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Estimating the critical determinants of a class of three-dimensional star bodies

Werner Georg Nowak

Abstract. In the problem of (simultaneous) Diophantine approximation in \mathbb{R}^3 (in the spirit of Hurwitz's theorem), lower bounds for the critical determinant of the special three-dimensional body

$$K_2 : (y^2 + z^2)(x^2 + y^2 + z^2) \leq 1$$

play an important role; see [1], [6]. This article deals with estimates from below for the critical determinant $\Delta(K_c)$ of more general star bodies

$$K_c : (y^2 + z^2)^{c/2}(x^2 + y^2 + z^2) \leq 1,$$

where c is any positive constant. These are obtained by inscribing into K_c either a double cone, or an ellipsoid, or a double paraboloid, depending on the size of c .

1 Introduction

During the last couple of decades, not much research has been done in the subfield of the *Geometry of Numbers* (see, e.g., the monograph by Gruber & Lekkerkerker [4]) which is concerned with the evaluation, or at least estimation, of *critical determinants* $\Delta(K)$ of starbodies K in \mathbb{R}^s , $s \geq 2$. These are defined as $\Delta(K) = \inf |\det A|$, where A ranges over all nonsingular real $(s \times s)$ -matrices, such that the origin is the only point of the lattice $A\mathbb{Z}^s$ in the interior of K .

It is the author's aim to rouse new interest in this classic topic *a fortiori* in view of its close connection to *simultaneous Diophantine approximation* in the spirit of Hurwitz's theorem: This is discussed at length in the author's survey article [9], as well as in the author's papers [6], [7], [8], [10].

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In brief, for each positive integer $s \geq 2$, and $1 \leq \nu \leq \infty$, define $\theta_{s,\nu}$ as the supremum of all values C with the following property: *For every $\alpha \in \mathbb{R}^s \setminus \mathbb{Q}^s$, there exist infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{Z}_+$ with $\gcd(\mathbf{p}, q) = 1$, such that*

$$\left\| \alpha - \frac{1}{q} \mathbf{p} \right\|_{\nu} < \frac{1}{q(Cq)^{1/s}}. \quad (1)$$

Then it is known due to a famous result of Davenport [3] that

$$\theta_{s,\nu} = \Delta(K^{(s,\nu)}), \quad (2)$$

where

$$K^{(s,\nu)} = \left\{ (x_0, \dots, x_s) \in \mathbb{R}^{s+1} : |x_0| \|(x_1, \dots, x_s)\|_{\nu}^s \leq 1 \right\}.$$

However, the exact determination of $\theta_{s,\nu}$ has only been accomplished for $s = 1$ (Hurwitz's classic theorem: $\theta_{1,\nu} = \sqrt{5}$) and for $s = \nu = 2$: $\theta_{2,2} = \frac{1}{2}\sqrt{23}$ [2].

2 Objective of the present article

To fix notions, we concentrate on the most natural one of the unsolved cases concerning $\theta_{s,\nu}$, namely on our familiar three-dimensional space and the Euclidean norm. Armitage [1] proved that

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq (\Delta(K^*))^3 \Delta(K_2),$$

where

$$K^* : x^2(x^2 + y^2)^3 \leq 1$$

is a planar star body with $\Delta(K^*) \geq 1.159$, and

$$K_2 : (y^2 + z^2)(x^2 + y^2 + z^2) \leq 1. \quad (3)$$

Armitage proceeded to estimate $\Delta(K_2)$ by inscribing an ellipsoid¹

$$x^2 + 4y^2 + 4z^2 \leq 2\sqrt{3}.$$

Thus he obtained

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq 1.774 \dots$$

Around the turn of the millennium, the present author [6] replaced this ellipsoid by the double paraboloid

$$|x| \leq (1 + \sqrt{2})(1 - y^2 - z^2),$$

and evaluated the critical determinant of the latter. This gave the overall improvement

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq 1.879 \dots$$

¹As it is common in the Geometry of Numbers, we will throughout use the terms *ellipsoid*, *paraboloid*, *cone*, ... for bodies, not for the boundary surfaces.

It is the aim of the present article to view the body K_2 as a member of a more general family of star-bodies²

$$K_c : \overline{(y^2 + z^2)^{c/2} (x^2 + y^2 + z^2)} \leq 1, \quad (4)$$

where c is an arbitrary fixed positive constant. Our objective is to deduce a lower bound for $\Delta(K_c)$, depending on c , for every $c > 0$.

We start with a brief survey of the bounds established, postponing a more detailed representation of the results to Table 2 at the end.

c	1	1.2	1.4	1.6	1.8	2
$\Delta(K_c) \geq$	0.9186	0.9612	1.0130	1.0780	1.1428	1.2071
c	2.2	2.4	2.6	2.8	3	
$\Delta(K_c) \geq$	1.2712	1.3358	1.4139	1.4917	1.5693	

Table 1: Lower bounds for $\Delta(K_c)$ obtained, for a couple of values c .

3 Strategy of proof and auxiliary results

There is no direct approach to estimate the critical determinant of a non-convex unbounded starbody like K_c . However, for convex (and \mathbf{o} -symmetric) bodies in \mathbb{R}^3 the situation is considerably better. For this case, Minkowski [5] has established a general theorem which tells us how in this case the critical lattices³ necessarily look like; see also [4, p. 342, Theorem 3]. On the basis of this result, Minkowski was able to evaluate $\Delta(\mathcal{O}) = \frac{19}{108}$ for the octahedron

$$\mathcal{O} : |x| + |y| + |z| \leq 1.$$

Similarly, Ollerenshaw [11] showed that

$$\Delta(\mathcal{B}_3) = \frac{1}{\sqrt{2}} \quad (5)$$

for the origin-centered unit ball \mathcal{B}_3 in \mathbb{R}^3 . Furthermore, Whitworth [12] considered the double cone

$$\mathcal{C} : |x| + \sqrt{y^2 + z^2} \leq 1$$

and obtained

$$\Delta(\mathcal{C}) = \frac{\sqrt{6}}{8}. \quad (6)$$

Finally, the author [6] was able to show for the double paraboloid

$$\mathcal{P} : |x| + y^2 + z^2 \leq 1$$

²Obviously, no loss of generality is implied by the fact that only one of the exponents of the two brackets is assumed to vary.

³A lattice $A\mathbb{Z}^3$ is called *critical* for a body B if $|\det A| = \Delta(B)$ and \mathbf{o} is the only lattice point in the interior of B .

that

$$\Delta(\mathcal{P}) = \frac{1}{2}. \quad (7)$$

Our argument will be based on the idea to inscribe into K_c one of the three last-mentioned convex bodies, depending on the value of c , and to use the results (5)–(7). In fact, for a certain interval around $c = 2$, the choice of a paraboloid will turn out to be optimal, while for smaller values of c an ellipsoid will be the best choice, and for larger c the double cone will be most appropriate.

4 The details of the analysis

Lemma 1. For fixed c , $0 < c < 4$, let

$$\lambda_0 := \left(\frac{6}{4-c} \right)^{2/(2+c)}. \quad (8)$$

For any $\lambda > 0$, the ellipsoid

$$\mathcal{E}_c(\lambda) : \frac{x^2}{(1 + \frac{1}{2}c)\lambda^{c/2}} + \frac{(y^2 + z^2)}{1 + \frac{1}{2}c} \left(\frac{1}{2}c\lambda + \lambda^{-c/2} \right) \leq 1$$

is completely contained in K_c and has critical determinant

$$\Delta(\mathcal{E}_c(\lambda)) = \frac{(1 + \frac{1}{2}c)^{3/2}}{\sqrt{2}} \frac{\lambda^{c/4}}{\frac{1}{2}c\lambda + \lambda^{-c/2}}. \quad (9)$$

For any fixed c , $0 < c < 4$, this expression attains its maximum for $\lambda = \lambda_0$, as given in (8). Hence $\Delta(K_c) \geq \Delta(\mathcal{E}_c)$ with $\mathcal{E}_c := \mathcal{E}_c(\lambda_0)$.

Proof. Let $r = \sqrt{y^2 + z^2}$ for short. Then, by the mean inequality with weights,

$$r^c(r^2 + x^2) = (\lambda r^2)^{c/2} \frac{r^2 + x^2}{\lambda^{c/2}} \leq \left(\frac{\frac{1}{2}c\lambda r^2 + (r^2 + x^2)\lambda^{-c/2}}{1 + \frac{1}{2}c} \right)^{1+c/2}.$$

From this, $K_c \supset \mathcal{E}_c(\lambda)$ is immediate. By (5), and an obvious linear substitution, (9) readily follows. Differentiating the right hand side of (9) with respect to λ and equating to zero, the choice $\lambda = \lambda_0$, as given in (8), turns out to be optimal. \square

Lemma 2. For any $c > 0$, define $r_0(c)$ as the unique⁴ solution in $(0, 1)$ of the equation

$$2r_0^{c+2} - (c+2)r_0 + c = 0. \quad (10)$$

Put further

$$x_0(c) := \frac{\sqrt{r_0(c)^{-c} - r_0(c)^2}}{1 - r_0(c)}. \quad (11)$$

⁴Let L denote the left-hand side of (10), then $\frac{dL}{dr_0} = 0$ iff $r_0 = r_* := 2^{-1/(c+1)}$. Hence L decreases on $[0, r_*]$ from c to $-r_*(c+1)$, and increases on $[r_*, 1]$ from $-r_*(c+1)$ to 0. Hence the uniqueness of $r_0(c)$.

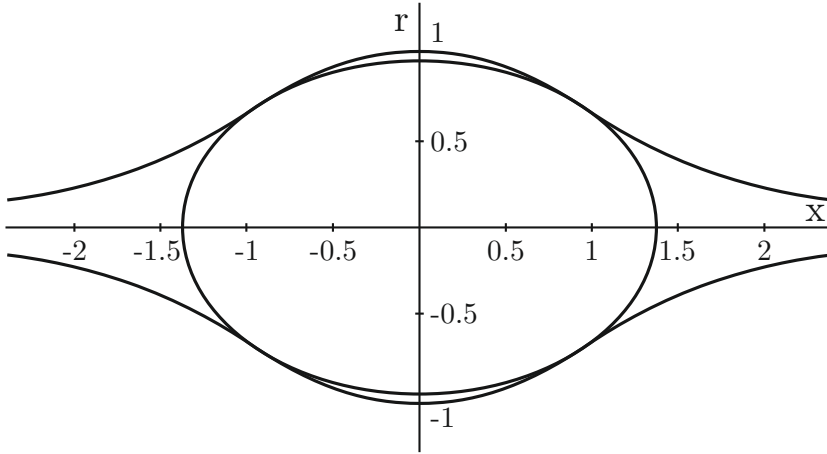


Figure 1: $c = 1$: The body K_1 and an optimal inscribed ellipsoid, in front view

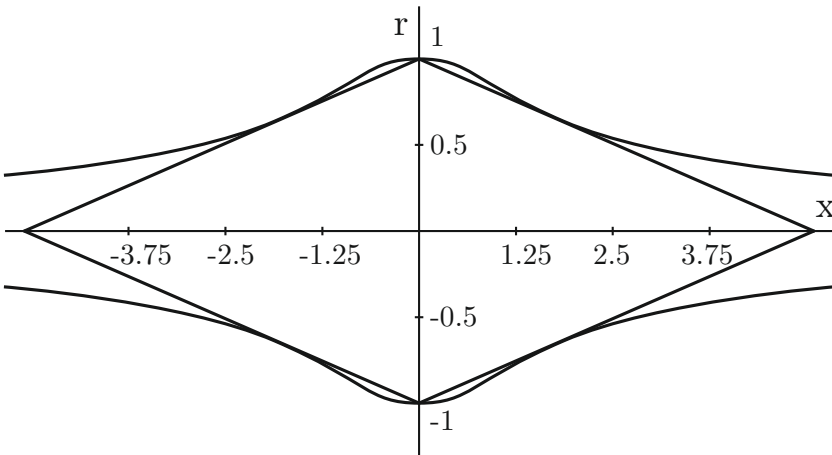


Figure 2: $c = 3$: The body K_3 and an optimal inscribed double cone, in front view

Then the double cone

$$\mathcal{C}_c : \frac{|x|}{x_0(c)} + \sqrt{y^2 + z^2} \leq 1$$

is completely contained in K_c and has critical determinant

$$\Delta(\mathcal{C}_c) = \frac{\sqrt{6}}{8} x_0(c). \tag{12}$$

Proof. Since both K_c and \mathcal{C}_c are bodies of rotation, with respect to the x -axis, it suffices to discuss the situation in front view - in a (x, r) -plane, say, $r = \sqrt{y^2 + z^2}$.

By symmetry, we may restrict the calculations to $x \geq 0, r \geq 0$. The curve k_c whose rotation generates ∂K_c is given by $r^c(x^2 + r^2) = 1$. Solving for x gives

$$x = \xi(r) := \sqrt{r^{-c} - r^2}.$$

$\partial \mathcal{C}_c$ is generated by the tangent

$$T : x - \xi(r_0) = \xi'(r_0)(r - r_0) \tag{13}$$

which contains the point $(x, r) = (0, 1)$. Inserting this into (13) and carrying out some bulky analysis, we arrive at (10). Since $(x_0, 0)$ is the point of intersection of T with the x -axis, (11) follows by one more routine calculation. Finally, (6) readily implies (12).

By the way, the point of inflection of k_c is $(\xi(r_W(c)), r_W(c))$ with

$$r_W(c) = \left(\frac{c}{2(c+1)} \right)^{1/(c+2)}.$$

It is easily checked that throughout $r_W(c) > r_0(c)$. □

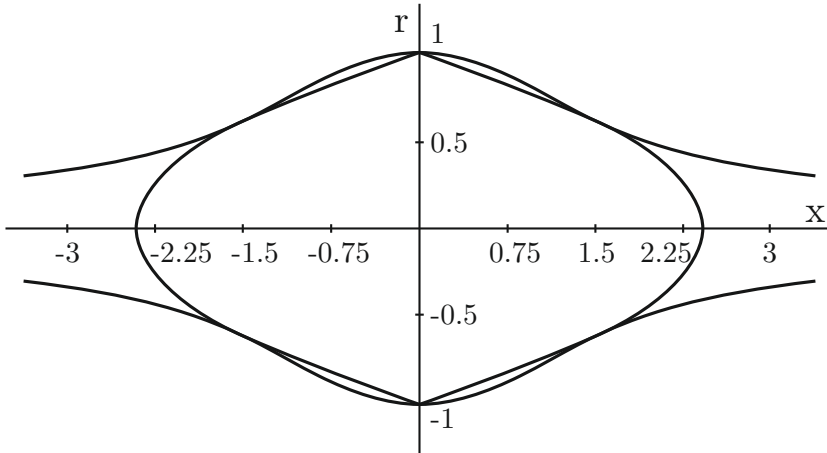


Figure 3: $c = 2$: The body K_2 and an optimal inscribed double paraboloid, in front view

Lemma 3. For any $c > 0$, define $r_1(c)$ as the unique⁵ solution in $(0, 1)$ of the equation

$$2r_1^{c+2}(r_1^2 + 1) + c(1 - r_1^2) - 4r_1^2 = 0. \tag{14}$$

Put further

$$\alpha(c) := \frac{\sqrt{r_1(c)^{-c} - r_1(c)^2}}{1 - r_1(c)^2}. \tag{15}$$

⁵A similar argument applies as in footnote 4 in Lemma 2.

Then the double paraboloid

$$\mathcal{P}_c : |x| \leq \alpha(c)(1 - y^2 - z^2)$$

is completely contained in K_c and has critical determinant

$$\Delta(\mathcal{P}_c) = \frac{\alpha(c)}{2}. \tag{16}$$

Proof. Again we consider the situation in front view, in (x, r) -variables, $x, r \geq 0$. The aim is to choose $\alpha = \alpha(c)$ so that the parabola $p_c : x = \alpha(1 - r^2)$ and the curve k_c have one point (x_1, r_1) in common (in the first quadrant), where also the derivative $x'_1 = \left. \frac{dx}{dr} \right|_{r=r_1}$ has the same value. In this way we get:

$$r_1^c(r_1^2 + x_1^2) = 1, \tag{17}$$

$$(c + 2)r_1^{c+1} + cr_1^{c-1}x_1^2 + 2r_1^c x_1 x'_1 = 0, \tag{18}$$

$$x_1 = \alpha(1 - r_1^2), \tag{19}$$

$$x'_1 = -2\alpha r_1. \tag{20}$$

Dividing (20) by (19), we conclude that

$$x'_1 = -\frac{2r_1}{1 - r_1^2} x_1.$$

Using this in (18), we get

$$(c + 2)r_1^{c+1} + x_1^2 \left(cr_1^{c-1} - \frac{4r_1}{1 - r_1^2} \right) = 0. \tag{21}$$

Solving (17) for x_1^2 and using this in (21), we obtain an equation in the single unknown r_1 which, after simplifying, is just (14). Further, (17) and (19) readily imply (15). Finally, (16) is immediate from (7).

Again, it is easily checked numerically that throughout $r_W(c) > r_1(c)$. □

We are now in a position to summarize the results obtained.

Theorem 1. For $0 < c < 4$, the critical determinant of the starbody

$$K_c : (y^2 + z^2)^{c/2}(x^2 + y^2 + z^2) \leq 1$$

can be estimated from below by

$$\Delta(K_c) \geq \max(\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)).$$

Here, $\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)$ are given in Lemmas 1–3.

Further, for $c \geq 4$,

$$\Delta(K_c) \geq \Delta(\mathcal{C}_c).$$

Remark 1. As can be seen from the table below, for $c \in \{1, 1.2\}$, the sharpest lower bound for $\Delta(K_c)$ can be obtained by inscribing an ellipsoid. For $c \in \{1.4, 1.6, 1.8, 2, 2.2\}$, inscribing a double paraboloid yields the best result, while for $c \in \{2.4, 2.6, 2.8, 3\}$, an inscribed double cone is the best choice.

c	$\Delta(\mathcal{E}_c)$	$\Delta(\mathcal{P}_c)$	$\Delta(\mathcal{C}_c)$
1	0.9186	0.8810	0.7785
1.2	0.9612	0.9473	0.8601
1.4	1.0045	1.0130	0.9408
1.6	1.0485	1.0780	1.0207
1.8	1.0935	1.1428	1.1000
2	1.1398	1.2071	1.1790
2.2	1.1875	1.2712	1.2576
2.4	1.2371	1.3351	1.3358
2.6	1.2890	1.3988	1.4139
2.8	1.3437	1.4623	1.4917
3	1.4019	1.5257	1.5693

Table 2: The critical determinants of $\mathcal{E}_c, \mathcal{P}_c, \mathcal{C}_c$, for $1 \leq c \leq 3$, in step lengths of 0.2.

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