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## Nonlinear $*$ -Lie higher derivations of standard operator algebras

*Mohammad Ashraf, Shakir Ali, Bilal Ahmad Wani*

**Abstract.** Let  $\mathcal{H}$  be an infinite-dimensional complex Hilbert space and  $\mathfrak{A}$  be a standard operator algebra on  $\mathcal{H}$  which is closed under the adjoint operation. It is shown that every nonlinear  $*$ -Lie higher derivation  $\mathcal{D} = \{\delta_n\}_{n \in \mathbb{N}}$  of  $\mathfrak{A}$  is automatically an additive higher derivation on  $\mathfrak{A}$ . Moreover,  $\mathcal{D} = \{\delta_n\}_{n \in \mathbb{N}}$  is an inner  $*$ -higher derivation.

### 1 Introduction

Let  $\mathfrak{A}$  be an algebra over a commutative ring  $R$ . Recall that an  $R$ -linear mapping  $d: \mathfrak{A} \rightarrow \mathfrak{A}$  is called a derivation if  $d(AB) = d(A)B + Ad(B)$  for all  $A, B \in \mathfrak{A}$ ; in particular,  $d$  is called an inner derivation if there exists some  $X \in \mathfrak{A}$  such that  $d(A) = AX - XA$  for all  $A \in \mathfrak{A}$ . An  $R$ -linear mapping  $d: \mathfrak{A} \rightarrow \mathfrak{A}$  is called a Lie derivation if  $d([A, B]) = [d(A), B] + [A, d(B)]$  for all  $A, B \in \mathfrak{A}$ , where  $[A, B] = AB - BA$  is the usual Lie product. Furthermore, without linearity/additivity assumption, if  $d$  satisfies  $d([A, B]) = [d(A), B] + [A, d(B)]$  for all  $A, B \in \mathfrak{A}$ , then  $d$  is called a nonlinear Lie derivation. The question of characterizing Lie derivations and revealing the relationship between derivations and Lie derivations have been studied by many authors (see [1], [2], [5], [6], [7], [8], [11], [12], [15], [18]).

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Let  $\mathfrak{A}$  be an associative  $*$ -algebra over the complex field  $\mathbb{C}$ . A mapping  $d: \mathfrak{A} \rightarrow \mathfrak{A}$  is said to be an additive  $*$ -derivation if it is an additive derivation and satisfies  $d(A)^* = d(A^*)$  for all  $A \in \mathfrak{A}$ . Further, if  $d: \mathfrak{A} \rightarrow \mathfrak{A}$  is a map (not necessarily linear) which satisfies  $d([A, B]_*) = [d(A), B]_* + [A, d(B)]_*$  for all  $A, B \in \mathfrak{A}$ , where  $[A, B]_* = AB - BA^*$ , then  $d$  is known as a nonlinear  $*$ -Lie derivation of  $\mathfrak{A}$ .

In [16] Yu and Zhang showed that every nonlinear  $*$ -Lie derivation from a factor von Neumann algebra on an infinite-dimensional Hilbert space into itself is an additive  $*$ -derivation. It is to be noted that a factor von Neumann algebra is a von Neumann algebra whose centre is trivial. In [4] Wu Jing proved that every nonlinear  $*$ -Lie derivation on standard operator algebra is automatically linear. Moreover, it is an inner  $*$ -derivation .

Let us recall some basic facts related to Lie higher derivations and  $*$ -Lie higher derivations of an associative algebra. Many different kinds of higher derivations, which consist of a family of some additive mappings, have been widely studied in commutative and noncommutative rings. Let  $\mathbb{N}$  be the set of non-negative integers and  $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$  be a family of linear mappings  $d_n: \mathfrak{A} \rightarrow \mathfrak{A}$  such that  $d_0 = \text{id}_{\mathfrak{A}}$ , the identity map on  $\mathfrak{A}$ . Then  $\mathcal{D}$  is called

- (i) a higher derivation on  $\mathfrak{A}$  if for every  $n \in \mathbb{N}$ ,

$$d_n(AB) = \sum_{i+j=n} d_i(A)d_j(B)$$

for all  $A, B \in \mathfrak{A}$ .

- (ii) a Lie higher derivation on  $\mathfrak{A}$  if for every  $n \in \mathbb{N}$ ,

$$d_n([A, B]) = \sum_{i+j=n} [d_i(A), d_j(B)]$$

for all  $A, B \in \mathfrak{A}$ .

- (iii) a  $*$ -Lie higher derivation on  $\mathfrak{A}$  if for every  $n \in \mathbb{N}$ ,

$$d_n([A, B]_*) = \sum_{i+j=n} [d_i(A), d_j(B)]_*$$

for all  $A, B \in \mathfrak{A}$ .

- (iv) an inner higher derivation on  $\mathfrak{A}$  if there exist two sequences  $\{X_n\}_{n \in \mathbb{N}}$  and  $\{Y_n\}_{n \in \mathbb{N}}$  in  $\mathfrak{A}$  satisfying the conditions

$$X_0 = Y_0 = 1 \quad \text{and} \quad \sum_{i=0}^n X_i Y_{n-i} = \delta_{n0} = \sum_{i=0}^n Y_i X_{n-i}$$

such that  $d_n(A) = \sum_{i=0}^n X_i A Y_{n-i}$ , for all  $A \in \mathfrak{A}$  and for every  $n \in \mathbb{N}$ , where  $\delta_{n0}$  is the Kronecker sign.

If the linear assumption in the above definitions is dropped, then the corresponding higher derivation, Lie higher derivation and  $\ast$ -Lie higher derivation is said to be nonlinear higher derivation, nonlinear Lie higher derivation and nonlinear  $\ast$ -Lie higher derivation respectively. Moreover, if  $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$  is assumed to be the family of additive mappings, then in the above definition higher derivation, Lie higher derivation and  $\ast$ -Lie higher derivation is said to be additive higher derivation, additive Lie higher derivation and additive  $\ast$ -Lie higher derivation respectively. Note that  $d_1$  is always a derivation, Lie derivation and  $\ast$ -Lie derivation if  $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$  is a higher derivation, Lie higher derivation and  $\ast$ -Lie higher derivation respectively.

The objective of this article is to investigate nonlinear  $\ast$ -Lie higher derivations on standard operator algebras which are closed under adjoint operation in infinite-dimensional complex Hilbert spaces. Many researchers have made important contributions to the related topics (see [3], [9], [13]). Xiao [14] proved that every nonlinear Lie higher derivation of triangular algebras is the sum of an additive higher derivation and a nonlinear functional vanishing on all commutators. Qi and Hou [10] gave a characterization of Lie higher derivations on nest algebras. Zhang et al., [17] showed that every nonlinear  $\ast$ -Lie higher derivation on factor von Neumann algebra is linear. Motivated by the above work in this article, we study nonlinear  $\ast$ -Lie higher derivations on standard operator algebras .

## 2 Nonlinear $\ast$ -Lie higher derivations

Throughout this paper,  $\mathbb{R}$  and  $\mathbb{C}$  represents the set of real numbers and complex numbers respectively and  $\mathcal{H}$  represents a complex Hilbert space. By  $\mathcal{B}(\mathcal{H})$  we mean the algebra of all bounded linear operators on  $\mathcal{H}$ . Denote by  $\mathcal{F}(\mathcal{H})$  the subalgebra of bounded finite rank operators. It is to be noted that  $\mathcal{F}(\mathcal{H})$  forms a  $\ast$ -closed ideal in  $\mathcal{B}(\mathcal{H})$ . An algebra  $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$  is said to be standard operator algebra in case  $\mathcal{F}(\mathcal{H}) \subset \mathfrak{A}$ . An operator  $P \in \mathcal{B}(\mathcal{H})$  is said to be a projection provided  $P^\ast = P$  and  $P^2 = P$ . Note that, different from von Neumann algebras which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra  $\mathfrak{A}$  is prime if  $A\mathfrak{A}B = 0$  implies either  $A = 0$  or  $B = 0$ . It is to be noted that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Motivated by the work of Jing [4], we have obtained the following main result.

**Theorem 1.** *Let  $\mathcal{H}$  be an infinite-dimensional complex Hilbert space and  $\mathfrak{A}$  be a standard operator algebra on  $\mathcal{H}$  containing identity operator  $I$ . If  $\mathfrak{A}$  is closed under the adjoint operation, then every nonlinear  $\ast$ -Lie higher derivation  $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$  from  $\mathfrak{A}$  to  $\mathcal{B}(\mathcal{H})$  is an additive  $\ast$ -higher derivation.*

Now take a projection  $P_1 \in \mathfrak{A}$  and let  $P_2 = I - P_1$ . We write  $\mathfrak{A}_{jk} = P_j\mathfrak{A}P_k$  for  $j, k = 1, 2$ . Then by Peirce decomposition of  $\mathfrak{A}$  we have  $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$ . Note that any operator  $A \in \mathfrak{A}$  can be expressed as  $A = A_{11} + A_{12} + A_{21} + A_{22}$ , and  $A_{jk}^\ast \in \mathfrak{A}_{kj}$  for any  $A_{jk} \in \mathfrak{A}_{jk}$ .

We facilitate our discussion with the following known results.

**Lemma 1.** *[4, Lemma 2.1] Let  $\mathfrak{A}$  be a standard operator algebra containing identity operator  $I$  in a complex Hilbert space which is closed under the adjoint operation. If  $AB = BA^\ast$  holds true for all  $B \in \mathfrak{A}$ , then  $A \in \mathbb{R}I$ .*

**Lemma 2.** [4, Proposition 2.7] *Let  $\mathfrak{A}$  be a standard operator algebra containing identity operator  $I$  in a complex Hilbert space which is closed under the adjoint operation. For any  $A \in \mathfrak{A}$ ,*

$$(i) [iP_1, A]_* = 0 \text{ implies } A_{11} = A_{12} = A_{21} = 0.$$

$$(ii) [iP_2, A]_* = 0 \text{ implies } A_{12} = A_{21} = A_{22} = 0.$$

$$(iii) [i(P_2 - P_1), A]_* = 0 \text{ implies } A_{11} = A_{22} = 0.$$

Now we shall use the hypothesis of Theorem 1 freely without any specific mention in proving the following lemmas.

**Lemma 3.**  $d_n(0) = 0$  for each  $n \in \mathbb{N}$ .

*Proof.* We proceed by induction on  $n \in \mathbb{N}$  with  $n \geq 1$ . If  $n = 1$ , by [4, Lemma 2.2], the result is true. Now assume that the result is true for  $k < n$ , i.e.,  $d_k(0) = 0$ . Our aim is to show that  $d_n$  satisfies the similar property. Observe that

$$d_n(0) = d_n([0, 0]_*) = \sum_{i+j=n} [d_i(0), d_j(0)]_* = [d_n(0), 0]_* + [0, d_n(0)]_* = 0.$$

□

**Lemma 4.**  $d_n$  has the following properties:

$$(i) \text{ For any } \lambda \in \mathbb{R}, d_n(\lambda I) \in \mathbb{R}I.$$

$$(ii) \text{ For any } A \in \mathfrak{A} \text{ with } A = A^*, d_n(A) = d_n(A^*) = d_n(A)^*.$$

$$(iii) \text{ For any } \lambda \in \mathbb{C}, d_n(\lambda I) \in \mathbb{C}I.$$

*Proof.* We proceed by induction on  $n \in \mathbb{N}$  with  $n \geq 1$ . By Lemmas 2.3, 2.4 & 2.5 of [4] the result is true for  $n = 1$ .

Assume that the result is true for  $k < n$ , i.e.,

$$d_k(\lambda I) \in \mathbb{R}I, d_k(A) = d_k(A^*) = d_k(A)^*, d_k(\lambda I) \in \mathbb{C}I.$$

Our aim is to show that  $d_n$  satisfies the similar property. By the induction hypothesis;

$$(i) \text{ For any } \lambda \in \mathbb{R}, \text{ since } d_k(\lambda I) \in \mathbb{R}I, \text{ i.e., } d_k(\lambda I) = d_k(\lambda I)^* \in \mathbb{R}I$$

$$\begin{aligned} 0 &= d_n([\lambda I, A]_*) = [d_n(\lambda I), A]_* + [\lambda I, d_n(A)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(\lambda I), d_j(A)]_* \\ &= d_n(\lambda I)A - Ad_n(\lambda I)^*. \end{aligned}$$

This gives us that  $d_n(\lambda I)A = Ad_n(\lambda I)^*$ . By Lemma 1, we have  $d_n(\lambda I) \in \mathbb{R}I$ .

(ii) Using (i), we have for  $A = A^*$

$$\begin{aligned} 0 &= d_n([A, I]_*) = [d_n(A), I]_* + [A, d_n(I)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(A), d_j(I)]_* \\ &= d_n(A) - d_n(A)^*. \end{aligned}$$

(iii) For any  $\lambda \in \mathbb{C}$  and  $A \in \mathfrak{A}$  with  $A = A^*$ , applying (ii), we see that

$$\begin{aligned} 0 &= d_n([A, \lambda I]_*) = [d_n(A), \lambda I]_* + [A, d_n(\lambda I)]_* + \sum_{\substack{i+j=n \\ 0 < i, j \leq n-1}} [d_i(A), d_j(\lambda I)]_* \\ &= Ad_n(\lambda I) - d_n(\lambda I)A. \end{aligned}$$

This yields that  $d_n(\lambda I)A = Ad_n(\lambda I)$  for all  $A \in \mathfrak{A}$  with  $A = A^*$ , and hence  $d_n(\lambda I) \in \mathbb{C}I$ . □

**Lemma 5.**  $d_n(\frac{1}{2}iI) = 0$  for each  $n \in \mathbb{N}$  with  $n \geq 1$  and  $d_n(iA) = id_n(A)$  for all  $A \in \mathfrak{A}$ .

*Proof.* The result is true for  $n = 1$  by [4, Lemma 2.6]. Assume that the result is true for  $k < n$ , i.e.,  $d_k(\frac{1}{2}iI) = 0$ . Now we compute

$$\begin{aligned} d_n\left(-\frac{1}{2}I\right) &= d_n\left(\left[\frac{1}{2}iI, \frac{1}{2}iI\right]_*\right) \\ &= \left[d_n\left(\frac{1}{2}iI\right), \frac{1}{2}iI\right]_* + \left[\frac{1}{2}iI, d_n\left(\frac{1}{2}iI\right)\right] + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(\frac{1}{2}iI\right), d_q\left(\frac{1}{2}iI\right)\right]_* \\ &= id_n\left(\frac{1}{2}iI\right) + \frac{1}{2}i\left\{d_n\left(\frac{1}{2}iI\right) - d_n\left(\frac{1}{2}iI\right)\right\}^*. \end{aligned}$$

Since both  $d_n(-\frac{1}{2}I)$  and  $\frac{1}{2}i\{d_n(\frac{1}{2}iI) - d_n(\frac{1}{2}iI)\}^*$  are self-adjoint,  $id_n(\frac{1}{2}iI)$  is also self-adjoint, and hence it follows that

$$d_n\left(\frac{1}{2}iI\right) = -d_n\left(\frac{1}{2}iI\right)^*.$$

Thus, the above computation gives that

$$d_n\left(-\frac{1}{2}I\right) = 2id_n\left(\frac{1}{2}iI\right). \quad (1)$$

Similarly, we can obtain from the fact  $[-\frac{1}{2}iI, -\frac{1}{2}iI] = \frac{1}{2}I$  that  $d_n(-\frac{1}{2}iI)^* = -d_n(-\frac{1}{2}iI)$  and  $d_n(-\frac{1}{2}I) = -2id_n(-\frac{1}{2}iI)$ . Thus  $d_n(-\frac{1}{2}iI) = -d_n(\frac{1}{2}iI)$ . Now we

compute

$$\begin{aligned}
d_n\left(\frac{1}{2}iI\right) &= d_n\left(\left[-\frac{1}{2}iI, -\frac{1}{2}I\right]_*\right) \\
&= \left[d_n\left(-\frac{1}{2}iI\right), -\frac{1}{2}I\right]_* + \left[-\frac{1}{2}iI, d_n\left(-\frac{1}{2}I\right)\right]_* \\
&\quad + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(-\frac{1}{2}iI\right), d_q\left(\frac{1}{2}I\right)\right]_* \\
&= -id_n\left(-\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right) = d_n\left(\frac{1}{2}iI\right) - id_n\left(-\frac{1}{2}I\right).
\end{aligned}$$

It follows that  $d_n(-\frac{1}{2}I) = 0$ , and so, by the equality (1), we have  $d_n(\frac{1}{2}iI) = 0$ . Now, for any  $A \in \mathfrak{A}$ , we have by induction hypothesis

$$\begin{aligned}
d_n(iA) &= d_n\left(\left[\frac{1}{2}iI, A\right]_*\right) \\
&= \left[d_n\left(\frac{1}{2}iI\right), -A\right]_* + \left[\frac{1}{2}iI, d_n(A)\right]_* + \sum_{\substack{p+q=n \\ 0 < p, q \leq n-1}} \left[d_p\left(\frac{1}{2}iI\right), d_q(A)\right]_* \\
&= id_n(A).
\end{aligned}$$

□

**Lemma 6.** For any  $A_{12} \in \mathfrak{A}_{12}$  and  $B_{21} \in \mathfrak{A}_{21}$ ,

$$d_n(A_{12} + B_{21}) = d_n(A_{12}) + d_n(B_{21}).$$

*Proof.* We proceed by induction on  $n \in \mathbb{N}$  with  $n \geq 1$ . By [4, Lemma 2.8] the result is true for  $n = 1$ .

Assume that the result is true for  $k < n$ , i.e.,  $d_k(A_{12} + B_{21}) = d_k(A_{12}) + d_k(B_{21})$ .

Let  $M = d_n(A_{12} + B_{21}) - d_n(A_{12}) - d_n(B_{21})$ . We now show that  $M = 0$ .

By the induction hypothesis, we have

$$\begin{aligned}
0 &= d_n\left([i(P_2 - P_1), A_{12} + B_{21}]_*\right) \\
&= \left[d_n\left(i(P_2 - P_1)\right), A_{12} + B_{21}\right]_* + \left[i(P_2 - P_1), d_n(A_{12} + B_{21})\right]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} \left[d_r\left(i(P_2 - P_1)\right), d_s(A_{12} + B_{21})\right]_* \\
&= \left[d_n\left(i(P_2 - P_1)\right), A_{12} + B_{21}\right]_* + \left[i(P_2 - P_1), d_n(A_{12} + B_{21})\right]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} \left[d_r\left(i(P_2 - P_1)\right), d_s(A_{12}) + d_s(B_{21})\right]_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
 0 &= d_n([i(P_2 - P_1), A_{12}]_*) + d_n([i(P_2 - P_1), B_{21}]_*) \\
 &= [d_n(i(P_2 - P_1)), A_{12}]_* + [i(P_2 - P_1), d_n(A_{12})]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(A_{12})]_* + [d_n(i(P_2 - P_1)), B_{21}]_* \\
 &\quad + [i(P_2 - P_1), d_n(B_{21})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(B_{21})]_* \\
 &= [d_n(i(P_2 - P_1)), A_{12} + B_{21}]_* + [i(P_2 - P_1), d_n(A_{12}) + d_n(B_{21})]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(i(P_2 - P_1)), d_s(A_{12}) + d_s(B_{21})]_*.
 \end{aligned}$$

Comparing the above two equations, we arrive at  $[i(P_2 - P_1), M]_* = 0$ . It follows from Lemma 2 that  $M_{11} = M_{22} = 0$ . Now we calculate  $d_n(A_{12} - A_{12}^*)$  in two ways

$$\begin{aligned}
 d_n(A_{12} - A_{12}^*) &= d_n([A_{12} + B_{21}, P_2]_*) \\
 &= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12} + B_{21}), d_s(P_2)]_* \\
 &= [d_n(A_{12} + B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 d_n(A_{12} - A_{12}^*) &= d_n([A_{12}, P_2]_*) + d_n([B_{21}, P_2]_*) \\
 &= [d_n(A_{12}), P_2]_* + [A_{12}, d_n(P_2)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}), d_s(P_2)]_* \\
 &\quad + [d_n(B_{21}), P_2]_* + [B_{21}, d_n(P_2)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(B_{21}), d_s(P_2)]_* \\
 &= [d_n(A_{12}) + d_n(B_{21}), P_2]_* + [A_{12} + B_{21}, d_n(P_2)]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{12}) + d_r(B_{21}), d_s(P_2)]_*.
 \end{aligned}$$



The above two identities give us that  $[M, P_2]_* = 0$ . But

$$[M, P_2]_* = MP_2 - P_2M^* = (M_{12} + M_{21})P_2 - P_2(M_{12}^* + M_{21}^*) = M_{12} - M_{12}^*.$$

Hence it follows that  $M_{12} = 0$ .

Similarly, using the fact that

$$\begin{aligned} d_n(B_{21} - B_{21}^*) &= d_n([A_{12} + B_{21}, P_1]_*) \\ &= d_n([A_{12}, P_1]_*) + d_n([B_{21}, P_1]_*), \end{aligned}$$

one can show that  $M_{21} = 0$ . □

**Lemma 7.** For any  $A_{11} \in \mathfrak{A}_{11}$ ,  $B_{12} \in \mathfrak{A}_{12}$ ,  $C_{21} \in \mathfrak{A}_{21}$  and  $D_{22} \in \mathfrak{A}_{22}$ ;

- (i)  $d_n(A_{11} + B_{12} + C_{21}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})$ .
- (ii)  $d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22})$ .

*Proof.* (i) We proceed by induction on  $n \in \mathbb{N}$  with  $n \geq 1$ . By [4, Lemma 2.9] the result is true for  $n = 1$ .

Assume that the result is true for  $k < n$ , that is,

$$d_k(A_{11} + B_{12} + C_{21}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}).$$

Let

$$M = d_n(A_{11} + B_{12} + C_{21}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}).$$

We now show that  $M = 0$ .

By the induction hypothesis, we have by Lemma 6,

$$\begin{aligned} d_n(iB_{12}) + d_n(iC_{21}) &= d_n(iB_{12} + iC_{21}) \\ &= d_n([iP_2, A_{11} + B_{12} + C_{21}]_*) \\ &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* \\ &\quad + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11} + B_{12} + C_{21})]_* \\ &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* \\ &\quad + [iP_2, d_n(A_{11} + B_{12} + C_{21})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_* \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 d_n(iB_{12}) + d_n(iC_{21}) &= d_n([iP_2, A_{11}]_*) + d_n([iP_2, B_{21}]_*) + d_n([iP_2, C_{21}]_*) \\
 &= [d_n(iP_2), A_{11}]_* + [iP_2, d_n(A_{11})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11})]_* \\
 &\quad + [d_n(iP_2), B_{12}]_* + [iP_2, d_n(B_{12})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(B_{12})]_* \\
 &\quad + [d_n(iP_2), C_{21}]_* + [iP_2, d_n(C_{21})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(C_{21})]_* \\
 &= [d_n(iP_2), A_{11} + B_{12} + C_{21}]_* + [iP_2, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]_* \\
 &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_2), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_*.
 \end{aligned}$$

Comparing the above two equalities, we have  $[iP_2, M]_* = 0$  and hence it follows from Lemma 2 (ii), that  $M_{12} = M_{21} = M_{22} = 0$ .

We now show that  $M_{11} = 0$ . Note that

$$[i(P_2 - P_1), B_{12}]_* = [i(P_2 - P_1), C_{21}]_* = 0.$$

We have

$$\begin{aligned}
 d_n([i(P_2 - P_1), A_{11} + B_{12} + C_{21}]_*) &= d_n([i(P_2 - P_1), A_{11}]_*) \\
 &\quad + d_n([i(P_2 - P_1), B_{12}]_*) + d_n([i(P_2 - P_1), C_{21}]_*).
 \end{aligned}$$

Using the similar arguments as used above, we get  $[i(P_2 - P_1), M]_* = 0$ . Therefore by Lemma 2,  $M_{11} = 0$ . Hence we are done.

- (ii) Considering  $d_n([iP_1, B_{12} + C_{21} + D_{22}]_*)$  and  $d_n([i(P_2 - P_1), B_{12} + C_{21} + D_{22}]_*)$ , with the similar argument as in (i), one can obtain

$$d_n(B_{12} + C_{21} + D_{22}) = d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

□

**Lemma 8.** For any  $A_{11} \in \mathfrak{A}_{11}$ ,  $B_{12} \in \mathfrak{A}_{12}$ ,  $C_{21} \in \mathfrak{A}_{21}$  and  $D_{22} \in \mathfrak{A}_{22}$ ;

$$d_n(A_{11} + B_{12} + C_{21} + D_{22}) = d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21}) + d_n(D_{22}).$$

*Proof.* By [4, Lemma 2.10], the result is true for  $n = 1$ . Assume that the result is true for  $k < n$ , i.e.,

$$d_k(A_{11} + B_{12} + C_{21} + D_{22}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}) + d_k(D_{22}).$$

Our aim is to show that the result is true for every  $n \in \mathbb{N}$ . Let

$$M = d_n(A_{11} + B_{12} + C_{21} + D_{22}) - d_n(A_{11}) - d_n(B_{12}) - d_n(C_{21}) - d_n(D_{22}).$$

Note that  $[iP_1, D_{22}]_* = 0$ , by induction hypothesis, we have

$$\begin{aligned}
d_n([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*) &= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21} + D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21}) + d_s(D_{22})]_*.
\end{aligned}$$

On the other hand, we have by (i) of Lemma 7,

$$\begin{aligned}
d_n([iP_1, A_{11} + B_{12} + C_{21} + D_{22}]_*) &= d_n([iP_1, A_{11} + B_{12} + C_{21}]_*) + d_n([iP_1, D_{22}]_*) \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_* \\
&+ [iP_1, d_n(A_{11} + B_{12} + C_{21})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11} + B_{12} + C_{21})]_* \\
&+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(D_{22})]_* \\
&= [d_n(iP_1), A_{11} + B_{12} + C_{21}]_* \\
&+ [iP_1, d_n(A_{11}) + d_n(B_{12}) + d_n(C_{21})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(A_{11}) + d_s(B_{12}) + d_s(C_{21})]_* \\
&+ [d_n(iP_1), D_{22}]_* + [iP_1, d_n(D_{22})]_* \\
&+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_1), d_s(D_{22})]_*.
\end{aligned}$$

Comparing the above two equalities, it follows that  $[iP_1, M] = 0$ , and hence by Lemma 2,  $M_{11} = M_{12} = M_{21} = 0$ . Using the fact that  $[iP_2, A_{11}] = 0$  and the above similar arguments, we obtain  $[iP_2, M]_* = 0$  which leads to  $M_{22} = 0$ . This completes the proof.  $\square$

**Lemma 9.** For any  $A_{jk}, B_{jk} \in \mathfrak{A}_{jk}$ , where  $j, k \in 1, 2$ , we have

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(B_{jk})$$

*Proof.* We separate the proof in two distinct cases.

**Case I:**  $j \neq k$

On one side, by Lemma 8, we have

$$\begin{aligned} d_n(iA_{jk} + iB_{jk} + iA_{jk}^* + iB_{jk}A_{jk}^*) \\ = d_n(iA_{jk} + iB_{jk}) + d_n(iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*). \end{aligned}$$

On the other hand, using Lemmas 6 and 8, by induction, we have

$$\begin{aligned} d_n(iA_{jk} + iB_{jk} + iA_{jk}^* + iB_{jk}A_{jk}^*) &= d_n([iP_j + iA_{jk}, P_k + B_{jk}]_*) \\ &= [d_n(iP_j + iA_{jk}), P_k + B_{jk}]_* + [iP_j + iA_{jk}, d_n(P_k + B_{jk})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_j + iA_{jk}), d_s(P_k + B_{jk})]_* \\ &= [d_n(iP_j) + d_n(iA_{jk}), P_k + B_{jk}]_* \\ &\quad + [iP_j + iA_{jk}, d_n(P_k) + d_n(B_{jk})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_j) + d_r(iA_{jk}), d_s(P_k) + d_s(B_{jk})]_* \\ &= d_n([iP_j, P_k]_*) + d_n([iP_j, B_{jk}]_*) \\ &\quad + d_n([iA_{jk}, P_k]_*) + d_n([iA_{jk}, B_{jk}]_*) \\ &= d_n(iB_{jk}) + d_n(iA_{jk} + iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*) \\ &= d_n(iB_{jk}) + d_n(iA_{jk}) + d_n(iA_{jk}^*) + d_n(iB_{jk}A_{jk}^*). \end{aligned}$$

Comparing the above two equalities, we can conclude that

$$d_n(A_{jk} + B_{jk}) = d_n(A_{jk}) + d_n(A_{jk}^*).$$

**Case II:**  $j = k$ .

Let  $A_{jj}, B_{jj} \in \mathfrak{A}_{jj}$  and  $n \in \{1, 2\}$  with  $n \neq j$ . We have

$$\begin{aligned} 0 &= d_n([iP_n, A_{jj} + B_{jj}]_*) \\ &= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj} + B_{jj})]_* \\ &= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj} + B_{jj})]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})]_*. \end{aligned}$$

On the other hand we have,

$$\begin{aligned}
0 &= d_n([iP_n, A_{jj}]_*) + d_n([iP_n, B_{jj}]_*) \\
&= [d_n(iP_n), A_{jj}]_* + [iP_n, d_n(A_{jj})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj})]_* \\
&\quad + [d_n(iP_n), B_{jj}]_* + [iP_n, d_n(B_{jj})]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(B_{jj})]_* \\
&= [d_n(iP_n), A_{jj} + B_{jj}]_* + [iP_n, d_n(A_{jj}) + d_n(B_{jj})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iP_n), d_s(A_{jj}) + d_s(B_{jj})]_*.
\end{aligned}$$

Take  $M = d_n(A_{jj} + B_{jj}) - d_n(A_{jj}) - d_n(B_{jj})$ . The above computation yields that  $[iP_n, M]_* = 0$ . By Lemma 2, we have  $M_{nj} = M_{jn} = M_{nn} = 0$ . We now show that  $M_{jj} = 0$ . For any  $C_{jn} \in \mathfrak{A}_{jn}$ , using Case I, we compute

$$\begin{aligned}
d_n([A_{jj} + B_{jj}, C_{jn}]_*) &= [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj} + B_{jj}), d_s(C_{jn})]_* \\
&= [d_n(A_{jj} + B_{jj}), C_{jn}]_* + [A_{jj} + B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj}) + d_r(B_{jj}), d_s(C_{jn})]_*.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d_n([A_{jj} + B_{jj}, C_{jn}]_*) &= d_n(A_{jj}C_{jn} + B_{jj}C_{jn}) \\
&= d_n(A_{jj}C_{jn}) + d_n(B_{jj}C_{jn}) \\
&= d_n([A_{jj}, C_{jn}]_*) + d_n([B_{jj}, C_{jn}]_*) \\
&= [d_n(A_{jj}), C_{jn}]_* + [A_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A_{jj}), d_s(C_{jn})]_* \\
&\quad + [d_n(B_{jj}), C_{jn}]_* + [B_{jj}, d_n(C_{jn})]_* \\
&\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(B_{jj}), d_s(C_{jn})]_*.
\end{aligned}$$

Comparing the above two equalities, we obtain  $[M, C_{jn}]_* = 0$  which leads to  $M_{jj}C_{jn} = 0$ . Since  $\mathfrak{A}$  is prime, we see that  $M_{jj} = 0$ , which completes the proof. □

**Lemma 10.**  $d_n$  is an additive  $*$ -higher derivation on  $\mathfrak{A}$ .

*Proof.* We first show that  $d_n$  is additive. For arbitrary  $A, B \in \mathfrak{A}$ , we write  $A = \sum_{j,k=1}^2 A_{jk}$  and  $B = \sum_{j,k=1}^2 B_{jk}$ . It follows from Lemmas 8 and 9 that

$$\begin{aligned} d_n(A+B) &= d_n \left\{ \sum_{j,k=1}^2 (A_{jk} + B_{jk}) \right\} \\ &= \sum_{j,k=1}^2 d_n(A_{jk} + B_{jk}) \\ &= \sum_{j,k=1}^2 (d_n(A_{jk}) + d_n(B_{jk})) \\ &= d_n \left( \sum_{j,k=1}^2 A_{jk} \right) + d_n \left( \sum_{j,k=1}^2 B_{jk} \right) \\ &= d_n(A) + d_n(B). \end{aligned}$$

We now show that  $d_n(A^*) = d_n(A)^*$ .

For any  $A \in \mathfrak{A}$ , it follows from Lemmas 4 and 5 that

$$\begin{aligned} d_n(A^*) &= d_n(\Re A - i\Im A) = d_n(\Re A) - d_n(i\Im A) \\ &= d_n(\Re A) - id_n(\Im A) = d_n(\Re A)^* - id_n(\Im A)^* \\ &= d_n(\Re A)^* + (id_n(\Im A))^* = d_n(\Re A)^* + d_n(i\Im A)^* \\ &= (d_n(\Re A + i\Im A))^* = d_n(A)^*. \end{aligned}$$

To complete the proof, we need to show that  $d_n$  is a higher derivation on  $\mathfrak{A}$ .

Since  $d_n$  is additive, it follows from Lemma 5, that  $d_n(iI) = 0$ . It is to be noted that  $[iI + A, B]_* = 2iB + AB - BA^*$ .

$$\begin{aligned} d_n(2iB) + d_n(AB) - d_n(BA^*) &= d_n([iI + A, B]_*) \\ &= [d_n(iI + A), B]_* + [iI + A, d_n(B)]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iI + A), d_s(B)]_* \\ &= [d_n(iI) + d_n(A), B]_* + [iI + A, d_n(B)]_* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(iI) + d_r(A), d_s(B)]_* \\ &= [d_n(A), B]_* + [iI + A, d_n(B)]_* + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} [d_r(A), d_s(B)]_* \\ &= d_n(A)B - Bd_n(A)^* + 2id_n(B) + Ad_n(B) - d_n(B)A^* \\ &\quad + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) - d_s(B)d_r(A)^*). \end{aligned}$$

It follows that

$$\begin{aligned} d_n(AB) - d_n(BA^*) &= d_n(A)B - Bd_n(A)^* + Ad_n(B) - d_n(B)A^* \\ &+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) - d_s(B)d_r(A)^*). \end{aligned}$$

Replacing  $A$  by  $iA$  in the above equality, we get

$$\begin{aligned} d_n(AB) + d_n(BA^*) &= d_n(A)B + Bd_n(A)^* + Ad_n(B) + d_n(B)A^* \\ &+ \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} (d_r(A)d_s(B) + d_s(B)d_r(A)^*). \end{aligned}$$

Thus we have,

$$\begin{aligned} d_n(AB) &= d_n(A)B + Ad_n(B) + \sum_{\substack{r+s=n \\ 0 < r, s \leq n-1}} d_r(A)d_s(B) \\ &= \sum_{r+s=n} d_r(A)d_s(B). \end{aligned}$$

This shows that  $d_n$  is an additive higher derivation with  $d_n(A^*) = d_n(A)^*$ . Hence  $d_n$  is an additive  $*$ -higher derivation on  $\mathfrak{A}$ , which completes the proof.  $\square$

Note that every additive derivation  $d: \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$  is an inner derivation (see [12]). Nowicki [9] proved that if every additive (linear) derivation of  $\mathfrak{A}$  is inner, then every additive (linear) higher derivation of  $\mathfrak{A}$  is inner (see also [13]). So by Theorem 1, the following corollary is immediate.

**Corollary 1.** *Let  $\mathcal{H}$  be an infinite-dimensional complex Hilbert space and  $\mathfrak{A}$  be a standard operator algebra on  $\mathcal{H}$  containing identity operator  $I$ . If  $\mathfrak{A}$  is closed under the adjoint operation, then every nonlinear  $*$ -Lie higher derivation  $\mathcal{D} = \{d_n\}_{n \in \mathbb{N}}$  is inner with  $d_n(A^*) = d_n(A)^*$  for each  $A \in \mathfrak{A}$  and every  $n \in \mathbb{N}$ .*

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