix} \middle| q \right]_{2n}. \end{aligned} \quad (14)$$

*Proof.* Multiplying across (14) by  $(q^{n+1}b; q)_n (q^{n+1}bcd; q)_n$ , we may rewrite the resulting equation equivalently as

$$\begin{aligned} & \sum_{k=0}^{2n} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \frac{[b, c, d, q^{-1-3n}/bcd; q]_k (q^{n+1}b; q)_n (q^{n+1}bcd; q)_n}{[q^{-2n}/b, q^{-2n}/c, q^{-2n}/d, q^{1+n}bcd; q]_k} q^{k(1+2\delta)} \\ &= (-q^{1+2n})^{\delta-1} \frac{(q^{n+1}; q)_{n+1} (q^{n+1}bc; q)_n (q^{n+1}bd; q)_n (q^{n+1}cd; q)_n}{(q^{n+1}c; q)_n (q^{n+1}d; q)_n}. \end{aligned} \quad (15)$$

Observing that

$$\begin{aligned} & \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{-2n}/b; q)_k} \frac{(q^{-1-3n}/bcd; q)_k (q^{n+1}bcd; q)_n}{(q^{n+1}bcd; q)_k} \\ &= \left( \frac{q^{-n-1}}{cd} \right)^k \frac{(b; q)_k (q^{n+1}b; q)_n}{(q^{2n+1-k}b; q)_k} \frac{(q^{2+3n-k}bcd; q)_k (q^{n+1}bcd; q)_n}{(q^{n+1}bcd; q)_k} \\ &= \left( \frac{q^{-n-1}}{cd} \right)^k \frac{(b; q)_k (b; q)_{1+2n-k}}{(b; q)_{n+1}} \frac{(bcd; q)_{2n+1} (bcd; q)_{3n+2}}{(bcd; q)_{n+k+1} (bcd; q)_{3n+2-k}} \end{aligned}$$

we assert that both sides of (15) are polynomials of degree  $2n$  in  $b$ . In order to prove (14), it suffices to show that the equality holds for  $2n + 1$  distinct values of  $b$ .

First, for  $b = 1$  in (14), the corresponding  ${}_5\phi_4$ -series reduces to  $1 - q^{(2n+1)(2\delta-1)}$  because only the two extreme terms survive. It is trivial to check that the right member has the same value in this case.

Let  $\mathcal{S}(b)$  be the  ${}_5\phi_4$ -series displayed in (14). Then for  $b = q^{-m-n}/c$  with  $1 \leq m \leq n$ , the right member equals zero. The corresponding left member can be written as the expression

$$\begin{aligned} \mathcal{S}(q^{-m-n}/c) &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} c, & d, & q^{-m-n}/c, & q^{m-2n-1}/d \\ q^{m-n}c, & q^{1-m}d, & q^{-2n}/c, & q^{-2n}/d \end{matrix} \middle| q \right]_k q^{k+2k\delta} \\ &= \sum_{k=0}^{2n+1} \frac{(q^{-1-2n}; q)_k}{(q; q)_k} \left[ \begin{matrix} q^{k+m-n}c, & q^{k-2n}/c \\ q^{m-n}c, & q^{-2n}/c \end{matrix} \middle| q \right]_{n-m} \\ &\quad \times \left[ \begin{matrix} q^{1+k-m}d, & q^{k-2n}/d \\ q^{1-m}d, & q^{-2n}/d \end{matrix} \middle| q \right]_{m-1} q^{k+2k\delta}. \end{aligned}$$

In view of (1) and (2), the last sum vanishes for  $1 \leq m \leq n$  because it results in a multiple of the  $(2n + 1)$ th  $q$ -derivative of the polynomial

$$x^{2\delta} [q^{m-n}xc, q^{-2n}x/c; q]_{n-m} [q^{1-m}xd, q^{-2n}x/d; q]_{m-1}$$

of degree  $2\delta - 2 + 2n < 2n + 1$  in  $x$ .

Analogously, (14) is valid also for  $b = q^{-m-n}/d$  with  $1 \leq m \leq n$  for its symmetry with respect to  $c$  and  $d$ . In conclusion, we have shown (14) for  $2n + 1$  distinct values of  $b \in \{1\} \cup \{q^{-m-n}/c\}_{1 \leq m \leq n} \cup \{q^{-m-n}/d\}_{1 \leq m \leq n}$ , which completes the proof.  $\square$

## 6 Gasper's $q$ -Karlsson-Minton Formula

Finally, we examine the  $q$ -analogue of Gasper [22] (see also Chu [13]) for a classical hypergeometric sum due to Minton [29] and subsequently extended by Karlsson [27].

**Theorem 9.** For nonnegative integers  $m_i$  and  $n$  with  $n \geq \sum_{i=1}^{\ell} m_i$ , we have

$${}_{\ell+2}\phi_{\ell+1} \left[ \begin{matrix} q^{-n}, & \lambda, & \{q^{m_i}a_i\}_{i=1}^{\ell} \\ q\lambda, & \{a_i\}_{i=1}^{\ell} \end{matrix} \middle| q; q \right] = \lambda^n \frac{(q; q)_n}{(q\lambda; q)_n} \prod_{i=1}^{\ell} \frac{(a_i/\lambda; q)_{m_i}}{(a_i; q)_{m_i}}. \quad (16)$$

Its nonterminating form and extensions can be found in Gasper [22] and Chu [13], [15]. However, we believe that the proof given here is the simplest.

*Proof.* According to the relation

$$\frac{(q^{m_i}a_i; q)_k}{(a_i; q)_k} = \frac{(q^k a_i; q)_{m_i}}{(a_i; q)_{m_i}}$$

we may express (16) equivalently as the equality

$$\sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{\prod_{i=1}^{\ell} (q^k a_i; q)_{m_i}}{1 - q^k \lambda} = \lambda^n \frac{(q; q)_n}{(\lambda; q)_{n+1}} \prod_{i=1}^{\ell} (a_i/\lambda; q)_{m_i}. \tag{17}$$

Writing the last sum in terms of  $q$ -derivatives (1) and then evaluating it, by (3), as

$$\mathcal{D}^n \frac{\prod_{i=1}^{\ell} (a_i x; q)_{m_i}}{1 - \lambda x} \Big|_{x=1} = \lambda^n \frac{(q; q)_n}{(\lambda; q)_{n+1}} \prod_{i=1}^{\ell} (a_i/\lambda; q)_{m_i},$$

we confirm (17) and so Gasper’s summation formula (16). □

### 7 Concluding comments

It should be pointed out that the approach presented in this paper works only for some  $q$ -series identities. For example, we have failed to verify the following  $q$ -Whipple formula due to Andrews [2, Theorem 2] (cf. Chu [19, Corollary 10] and Verma-Jain [32, Eq. 1.2]):

$${}_4\phi_3 \left[ \begin{matrix} q^{-n}, q^{1+n}, \sqrt{c}, -\sqrt{c} \\ -q, e, qc/e \end{matrix} \middle| q; q \right] = q^{\binom{n+1}{2}} \frac{(q^{-n}e; q^2)_n (q^{1-n}c/e; q^2)_n}{(e; q)_n (qc/e; q)_n},$$

even though this will evidently become a polynomial identity of degree  $n$  in  $c$  if multiplying it by the factorial  $(qc/e; q)_n$ .

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