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# A Study on $\phi$ -recurrence $\tau$ -curvature tensor in $(k, \mu)$ -contact metric manifolds

Gurupadavva Ingalahalli, C.S. Bagewadi

**Abstract.** In this paper we study  $\phi$ -recurrence  $\tau$ -curvature tensor in  $(k, \mu)$ -contact metric manifolds.

## 1 Introduction

In [11], S. Tanno introduced the notion of  $k$ -nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field  $\xi$  of the contact metric manifold belongs to the distribution. The contact metric manifold with  $\xi$  belonging to the  $k$ -nullity distribution is called  $N(k)$ -contact metric manifold and such a manifold is also studied by various authors. Generalizing this notion in 1995, D.E. Blair, T. Koufogiorgos and B.J. Papantoniou [2] introduced the notion of a contact metric manifold with  $\xi$  belonging to the  $(k, \mu)$ -nullity distribution, where  $k$  and  $\mu$  are real constants. In particular, if  $\mu = 0$  then the notion of  $(k, \mu)$ -nullity distribution reduces to the notion of  $k$ -nullity distribution.

In [13], M.M. Tripathi and et al. introduced the  $\tau$ -curvature tensor which consists of known curvatures like conformal, concircular, projective,  $M$ -projective,  $W_i$ -curvature tensor ( $i = 0, \dots, 9$ ) and  $W_j^*$ -curvature tensor ( $j = 0, 1$ ). Further, in [14] and [15] M.M. Tripathi and et al. studied  $\tau$ -curvature tensor in  $K$ -contact, Sasakian and semi-Riemannian manifolds. Later in [6] the authors studied some properties of  $\tau$ -curvature tensor and they obtained some interesting results.

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The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. In the context of contact geometry the notion of  $\phi$ -symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [3] with several examples. As a weaker version of local symmetry, T. Takahashi [12] introduced the notion of locally  $\phi$ -symmetry on a Sasakian manifold. Generalizing the notion of  $\phi$ -symmetry De et al. [5] introduced the notion of  $\phi$ -recurrent Sasakian manifold. In [4], the authors studied  $\phi$ -recurrent  $N(k)$ -contact metric manifolds. Motivated by all these work in this paper we study the  $\phi$ -recurrent  $\tau$ -curvature tensor in  $(k, \mu)$ -contact metric manifold.

## 2 Preliminaries

A  $(2n + 1)$ -dimensional differential manifold  $M$  is said to have an almost contact structure  $(\phi, \xi, \eta)$  if it carries a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and 1-form  $\eta$  satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \quad (1)$$

Let  $g$  be a compatible Riemannian metric with almost contact structure  $(\phi, \xi, \eta)$  such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

$$g(\phi X, Y) = -g(X, \phi Y) \quad g(X, \xi) = \eta(X), \quad (3)$$

where  $X, Y$  are vector fields defined on  $M$ . Then the structure  $(\phi, \xi, \eta, g)$  on  $M$  is said to have an almost contact metric structure and the manifold  $M$  equipped with this structure is called an almost contact metric manifold.

An almost contact metric structure  $(\phi, \xi, \eta, g)$  becomes a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields  $X, Y$  on  $M$ .

Given a contact metric manifold  $M(\phi, \xi, \eta, g)$ , we define a  $(1, 1)$  tensor field  $h$  by  $h = \frac{1}{2}\mathfrak{L}\phi$ , where  $\mathfrak{L}$  denotes the Lie differentiation. Then  $h$  is symmetric and satisfies  $h\phi = -\phi h$ . Also we have  $\text{tr}(h) = \text{tr}(\phi h) = 0$  and  $h\xi = 0$ . Moreover, if  $\nabla$  denotes the Riemannian connection on  $M$ , then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi hX. \quad (4)$$

In contact metric manifold  $M(\phi, \xi, \eta, g)$ , the  $(k, \mu)$ -nullity distribution is

$$p \rightarrow N_p(k, \mu) = \left\{ Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY] \right\}, \quad (5)$$

for all vector fields  $X, Y \in T_p M$  and  $k, \mu$  are real numbers and  $R$  is the curvature tensor. Hence, if the characteristic vector field  $\xi$  belongs to the  $(k, \mu)$ -nullity distribution, then we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (6)$$

Thus a contact metric manifold satisfying (6) is called a  $(k, \mu)$ -contact metric manifold. In particular, if  $\mu = 0$ , then the notion of  $(k, \mu)$ -nullity distribution reduces to the notion of  $k$ -nullity distribution introduced by S. Tanno [11]. In a  $(k, \mu)$ -contact metric manifold the following relations hold [2], [9]:

$$h^2 = (k - 1)\phi^2, \quad k \leq 1, \quad (7)$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)[X + hX],$$

$$\begin{aligned} (\nabla_X h)Y &= [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi \\ &\quad + \eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY, \end{aligned}$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],$$

$$\begin{aligned} \eta(R(X, Y)Z) &= k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &\quad + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)], \end{aligned}$$

$$S(X, \xi) = 2nk\eta(X), \quad (8)$$

$$\begin{aligned} S(X, Y) &= [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) \\ &\quad + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned} \quad (9)$$

$$\begin{aligned} QX &= [2(n - 1) - n\mu]X + [2(n - 1) + \mu]hX \\ &\quad + [2(1 - n) + n(2k + \mu)]\eta(X)\xi, \quad n \geq 1, \end{aligned} \quad (10)$$

$$r = 2n[2n - 2 + k - n\mu],$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \quad (11)$$

where  $S$  is the Ricci tensor of type  $(0, 2)$ ,  $Q$  is the Ricci operator, that is,  $S(X, Y) = g(QX, Y)$  and  $r$  is the scalar curvature of the manifold. From (3), it follows that

$$(\nabla_X \eta)Y = g(X + hX, \phi Y).$$

**Definition 1.** A  $(k, \mu)$ -contact metric manifold  $M$  is said to be locally  $\phi$ -symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ . This notion was introduced by T. Takahashi [12] for Sasakian manifolds.

A field that is at every point and for every direction proportional to its covariant differential is called recurrent. Based on this concept we define the following definition:

**Definition 2.** A  $(k, \mu)$ -contact metric manifold  $M$  is said to be  $\phi$ -recurrent if and only if there exists a non zero 1-form  $A$  such that

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z,$$

for all arbitrary vector fields  $X, Y, Z, W$  which are not necessarily orthogonal to  $\xi$ .

If the 1-form  $A$  vanishes identically, then the manifold is said to be a locally  $\phi$ -symmetric manifold.

**Definition 3.** A  $(k, \mu)$ -contact metric manifold  $M$  is said to be  $\phi$ - $\tau$ -recurrent if and only if there exists a non zero 1-form  $A$  such that

$$\phi^2((\nabla_W \tau)(X, Y)Z) = A(W)\tau(X, Y)Z,$$

for all arbitrary vector fields  $X, Y, Z, W$  which are not necessarily orthogonal to  $\xi$ .

The  $\tau$ -curvature tensor [13] is given by

$$\begin{aligned} \tau(X, Y)Z &= a_0R(X, Y)Z + a_1S(Y, Z)X + a_2S(X, Z)Y + a_3S(X, Y)Z \\ &\quad + a_4g(Y, Z)QX + a_5g(X, Z)QY + a_6g(X, Y)QZ \\ &\quad + a_7r[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (12)$$

where  $a_0, \dots, a_7$  are all constants on  $M$ . For different values of  $a_0, \dots, a_7$  the  $\tau$ -curvature tensor reduces to the curvature tensor  $R$ , quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor,  $M$ -projective curvature tensor,  $W_i$ -curvature tensors ( $i = 0, \dots, 9$ ),  $W_j^*$ -curvature tensors ( $j = 0, 1$ ).

### 3 $\phi$ - $\tau$ -recurrent $(k, \mu)$ -contact metric manifold

In this section, we define  $\phi$ - $\tau$ -recurrent  $(k, \mu)$ -contact metric manifold

$$\phi^2((\nabla_W \tau)(X, Y)Z) = A(W)\tau(X, Y)Z, \quad (13)$$

for all vector fields  $X, Y, Z, W$ . By using (9) and (10) in (12), we get

$$\begin{aligned} \tau(X, Y)Z &= a_0R(X, Y)Z + a_1[\alpha g(Y, Z)X + \beta g(hY, Z)X + \gamma \eta(Y)\eta(Z)X] \\ &\quad + a_2[\alpha g(X, Z)Y + \beta g(hX, Z)Y + \gamma \eta(X)\eta(Z)Y] \\ &\quad + a_3[\alpha g(X, Y)Z + \beta g(hX, Y)Z + \gamma \eta(X)\eta(Y)Z] \\ &\quad + a_4g(Y, Z)[\alpha X + \beta hX + \gamma \eta(X)\xi] \\ &\quad + a_5g(X, Z)[\alpha Y + \beta hY + \gamma \eta(Y)\xi] \\ &\quad + a_6g(X, Y)[\alpha Z + \beta hZ + \gamma \eta(Z)\xi] \\ &\quad + a_7r[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (14)$$

where  $\alpha = [2(n-1) - n\mu]$ ,  $\beta = [2(n-1) + \mu]$  and  $\gamma = [2(1-n) + n(2k + \mu)]$ .

Differentiating (14) with respect to  $W$ , we obtain

$$\begin{aligned}
(\nabla_W \tau)(X, Y)Z &= a_0(\nabla_W R)(X, Y)Z \\
&+ a_1 \left[ \beta g((\nabla_W h)Y, Z)X + \gamma \{ (\nabla_W \eta)Y \eta(Z)X + (\nabla_W \eta)Z \eta(Y)X \} \right] \\
&+ a_2 \left[ \beta g((\nabla_W h)X, Z)Y + \gamma \{ (\nabla_W \eta)X \eta(Z)Y + (\nabla_W \eta)Z \eta(X)Y \} \right] \\
&+ a_3 \left[ \beta g((\nabla_W h)X, Y)Z + \gamma \{ (\nabla_W \eta)X \eta(Y)Z + (\nabla_W \eta)Y \eta(X)Z \} \right] \\
&+ a_4 g(Y, Z) \left[ \beta (\nabla_W h)X + \gamma \{ (\nabla_W \eta)(X)\xi + \eta(X)\nabla_W \xi \} \right] \\
&+ a_5 g(X, Z) \left[ \beta (\nabla_W h)Y + \gamma \{ (\nabla_W \eta)(Y)\xi + \eta(Y)\nabla_W \xi \} \right] \\
&+ a_6 g(X, Y) \left[ \beta (\nabla_W h)Z + \gamma \{ (\nabla_W \eta)(Z)\xi + \eta(Z)\nabla_W \xi \} \right] \\
&+ a_7 (\nabla_W r)[g(Y, Z)X - g(X, Z)Y].
\end{aligned} \tag{15}$$

By virtue of (1), (13), we have

$$-(\nabla_W \tau)(X, Y)Z + \eta((\nabla_W \tau)(X, Y)Z)\xi = A(W)\tau(X, Y)Z. \tag{16}$$

By taking an inner product with  $U$ , we obtain

$$-g((\nabla_W \tau)(X, Y)Z, U) + \eta((\nabla_W \tau)(X, Y)Z)g(\xi, U) = A(W)g(\tau(X, Y)Z, U). \tag{17}$$

Let  $\{e_i : i = 1, 2, \dots, 2n + 1, \}$  be an orthonormal basis of the tangent space at any point of the manifold. Putting  $X = U = e_i$  in (17) and taking summation over  $i$ , we get

$$-g((\nabla_W \tau)(e_i, Y)Z, e_i) + \eta((\nabla_W \tau)(e_i, Y)Z)g(\xi, e_i) = A(W)g(\tau(e_i, Y)Z, e_i). \tag{18}$$

By using (15) in (18), we obtain

$$\begin{aligned}
&-a_0(\nabla_W S)(Y, Z) - [2na_1 + a_2 + a_5] \left[ \beta \left\{ [(1-k)g(W, \phi Y) + g(W, h\phi Y)]\eta(Z) \right. \right. \\
&+ \eta(Y)g(h(\phi W + \phi hW), Z) - \mu\eta(W)g(\phi hY, Z) \left. \right\} + \gamma \left\{ g(W + hW, \phi Y)\eta(Z) \right. \\
&+ \eta(Y)g(W + hW, \phi Z) \left. \right\} \left. \right] - (a_3 + a_6) \left[ \beta \left\{ [(1-k)g(W, \phi Z) + g(W, h\phi Z)]\eta(Y) \right. \right. \\
&+ \eta(Z)g(h(\phi W + \phi hW), Y) - \mu\eta(W)g(\phi hZ, Y) \left. \right\} + \gamma \left\{ g(W + hW, \phi Z)\eta(Y) \right. \\
&+ \eta(Z)g(W + hW, \phi Y) \left. \right\} \left. \right] - a_7(\nabla_W r)[2ng(Y, Z)] + a_0\eta((\nabla_W R)(\xi, Y)Z) \\
&+ a_2[\beta g(h(\phi W + \phi hW), Z)\eta(Y) + \gamma g(W + hW, \phi Z)\eta(Y)] \\
&+ a_3[\beta g(h(\phi W + \phi hW), Y)\eta(Z) + \gamma g(W + hW, \phi Y)\eta(Z)] \\
&+ a_5[\beta \{(1-k)g(W, \phi Y) + g(W, h\phi Y)\}\eta(Z) + \gamma\eta(Z)g(W + hW, \phi Y)] \\
&+ a_6[\beta \{(1-k)g(W, \phi Z) + g(W, h\phi Z)\}\eta(Y) + \gamma\eta(Y)g(W + hW, \phi Z)] \\
&+ a_7(\nabla_W r)[g(Y, Z) - \eta(Y)\eta(Z)] \\
&= A(W)[a_4 + 2na_7]rg(Y, Z) + A(W)[a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y, Z).
\end{aligned} \tag{19}$$

Putting  $Z = \xi$  in (19) and on simplification, we get

$$\begin{aligned} & -a_0(\nabla_W S)(Y, \xi) - (2na_1 + a_2 + a_6)[\beta\{(1-k)g(W, \phi Y) + g(W, h\phi Y)\} \\ & + \gamma g(W + hW, \phi Y)] - 2na_7(\nabla_W r)\eta(Y) \\ & = A(W)\eta(Y)[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r]. \end{aligned} \quad (20)$$

We know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \quad (21)$$

By using (4), (8) in (21), we obtain

$$(\nabla_W S)(Y, \xi) = S(Y, \phi W) + S(Y, \phi hW) - 2nkg(Y, \phi W) - 2nkg(Y, \phi hW). \quad (22)$$

Substituting (22) in (20), we get

$$\begin{aligned} & -a_0\{S(Y, \phi W) + S(Y, \phi hW) - 2nkg(Y, \phi W) - 2nkg(Y, \phi hW)\} \\ & - (2na_1 + a_2 + a_6)[\beta\{(1-k)g(W, \phi Y) + g(W, h\phi Y)\} + \gamma g(W + hW, \phi Y)] \\ & - 2na_7(\nabla_W r)\eta(Y) \\ & = A(W)\eta(Y)[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r]. \end{aligned} \quad (23)$$

Replacing  $Y$  by  $\phi Y$  in (23) and simplifying, we have

$$\begin{aligned} & -a_0S(Y, W) - a_0S(Y, hW) + [2a_0\beta + 2nka_0 + (2na_1 + a_2 + a_6)(\beta + \gamma)]g(hW, Y) \\ & + [2nka_0 + (2na_1 + a_2)\{\beta(1-k) + \gamma\} + a_6\gamma - (2a_0 + a_6)\beta(k-1)]g(W, Y) \\ & + [(2a_0 + a_6)\beta(k-1) - (2na_1 + a_2)\{\beta(1-k) + \gamma\} - a_6\gamma]\eta(W)\eta(Y) = 0. \end{aligned} \quad (24)$$

Replacing  $W$  by  $hW$  in (24) and by virtue of (7), (9) and on simplification, we get

$$\begin{aligned} & -a_0S(Y, hW) + a_0(k-1)S(Y, W) \\ & + (k-1)[2a_0\beta + (2na_1 + a_2 + a_6)(\beta + \gamma)]\eta(W)\eta(Y) \\ & + [2nka_0 + (2na_1 + a_2)\{\beta(1-k) + \gamma\} + a_6\gamma - (2a_0 + a_6)\beta(k-1)]g(hW, Y) \\ & - (k-1)[2a_0\beta + 2nka_0 + (2na_1 + a_2 + a_6)(\beta + \gamma)]g(W, Y) = 0. \end{aligned} \quad (25)$$

Subtracting (24) and (25) and by virtue of (9), we obtain

$$g(Y, hW) = \frac{E}{F}g(Y, W) + \frac{G}{F}\eta(Y)\eta(W), \quad (26)$$

where

$$\begin{aligned} E &= [a_0(\beta - 2nk) + (2na_1 + a_2 + a_6)\gamma] \\ F &= [a_0(2\beta - \alpha) + \beta(2na_1 + a_2)] \end{aligned}$$

and

$$G = \gamma(-a_0 + 2na_1 + a_2 + a_6).$$

By substituting (26) in (24), we get

$$S(Y, W) = \left[ \frac{NE}{a_0 F} + \frac{P}{a_0} \right] g(Y, W) + \left[ \frac{GN}{F a_0} + \frac{Q}{a_0} \right] \eta(Y)\eta(W),$$

where

$$N = [2nk - \alpha + 2\beta]a_0 + (2na_1 + a_2 + a_6)(\beta + \gamma),$$

$$P = [2nka_0 - \beta(k - 1)[a_0 + 2na_1 + a_2 + a_6] + \gamma[2na_1 + a_2 + a_6]]$$

and

$$Q = [\beta(k - 1)[a_0 + 2na_1 + a_2 + a_6] - \gamma[2na_1 + a_2 + a_6]].$$

Hence, we state the following:

**Theorem 1.** *A  $\phi$ - $\tau$ -recurrent  $(k, \mu)$ -contact metric manifold is an  $\eta$ -Einstein manifold with  $a_0 \neq 0$ .*

#### 4 $\eta$ - $\tau$ -Ricci-recurrent $(k, \mu)$ -contact metric manifold

**Definition 4.** A  $(k, \mu)$ -contact metric manifold  $M$  is said to be  $\eta$ - $\tau$ -Ricci-recurrent if it satisfies the condition

$$(\nabla_X S_\tau)(\phi Y, \phi Z) = A(X)S_\tau(\phi Y, \phi Z), \quad (27)$$

for all vector fields  $X, Y, Z$  on  $M$

From (12), we have

$$S_\tau(Y, Z) = [a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y, Z) + r[a_4 + 2na_7]g(Y, Z). \quad (28)$$

Replacing  $Y = \phi Y$  and  $Z = \phi Z$  in (28), we obtain

$$S_\tau(\phi Y, \phi Z) = [a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]S(\phi Y, \phi Z) + [a_4 + 2na_7]rg(\phi Y, \phi Z). \quad (29)$$

Differentiating (29) with respect to  $X$ , we get

$$(\nabla_X S_\tau)(\phi Y, \phi Z) = [a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6](\nabla_X S)(\phi Y, \phi Z) + [a_4 + 2na_7](\nabla_X r)g(\phi Y, \phi Z). \quad (30)$$

By using (30) and (29) in (27), we have

$$L(\nabla_X S)(\phi Y, \phi Z) + M(\nabla_X r)g(\phi Y, \phi Z) = A(X)\{LS(\phi Y, \phi Z) + Mrg(\phi Y, \phi Z)\}, \quad (31)$$

where  $L = [a_0 + (2n + 1)a_1 + a_2 + a_3 + a_5 + a_6]$  and  $M = [a_4 + 2na_7]$ .

Now, differentiating (11), we have

$$\begin{aligned} (\nabla_X S)(\phi Y, \phi Z) &= (\nabla_X S)(Y, Z) - 2nk[-\eta(Z)g(Y, \phi X) - \eta(Z)g(Y, \phi hX) \\ &\quad - \eta(Y)g(Z, \phi X) - \eta(Y)g(Z, \phi hX)] \\ &\quad - 2[2n - 2 + \mu][(1 - k)g(X, \phi Y)\eta(Z) \\ &\quad + g(X, h\phi Y)\eta(Z) + \eta(Y)g(h(\phi X + \phi hX), Z) \\ &\quad - \mu\eta(X)g(\phi hY, Z)] \\ &\quad + \eta(Y)S(X + hX, \phi Z) + \eta(Z)S(\phi Y, X + hX). \end{aligned} \quad (32)$$



Substituting (32) in (31) and on simplification, we obtain

$$\begin{aligned}
(\nabla_X S)(Y, Z) &= -2nk[\eta(Z)g(Y, \phi X) + \eta(Z)g(Y, \phi hX) + \eta(Y)g(Z, \phi X) \\
&\quad + \eta(Y)g(Z, \phi hX)] + 2[2n - 2 + \mu][(1 - k)g(X, \phi Y)\eta(Z) \\
&\quad + g(X, h\phi Y)\eta(Z) + \eta(Y)g(h(\phi X + \phi hX), Z) - \mu\eta(X)g(\phi hY, Z)] \\
&\quad - \eta(Y)S(X + hX, \phi Z) - \eta(Z)S(\phi Y, X + hX) \\
&\quad - \frac{M}{L}(\nabla_X r)g(\phi Y, \phi Z) + A(X)\{S(Y, Z) - 2nk\eta(Y)\eta(Z) \\
&\quad - 2[2n - 2 + \mu]g(hY, Z) + \frac{Mr}{L}g(\phi Y, \phi Z)\}.
\end{aligned} \tag{33}$$

Let  $\{e_i : i = 1, 2, \dots, 2n + 1\}$  be an orthonormal frame field at any point of the manifold. Then contracting  $Y$  and  $Z$  in (33), we have

$$dr(X) = A(X) \left[ r - \frac{2nkL}{(L + 2nM)} \right]. \tag{34}$$

Again, contracting over  $X$  and  $Z$  in (33), we get

$$\begin{aligned}
\frac{1}{2}dr(Y) &= \left[ -2nk \operatorname{tr}(\phi) + 2(2n - 2 + \mu) \operatorname{tr}(h\phi h) - \operatorname{tr}(Q\phi) - \operatorname{tr}(Qh\phi) + \frac{(\xi r)M}{L} \right] \eta(Y) \\
&\quad - \left[ 2nk + \frac{rM}{L} \right] A(\xi)\eta(Y) - \frac{M}{L}dr(Y) + S(Y, \rho) - 2(2n - 2 + \mu)A(hY) \\
&\quad + \frac{rM}{L}A(Y).
\end{aligned} \tag{35}$$

By using (34) in (35) and on simplification, we get

$$\begin{aligned}
A(Y) \left[ \frac{r}{2} - \frac{nk(L + 2M)}{(L + 2nM)} \right] &= \left[ -2nk \operatorname{tr}(\phi) + 2(2n - 2 + \mu) \operatorname{tr}(h\phi h) \right. \\
&\quad \left. - \operatorname{tr}(Q\phi) - \operatorname{tr}(Qh\phi) - \frac{2nk(L + M + 2nM)}{(L + 2nM)} A(\xi) \right] \eta(Y) \\
&\quad + S(Y, \rho) - 2(2n - 2 + \mu)A(hY). \tag{36}
\end{aligned}$$

Replacing  $Y = hY$  in (36) and by virtue of (9), we obtain

$$A(hY) = \frac{2(L + 2nM)}{[(r - 2\alpha)(L + 2nM) - 2nk(L + 2M)]} [\beta(k - 1)\{A(Y) - A(\xi)\eta(Y)\}], \tag{37}$$

where  $\alpha = [2(n - 1) - n\mu]$  and  $\beta = [2(n - 1) + \mu]$ .

Substituting (37) in (36), we have

$$\begin{aligned}
 A(Y) & \left[ \frac{r}{2} - \frac{nk(L+2M)}{(L+2nM)} + \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} \right] \\
 & = [-2nk \operatorname{tr}(\phi) + 2\beta \operatorname{tr}(h\phi h) - \operatorname{tr}(Q\phi) - \operatorname{tr}(Qh\phi)]\eta(Y) \\
 & + \left[ \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} - \frac{2nk(L+M+2nM)}{(L+2nM)} \right] A(\xi)\eta(Y) \\
 & \quad + S(Y, \rho). \quad (38)
 \end{aligned}$$

Putting  $Y = \xi$  in (38), we get

$$A(\xi) \left[ \frac{r}{2} - \frac{nkL}{(L+2nM)} \right] = [-2nk \operatorname{tr}(\phi) + 2\beta \operatorname{tr}(h\phi h) - \operatorname{tr}(Q\phi) - \operatorname{tr}(Qh\phi)]. \quad (39)$$

From (39) and (38), we have

$$\begin{aligned}
 S(Y, \rho) & = \left[ \frac{r}{2} - \frac{nk(L+2M)}{(L+2nM)} + \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} \right] g(Y, \rho) \\
 & + \left[ -\frac{r}{2} + \frac{nkL}{(L+2nM)} - \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} \right. \\
 & \quad \left. + \frac{2nk(L+M+2nM)}{(L+2nM)} \right] \eta(Y)\eta(\rho). \quad (40)
 \end{aligned}$$

From (40), we have

$$\begin{aligned}
 QY & = \left[ \frac{r}{2} - \frac{nk(L+2M)}{(L+2nM)} + \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} \right] Y \\
 & + \left[ -\frac{r}{2} + \frac{nkL}{(L+2nM)} - \frac{4\beta^2(L+2nM)(k-1)}{[(r-2\alpha)(L+2nM) - 2nk(L+2M)]} \right. \\
 & \quad \left. + \frac{2nk(L+M+2nM)}{(L+2nM)} \right] \eta(Y)\xi. \quad (41)
 \end{aligned}$$

Hence, we state the following:

**Theorem 2.** *If the Ricci tensor of a  $(k, \mu)$ -contact metric manifold is  $\eta$ - $\tau$ -Ricci-recurrent then its Ricci tensor along the associated vector field of the 1-form is given by (40) and the eigen value of the Ricci tensor with respect to the characteristic vector  $\xi$  is given by (41).*

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## References

- [1] D.E. Blair: *Contact metric manifolds in Riemannian geometry*. Springer-Verlag, Berlin-New-York (1976). Lecture Notes in Mathematics 509
- [2] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou: Contact metric manifolds satisfying a nullity condition. *Israel J. Math.* 91 (1995) 189–214.
- [3] E. Boeckx, P. Buecken, L. Vanhecke:  $\phi$ -symmetric contact metric spaces. *Glasgow Math. J.* 41 (1999) 409–416.
- [4] U.C. De, A.K. Gazi: On  $\phi$ -recurrent  $N(k)$ -contact metric manifolds. *Math. J. Okayama Univ.* 50 (2008) 101–112.
- [5] U.C. De, A.A. Shaikh, S. Biswas: On  $\phi$ -recurrent Sasakian manifolds. *Novi Sad J. Math.* 33 (2003) 13–48.
- [6] H.G. Nagaraja, G. Somashekhara:  $\tau$ -curvature tensor in  $(k, \mu)$ -contact metric manifolds. *Mathematica Aeterna* 2 (6) (2012) 523–532.
- [7] B.J. Papantoniou: Contact Riemannian manifolds satisfying  $R(\xi, X) \cdot R = 0$  and  $\xi \in (k, \mu)$ -nullity distribution. *Yokohama Math. J.* 40 (2) (1993) 149–161.
- [8] C.R. Premalatha, H.G. Nagaraja: On Generalized  $(k, \mu)$ -space forms. *Journal of Tensor Society* 7 (2013) 29–38.
- [9] A.A. Shaikh, K.K. Baishya: On  $(k, \mu)$ -contact metric manifolds. *Differential Geometry – Dynamical Systems* 8 (2006) 253–261.
- [10] R. Sharma, D.E. Blair: Conformal motion of contact manifolds with characteristic vector field in the  $k$ -nullity distribution. *Illinois J. Math.* 42 (1998) 673–677.
- [11] S. Tanno: Ricci curvatures of contact Riemannian manifolds. *Tohoku Math. J.* 40 (1988) 441–448.
- [12] T. Takahashi: Sasakian  $\phi$ -symmetric spaces. *Tohoku Math. J.* 29 (1977) 91–113.
- [13] M.M. Tripathi, P. Gupta:  $\tau$ -curvature tensor on a semi-Riemannian manifold. *J. Adv. Math. Stud.* 4 (1) (2011) 117–129.
- [14] M.M. Tripathi, P. Gupta: On  $\tau$ -curvature tensor in K-contact and Sasakian manifolds. *International Electronic Journal of Geometry* 4 (2011) 32–47.
- [15] M.M. Tripathi, P. Gupta:  $(N(k), \xi)$ -semi-Riemannian manifolds: Semisymmetries. *arXiv:1202.6138v[math.DG]* 28 (2012).

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