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► **To cite this version:**

V. A. Rukavishnikov, E. V. Matveeva, E. I. Rukavishnikova. The Properties of the Weighted Space  $Hk_{2,\alpha}(\Omega)$  and Weighted Set  $Wk_{2,\alpha}(\Omega, \delta)$ . Communications in Mathematics, 2018, Volume 26 (2018), Issue 1 (1), pp.31 - 45. 10.2478/cm-2018-0004 . hal-03664948

**HAL Id: hal-03664948**

**<https://hal.science/hal-03664948v1>**

Submitted on 11 May 2022

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# The Properties of the Weighted Space $H_{2,\alpha}^k(\Omega)$ and Weighted Set $W_{2,\alpha}^k(\Omega, \delta)$

V.A. Rukavishnikov, E.V. Matveeva, E.I. Rukavishnikova

**Abstract.** We study the properties of the weighted space  $H_{2,\alpha}^k(\Omega)$  and weighted set  $W_{2,\alpha}^k(\Omega, \delta)$  for boundary value problem with singularity.

## 1 Introduction

A boundary value problem is said to possess a strong singularity if its solution  $u$  does not belong to the Sobolev space  $W_2^1(H^1)$  or, in other words, the Dirichlet integral of the solution  $u$  diverges.

Boundary value problems with strong singularity caused by the singularity in the initial data or by the internal properties of the solution are found in the physics of plasma and gas discharge, electrodynamics, nuclear physics, nonlinear optics, and other branches of physics. In some particular cases, numerical methods for problems of electrodynamics and quantum mechanics with strong singularity were constructed, based on separation of singular and regular components, mesh refinement near singular points, multiplicative extraction of singularities, etc. (see, e.g., [2]–[3], [8], [9], [11]).

Boundary value problems with weak singularity of a solution ( $u \in H^1$ ) is caused by the presence of corner points on the boundary of a domain and by a change in the type of boundary conditions are found in different mathematical models. By using special methods for extracting the singular part of the solution near corner points or applying grids refined toward the singularity point, it is possible

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2010 MSC: 46E35

Key words: weighted functional spaces, weighted functional sets, weighted Sobolev spaces.

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to construct high order accurate finite-element schemes (see, e.g., [1], [5]–[7], [10], [12]–[16], [27]).

In [18], it was suggested to define the solution of boundary value problem for second-order elliptic equation with singularity on a finite set of points belonging to boundary of a two-dimensional domain as an  $R_\nu$ -generalized solutions in the weighted Sobolev space. Such a new concept of solution led to the distinction of two classes of boundary value problems: problems with coordinated and uncoordinated degeneracy of input data; it also made it possible to study the existence and uniqueness of solutions as well as its coercivity and differential properties in the weighted spaces and weighted sets (see [18]–[21]). In [22]–[26], we constructed and investigated the finite-element method for different boundary value problems.

For investigation of the weighted finite element methods with high-degree accuracy for singular boundary value problems with coordinated and uncoordinated degeneration of input data we need to know properties of the weighted space  $H_{2,\alpha}^k(\Omega)$  and weighted set  $W_{2,\alpha}^k(\Omega, \delta)$ . In the present paper we study the properties of  $H_{2,\alpha}^k(\Omega)$  and  $W_{2,\alpha}^k(\Omega, \delta)$ .

## 2 Basic notations

We denote the two-dimensional Euclidean space by  $\mathbb{R}^2$  with  $x = (x_1, x_2)$  and  $dx = dx_1 dx_2$ . Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$ , and let  $\bar{\Omega}$  be the closure of  $\Omega$ , i.e.  $\bar{\Omega} = \Omega \cup \partial\Omega$ . We denote by  $\bigcup_{i=1}^n \tau_i$  a set of points  $\tau_i$ ,  $i = 1, \dots, n$ , belonging to  $\partial\Omega$ , including the points of intersection of its smooth pieces.

Let  $O_i^\delta$  be a disk of radius  $\delta > 0$  with its center in  $\tau_i$ ,  $i = 1, \dots, n$ , i.e.

$$O_i^\delta = \{x : \|x - \tau_i\| \leq \delta\},$$

and suppose that  $O_i^\delta \cap O_j^\delta = \emptyset$ ,  $i \neq j$ . Let  $\Omega' = \bigcup_{i=1}^n \Omega_i$ , where  $\Omega_i = \Omega \cap O_i^\delta$ ,  $i = 1, \dots, n$ .

Let  $\rho(x)$  be a function that is positive everywhere, except in  $\bigcup_{i=1}^n \tau_i$ , and satisfies the following conditions:

1.  $\rho(x) = \delta$  for  $x \in \Omega \setminus \bigcup_{i=1}^n O_i^\delta$ ,
2.  $\rho(x) = [(x_1 - x_1^{(i)})^2 + (x_2 - x_2^{(i)})^2]^{1/2}$ ,  $(x_1^{(i)}, x_2^{(i)}) = \tau_i$ , for  $x \in \Omega_i$ ,  $i = 1, \dots, n$ .

Moreover, it is assumed that

$$\left| \frac{\partial^{|\lambda|} \rho^k(x)}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}} \right| \leq \sigma \cdot \rho^{k-|\lambda|}(x).$$

We introduce the weighted spaces  $H_{2,\alpha}^k(\Omega)$  and  $W_{2,\alpha}^k(\Omega)$  with norms:

$$\|u\|_{H_{2,\alpha}^k(\Omega)} = \left( \sum_{|\lambda| \leq k} \int_{\Omega} \rho^{2(\alpha+|\lambda|-k)} |D^\lambda u|^2 dx \right)^{1/2},$$

$$\|u\|_{W_{2,\alpha}^k(\Omega)} = \left( \sum_{|\lambda| \leq k} \int_{\Omega} \rho^{2\alpha} |D^\lambda u|^2 dx \right)^{1/2}.$$

Here  $D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}}$ ,  $\lambda = (\lambda_1, \lambda_2)$  and  $|\lambda| = \lambda_1 + \lambda_2$ ;  $\lambda_1, \lambda_2$  are integer nonnegative numbers,  $\alpha, \sigma$  are some real nonnegative numbers,  $k$  is an integer nonnegative number. For  $k = 0$  we use the notation  $H_{2,\alpha}^0(\Omega) = W_{2,\alpha}^0(\Omega) = L_{2,\alpha}(\Omega)$ .

By  $W_{2,\alpha+l-1}^l(\Omega, \delta)$  for  $l \geq 1$  we denote a set of functions satisfying the following conditions:

- (a)  $|D^k u(x)| \leq c_1 \cdot \gamma^k \cdot k! \cdot (\rho^{\alpha+k}(x))^{-1}$  for  $x \in \Omega'$ , where  $k = 1, \dots, l$ , the constants  $c_1, \gamma \geq 1$  do not depend on  $k$ ;
- (b)  $\|u\|_{L_{2,\alpha}(\Omega \setminus \Omega')} \geq c_2 > 0$ ,  $c_2 = \text{const}$ ; with the squared norm

$$\|u\|_{W_{2,\alpha+l-1}^l(\Omega, \delta)}^2 = \sum_{|\lambda| \leq l} \|\rho^{\alpha+l-1} |D^\lambda u|\|_{L_2(\Omega)}^2.$$

Let  $L_{2,\alpha}(\Omega, \delta)$  be the set of functions satisfying conditions (a), (b) the norm

$$\|u\|_{L_{2,\alpha}(\Omega, \delta)} = \left( \int_{\Omega} \rho^{2\alpha} u^2 dx \right)^{1/2}.$$

Denote by  $r$  and  $\theta$  the polar coordinates in  $\Omega_i$ ,  $i = 1, \dots, n$ .

Introduce the space  $\mathcal{L}_{2,\alpha}(\Omega)$  with the squared norm

$$\|u(r, \theta)\|_{\mathcal{L}_{2,\alpha}(\Omega)}^2 = \int_{\Omega'} r^{2\alpha} u^2(r, \theta) ds + \int_{\Omega \setminus \Omega'} \delta^{2\alpha} u^2 dx,$$

where  $ds = r dr d\theta$ ,  $\alpha$  is a nonnegative real number.

By  $\mathcal{W}_{2,\alpha+l-1}^l(\Omega, \delta)$ , where  $\alpha$  is a nonnegative number and  $l$  is a nonnegative integer we denote the set of functions satisfying the following conditions:

- (a')  $|D^\lambda u(r, \theta)| \leq \tilde{c}_1 \cdot \tilde{\gamma}^{|\lambda|} \cdot |\lambda|! \cdot (r^{\alpha+\lambda_1})^{-1}$  for  $(r, \theta) \in \Omega'$ , where

$$D^\lambda u = \frac{\partial^{|\lambda|} u}{\partial r^{\lambda_1} \partial \theta^{\lambda_2}} = u_{r^{\lambda_1} \theta^{\lambda_2}},$$

$|\lambda| = 1, \dots, l$ ,  $\tilde{c}_1, \tilde{\gamma} \geq 1$  are constants independent of  $\lambda_1$ ;

- (b')  $\|u\|_{\mathcal{L}_{2,\alpha}(\Omega \setminus \Omega')} \geq \tilde{c}_2 > 0$ ; with the squared norm

$$\|u\|_{\mathcal{W}_{2,\alpha+l-1}^l(\Omega,\delta)}^2 = \sum_{|\lambda|\leq l} \|\rho^{\alpha+l-1-\lambda_2}|D^\lambda u|\|_{L_2(\Omega)}^2.$$

Let  $\mathcal{L}_{2,\alpha}(\Omega, \delta)$  be the set of functions satisfying conditions (a'), (b') with the squared norm

$$\|u(r, \theta)\|_{\mathcal{L}_{2,\alpha}(\Omega,\delta)}^2 = \int_{\dot{\Omega}'} r^{2\alpha} u^2(r, \theta) ds + \int_{\Omega \setminus \dot{\Omega}'} \delta^{2\alpha} u^2 dx.$$

### 3 The properties of the weighted space $H_{2,\alpha}^k(\Omega)$

**Lemma 1.** a) Let  $u \in H_{2,\alpha}^1(\Omega)$ , then  $\rho^\alpha u \in W_{2,0}^1(\Omega)$ ,  $\rho^{\alpha-1}u \in L_{2,0}(\Omega)$  and

$$|\rho^\alpha u|_{W_{2,0}^1(\Omega)} + \|\rho^{\alpha-1}u\|_{L_{2,0}(\Omega)} \leq c_3 \|u\|_{H_{2,\alpha}^1(\Omega)}, \quad (1)$$

where  $c_3$  is a positive constant independent of  $u$ .

b) Let  $\rho^\alpha u \in W_{2,0}^1(\Omega)$  and  $\rho^{\alpha-1}u \in L_{2,0}(\Omega)$ , then  $u \in H_{2,\alpha}^1(\Omega)$  and there exist positive constant  $c_4$  and  $c_5$  independent of  $u$  such that

$$c_4 |\rho^\alpha u|_{W_{2,0}^1(\Omega)} + c_5 \|\rho^{\alpha-1}u\|_{L_{2,0}(\Omega)} \geq \|u\|_{H_{2,\alpha}^1(\Omega)}, \quad (2)$$

is valid.

*Proof.* a) We estimate the first term in the left-hand side of (1)

$$\begin{aligned} |\rho^\alpha u|_{W_{2,0}^1(\Omega)}^2 &= \sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial}{\partial x_i} (\rho^\alpha u) \right)^2 dx = \sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial \rho^\alpha}{\partial x_i} u + \rho^\alpha \frac{\partial u}{\partial x_i} \right)^2 dx \\ &\leq 2 \sum_{i=1}^2 \int_{\Omega} \left[ \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right)^2 + \rho^{2\alpha} \left( \frac{\partial u}{\partial x_i} \right)^2 \right] dx \\ &\leq 4\delta^2 \alpha^2 \|u\|_{L_{2,\alpha-1}(\Omega)}^2 + 2|u|_{W_{2,\alpha}^1(\Omega)}^2 \\ &\leq \max\{4\delta^2 \alpha^2, 2\} \|u\|_{H_{2,\alpha}^1(\Omega)}^2. \end{aligned} \quad (3)$$

By Lemma 1 a)  $u$  belongs to  $H_{2,\alpha}^1(\Omega)$ ; therefore the second term in the left-hand side of (1) is estimated via the norm  $\|u\|_{H_{2,\alpha}^1(\Omega)}^2$ .

Then from this statement and estimate (3) we establish (1).

b) We write the obvious inequality

$$|\rho^\alpha u|_{W_{2,0}^1(\Omega)}^2 \geq \sum_{i=1}^2 \int_{\Omega} \left[ \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right)^2 - 2 \left| \frac{\partial \rho^\alpha}{\partial x_i} u \rho^\alpha \frac{\partial u}{\partial x_i} \right| + \rho^{2\alpha} \left( \frac{\partial u}{\partial x_i} \right)^2 \right] dx. \quad (4)$$

We estimate the second term in the right-hand side of (4) using the Cauchy-Schwarz inequality,  $\varepsilon$ -inequality and we have

$$\begin{aligned} 2 \sum_{i=1}^2 \int_{\Omega} \left| \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right) \left( \rho^\alpha \frac{\partial u}{\partial x_i} \right) \right| dx \\ \leq \varepsilon \sum_{i=1}^2 \int_{\Omega} \rho^{2\alpha} \left( \frac{\partial u}{\partial x_i} \right)^2 dx + \frac{1}{\varepsilon} \sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right)^2 dx, \end{aligned}$$

where  $\varepsilon > 0$ .

Using this inequality we strengthen (4)

$$|\rho^\alpha u|_{W_{2,0}^1(\Omega)}^2 \geq (1 - \varepsilon) |u|_{H_{2,\alpha}^1(\Omega)}^2 - \left( \frac{1}{\varepsilon} - 1 \right) \sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right)^2 dx. \quad (5)$$

Then, taking into account the inequality

$$\sum_{i=1}^2 \int_{\Omega} \left( \frac{\partial \rho^\alpha}{\partial x_i} u \right)^2 dx \leq 4\delta^2 \alpha^2 \|u\|_{L_{2,\alpha-1}(\Omega)}^2,$$

choosing  $\varepsilon < 1$ , from (5) we get the estimate

$$\frac{1}{1 - \varepsilon} |\rho^\alpha u|_{W_{2,0}^1(\Omega)}^2 + \frac{4\delta^2 \alpha^2}{\varepsilon} \|\rho^{\alpha-1} u\|_{L_{2,0}(\Omega)}^2 \geq |u|_{H_{2,\alpha}^1(\Omega)}^2. \quad (6)$$

We add to both sides of (6)  $\|u\|_{L_{2,\alpha-1}(\Omega)}^2$  and apply the inequality  $(|a| + |b|)^2 \geq a^2 + b^2$  to its right-hand side. Then we extract the square root from both sides of this relation and obtain (2).  $\square$

**Theorem 1.** a) If  $u \in H_{2,\alpha}^k(\Omega)$ , then  $\rho^{\alpha-(k-s)} u \in W_{2,0}^s(\Omega)$ ,  $s = 1, \dots, k$  and

$$|\rho^\alpha u|_{W_{2,0}^k(\Omega)} + \dots + |\rho^{\alpha-k} u|_{L_{2,0}(\Omega)} \leq c_6 \|u\|_{H_{2,\alpha}^k(\Omega)}, \quad (7)$$

where  $c_6$  is a positive constant independent of  $u$ .

b) If  $\rho^{\alpha-(k-s)} u \in W_{2,0}^s(\Omega)$ ,  $s = 1, \dots, k$ , then  $u \in H_{2,\alpha}^k(\Omega)$  and there exist positive constants  $c_0^*, \dots, c_k^*$  independent of  $u$  such that the inequality

$$c_k^* |\rho^\alpha u|_{W_{2,0}^k(\Omega)} + \dots + c_0^* |\rho^{\alpha-k} u|_{L_{2,0}(\Omega)} \geq \|u\|_{H_{2,\alpha}^k(\Omega)} \quad (8)$$

is valid.

*Proof.* The theorem will be proved by induction on  $k_1$  ( $0 \leq k_1 < k$ ).

- a) By the condition of the theorem  $u \in H_{2,\alpha}^k(\Omega)$ . It is obvious, that  $\rho^{\alpha-k}u \in L_{2,0}(\Omega)$ , i.e. for  $k_1 = 0$  the statement of theorem a) is proved.

Assume that for some number  $k_1$  ( $0 \leq k_1 < k$ ) the functions  $\rho^{\alpha-(k-s)}u$  belong to the space  $W_{2,0}^s(\Omega)$  for all  $s$  ( $0 \leq s \leq k_1$ ) and the inequality

$$\left| \rho^{\alpha-(k-k_1)}u \right|_{W_{2,0}^{k_1}(\Omega)} + \dots + \left| \rho^{\alpha-k}u \right|_{L_{2,0}(\Omega)} \leq c_7 \|u\|_{H_{2,\alpha}^{k_1-(k-k_1)}(\Omega)} \quad (9)$$

holds.

We show that  $\rho^{\alpha-(k-k_1-1)}u \in W_{2,0}^{k_1+1}(\Omega)$ . To this end, we estimate the seminorm  $\left| \rho^{\alpha-(k-k_1-1)}u \right|_{W_{2,0}^{k_1+1}(\Omega)}$ . We have

$$\begin{aligned} & \left| \rho^{\alpha-(k-k_1-1)}u \right|_{W_{2,0}^{k_1+1}(\Omega)}^2 \\ &= \sum_{|\lambda|=k_1+1} \int_{\Omega} \left[ \sum_{i=0}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right]^2 dx. \end{aligned} \quad (10)$$

For any  $\lambda = (\lambda_1, \lambda_2)$  and  $|\lambda| = k_1 + 1$  the inequality

$$\begin{aligned} & \int_{\Omega} \left[ \sum_{i=0}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right]^2 dx \\ & \leq (k_1 + 2) \int_{\Omega} \left[ \sum_{i=0}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right]^2 dx \end{aligned} \quad (11)$$

is valid. Here  $C_{k_1+1}^i$  is a combination of  $k_1 + 1$  things  $i$  at a time.

It is easy to note that

$$\left| D^i \rho^{\alpha-(k-k_1-1)}(x) \right| \leq c_8 \rho^{\alpha-(k-k_1-1)-i}(x), \quad x \in \bar{\Omega}. \quad (12)$$

From (10) and (11), (12) we have

$$\left| \rho^{\alpha-(k-k_1-1)}u \right|_{W_{2,0}^{k_1+1}(\Omega)} \leq c_9 \|u\|_{H_{2,\alpha}^{k_1+1-(k-k_1-1)}(\Omega)}$$

and  $\rho^{\alpha-(k-k_1-1)}u \in W_{2,0}^{k_1+1}(\Omega)$ . By virtue of induction the statement of Theorem 1 a) is proved.

- b) If  $0 \leq s \leq k$  then according to conditions of the Theorem 1 b) the functions  $\rho^{\alpha-(k-s)}u$  belong to the spaces  $W_{2,0}^s(\Omega)$ .

For  $s = 0$   $\rho^{\alpha-k}u \in W_{2,0}^0(\Omega)$  or  $u \in L_{2,\alpha-k}(\Omega)$ , i.e. for  $k_1 = 0$  the statement of the Theorem 1 b) is valid.

We suppose that  $u \in H_{2,\alpha-(k-k_1)}^{k_1}(\Omega)$  for some number  $k_1$  ( $0 \leq k_1 < k$ ) and the inequality

$$\tilde{c}_{k_1}^* \left| \rho^{\alpha-(k-k_1)} u \right|_{W_{2,0}^{k_1}(\Omega)} + \dots + \tilde{c}_0^* \left| \rho^{\alpha-k} u \right|_{L_{2,0}(\Omega)} \geq \|u\|_{H_{2,\alpha-(k-k_1)}^{k_1}(\Omega)} \quad (13)$$

holds. Here  $\tilde{c}_{k_1}^*, \dots, \tilde{c}_0^*$  are positive constants independent of  $u$ .

We show that the function  $u$  belongs to the space  $H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)$ . To this end we estimate the seminorm  $|u|_{H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)}$ . The proof of this estimate is similar to the proof of Lemma 1 b). Thus

$$\begin{aligned} & \left| \rho^{\alpha-(k-k_1-1)} u \right|_{W_{2,0}^{k_1+1}(\Omega)}^2 \\ &= \sum_{|\lambda|=k_1+1} \int_{\Omega} \left[ \sum_{i=0}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right]^2 dx \\ &\geq |u|_{H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)}^2 - 2 \sum_{|\lambda|=k_1+1} \left\{ \left| \rho^{\alpha-(k-k_1-1)} D^{\lambda} u \right|_{L_{2,0}(\Omega)} \times \right. \\ &\quad \left. \times \left[ \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right)^2 dx \right]^{1/2} \right\} \\ &+ \sum_{|\lambda|=k_1+1} \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right)^2 dx. \end{aligned} \quad (14)$$

By means of the Cauchy-Schwarz inequality,  $\varepsilon$ -inequality we estimate the second term in the right-hand side of (14)

$$\begin{aligned} & 2 \sum_{|\lambda|=k_1+1} \left\{ \left| \rho^{\alpha-(k-k_1-1)} D^{\lambda} u \right|_{L_{2,0}(\Omega)} \times \right. \\ &\quad \left. \times \left[ \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right)^2 dx \right]^{1/2} \right\} \\ &\leq \varepsilon |u|_{H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)}^2 \\ &\quad + \frac{1}{\varepsilon} \sum_{|\lambda|=k_1+1} \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{k_1+1-i} u \right)^2 dx, \end{aligned}$$

where  $\varepsilon > 0$ .

Using the last inequality and choosing  $\varepsilon$  such that  $\varepsilon < 1$  we write (14) in the



form

$$|u|_{H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)}^2 \leq \frac{1}{1-\varepsilon} \left| \rho^{\alpha-(k-k_1-1)} u \right|_{W_{2,0}^{k_1+1}(\Omega)}^2 + \frac{1}{\varepsilon} \sum_{|\lambda|=k_1+1} \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right)^2 dx. \quad (15)$$

Using inequalities  $\left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2$  and (12) we estimate the second term in the right-hand side (15). As a result we have

$$\frac{1}{\varepsilon} \sum_{|\lambda|=k_1+1} \int_{\Omega} \left( \sum_{i=1}^{k_1+1} C_{k_1+1}^i \left( D^i \rho^{\alpha-(k-k_1-1)} \right) D^{\lambda-i} u \right)^2 dx \leq c_{10} \|u\|_{H_{2,\alpha-(k-k_1)}^{k_1}(\Omega)}^2. \quad (16)$$

Taking into account that  $u$  belongs to  $H_{2,\alpha-(k-k_1)}^{k_1}(\Omega)$ , the conditions of Theorem 1 b) and (16), (15), (13) we obtain the estimate

$$|u|_{H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)} \leq \tilde{c}_{k_1+1}^* \left| \rho^{\alpha-(k-k_1-1)} u \right|_{W_{2,0}^{k_1+1}(\Omega)} + \dots + \tilde{c}_0^* \left| \rho^{\alpha-k} u \right|_{L_{2,0}(\Omega)}$$

and  $u \in H_{2,\alpha-(k-k_1-1)}^{k_1+1}(\Omega)$ . Thus the statement of theorem 1 b) is proved.  $\square$

#### 4 Properties of the functions of the set of $W_{2,\alpha}^k(\Omega, \delta)$

**Lemma 2.** For any function  $u$  in the set  $W_{2,\alpha}^1(\Omega, \delta)$  there exist a parameter  $\alpha^*$  that

$$\|u\|_{L_{2,\alpha^*-1}(\Omega', \delta)} \leq c_{11} \|u\|_{L_{2,\alpha^*}(\Omega, \delta)}, \quad (17)$$

where  $0 < c_{11} < 1$ .

*Proof.* Taking into account condition (a), one can show that for  $\alpha_1 > \alpha + 1$  we have

$$\|u\|_{L_{2,\alpha_1-1}(\Omega_i, \delta)}^2 = \int_{\Omega_i} \rho^{2(\alpha_1-1)} u^2 dx \leq c_1^2 \int_{\Omega_i} \rho^{-2\alpha} \rho^{2(\alpha_1-1)} dx \leq \frac{c_1^2 c_{12}^{(i)} \delta^{2(\alpha_1-\alpha)}}{2(\alpha_1-\alpha)},$$

where  $c_{12}^{(i)}$  is a constant. Here, the index  $i$  is determined in  $\Omega_i$ . Then

$$\|u\|_{L_{2,\alpha_1-1}(\Omega', \delta)}^2 \leq \frac{c_1^2 \cdot c_{12} \delta^{2(\alpha_1-\alpha)}}{2(\alpha_1-\alpha)}. \quad (18)$$

From the second condition (b), for  $u \in W_{2,\alpha}^1(\Omega, \delta)$  we write the inequality

$$\begin{aligned} \|u\|_{L_{2,\alpha_1}(\Omega,\delta)}^2 &\geq \|u\|_{L_{2,\alpha_1}(\Omega\setminus\Omega',\delta)}^2 = \int_{\Omega\setminus\Omega'} \rho^{2\alpha_1} u^2 dx \\ &= \int_{\Omega\setminus\Omega'} \rho^{2(\alpha_1-\alpha)} \rho^{2\alpha} u^2 dx = \delta^{2(\alpha_1-\alpha)} \|u\|_{L_{2,\alpha}(\Omega\setminus\Omega',\delta)}^2 \geq c_2^2 \delta^{2(\alpha_1-\alpha)}. \end{aligned} \quad (19)$$

Obviously there is  $\alpha_1$ , which is denoted by  $\alpha^*$ , such that from (18), (19) estimate (17) follows with constant  $c_{11}$  ( $0 < c_{11} < 1$ ).  $\square$

**Remark 1.** For any function  $u$  in the set  $W_{2,\alpha}^{k+1}(\Omega, \delta)$  there exists a parameter  $\alpha^*$ , that

$$\|u\|_{W_{2,\alpha^*-1}^k(\Omega',\delta)} \leq c_{13} \|u\|_{W_{2,\alpha^*}^k(\Omega,\delta)},$$

where  $0 < c_{13} < 1$ .

**Remark 2.** If  $u \in W_{2,\alpha^*}^1(\Omega, \delta)$ , then  $\rho^{\alpha^*} u \in W_{2,0}^1(\Omega, \delta)$  and

$$|\rho^{\alpha^*} u|_{W_{2,0}^1(\Omega,\delta)} \leq c_{14} \|u\|_{W_{2,\alpha^*}^1(\Omega,\delta)},$$

where  $c_{14}$  is a positive constant independent of  $u$ .

**Lemma 3.** a) Let  $\rho^{\alpha^*+1} u \in W_{2,0}^2(\Omega, \delta)$  and  $u \in W_{2,\alpha^*}^1(\Omega, \delta)$ .

Then  $u \in W_{2,\alpha^*+1}^2(\Omega, \delta)$  and there exist positive constants  $c_{15}$  and  $c_{16}$  independent of  $u$  such that

$$|u|_{W_{2,\alpha^*+1}^2(\Omega,\delta)}^2 \leq c_{15} |\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega,\delta)}^2 + c_{16} \left( \|u\|_{W_{2,\alpha^*}^1(\Omega',\delta)}^2 + \|u\|_{L_{2,\alpha^*}(\Omega,\delta)}^2 \right).$$

b) Let  $u \in W_{2,\alpha^*}^1(\Omega, \delta)$  and  $|u|_{W_{2,\alpha^*+1}^2(\Omega,\delta)}$  is a finite seminorm.

Then  $\rho^{\alpha^*+1} u \in W_{2,0}^2(\Omega, \delta)$  and there exist positive constants  $c_{17}$  and  $c_{18}$  independent of  $u$  such that

$$\left| \rho^{\alpha^*+1} u \right|_{W_{2,0}^2(\Omega,\delta)}^2 \leq c_{17} |u|_{W_{2,\alpha^*+1}^2(\Omega,\delta)}^2 + c_{18} \left( |u|_{W_{2,\alpha^*}^1(\Omega',\delta)}^2 + \|u\|_{L_{2,\alpha^*}(\Omega,\delta)}^2 \right). \quad (20)$$

*Proof.* a) We write the inequality

$$\begin{aligned} |\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega,\delta)}^2 &\geq \int_{\Omega} \sum_{l=1}^2 \left\{ \left( \frac{\partial^2 u}{\partial x_l^2} \right)^2 \rho^{2(\alpha^*+1)} - 2 \left| \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right| \times \right. \\ &\times \left| \frac{\partial^2 u}{\partial x_l^2} \rho^{\alpha^*+1} \right| + \left( \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right)^2 + \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right)^2 \\ &- 2 \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right| \left| \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right| \\ &\left. + \left( \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right)^2 \right\} dx. \quad (21) \end{aligned}$$

We estimate the second and fifth terms in the right-hand side of (21) using  $\varepsilon$ -inequality and we have

$$\begin{aligned} & 2 \sum_{l=1}^2 \int_{\Omega} \left| \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right| \left| \frac{\partial^2 u}{\partial x_l^2} \rho^{\alpha^*+1} \right| dx \leq \\ & \leq \varepsilon \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial^2 u}{\partial x_l^2} \rho^{\alpha^*+1} \right)^2 dx + \frac{1}{\varepsilon} \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right)^2 dx, \end{aligned} \quad (22)$$

$$\begin{aligned} & 2 \sum_{l=1}^2 \int_{\Omega} \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right| \left| \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right| dx \\ & \leq \varepsilon \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right)^2 dx \\ & \quad + \frac{1}{\varepsilon} \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right)^2 dx \end{aligned} \quad (23)$$

where  $\varepsilon > 0$ .

Using right-hand sides of these inequalities we strengthen (21)

$$\begin{aligned} |\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega,\delta)}^2 & \geq (1-\varepsilon) \int_{\Omega} \left\{ \sum_{l=1}^2 \left( \frac{\partial^2 u}{\partial x_l^2} \rho^{\alpha^*+1} \right)^2 + \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right)^2 \right\} dx \\ & \quad + \left( 1 - \frac{1}{\varepsilon} \right) \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right)^2 dx \\ & \quad + \left( 1 - \frac{1}{\varepsilon} \right) \int_{\Omega} \left( \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right)^2 dx. \end{aligned} \quad (24)$$

By means of the inequality  $\left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2$  we estimate the second and third terms in the right-hand side of 24

$$\begin{aligned} & \sum_{l=1}^2 \int_{\Omega} \left( \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right)^2 dx \\ & \leq 8(\alpha^* + 1)^2 \sum_{l=1}^2 \int_{\Omega'} \rho^{2\alpha^*} \left( \frac{\partial u}{\partial x_l} \right)^2 dx + 4(\alpha^* + 1)^2 (\alpha^*)^2 \int_{\Omega'} \rho^{2(\alpha^*-1)} u^2 dx \\ & \leq 8(\alpha^* + 1)^2 |u|_{W_{2,\alpha^*}^1(\Omega',\delta)}^2 + 4(\alpha^* + 1)^2 (\alpha^*)^2 c_{11}^2 \|u\|_{L_{2,\alpha^*}(\Omega,\delta)}^2, \end{aligned} \quad (25)$$

$$\int_{\Omega} \left( \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right)^2 dx$$

$$\leq 3(\alpha^* + 1)^2 |u|_{W_{2,\alpha^*}^1(\Omega', \delta)}^2 + 3(\alpha^* + 1)^2 (\alpha^*)^2 c_{11}^2 \|u\|_{L_{2,\alpha^*}(\Omega, \delta)}^2. \quad (26)$$

Using inequalities (25) and (26) we write (24) in the form

$$|\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega, \delta)}^2 \geq (1 - \varepsilon) |u|_{W_{2,\alpha^*+1}^2(\Omega, \delta)}^2$$

$$- \frac{1 - \varepsilon}{\varepsilon} (\alpha^* + 1)^2 \left( 11 |u|_{W_{2,\alpha^*}^1(\Omega', \delta)}^2 + 7(\alpha^*)^2 c_{11}^2 \|u\|_{L_{2,\alpha^*}(\Omega, \delta)}^2 \right).$$

Choosing  $\varepsilon < 1$  we obtain

$$|u|_{W_{2,\alpha^*+1}^2(\Omega, \delta)}^2 \leq \frac{1}{1 - \varepsilon} |\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega, \delta)}^2$$

$$+ \frac{1}{\varepsilon} (\alpha^* + 1)^2 \max\{11, 7(\alpha^*)^2 c_{11}^2\} \left( |u|_{W_{2,\alpha^*}^1(\Omega', \delta)}^2 + \|u\|_{L_{2,\alpha^*}(\Omega, \delta)}^2 \right).$$

Thus the statement of Lemma 1 a) is proved.

b) We write the inequality

$$|\rho^{\alpha^*+1} u|_{W_{2,0}^2(\Omega, \delta)}^2 \leq \sum_{l=1}^2 \int_{\Omega} \left\{ \left( \frac{\partial^2 u}{\partial x_l^2} \right)^2 \rho^{2(\alpha^*+1)} \right.$$

$$+ 2 \left| \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right| \left| \frac{\partial^2 u}{\partial x_l^2} \rho^{\alpha^*+1} \right| + \left( \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_l^2} u + 2 \frac{\partial \rho^{\alpha^*+1}}{\partial x_l} \frac{\partial u}{\partial x_l} \right)^2 \left. \right\} dx$$

$$+ \int_{\Omega} \left\{ \left( \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right)^2 + \right.$$

$$+ 2 \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \rho^{\alpha^*+1} \right| \left| \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right|$$

$$\left. + \left( \frac{\partial u}{\partial x_1} \frac{\partial \rho^{\alpha^*+1}}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial \rho^{\alpha^*+1}}{\partial x_1} + \frac{\partial^2 \rho^{\alpha^*+1}}{\partial x_1 \partial x_2} u \right)^2 \right\} dx.$$

Then from this statement and estimates (22), (23), (25) and (26) we establish (20).  $\square$

**Theorem 2.** Let  $\rho^p u_{r^k} \in \mathcal{W}_{2,\alpha^*+1}^2(\Omega, \delta)$  and  $u \in \mathcal{W}_{2,\alpha^*+p+m-(k+1)}^m(\Omega, \delta)$  for  $m = 2, \dots, k+1$ . Then  $u \in \mathcal{W}_{2,\alpha^*+p+1}^{k+2}(\Omega, \delta)$  and there exist positive constants  $c_{19}$  and  $c_{20}$  independent of  $u$  such that the estimate

$$|u|_{\mathcal{W}_{2,\alpha^*+p+1}^{k+2}(\Omega, \delta)}^2 \leq c_{19} |\rho^p u_{r^k}|_{\mathcal{W}_{2,\alpha^*+1}^2(\Omega, \delta)}^2 + c_{20} \sum_{\substack{\lambda_1=0 \\ |\lambda|=k+2}}^{k+1} \|\rho^{\alpha^*+p+1-\lambda_2} |D^\lambda u|\|_{\mathcal{L}_2(\Omega, \delta)}^2$$

is valid.

*Proof.* We lower estimate seminorm function  $\rho^p u_{r^k}$  in the set  $\mathcal{W}_{2, \alpha^*+1}^2(\Omega, \delta)$ . We obtain

$$\begin{aligned}
|\rho^p u_{r^k}|_{\mathcal{W}_{2, \alpha^*+1}^2(\Omega, \delta)}^2 &= \int_{\Omega'} r^{2(\alpha^*+1)} \left\{ \left( \frac{\partial^2}{\partial r^2} (r^p u_{r^k}) \right)^2 \right. \\
&\quad \left. + \left( \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} (r^p u_{r^k}) \right)^2 + \left( \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (r^p u_{r^k}) \right)^2 \right\} ds + \int_{\Omega \setminus \Omega'} \delta^{2(\alpha^*+1)} |D^2 u|^2 dx \\
&= \int_{\Omega'} r^{2(\alpha^*+1)} \left\{ (r^p u_{r^{k+2}} + 2pr^{p-1} u_{r^{k+1}} + p(p-1)r^{p-2} u_{r^k})^2 \right. \\
&\quad \left. + (r^{p-1} u_{r^{k+1}\theta} + pr^{p-2} u_{r^k\theta})^2 + (r^{p-2} u_{r^k\theta^2})^2 \right\} ds + \int_{\Omega \setminus \Omega'} \delta^{2(\alpha^*+1)} |D^2 u|^2 dx \\
&\geq \int_{\Omega'} r^{2(\alpha^*+1)} \left\{ r^{2p} u_{r^{k+2}}^2 - 2|r^p u_{r^{k+2}}| |2pr^{p-1} u_{r^{k+1}} + p(p-1)r^{p-2} u_{r^k}| + \right. \\
&\quad \left. + (2pr^{p-1} u_{r^{k+1}} + p(p-1)r^{p-2} u_{r^k})^2 \right\} ds + \int_{\Omega'} r^{2(\alpha^*+1)} \left\{ r^{2(p-1)} u_{r^{k+1}\theta}^2 \right. \\
&\quad \left. - 2|r^{p-1} u_{r^{k+1}\theta}| |pr^{p-2} u_{r^k\theta}| + p^2 r^{2(p-2)} u_{r^k\theta}^2 + r^{2(p-2)} u_{r^k\theta^2}^2 \right\} ds. \tag{27}
\end{aligned}$$

Using  $\varepsilon$ -inequality we estimate the second and sixth terms in the right-hand side of (27)

$$\begin{aligned}
&2 \int_{\Omega'} r^{2(\alpha^*+1)} |r^p u_{r^{k+2}}| |2pr^{p-1} u_{r^{k+1}} + p(p-1)r^{p-2} u_{r^k}| ds \\
&\leq \varepsilon \int_{\Omega'} r^{2(\alpha^*+p+1)} u_{r^{k+2}}^2 ds + \frac{1}{\varepsilon} \int_{\Omega'} r^{2(\alpha^*+1)} (2pr^{p-1} u_{r^{k+1}} + p(p-1)r^{p-2} u_{r^k})^2 ds,
\end{aligned}$$

$$\begin{aligned}
&2 \int_{\Omega'} r^{2(\alpha^*+1)} |r^{p-1} u_{r^{k+1}\theta}| |pr^{p-2} u_{r^k\theta}| \\
&\leq \varepsilon \int_{\Omega'} r^{2(\alpha^*+p)} u_{r^{k+1}\theta}^2 ds + \frac{p^2}{\varepsilon} \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r^k\theta}^2 ds.
\end{aligned}$$

Substituting the right-hand side of this inequality with a minus sign in (27), we

obtain

$$\begin{aligned}
|\rho^p u_{r,k}|_{\mathcal{W}_{2,\alpha^*+1}^2(\Omega,\delta)}^2 &\geq (1-\varepsilon) \int_{\Omega'} r^{2(\alpha^*+p+1)} u_{r,k+2}^2 \, ds + (1-\varepsilon) \int_{\Omega'} r^{2(\alpha^*+p)} u_{r,k+1\theta}^2 \, ds \\
&\quad + \left(1 - \frac{1}{\varepsilon}\right) \int_{\Omega'} r^{2(\alpha^*+1)} \left(2pr^{p-1} u_{r,k+1} + p(p-1)r^{p-2} u_{r,k}\right)^2 \, ds \\
&\quad + p^2 \left(1 - \frac{1}{\varepsilon}\right) \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r,k\theta}^2 \, ds + \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r,k\theta^2}^2 \, ds.
\end{aligned} \tag{28}$$

Taking into account that

$$\begin{aligned}
&\int_{\Omega'} r^{2(\alpha^*+1)} \left(2pr^{p-1} u_{r,k+1} + p(p-1)r^{p-2} u_{r,k}\right)^2 \, ds \\
&\leq 8p^2 \int_{\Omega'} r^{2(\alpha^*+p)} u_{r,k+1}^2 \, ds + 2p^2(p-1)^2 \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r,k}^2 \, ds,
\end{aligned}$$

from inequality (28) we get

$$\begin{aligned}
|\rho^p u_{r,k}|_{\mathcal{W}_{2,\alpha^*+1}^2(\Omega,\delta)}^2 &\geq (1-\varepsilon) \int_{\Omega'} r^{2(\alpha^*+p+1)} \left[ u_{r,k+2}^2 + \left(\frac{1}{r} u_{r,k+1\theta}^2\right)^2 + \left(\frac{1}{r^2} u_{r,k\theta^2}^2\right)^2 \right] \, ds \\
&\quad - \left(1 - \frac{1}{\varepsilon}\right) \left[ 8p^2 \int_{\Omega'} r^{2(\alpha^*+p)} u_{r,k+1}^2 \, ds + 2p^2(p-1)^2 \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r,k}^2 \, ds \right. \\
&\quad \left. + p^2 \int_{\Omega'} r^{2(\alpha^*+p-1)} u_{r,k\theta}^2 \, ds \right].
\end{aligned}$$

Choosing  $\varepsilon < 1$  we obtain

$$\begin{aligned}
&\int_{\Omega'} r^{2(\alpha^*+p+1)} \left[ u_{r,k+2}^2 + \left(\frac{1}{r} u_{r,k+1\theta}^2\right)^2 + \left(\frac{1}{r^2} u_{r,k\theta^2}^2\right)^2 \right] \, ds \\
&\leq \frac{1}{1-\varepsilon} |\rho^p u_{r,k}|_{\mathcal{W}_{2,\alpha^*+p}^2(\Omega,\delta)}^2 + \frac{p^2}{\varepsilon} \int_{\Omega'} r^{2(\alpha^*+p)} \left[ 8u_{r,k+1}^2 + 2(p-1)^2 \frac{u_{r,k}^2}{r^2} + \frac{u_{r,k\theta}^2}{r^2} \right] \, ds.
\end{aligned}$$

Add to both sides of the last inequality the sum of the form

$$\sum_{\substack{\lambda_1=0 \\ |\lambda|=k+2}}^{k-1} \|\rho^{\alpha^*+p+1-\lambda_2} |D^\lambda u|\|_{\mathcal{L}_2(\Omega,\delta)}^2 + \int_{\Omega \setminus \Omega'} \delta^{2(\alpha^*+p+1)} |D^2 u|^2 \, dx.$$

Then

$$\begin{aligned}
|u|_{\mathcal{W}_{2,\alpha^*+p+1}^{k+2}(\Omega,\delta)}^2 &\leq \frac{1}{1-\varepsilon} |\rho^p u_{r^k}|_{\mathcal{W}_{2,\alpha^*+p}^2(\Omega,\delta)}^2 \\
&\quad + \frac{p^2}{\varepsilon} \int_{\Omega'} r^{2(\alpha^*+p)} \left[ 8u_{r^{k+1}}^2 + 2(p-1)^2 \frac{u_{r^k}^2}{r^2} + \frac{u_{r^k\theta}^2}{r^2} \right] ds \\
&\quad + \sum_{\substack{\lambda_1=0 \\ |\lambda|=k+2}}^{k-1} \|\rho^{\alpha^*+p+1-\lambda_2} |D^\lambda u|\|_{\mathcal{L}_2(\Omega,\delta)}^2 + \int_{\Omega \setminus \Omega'} \delta^{2(\alpha^*+p+1)} |D^2 u|^2 dx \\
&\leq \frac{1}{1-\varepsilon} |\rho^p u_{r^k}|_{\mathcal{W}_{2,\alpha^*+p}^2(\Omega,\delta)}^2 + c_{20} \sum_{\substack{\lambda_1=0 \\ |\lambda|=k+2}}^{k+1} \|\rho^{\alpha^*+p+1-\lambda_2} |D^\lambda u|\|_{\mathcal{L}_2(\Omega,\delta)}^2,
\end{aligned}$$

where  $c_{20} = 2 \max \left\{ \frac{8p^2}{\varepsilon}, \frac{(p-1)^2 p^2}{\varepsilon}, 1 \right\}$ . □

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Received: 12 February, 2018

Accepted for publication: 2 April, 2018

Communicated by: Ari Laptev