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# Estimating the critical determinants of a class of three-dimensional star bodies

Werner Georg Nowak

**Abstract.** In the problem of (simultaneous) Diophantine approximation in  $\mathbb{R}^3$  (in the spirit of Hurwitz's theorem), lower bounds for the critical determinant of the special three-dimensional body

$$K_2 : (y^2 + z^2)(x^2 + y^2 + z^2) \leq 1$$

play an important role; see [1], [6]. This article deals with estimates from below for the critical determinant  $\Delta(K_c)$  of more general star bodies

$$K_c : (y^2 + z^2)^{c/2}(x^2 + y^2 + z^2) \leq 1,$$

where  $c$  is any positive constant. These are obtained by inscribing into  $K_c$  either a double cone, or an ellipsoid, or a double paraboloid, depending on the size of  $c$ .

## 1 Introduction

During the last couple of decades, not much research has been done in the subfield of the *Geometry of Numbers* (see, e.g., the monograph by Gruber & Lekkerkerker [4]) which is concerned with the evaluation, or at least estimation, of *critical determinants*  $\Delta(K)$  of starbodies  $K$  in  $\mathbb{R}^s$ ,  $s \geq 2$ . These are defined as  $\Delta(K) = \inf |\det A|$ , where  $A$  ranges over all nonsingular real  $(s \times s)$ -matrices, such that the origin is the only point of the lattice  $A\mathbb{Z}^s$  in the interior of  $K$ .

It is the author's aim to rouse new interest in this classic topic *a fortiori* in view of its close connection to *simultaneous Diophantine approximation* in the spirit of Hurwitz's theorem: This is discussed at length in the author's survey article [9], as well as in the author's papers [6], [7], [8], [10].

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In brief, for each positive integer  $s \geq 2$ , and  $1 \leq \nu \leq \infty$ , define  $\theta_{s,\nu}$  as the supremum of all values  $C$  with the following property: *For every  $\alpha \in \mathbb{R}^s \setminus \mathbb{Q}^s$ , there exist infinitely many  $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{Z}_+$  with  $\gcd(\mathbf{p}, q) = 1$ , such that*

$$\left\| \alpha - \frac{1}{q} \mathbf{p} \right\|_{\nu} < \frac{1}{q(Cq)^{1/s}}. \quad (1)$$

Then it is known due to a famous result of Davenport [3] that

$$\theta_{s,\nu} = \Delta(K^{(s,\nu)}), \quad (2)$$

where

$$K^{(s,\nu)} = \{(x_0, \dots, x_s) \in \mathbb{R}^{s+1} : |x_0| \|(x_1, \dots, x_s)\|_{\nu}^s \leq 1\}.$$

However, the exact determination of  $\theta_{s,\nu}$  has only been accomplished for  $s = 1$  (Hurwitz's classic theorem:  $\theta_{1,\nu} = \sqrt{5}$ ) and for  $s = \nu = 2$ :  $\theta_{2,2} = \frac{1}{2}\sqrt{23}$  [2].

## 2 Objective of the present article

To fix notions, we concentrate on the most natural one of the unsolved cases concerning  $\theta_{s,\nu}$ , namely on our familiar three-dimensional space and the Euclidean norm. Armitage [1] proved that

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq (\Delta(K^*))^3 \Delta(K_2),$$

where

$$K^* : x^2(x^2 + y^2)^3 \leq 1$$

is a planar star body with  $\Delta(K^*) \geq 1.159$ , and

$$K_2 : (y^2 + z^2)(x^2 + y^2 + z^2) \leq 1. \quad (3)$$

Armitage proceeded to estimate  $\Delta(K_2)$  by inscribing an ellipsoid<sup>1</sup>

$$x^2 + 4y^2 + 4z^2 \leq 2\sqrt{3}.$$

Thus he obtained

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq 1.774 \dots$$

Around the turn of the millennium, the present author [6] replaced this ellipsoid by the double paraboloid

$$|x| \leq (1 + \sqrt{2})(1 - y^2 - z^2),$$

and evaluated the critical determinant of the latter. This gave the overall improvement

$$\theta_{3,2} = \Delta(K^{(3,2)}) \geq 1.879 \dots$$

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<sup>1</sup>As it is common in the Geometry of Numbers, we will throughout use the terms *ellipsoid*, *paraboloid*, *cone*, ... for bodies, not for the boundary surfaces.

It is the aim of the present article to view the body  $K_2$  as a member of a more general family of star-bodies<sup>2</sup>

$$K_c : (y^2 + z^2)^{c/2} (x^2 + y^2 + z^2) \leq 1, \quad (4)$$

where  $c$  is an arbitrary fixed positive constant. Our objective is to deduce a lower bound for  $\Delta(K_c)$ , depending on  $c$ , for every  $c > 0$ .

We start with a brief survey of the bounds established, postponing a more detailed representation of the results to Table 2 at the end.

$c$	1	1.2	1.4	1.6	1.8	2
$\Delta(K_c) \geq$	0.9186	0.9612	1.0130	1.0780	1.1428	1.2071
$c$	2.2	2.4	2.6	2.8	3	
$\Delta(K_c) \geq$	1.2712	1.3358	1.4139	1.4917	1.5693	

Table 1: Lower bounds for  $\Delta(K_c)$  obtained, for a couple of values  $c$ .

### 3 Strategy of proof and auxiliary results

There is no direct approach to estimate the critical determinant of a non-convex unbounded starbody like  $K_c$ . However, for convex (and  $\mathbf{o}$ -symmetric) bodies in  $\mathbb{R}^3$  the situation is considerably better. For this case, Minkowski [5] has established a general theorem which tells us how in this case the critical lattices<sup>3</sup> necessarily look like; see also [4, p. 342, Theorem 3]. On the basis of this result, Minkowski was able to evaluate  $\Delta(\mathcal{O}) = \frac{19}{108}$  for the octahedron

$$\mathcal{O} : |x| + |y| + |z| \leq 1.$$

Similarly, Ollerenshaw [11] showed that

$$\Delta(\mathcal{B}_3) = \frac{1}{\sqrt{2}} \quad (5)$$

for the origin-centered unit ball  $\mathcal{B}_3$  in  $\mathbb{R}^3$ . Furthermore, Whitworth [12] considered the double cone

$$\mathcal{C} : |x| + \sqrt{y^2 + z^2} \leq 1$$

and obtained

$$\Delta(\mathcal{C}) = \frac{\sqrt{6}}{8}. \quad (6)$$

Finally, the author [6] was able to show for the double paraboloid

$$\mathcal{P} : |x| + y^2 + z^2 \leq 1$$

<sup>2</sup>Obviously, no loss of generality is implied by the fact that only one of the exponents of the two brackets is assumed to vary.

<sup>3</sup>A lattice  $A\mathbb{Z}^3$  is called *critical* for a body  $B$  if  $|\det A| = \Delta(B)$  and  $\mathbf{o}$  is the only lattice point in the interior of  $B$ .

that

$$\Delta(\mathcal{P}) = \frac{1}{2}. \quad (7)$$

Our argument will be based on the idea to inscribe into  $K_c$  one of the three last-mentioned convex bodies, depending on the value of  $c$ , and to use the results (5)–(7). In fact, for a certain interval around  $c = 2$ , the choice of a paraboloid will turn out to be optimal, while for smaller values of  $c$  an ellipsoid will be the best choice, and for larger  $c$  the double cone will be most appropriate.

#### 4 The details of the analysis

**Lemma 1.** *For fixed  $c$ ,  $0 < c < 4$ , let*

$$\lambda_0 := \left( \frac{6}{4-c} \right)^{2/(2+c)}. \quad (8)$$

*For any  $\lambda > 0$ , the ellipsoid*

$$\mathcal{E}_c(\lambda) : \frac{x^2}{(1 + \frac{1}{2}c)\lambda^{c/2}} + \frac{(y^2 + z^2)}{1 + \frac{1}{2}c} \left( \frac{1}{2}c\lambda + \lambda^{-c/2} \right) \leq 1$$

*is completely contained in  $K_c$  and has critical determinant*

$$\Delta(\mathcal{E}_c(\lambda)) = \frac{(1 + \frac{1}{2}c)^{3/2}}{\sqrt{2}} \frac{\lambda^{c/4}}{\frac{1}{2}c\lambda + \lambda^{-c/2}}. \quad (9)$$

*For any fixed  $c$ ,  $0 < c < 4$ , this expression attains its maximum for  $\lambda = \lambda_0$ , as given in (8). Hence  $\Delta(K_c) \geq \Delta(\mathcal{E}_c)$  with  $\mathcal{E}_c := \mathcal{E}_c(\lambda_0)$ .*

*Proof.* Let  $r = \sqrt{y^2 + z^2}$  for short. Then, by the mean inequality with weights,

$$r^c(r^2 + x^2) = (\lambda r^2)^{c/2} \frac{r^2 + x^2}{\lambda^{c/2}} \leq \left( \frac{\frac{1}{2}c\lambda r^2 + (r^2 + x^2)\lambda^{-c/2}}{1 + \frac{1}{2}c} \right)^{1+c/2}.$$

From this,  $K_c \supset \mathcal{E}_c(\lambda)$  is immediate. By (5), and an obvious linear substitution, (9) readily follows. Differentiating the right hand side of (9) with respect to  $\lambda$  and equating to zero, the choice  $\lambda = \lambda_0$ , as given in (8), turns out to be optimal.  $\square$

**Lemma 2.** *For any  $c > 0$ , define  $r_0(c)$  as the unique<sup>4</sup> solution in  $(0, 1)$  of the equation*

$$2r_0^{c+2} - (c+2)r_0 + c = 0. \quad (10)$$

*Put further*

$$x_0(c) := \frac{\sqrt{r_0(c)^{-c} - r_0(c)^2}}{1 - r_0(c)}. \quad (11)$$

---

<sup>4</sup>Let  $L$  denote the left-hand side of (10), then  $\frac{dL}{dr_0} = 0$  iff  $r_0 = r_* := 2^{-1/(c+1)}$ . Hence  $L$  decreases on  $[0, r_*]$  from  $c$  to  $-r_*(c+1)$ , and increases on  $[r_*, 1]$  from  $-r_*(c+1)$  to 0. Hence the uniqueness of  $r_0(c)$ .

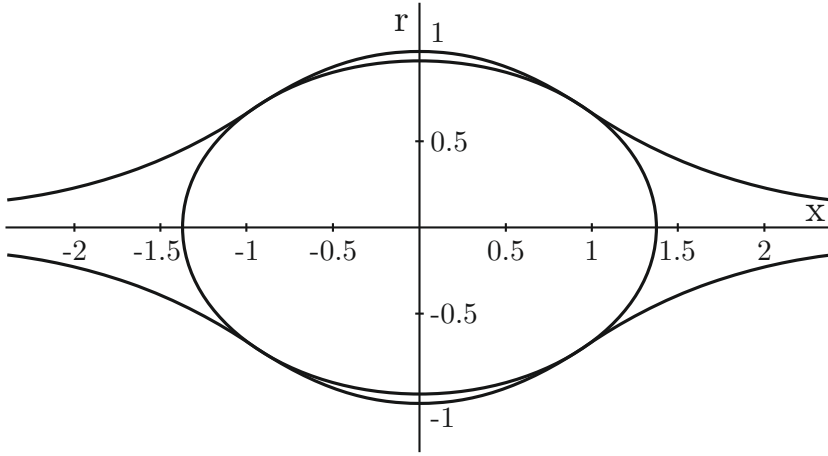


Figure 1:  $c = 1$ : The body  $K_1$  and an optimal inscribed ellipsoid, in front view

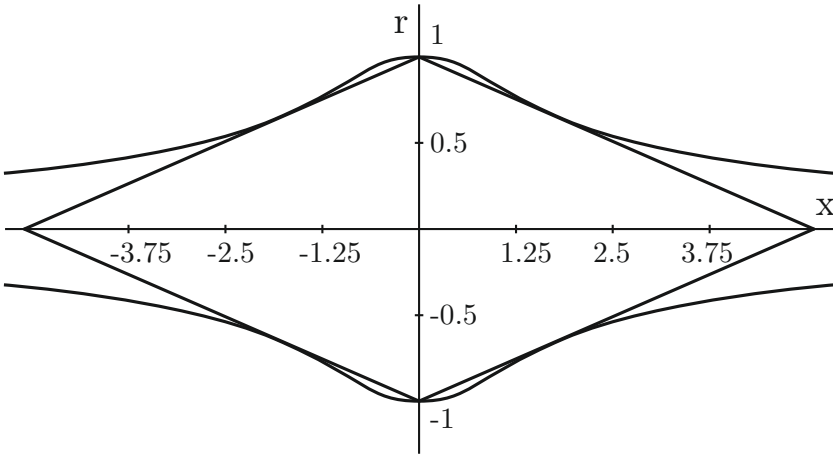


Figure 2:  $c = 3$ : The body  $K_3$  and an optimal inscribed double cone, in front view

Then the double cone

$$\mathcal{C}_c : \frac{|x|}{x_0(c)} + \sqrt{y^2 + z^2} \leq 1$$

is completely contained in  $K_c$  and has critical determinant

$$\Delta(\mathcal{C}_c) = \frac{\sqrt{6}}{8} x_0(c). \quad (12)$$

*Proof.* Since both  $K_c$  and  $\mathcal{C}_c$  are bodies of rotation, with respect to the  $x$ -axis, it suffices to discuss the situation in front view - in a  $(x, r)$ -plane, say,  $r = \sqrt{y^2 + z^2}$ .

By symmetry, we may restrict the calculations to  $x \geq 0, r \geq 0$ . The curve  $k_c$  whose rotation generates  $\partial K_c$  is given by  $r^c(x^2 + r^2) = 1$ . Solving for  $x$  gives

$$x = \xi(r) := \sqrt{r^{-c} - r^2}.$$

$\partial \mathcal{C}_c$  is generated by the tangent

$$T: \quad x - \xi(r_0) = \xi'(r_0)(r - r_0) \quad (13)$$

which contains the point  $(x, r) = (0, 1)$ . Inserting this into (13) and carrying out some bulky analysis, we arrive at (10). Since  $(x_0, 0)$  is the point of intersection of  $T$  with the  $x$ -axis, (11) follows by one more routine calculation. Finally, (6) readily implies (12).

By the way, the point of inflection of  $k_c$  is  $(\xi(r_W(c)), r_W(c))$  with

$$r_W(c) = \left( \frac{c}{2(c+1)} \right)^{1/(c+2)}.$$

It is easily checked that throughout  $r_W(c) > r_0(c)$ . □

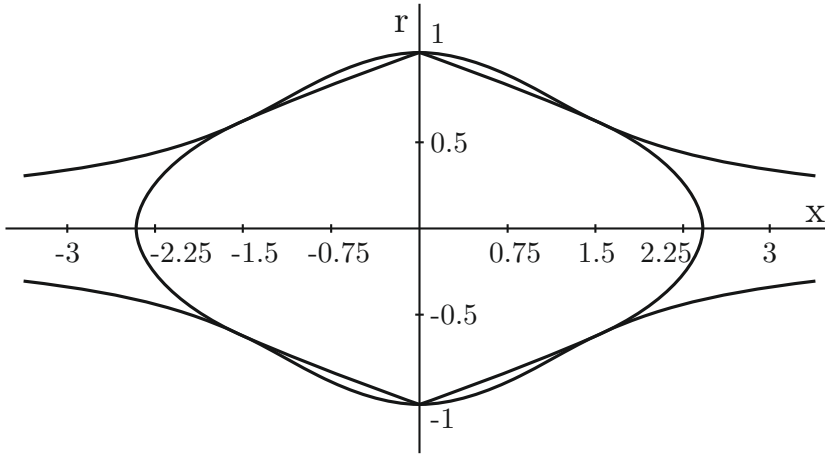


Figure 3:  $c = 2$ : The body  $K_2$  and an optimal inscribed double paraboloid, in front view

**Lemma 3.** For any  $c > 0$ , define  $r_1(c)$  as the unique<sup>5</sup> solution in  $(0, 1)$  of the equation

$$2r_1^{c+2}(r_1^2 + 1) + c(1 - r_1^2) - 4r_1^2 = 0. \quad (14)$$

Put further

$$\alpha(c) := \frac{\sqrt{r_1(c)^{-c} - r_1(c)^2}}{1 - r_1(c)^2}. \quad (15)$$

<sup>5</sup>A similar argument applies as in footnote 4 in Lemma 2.

Then the double paraboloid

$$\mathcal{P}_c : |x| \leq \alpha(c)(1 - y^2 - z^2)$$

is completely contained in  $K_c$  and has critical determinant

$$\Delta(\mathcal{P}_c) = \frac{\alpha(c)}{2}. \quad (16)$$

*Proof.* Again we consider the situation in front view, in  $(x, r)$ -variables,  $x, r \geq 0$ . The aim is to choose  $\alpha = \alpha(c)$  so that the parabola  $p_c : x = \alpha(1 - r^2)$  and the curve  $k_c$  have one point  $(x_1, r_1)$  in common (in the first quadrant), where also the derivative  $x'_1 = \left. \frac{dx}{dr} \right|_{r=r_1}$  has the same value. In this way we get:

$$r_1^c(r_1^2 + x_1^2) = 1, \quad (17)$$

$$(c + 2)r_1^{c+1} + cr_1^{c-1}x_1^2 + 2r_1^c x_1 x'_1 = 0, \quad (18)$$

$$x_1 = \alpha(1 - r_1^2), \quad (19)$$

$$x'_1 = -2\alpha r_1. \quad (20)$$

Dividing (20) by (19), we conclude that

$$x'_1 = -\frac{2r_1}{1 - r_1^2} x_1.$$

Using this in (18), we get

$$(c + 2)r_1^{c+1} + x_1^2 \left( cr_1^{c-1} - \frac{4r_1}{1 - r_1^2} \right) = 0. \quad (21)$$

Solving (17) for  $x_1^2$  and using this in (21), we obtain an equation in the single unknown  $r_1$  which, after simplifying, is just (14). Further, (17) and (19) readily imply (15). Finally, (16) is immediate from (7).

Again, it is easily checked numerically that throughout  $r_W(c) > r_1(c)$ .  $\square$

We are now in a position to summarize the results obtained.

**Theorem 1.** For  $0 < c < 4$ , the critical determinant of the starbody

$$K_c : (y^2 + z^2)^{c/2}(x^2 + y^2 + z^2) \leq 1$$

can be estimated from below by

$$\Delta(K_c) \geq \max(\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)).$$

Here,  $\Delta(\mathcal{E}_c), \Delta(\mathcal{C}_c), \Delta(\mathcal{P}_c)$  are given in Lemmas 1–3.

Further, for  $c \geq 4$ ,

$$\Delta(K_c) \geq \Delta(\mathcal{C}_c).$$

**Remark 1.** As can be seen from the table below, for  $c \in \{1, 1.2\}$ , the sharpest lower bound for  $\Delta(K_c)$  can be obtained by inscribing an ellipsoid. For  $c \in \{1.4, 1.6, 1.8, 2, 2.2\}$ , inscribing a double paraboloid yields the best result, while for  $c \in \{2.4, 2.6, 2.8, 3\}$ , an inscribed double cone is the best choice.



$c$	$\Delta(\mathcal{E}_c)$	$\Delta(\mathcal{P}_c)$	$\Delta(\mathcal{C}_c)$
1	<b>0.9186</b>	0.8810	0.7785
1.2	<b>0.9612</b>	0.9473	0.8601
1.4	1.0045	<b>1.0130</b>	0.9408
1.6	1.0485	<b>1.0780</b>	1.0207
1.8	1.0935	<b>1.1428</b>	1.1000
2	1.1398	<b>1.2071</b>	1.1790
2.2	1.1875	<b>1.2712</b>	1.2576
2.4	1.2371	1.3351	<b>1.3358</b>
2.6	1.2890	1.3988	<b>1.4139</b>
2.8	1.3437	1.4623	<b>1.4917</b>
3	1.4019	1.5257	<b>1.5693</b>

Table 2: The critical determinants of  $\mathcal{E}_c, \mathcal{P}_c, \mathcal{C}_c$ , for  $1 \leq c \leq 3$ , in step lengths of 0.2.

## References

- [1] J.V. Armitage: [On a method of Mordell in the geometry of numbers](#). *Mathematika* 2 (2) (1955) 132–140.
- [2] H. Davenport, K. Mahler: [Simultaneous Diophantine approximation](#). *Duke Math. J.* 13 (1946) 105–111.
- [3] H. Davenport: On a theorem of Furtwängler. *J. London Math.Soc.* 30 (1955) 185–195.
- [4] P.M. Gruber, C.G. Lekkerkerker: *Geometry of numbers*. North Holland, Amsterdam (1987).
- [5] H. Minkowski: Dichteste gitterförmige Lagerung kongruenter Körper. *Nachr. Kön. Ges. Wiss. Göttingen* (1904) 311–355.
- [6] W.G. Nowak: The critical determinant of the double paraboloid and Diophantine approximation in  $\mathbb{R}^3$  and  $\mathbb{R}^4$ . *Math. Pannonica* 10 (1999) 111–122.
- [7] W.G. Nowak: Diophantine approximation in  $\mathbb{R}^s$ : On a method of Mordell and Armitage. In: *Algebraic number theory and Diophantine analysis. Proceedings of the conference held in Graz, Austria, August 30 to September 5, 1998*, W. de Gruyter, Berlin. (2000) 339–349.
- [8] W.G. Nowak: Lower bounds for simultaneous Diophantine approximation constants. *Comm. Math.* 22 (1) (2014) 71–76.
- [9] W.G. Nowak: Simultaneous Diophantine approximation: Searching for analogues of Hurwitz’s theorem. In: *T.M. Rassias and P.M. Pardalos (eds.), Essays in mathematics and its applications*. Springer, Switzerland (2016) 181–197.
- [10] W.G. Nowak: On the critical determinants of certain star bodies. *Comm. Math.* 25 (1) (2017) 5–11.
- [11] K. Ollerenshaw: The critical lattices of a sphere. *J. London Math. Soc.* 23 (1949) 297–299.
- [12] J.V. Whitworth: The critical lattices of the double cone. *Proc. London Math. Soc.* 2 (1) (1951) 422–443.

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