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Nonlinear $*$ -Lie higher derivations of standard operator algebras

Mohammad Ashraf, Shakir Ali, Bilal Ahmad Wani

Abstract. Let \mathcal{H} be an infinite-dimensional complex Hilbert space and \mathfrak{A} be a standard operator algebra on \mathcal{H} which is closed under the adjoint operation. It is shown that every nonlinear $*$ -Lie higher derivation $\mathcal{D} = \{\delta_n\}_{n \in \mathbb{N}}$ of \mathfrak{A} is automatically an additive higher derivation on \mathfrak{A} . Moreover, $\mathcal{D} = \{\delta_n\}_{n \in \mathbb{N}}$ is an inner $*$ -higher derivation.

1 Introduction

Let \mathfrak{A} be an algebra over a commutative ring R . Recall that an R -linear mapping $d: \mathfrak{A} \rightarrow \mathfrak{A}$ is called a derivation if $d(AB) = d(A)B + Ad(B)$ for all $A, B \in \mathfrak{A}$; in particular, d is called an inner derivation if there exists some $X \in \mathfrak{A}$ such that $d(A) = AX - XA$ for all $A \in \mathfrak{A}$. An R -linear mapping $d: \mathfrak{A} \rightarrow \mathfrak{A}$ is called a Lie derivation if $d([A, B]) = [d(A), B] + [A, d(B)]$ for all $A, B \in \mathfrak{A}$, where $[A, B] = AB - BA$ is the usual Lie product. Furthermore, without linearity/additivity assumption, if d satisfies $d([A, B]) = [d(A), B] + [A, d(B)]$ for all $A, B \in \mathfrak{A}$, then d is called a nonlinear Lie derivation. The question of characterizing Lie derivations and revealing the relationship between derivations and Lie derivations have been studied by many authors (see [1], [2], [5], [6], [7], [8], [11], [12], [15], [18]).

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Let \mathfrak{A} be an associative $*$ -algebra over the complex field \mathbb{C} . A mapping $d: \mathfrak{A} \rightarrow \mathfrak{A}$ is said to be an additive $*$ -derivation if it is an additive derivation and satisfies $d(A)^* = d(A^*)$ for all $A \in \mathfrak{A}$. Further, if $d: \mathfrak{A} \rightarrow \mathfrak{A}$ is a map (not necessarily linear) which satisfies $d([A, B]_*) = [d(A), B]_* + [A, d(B)]_*$ for all $A, B \in \mathfrak{A}$, where $[A, B]_* = AB - BA^*$, then d is known as a nonlinear $*$ -Lie derivation of \mathfrak{A} .

In [16] Yu and Zhang showed that every nonlinear $*$ -Lie derivation from a factor von Neumann algebra on an infinite-dimensional Hilbert space into itself is an additive $*$ -derivation. It is to be noted that a factor von Neumann algebra is a von Neumann algebra whose centre is trivial. In [4] Wu Jing proved that every nonlinear $*$ -Lie derivation on standard operator algebra is automatically linear. Moreover, it is an inner $*$ -derivation .

Let us recall some basic facts related to Lie higher derivations and $*$ -Lie higher derivations of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let \mathbb{N} be the set of non-negative integers and $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ be a family of linear mappings $d_n: \mathfrak{A} \rightarrow \mathfrak{A}$ such that $d_0 = \text{id}_{\mathfrak{A}}$, the identity map on \mathfrak{A} . Then \mathcal{D} is called

- (i) a higher derivation on \mathfrak{A} if for every $n \in \mathbb{N}$,

$$d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$$

for all $A, B \in \mathfrak{A}$.

- (ii) a Lie higher derivation on \mathfrak{A} if for every $n \in \mathbb{N}$,

$$d_n([A, B]) = \sum_{i+j=n} [d_i(A), d_j(B)]$$

for all $A, B \in \mathfrak{A}$.

- (iii) a $*$ -Lie higher derivation on \mathfrak{A} if for every $n \in \mathbb{N}$,

$$d_n([A, B]_*) = \sum_{i+j=n} [d_i(A), d_j(B)]_*$$

for all $A, B \in \mathfrak{A}$.

- (iv) an inner higher derivation on \mathfrak{A} if there exist two sequences $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ in \mathfrak{A} satisfying the conditions

$$X_0 = Y_0 = 1 \quad \text{and} \quad \sum_{i=0}^n X_i Y_{n-i} = \delta_{n0} = \sum_{i=0}^n Y_i X_{n-i}$$

such that $d_n(A) = \sum_{i=0}^n X_i A Y_{n-i}$, for all $A \in \mathfrak{A}$ and for every $n \in \mathbb{N}$, where δ_{n0} is the Kronecker sign.

If the linear assumption in the above definitions is dropped, then the corresponding higher derivation, Lie higher derivation and $*$ -Lie higher derivation is said to be nonlinear higher derivation, nonlinear Lie higher derivation and nonlinear $*$ -Lie higher derivation respectively. Moreover, if $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and $*$ -Lie higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive $*$ -Lie higher derivation respectively. Note that d_1 is always a derivation, Lie derivation and $*$ -Lie derivation if $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ is a higher derivation, Lie higher derivation and $*$ -Lie higher derivation respectively.

The objective of this article is to investigate nonlinear $*$ -Lie higher derivations on standard operator algebras which are closed under adjoint operation in infinite-dimensional complex Hilbert spaces. Many researchers have made important contributions to the related topics (see [3], [9], [13]). Xiao [14] proved that every nonlinear Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a nonlinear functional vanishing on all commutators. Qi and Hou [10] gave a characterization of Lie higher derivations on nest algebras. Zhang et al., [17] showed that every nonlinear $*$ -Lie higher derivation on factor von Neumann algebra is linear. Motivated by the above work in this article, we study nonlinear $*$ -Lie higher derivations on standard operator algebras .

2 Nonlinear $*$ -Lie higher derivations

Throughout this paper, \mathbb{R} and \mathbb{C} represents the set of real numbers and complex numbers respectively and \mathcal{H} represents a complex Hilbert space. By $\mathcal{B}(\mathcal{H})$ we mean the algebra of all bounded linear operators on \mathcal{H} . Denote by $\mathcal{F}(\mathcal{H})$ the subalgebra of bounded finite rank operators. It is to be noted that $\mathcal{F}(\mathcal{H})$ forms a $*$ -closed ideal in $\mathcal{B}(\mathcal{H})$. An algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is said to be standard operator algebra in case $\mathcal{F}(\mathcal{H}) \subset \mathfrak{A}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projection provided $P^* = P$ and $P^2 = P$. Note that, different from von Neumann algebras which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra \mathfrak{A} is prime if $A\mathfrak{A}B = 0$ implies either $A = 0$ or $B = 0$. It is to be noted that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Motivated by the work of Jing [4], we have obtained the following main result.

Theorem 1. *Let \mathcal{H} be an infinite-dimensional complex Hilbert space and \mathfrak{A} be a standard operator algebra on \mathcal{H} containing identity operator I . If \mathfrak{A} is closed under the adjoint operation, then every nonlinear $*$ -Lie higher derivation $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ from \mathfrak{A} to $\mathcal{B}(\mathcal{H})$ is an additive $*$ -higher derivation.*

Now take a projection $P_1 \in \mathfrak{A}$ and let $P_2 = I - P_1$. We write $\mathfrak{A}_{jk} = P_j\mathfrak{A}P_k$ for $j, k = 1, 2$. Then by Peirce decomposition of \mathfrak{A} we have $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$. Note that any operator $A \in \mathfrak{A}$ can be expressed as $A = A_{11} + A_{12} + A_{21} + A_{22}$, and $A_{jk}^* \in \mathfrak{A}_{kj}$ for any $A_{jk} \in \mathfrak{A}_{jk}$.

We facilitate our discussion with the following known results.

Lemma 1. *[4, Lemma 2.1] Let \mathfrak{A} be a standard operator algebra containing identity operator I in a complex Hilbert space which is closed under the adjoint operation. If $AB = BA^*$ holds true for all $B \in \mathfrak{A}$, then $A \in \mathbb{R}I$.*

Lemma 2. [4, Proposition 2.7] *Let \mathfrak{A} be a standard operator algebra containing identity operator I in a complex Hilbert space which is closed under the adjoint operation. For any $A \in \mathfrak{A}$,*

$$(i) [iP_1, A]_* = 0 \text{ implies } A_{11} = A_{12} = A_{21} = 0.$$

$$(ii) [iP_2, A]_* = 0 \text{ implies } A_{12} = A_{21} = A_{22} = 0.$$

$$(iii) [i(P_2 - P_1), A]_* = 0 \text{ implies } A_{11} = A_{22} = 0.$$

Now we shall use the hypothesis of Theorem 1 freely without any specific mention in proving the following lemmas.

Lemma 3. $d_n(0) = 0$ for each $n \in \mathbb{N}$.

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. If $n = 1$, by [4, Lemma 2.2], the result is true. Now assume that the result is true for $k < n$, i.e., $d_k(0) = 0$. Our aim is to show that d_n satisfies the similar property. Observe that

$$d_n(0) = d_n([0, 0]_*) = \sum_{i+j=n} [d_i(0), d_j(0)]_* = [d_n(0), 0]_* + [0, d_n(0)]_* = 0.$$

□

Lemma 4. d_n has the following properties:

$$(i) \text{ For any } \lambda \in \mathbb{R}, d_n(\lambda I) \in \mathbb{R}I.$$

$$(ii) \text{ For any } A \in \mathfrak{A} \text{ with } A = A^*, d_n(A) = d_n(A^*) = d_n(A)^*.$$

$$(iii) \text{ For any } \lambda \in \mathbb{C}, d_n(\lambda I) \in \mathbb{C}I.$$

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By Lemmas 2.3, 2.4 & 2.5 of [4] the result is true for $n = 1$.

Assume that the result is true for $k < n$, i.e.,

$$d_k(\lambda I) \in \mathbb{R}I, d_k(A) = d_k(A^*) = d_k(A)^*, d_k(\lambda I) \in \mathbb{C}I.$$

Our aim is to show that d_n satisfies the similar property. By the induction hypothesis;

$$(i) \text{ For any } \lambda \in \mathbb{R}, \text{ since } d_k(\lambda I) \in \mathbb{R}I, \text{ i.e., } d_k(\lambda I) = d_k(\lambda I)^* \in \mathbb{R}I$$

$$\begin{aligned} 0 &= d_n([\lambda I, A]_*) = [d_n(\lambda I), A]_* + [\lambda I, d_n(A)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(\lambda I), d_j(A)]_* \\ &= d_n(\lambda I)A - Ad_n(\lambda I)^*. \end{aligned}$$

This gives us that $d_n(\lambda I)A = Ad_n(\lambda I)^*$. By Lemma 1, we have $d_n(\lambda I) \in \mathbb{R}I$.

(ii) Using (i), we have for $A = A^*$

$$\begin{aligned} 0 &= d_n([A, I]_*) = [d_n(A), I]_* + [A, d_n(I)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(A), d_j(I)]_* \\ &= d_n(A) - d_n(A)^*. \end{aligned}$$

(iii) For any $\lambda \in \mathbb{C}$ and $A \in \mathfrak{A}$ with $A = A^*$, applying (ii), we see that

$$\begin{aligned} 0 &= d_n([A, \lambda I]_*) = [d_n(A), \lambda I]_* + [A, d_n(\lambda I)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(A), d_j(\lambda I)]_* \\ &= Ad_n(\lambda I) - d_n(\lambda I)A. \end{aligned}$$

This yields that $d_n(\lambda I)A = Ad_n(\lambda I)$ for all $A \in \mathfrak{A}$ with $A = A^*$, and hence $d_n(\lambda I) \in \mathbb{C}I$. □

Lemma 5. $d_n(\frac{1}{2}iI) = 0$ for each $n \in \mathbb{N}$ with $n \geq 1$ and $d_n(iA) = id_n(A)$ for all $A \in \mathfrak{A}$.

Proof. The result is true for $n = 1$ by [4, Lemma 2.6]. Assume that the result is true for $k < n$, i.e., $d_k(\frac{1}{2}iI) = 0$. Now we compute

$$\begin{aligned} d_n\left(-\frac{1}{2}I\right) &= d_n\left(\left[\frac{1}{2}iI, \frac{1}{2}iI\right]_*\right) \\ &= \left[d_n\left(\frac{1}{2}iI\right), \frac{1}{2}iI\right]_* + \left[\frac{1}{2}iI, d_n\left(\frac{1}{2}iI\right)\right] + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(\frac{1}{2}iI\right), d_q\left(\frac{1}{2}iI\right)\right]_* \\ &= id_n\left(\frac{1}{2}iI\right) + \frac{1}{2}i\left\{d_n\left(\frac{1}{2}iI\right) - d_n\left(\frac{1}{2}iI\right)\right\}^*. \end{aligned}$$

Since both $d_n(-\frac{1}{2}I)$ and $\frac{1}{2}i\{d_n(\frac{1}{2}iI) - d_n(\frac{1}{2}iI)\}^*$ are self-adjoint, $id_n(\frac{1}{2}iI)$ is also self-adjoint, and hence it follows that

$$d_n\left(\frac{1}{2}iI\right) = -d_n\left(\frac{1}{2}iI\right)^*.$$

Thus, the above computation gives that

$$d_n\left(-\frac{1}{2}I\right) = 2id_n\left(\frac{1}{2}iI\right). \quad (1)$$

Similarly, we can obtain from the fact $[-\frac{1}{2}iI, -\frac{1}{2}iI] = \frac{1}{2}I$ that $d_n(-\frac{1}{2}iI)^* = -d_n(-\frac{1}{2}iI)$ and $d_n(-\frac{1}{2}I) = -2id_n(-\frac{1}{2}iI)$. Thus $d_n(-\frac{1}{2}iI) = -d_n(\frac{1}{2}iI)$. Now we

compute

$$\begin{aligned}
d_n\left(\frac{1}{2}iI\right) &= d_n\left(\left[-\frac{1}{2}iI, -\frac{1}{2}I\right]_*\right) \\
&= \left[d_n\left(-\frac{1}{2}iI\right), -\frac{1}{2}I\right]_* + \left[-\frac{1}{2}iI, d_n\left(-\frac{1}{2}I\right)\right]_* \\
&\quad + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(-\frac{1}{2}iI\right), d_q\left(\frac{1}{2}I\right)\right]_* \\
&= -id_n\left(-\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right) = d_n\left(\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right).
\end{aligned}$$

It follows that $d_n(-\frac{1}{2}I) = 0$, and so, by the equality (1), we have $d_n(\frac{1}{2}iI) = 0$. Now, for any $A \in \mathfrak{A}$, we have by induction hypothesis

$$\begin{aligned}
d_n(iA) &= d_n\left(\left[\frac{1}{2}iI, A\right]_*\right) \\
&= \left[d_n\left(\frac{1}{2}iI\right), -A\right]_* + \left[\frac{1}{2}iI, d_n(A)\right]_* + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(\frac{1}{2}iI\right), d_q(A)\right]_* \\
&= id_n(A).
\end{aligned}$$

□

Lemma 6. For any $A_{12} \in \mathfrak{A}_{12}$ and $B_{21} \in \mathfrak{A}_{21}$,

$$d_n(A_{12} + B_{21}) = d_n(A_{12}) + d_n(B_{21}).$$

Proof. We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By [4, Lemma 2.8] the result is true for $n = 1$.

Assume that the result is true for $k < n$, i.e., $d_k(A_{12} + B_{21}) = d_k(A_{12}) + d_k(B_{21})$.

Let $M = d_n(A_{12} + B_{21}) - d_n(A_{12}) - d_n(B_{21})$. We now show that $M = 0$.

By the induction hypothesis, we have

$$\begin{aligned}
0 &= d_n\left([i(P_2 - P_1), A_{12} + B_{21}]_*\right) \\
&= \left[d_n\left(i(P_2 - P_1)\right), A_{12} + B_{21}\right]_* + \left[i(P_2 - P_1), d_n(A_{12} + B_{21})\right]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} \left[d_r\left(i(P_2 - P_1)\right), d_s(A_{12} + B_{21})\right]_* \\
&= \left[d_n\left(i(P_2 - P_1)\right), A_{12} + B_{21}\right]_* + \left[i(P_2 - P_1), d_n(A_{12} + B_{21})\right]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} \left[d_r\left(i(P_2 - P_1)\right), d_s(A_{12}) + d_s(B_{21})\right]_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
0 &= d_n([i(P_2 - P_1), A_{12}]_*) + d_n([i(P_2 - P_1), B_{21}]_*) \\
&= [d_n(i(P_2 - P_1)), A_{12}]_* + [i(P_2 - P_1), d_n(A_{12})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(A_{12})]_* + [d_n(i(P_2 - P_1)), B_{21}]_* \\
&\quad + [i(P_2 - P_1), d_n(B_{21})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(B_{21})]_* \\
&= [d_n(i(P_2 - P_1)), A_{12} + B_{21}]_* + [i(P_2 - P_1), d_n(A_{12}) + d_n(B_{21})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(A_{12}) + d_s(B_{21})]_*.
\end{aligned}$$

Comparing the above two equations, we arrive at $[i(P_2 - P_1), M]_* = 0$. It follows from Lemma 2 that $M_{11} = M_{22} = 0$. Now we calculate $d_n(A_{12} - A_{12}^*)$ in two ways

$$\begin{aligned}
d_n(A_{12} - A_{12}^*) &= d_n([A_{12} + B_{21}, P_2]_*) \\
&= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12} + B_{21}), d_s(P_2)]_* \\
&= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d_n(A_{12} - A_{12}^*) &= d_n([A_{12}, P_2]_*) + d_n([B_{21}, P_2]_*) \\
&= [d_n(A_{12}), P_2]_* + [A_{12}, d_n(P_2)]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}), d_s(P_2)]_* \\
&\quad + [d_n(B_{21}), P_2]_* + [B_{21}, d_n(P_2)]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(B_{21}), d_s(P_2)]_* \\
&= [d_n(A_{12}) + d_n(B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
\end{aligned}$$

The above two identities give us that $[M, P_2]_* = 0$. But

$$[M, P_2]_* = MP_2 - P_2M^* = (M_{12} + M_{21})P_2 - P_2(M_{12}^* + M_{21}^*) = M_{12} - M_{12}^*.$$

Hence it follows that $M_{12} = 0$.

Similarly, using the fact that

$$\begin{aligned} d_n(B_{21} - B_{21}^*) &= d_n([A_{12} + B_{21}, P_1]_*) \\ &= d_n([A_{12}, P_1]_*) + d_n([B_{21}, P_1]_*), \end{aligned}$$

one can show that $M_{21} = 0$. □

Lemma 7. For any $A_{11} \in \mathfrak{A}_{11}$, $B_{12} \in \mathfrak{A}_{12}$, $C_{21} \in \mathfrak{A}_{21}$ and $D_{22} \in \mathfrak{A}_{22}$;

- (i) $d_n(A_{11} + B_{12} + C_{21}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})$.
- (ii) $d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22})$.

Proof. (i) We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By [4, Lemma 2.9] the result is true for $n = 1$.

Assume that the result is true for $k < n$, that is,

$$d_k(A_{11} + B_{12} + C_{21}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}).$$

Let

$$M = d_n(A_{11} + B_{12} + C_{21}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}).$$

We now show that $M = 0$.

By the induction hypothesis, we have by Lemma 6,

$$\begin{aligned} d_n(iB_{12}) + d_n(iC_{21}) &= d_n(iB_{12} + iC_{21}) \\ &= d_n([iP_2, A_{11} + B_{12} + C_{21}]_*) \\ &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* \\ &\quad + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11} + B_{12} + C_{21})]_* \\ &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* \\ &\quad + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_* \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 d_n(iB_{12}) + d_n(iC_{21}) &= d_n([iP_2, A_{11}]_*) + d_n([iP_2, B_{21}]_*) + d_n([iP_2, C_{21}]_*) \\
 &= [d_n(iP_2), A_{11}]_* + [iP_2, d_n(A_{11})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11})]_* \\
 &\quad + [d_n(iP_2), B_{12}]_* + [iP_2, d_n(B_{12})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(B_{12})]_* \\
 &\quad + [d_n(iP_2), C_{21}]_* + [iP_2, d_n(C_{21})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(C_{21})]_* \\
 &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* + [iP_2, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_*.
 \end{aligned}$$

Comparing the above two equalities, we have $[iP_2, M]_* = 0$ and hence it follows from Lemma 2 (ii), that $M_{12} = M_{21} = M_{22} = 0$.

We now show that $M_{11} = 0$. Note that

$$[i(P_2 - P_1), B_{12}]_* = [i(P_2 - P_1), C_{21}]_* = 0.$$

We have

$$\begin{aligned}
 d_n([i(P_2 - P_1), A_{11} + B_{12} + C_{21}]_*) &= d_n([i(P_2 - P_1), A_{11}]_*) \\
 &\quad + d_n([i(P_2 - P_1), B_{12}]_*) + d_n([i(P_2 - P_1), C_{21}]_*).
 \end{aligned}$$

Using the similar arguments as used above, we get $[i(P_2 - P_1), M]_* = 0$. Therefore by Lemma 2, $M_{11} = 0$. Hence we are done.

- (ii) Considering $d_n([iP_1, B_{12} + C_{21} + D_{22}]_*)$ and $d_n([i(P_2 - P_1), B_{12} + C_{21} + D_{22}]_*)$, with the similar argument as in (i), one can obtain

$$d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

□

Lemma 8. For any $A_{11} \in \mathfrak{A}_{11}$, $B_{12} \in \mathfrak{A}_{12}$, $C_{21} \in \mathfrak{A}_{21}$ and $D_{22} \in \mathfrak{A}_{22}$;

$$d_n(A_{11} + B_{12} + C_{21} + D_{22}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

Proof. By [4, Lemma 2.10], the result is true for $n = 1$. Assume that the result is true for $k < n$, i.e.,

$$d_k(A_{11} + B_{12} + C_{21} + D_{22}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}) + d_k(D_{22}).$$

Our aim is to show that the result is true for every $n \in \mathbb{N}$. Let

$$M = d_n(A_{11} + B_{12} + C_{21} + D_{22}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}) - d_n(D_{22}).$$

Note that $[iP_1, D_{22}]_* = 0$, by induction hypothesis, we have

$$\begin{aligned}
d_n([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*) &= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21}) + d_s(D_{22})]_*.
\end{aligned}$$

On the other hand, we have by (i) of Lemma 7,

$$\begin{aligned}
d_n([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*) &= d_n([iP_1, A_{11} + B_{12} + C_{21}]_*) + d_n([iP_1, D_{22}]_*) \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21})]_* \\
&+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(D_{22})]_* \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_* \\
&+ [iP_1, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_* \\
&+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(D_{22})]_*.
\end{aligned}$$

Comparing the above two equalities, it follows that $[iP_1, M] = 0$, and hence by Lemma 2, $M_{11} = M_{12} = M_{21} = 0$. Using the fact that $[iP_2, A_{11}] = 0$ and the above similar arguments, we obtain $[iP_2, M]_* = 0$ which leads to $M_{22} = 0$. This completes the proof. \square

Lemma 9. For any $A_{jk}, B_{jk} \in \mathfrak{A}_{jk}$, where $j, k \in 1, 2$, we have

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(B_{jk})$$

Proof. We separate the proof in two distinct cases.

Case I: $j \neq k$

On one side, by Lemma 8, we have

$$\begin{aligned} d_n(iA_{jk} + iB_{jk} + iA_{jk}^* + iB_{jk}A_{jk}^*) \\ = d_n(iA_{jk} + iB_{jk}) + d_n(iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*). \end{aligned}$$

On the other hand, using Lemmas 6 and 8, by induction, we have

$$\begin{aligned} d_n(iA_{jk} + iB_{jk} + iA_{jk}^* + iB_{jk}A_{jk}^*) &= d_n([iP_j + iA_{jk}, P_k + B_{jk}]_*) \\ &= [d_n(iP_j + iA_{jk}), P_k + B_{jk}]_* + [iP_j + iA_{jk}, d_n(P_k + B_{jk})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_j + iA_{jk}), d_s(P_k + B_{jk})]_* \\ &= [d_n(iP_j) + d_n(iA_{jk}), P_k + B_{jk}]_* \\ &\quad + [iP_j + iA_{jk}, d_n(P_k) + d_n(B_{jk})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_j) + d_r(iA_{jk}), d_s(P_k) + d_s(B_{jk})]_* \\ &= d_n([iP_j, P_k]_*) + d_n([iP_j, B_{jk}]_*) \\ &\quad + d_n([iA_{jk}, P_k]_*) + d_n([iA_{jk}, B_{jk}]_*) \\ &= d_n(iB_{jk}) + d_n(iA_{jk} + iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*) \\ &= d_n(iB_{jk}) + d_n(iA_{jk}) + d_n(iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*). \end{aligned}$$

Comparing the above two equalities, we can conclude that

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(A_{jk}^*).$$

Case II: $j = k$.

Let $A_{jj}, B_{jj} \in \mathfrak{A}_{jj}$ and $n \in \{1, 2\}$ with $n \neq j$. We have

$$\begin{aligned} 0 &= d_n([iP_n, A_{jj} + B_{jj}]_*) \\ &= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj} + B_{jj})]_* \\ &= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})]_*. \end{aligned}$$

On the other hand we have,

$$\begin{aligned}
0 &= d_n([iP_n, A_{jj}]_*) + d_n([iP_n, B_{jj}]_*) \\
&= [d_n(iP_n), A_{jj}]_* + [iP_n, d_n(A_{jj})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj})]_* \\
&\quad + [d_n(iP_n), B_{jj}]_* + [iP_n, d_n(B_{jj})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(B_{jj})]_* \\
&= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj}) + d_n(B_{jj})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})]_*.
\end{aligned}$$

Take $M = d_n(A_{jj} + B_{jj}) - d_n(A_{jj}) - d_n(B_{jj})$. The above computation yields that $[iP_n, M]_* = 0$. By Lemma 2, we have $M_{nj} = M_{jn} = M_{nn} = 0$. We now show that $M_{jj} = 0$. For any $C_{jn} \in \mathfrak{A}_{jn}$, using Case I, we compute

$$\begin{aligned}
d_n([A_{jj} + B_{jj}, C_{jn}]_*) &= [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj} + B_{jj}), d_s(C_{jn})]_* \\
&= [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj}) + d_r(B_{jj}), d_s(C_{jn})]_*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d_n([A_{jj} + B_{jj}, C_{jn}]_*) &= d_n(A_{jj}C_{jn} + B_{jj}C_{jn}) \\
&= d_n(A_{jj}C_{jn}) + d_n(B_{jj}C_{jn}) \\
&= d_n([A_{jj}, C_{jn}]_*) + d_n([B_{jj}, C_{jn}]_*) \\
&= [d_n(A_{jj}), C_{jn}]_* + [A_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj}), d_s(C_{jn})]_* \\
&\quad + [d_n(B_{jj}), C_{jn}]_* + [B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(B_{jj}), d_s(C_{jn})]_*.
\end{aligned}$$

Comparing the above two equalities, we obtain $[M, C_{jn}]_* = 0$ which leads to $M_{jj}C_{jn} = 0$. Since \mathfrak{A} is prime, we see that $M_{jj} = 0$, which completes the proof. \square

Lemma 10. d_n is an additive $*$ -higher derivation on \mathfrak{A} .

Proof. We first show that d_n is additive. For arbitrary $A, B \in \mathfrak{A}$, we write $A = \sum_{j,k=1}^2 A_{jk}$ and $B = \sum_{j,k=1}^2 B_{jk}$. It follows from Lemmas 8 and 9 that

$$\begin{aligned}
 d_n(A + B) &= d_n \left\{ \sum_{j,k=1}^2 (A_{jk} + B_{jk}) \right\} \\
 &= \sum_{j,k=1}^2 d_n(A_{jk} + B_{jk}) \\
 &= \sum_{j,k=1}^2 (d_n(A_{jk}) + d_n(B_{jk})) \\
 &= d_n \left(\sum_{j,k=1}^2 A_{jk} \right) + d_n \left(\sum_{j,k=1}^2 B_{jk} \right) \\
 &= d_n(A) + d_n(B).
 \end{aligned}$$

We now show that $d_n(A^*) = d_n(A)^*$.

For any $A \in \mathfrak{A}$, it follows from Lemmas 4 and 5 that

$$\begin{aligned}
 d_n(A^*) &= d_n(\Re A - i\Im A) = d_n(\Re A) - d_n(i\Im A) \\
 &= d_n(\Re A) - id_n(\Im A) = d_n(\Re A)^* - id_n(\Im A)^* \\
 &= d_n(\Re A)^* + (id_n(\Im A))^* = d_n(\Re A)^* + d_n(i\Im A)^* \\
 &= (d_n(\Re A + i\Im A))^* = d_n(A)^*.
 \end{aligned}$$

To complete the proof, we need to show that d_n is a higher derivation on \mathfrak{A} .

Since d_n is additive, it follows from Lemma 5, that $d_n(iI) = 0$. It is to be noted that $[iI + A, B]_* = 2iB + AB - BA^*$.

$$\begin{aligned}
 d_n(2iB) + d_n(AB) - d_n(BA^*) &= d_n([iI + A, B]_*) \\
 &= [d_n(iI + A), B]_* + [iI + A, d_n(B)]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iI + A), d_s(B)]_* \\
 &= [d_n(iI) + d_n(A), B]_* + [iI + A, d_n(B)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iI) + d_r(A), d_s(B)]_* \\
 &= [d_n(A), B]_* + [iI + A, d_n(B)]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A), d_s(B)]_* \\
 &= d_n(A)B - Bd_n(A)^* + 2id_n(B) + Ad_n(B) - d_n(B)A^* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) - d_s(B)d_r(A)^*).
 \end{aligned}$$

It follows that

$$\begin{aligned} d_n(AB) - d_n(BA^*) &= d_n(A)B - Bd_n(A)^* + Ad_n(B) - d_n(B)A^* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) - d_s(B)d_r(A)^*). \end{aligned}$$

Replacing A by iA in the above equality, we get

$$\begin{aligned} d_n(AB) + d_n(BA^*) &= d_n(A)B + Bd_n(A)^* + Ad_n(B) + d_n(B)A^* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) + d_s(B)d_r(A)^*). \end{aligned}$$

Thus we have,

$$\begin{aligned} d_n(AB) &= d_n(A)B + Ad_n(B) + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} d_r(A)d_s(B) \\ &= \sum_{r+s=n} d_r(A)d_s(B). \end{aligned}$$

This shows that d_n is an additive higher derivation with $d_n(A^*) = d_n(A)^*$. Hence d_n is an additive $*$ -higher derivation on \mathfrak{A} , which completes the proof. \square

Note that every additive derivation $d: \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ is an inner derivation (see [12]). Nowicki [9] proved that if every additive (linear) derivation of \mathfrak{A} is inner, then every additive (linear) higher derivation of \mathfrak{A} is inner (see also [13]). So by Theorem 1, the following corollary is immediate.

Corollary 1. *Let \mathcal{H} be an infinite-dimensional complex Hilbert space and \mathfrak{A} be a standard operator algebra on \mathcal{H} containing identity operator I . If \mathfrak{A} is closed under the adjoint operation, then every nonlinear $*$ -Lie higher derivation $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$ is inner with $d_n(A^*) = d_n(A)^*$ for each $A \in \mathfrak{A}$ and every $n \in \mathbb{N}$.*

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