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EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS

ANNE BOUTET DE MONVEL* AND LECH ZIELINSKI[†]

ABSTRACT. We obtain eigenvalue asymptotics for Jacobi matrices of various Jaynes-Cummings type.

1. The results

We consider a type of Jacobi matrices with unbounded entries related to some problems of quantum optics. See [1-4].

Let $\mathbb{N}^* = \{1, 2, ...\}$ be the set of positive integers and let l^2 denote the Hilbert space of square summable complex sequences $x = (x_n)_{n \in \mathbb{N}^*}$. Let c_{00} be the subspace of sequences for which $\{n \in \mathbb{N}^* \mid x_n \neq 0\}$ is finite. We fix a real valued sequence $(\beta_n)_{n \in \mathbb{N}^*}$ and consider a linear operator J acting on $(x_n)_{n \in \mathbb{N}^*} \in c_{00}$ according to the formula

(1.1)
$$(Jx)_n = \begin{cases} nx_n + \beta_n x_{n+1} + \beta_{n-1} x_{n-1} & n \ge 2, \\ x_1 + \beta_1 x_2 & n = 1. \end{cases}$$

Then it is easy to establish the following elementary fact.

Proposition 1. Assume that there exists $\rho > 0$ such that

(1.2)
$$\beta_n = \mathcal{O}(n^{1-\rho}).$$

Then the closure of the operator defined by (1.1) is a self-adjoint operator J, its spectrum is discrete and bounded from below. Let $(\lambda_n(J))_{n \in \mathbb{N}^*}$ denote the sequence of eigenvalues of Jrepeated according to their multiplicities and ordered so that $\lambda_n(J) \leq \lambda_{n+1}(J)$ for all $n \in \mathbb{N}^*$. Then the following estimate

$$\lambda_n(J) = n + \mathcal{O}(n^{1-\rho})$$

holds as $n \to \infty$.

The aim of this paper is to obtain sharper estimates of the asymptotic behaviour of $(\lambda_n(J))_{n\in\mathbb{N}^*}$ which can be deduced from additional assumptions made on the sequence $(\beta_n)_{n\in\mathbb{N}^*}$. Our first result is

Theorem 1. Assume that (1.2) holds with a certain $\rho > 0$ and

(1.4)
$$\beta_{n+1} - \beta_n = \mathcal{O}(n^{-\rho'})$$

holds with a certain $\rho' > 0$. Then one has the estimate

(1.5)
$$\lambda_n(J) = n + \mathcal{O}(n^{1-\rho-\rho'})$$

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Our second theorem depends on the behaviour of the sequence

(1.6)
$$\gamma_n = \begin{cases} \beta_{n-1}^2 - \beta_n^2 & n \ge 2, \\ -\beta_1^2 & n = 1. \end{cases}$$

Theorem 2. Assume that $(\beta_n)_{n \in \mathbb{N}^*}$ satisfies the hypotheses of Theorem 1. Let $(\gamma_n)_{n \in \mathbb{N}^*}$ be the sequence given by (1.6). If

(1.7)
$$\gamma_{n+1} - \gamma_n = \mathcal{O}(n^{-\rho_1})$$

holds with a certain $\rho_1 > 0$, then one has

(1.8)
$$\lambda_n(J) = n + \gamma_n + \mathcal{O}(n^{1-\rho-\rho_1}).$$

Remark. If it is possible to evaluate $\beta_n = b(n)$ by means of a function $b \in C^{\infty}((0, +\infty))$ satisfying the estimates

$$\begin{cases} b(\lambda) = \mathcal{O}(\lambda^{1-\rho}), \\ b'(\lambda) = \mathcal{O}(\lambda^{-\rho}), \end{cases}$$

then

$$\beta_{n+1} - \beta_n = \int_0^1 b'(n+s) ds = O(n^{-\rho}),$$

i.e., (1.4) holds with $\rho = \rho'$, and (1.5) takes the form

$$\lambda_n(J) = n + \mathcal{O}(n^{1-2\rho})$$

If moreover

$$b''(\lambda) = \mathcal{O}(\lambda^{-1-\rho}),$$

then

$$\begin{split} \gamma_{n+1} - \gamma_n &= -\int_0^1 \mathrm{d}s \int_0^1 b^{2\prime\prime} (n+s-s') \mathrm{d}s' \\ &\quad -2\int_0^1 \mathrm{d}s \int_0^1 (bb''+b'^2)(n+s-s') \mathrm{d}s', \\ &= \mathrm{O}(n^{-2\rho}), \end{split}$$

i.e., (1.7) holds with $\rho_1 = 2\rho$, and (1.8) takes the form

$$\lambda_n(J) = n + \gamma_n + \mathcal{O}(n^{1-3\rho}).$$

2. Proof of Proposition 1

Let $\mathcal{B}(l^2)$ denote the algebra of bounded operators in l^2 . Let $(\mathbf{e}_k)_{k\in\mathbb{N}^*}$ be the canonical basis of l^2 , i.e. $\mathbf{e}_k = (\delta_{k,n})_{n\in\mathbb{N}^*}$ where

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

We denote by Λ the self-adjoint operator on l^2 satisfying (2.1) $\Lambda e_n = ne_n \text{ for } n \in \mathbb{N}^*.$

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Proof of Proposition 1. The estimate (1.2) allows us to find a constant C > 0 such that

(2.2)
$$-C\Lambda^{1-\rho} \le J - \Lambda \le C\Lambda^{1-\rho}$$

holds in the sense of quadratic forms and it follows straightforwardly that Λ and J are both bounded from below and essentially self-adjoint on c_{00} .

Next we choose $\lambda > 0$ large enough and we observe that the operator

$$Q_{\lambda} = (J+\lambda)^{-1} - (\Lambda+\lambda)^{-1} = -(J+\lambda)^{-1}(J-\Lambda)(\Lambda+\lambda)^{-1}$$

satisfies $Q_{\lambda}\Lambda^{\rho} \in \mathcal{B}(l^2)$. However $\Lambda^{-\rho}$ is compact on l^2 , hence Q_{λ} is compact as well and the essential spectrum $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(\Lambda) = \emptyset$. Moreover (2.2) gives

(2.3)
$$\Lambda - C\Lambda^{1-\rho} \le J \le \Lambda + C\Lambda^{1-\rho}$$

and the min-max principle ensures

(2.4)
$$\lambda_n(\Lambda - C\Lambda^{1-\rho}) \le \lambda_n(J) \le \lambda_n(\Lambda + C\Lambda^{1-\rho}),$$

where

(2.5)
$$\lambda_n(\Lambda \pm C\Lambda^{1-\rho}) = n \pm Cn^{1-\rho}$$

is the *n*-th eigenvalue of $\Lambda \pm C \Lambda^{1-\rho}$. This completes the proof of (1.3).

3. NOTATIONS AND CONVENTIONS

Below we describe further notations and conventions.

3.1. For any application $q: \mathbb{N}^* \to \mathbb{R}$ we denote by $q(\Lambda)$ the self-adjoint operator satisfying

(3.1)
$$q(\Lambda)\mathbf{e}_n = q(n)\mathbf{e}_n \text{ for } n \in \mathbb{N}^*,$$

i.e. the domain of $q(\Lambda)$ is $D(q(\Lambda)) = \{(x_n)_{n \in \mathbb{N}^*} \mid (q(n)x_n)_{n \in \mathbb{N}^*} \in l^2\}.$

3.2. Let B_1 and B_2 be operators acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$, i.e., the subspace of sequences satisfying $x_n = O(n^{-s})$ for every $s \in \mathbb{R}$. Then for $m \in \mathbb{R}$ we write

$$(3.2) B_1 = B_2 + \mathcal{O}(\Lambda^m)$$

if and only if $\Lambda^{s-m}(B_1 - B_2)\Lambda^{-s} \in \mathcal{B}(l^2)$ holds for any $s \in \mathbb{R}$.

3.3. We also observe property

(3.3)
$$\begin{array}{c} B = O(\Lambda^m) \\ B' = O(\Lambda^{m'}) \end{array} \} \implies BB' = O(\Lambda^{m+m'}),$$

which follows immediately from the inequality

$$\|\Lambda^{s-m-m'}BB'\Lambda^{-s}\| \le \|\Lambda^{(s-m')-m}B\Lambda^{-(s-m')}\| \cdot \|\Lambda^{s-m'}B'\Lambda^{-s}\|,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{B}(l^2)$.

3.4. Further on all operators are acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ and are assumed to be closable on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$. Moreover we often write A + hc instead of $A + A^*$.

ℕ*,

3.5. Let $S \in \mathcal{B}(l^2)$ denote the shift operator satisfying

$$(3.4) Se_n = e_{n+1} ext{ for } n \in$$

let $b: \mathbb{N}^* \to \mathbb{R}$ be given by the formula

(3.5)
$$b(n) = \beta_n \text{ for } n \in \mathbb{N}^*$$

and set

(3.6)
$$J_1 = Sb(\Lambda) + b(\Lambda)S^* = Sb(\Lambda) + hc$$

Thus the operator J can be expressed

$$(3.7) J = \Lambda + J_1.$$

3.6. We introduce the closed operator A defined on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ by

(3.8)
$$A = Sb(\Lambda) - b(\Lambda)S^* = Sb(\Lambda) - hc.$$

4. Proof of theorem 1

We deduce Theorem 1 from

Proposition 2. The domain of A is the domain of the self-adjoint operator iA and

(4.1)
$$\tilde{J} = e^{-A} J e^{A} = \Lambda + O(\Lambda^{1-\rho-\rho'})$$

holds under the hypotheses of Theorem 1.

Proof of Proposition 2. See Section 6.

Proof of Theorem 1. It is easy to see that Theorem 1 follows from Proposition 2. Indeed, (4.1) implies

(4.2)
$$\Lambda - C\Lambda^{1-\rho-\rho'} \le \tilde{J} \le \Lambda + C\Lambda^{1-\rho-\rho'}$$

for a certain constant C > 0 and the min-max principle gives

(4.3)
$$\lambda_n(\Lambda - C\Lambda^{1-\rho-\rho'}) \le \lambda_n(\tilde{J}) \le \lambda_n(\Lambda + C\Lambda^{1-\rho-\rho'})$$

with $\lambda_n(\Lambda \pm C\Lambda^{1-\rho-\rho'}) = n \pm Cn^{1-\rho-\rho'}$. Hence (1.5) follows from the fact that J and \tilde{J} are unitary equivalent, which ensures $\lambda_n(J) = \lambda_n(\tilde{J})$ for all \mathbb{N}^* .

5. Proof of theorem 2

Similarly we can deduce the assertion of Theorem 2 from

Proposition 3. Under the hypotheses of Theorem 2 one has

(5.1)
$$e^{-A}Je^{A} = \Lambda + g(\Lambda) + O(\Lambda^{1-\rho-\rho_{1}}),$$

where $g: \mathbb{N}^* \to \mathbb{R}$ satisfies $g(n) = \gamma_n$ for $n \in \mathbb{N}^*$.

Proof of Proposition 3. See Section 7.

Proof of Theorem 2. Theorem 2 follows from Proposition 3. Indeed, (5.1) ensures existence of a constant C > 0 such that

$$\lambda_n(\Lambda + g(\Lambda) - C\Lambda^{1-\rho-\rho_1}) \le \lambda_n(\tilde{J}) \le \lambda_n(\Lambda + g(\Lambda) + C\Lambda^{1-\rho-\rho_1}),$$

where

$$\lambda_n(\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}) = n + g(n) \pm Cn^{1-\rho-\rho_1}$$

is the *n*-th eigenvalue of $\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}$, and (1.8) follows from $\lambda_n(J) = \lambda_n(\tilde{J})$. \Box

6. Proof of Proposition 2

We begin by a few simple lemmas.

Lemma 1. If $q: \mathbb{N}^* \to \mathbb{R}$ then the commutator of $q(\Lambda)$ with the shift operator S has the form (6.1) $[q(\Lambda), S] = (q(\Lambda + I) - q(\Lambda))S.$

Proof. Indeed, the direct computation gives

$$Sq(\Lambda)\mathbf{e}_{n} = Sq(n)\mathbf{e}_{n} = q(n)\mathbf{e}_{n+1} = q(\Lambda)S\mathbf{e}_{n}$$
$$q(\Lambda)S\mathbf{e}_{n} = q(\Lambda)\mathbf{e}_{n+1} = q(n+1)\mathbf{e}_{n+1} = q(\Lambda+I)S\mathbf{e}_{n}$$

for every $n \in \mathbb{N}^*$.

Lemma 2. Let A and J_1 be as in Section 3. Then

$$[\Lambda, A] = \Lambda A - A\Lambda = J_1.$$

Proof. Using Lemma 1 with q(n) = n we find

$$[\Lambda, S] = S,$$

hence

$$[\Lambda, Sb(\Lambda)] = [\Lambda, S]b(\Lambda) = Sb(\Lambda)$$

and

$$[\Lambda, A] = [\Lambda, Sb(\Lambda)] + hc = Sb(\Lambda) + (Sb(\Lambda))^* = J_1.$$

Lemma 3. Let g be as in Proposition 3. Then

$$(6.4) [J_1, A] = -2g(\Lambda).$$

Proof. To begin we observe that

(6.5)
$$g(\Lambda) = Sb(\Lambda)^2 S^* - b(\Lambda)^2$$

follows from $Sb(\Lambda)S^*e_n = Sb(\Lambda)e_{n-1} = b(n-1)e_n$ if $n \ge 2$ and $S^*e_1 = 0$. Then

$$[J_1, A] = [Sb(\Lambda) + b(\Lambda)S^*, Sb(\Lambda)] + hc$$
$$= [b(\Lambda)S^*, Sb(\Lambda)] + hc$$

and we complete the proof writing

$$[b(\Lambda)S^*, Sb(\Lambda)] = b(\Lambda)^2 - Sb(\Lambda)^2S^* = -g(\Lambda),$$

where we used $S^*S = I$ and (6.5).

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Proof of Proposition 2. The standard expansion formula gives

(6.6)
$$e^{A}\Lambda e^{-A} = \Lambda + [\Lambda, A] + \int_{0}^{1} (1-s)e^{sA} [[\Lambda, A], A]e^{-sA} ds$$

and (6.2) allows us to rewrite (6.6) in the form

(6.7)
$$e^{-A}Je^{A} = \Lambda - \int_{0}^{1} (1-s)e^{(s-1)A} [[\Lambda, A], A]e^{(1-s)A}ds$$

However (6.2), (6.4) and (1.4) imply

$$\left[[\Lambda, A], A \right] = [J_1, A] = -2g(\Lambda) = \mathcal{O}(\Lambda^{1-\rho-\rho'})$$

which completes the proof due to

Lemma 4. For every $m \in \mathbb{R}$ one has

(6.8)
$$\sup_{-1 \le s \le 1} \|\Lambda^m \mathrm{e}^{sA} \Lambda^{-m}\| < \infty$$

Proof. (a) To begin, we check that the estimate

(6.9)
$$[\Lambda^{\varepsilon}, A] = \mathcal{O}(\Lambda^{\varepsilon - \rho})$$

holds for every $\varepsilon > 0$. Indeed, using Lemma 1 with $q(n) = n^{\varepsilon}$ we find

$$\begin{split} \Lambda^{\varepsilon}, A] &= [\Lambda^{\varepsilon}, Sb(\Lambda)] + \mathrm{hc} \\ &= [\Lambda^{\varepsilon}, S]b(\Lambda) + \mathrm{hc} \\ &= \left((\Lambda + I)^{\varepsilon} - \Lambda^{\varepsilon} \right) Sb(\Lambda) + \mathrm{hc} \end{split}$$

Hence using property (3.3) and

$$Sb(\Lambda) = \mathcal{O}(\Lambda^{1-\rho}),$$
$$(\Lambda + I)^{\varepsilon} - \Lambda^{\varepsilon} = \mathcal{O}(\Lambda^{\varepsilon-1})$$

we obtain (6.9).

(b) Further on we assume $0 < \varepsilon \leq \rho$ and we show that

(6.10)
$$\mathcal{M}_{k\varepsilon} = \sup_{-1 \le s \le 1} ||\Lambda^{k\varepsilon} \mathrm{e}^{sA} \Lambda^{-k\varepsilon}|| < \infty$$

holds for every $k \in \mathbb{N}$. We introduce

$$R_{k\varepsilon}(s) = \Lambda^{(k+1)\varepsilon} e^{sA} \Lambda^{-(k+1)\varepsilon} - \Lambda^{k\varepsilon} e^{sA} \Lambda^{-k\varepsilon}$$

and observe that

$$R_{\varepsilon}(s) = \left[e^{s(1-t)A} \Lambda^{\varepsilon} e^{stA} \Lambda^{-\varepsilon} \right]_{t=0}^{t=1} = \int_{0}^{1} e^{s(1-t)A} [A, \Lambda^{\varepsilon}] e^{stA} \Lambda^{-\varepsilon} dt$$

allows us to estimate (6.9) allows us to estimate

$$\begin{aligned} \|R_{k\varepsilon}(s)\| &= \|\Lambda^{k\varepsilon}R_{\varepsilon}(s)\Lambda^{-k\varepsilon}\| \\ &\leq \mathcal{M}_{k\varepsilon}^{2}\|\Lambda^{k\varepsilon}[\Lambda^{\varepsilon},A]\Lambda^{-k\varepsilon}\| < \infty \end{aligned}$$

if $\mathcal{M}_{k\varepsilon} < \infty$.

7. Proof of Proposition 3

(a) To begin, we observe that

(7.1)
$$[g(\Lambda), A] = \mathcal{O}(\Lambda^{1-\rho-\rho_1})$$

follows from assumption (1.7). Indeed,

$$[g(\Lambda), A] = [g(\Lambda), Sb(\Lambda)] + hc$$

= $[g(\Lambda), S]b(\Lambda) + hc$
= $(g(\Lambda + I) - g(\Lambda))Sb(\Lambda) + hc$

hence using property (3.2) and

$$\begin{split} Sb(\Lambda) &= \mathcal{O}(\Lambda^{1-\rho}), \\ g(\Lambda+I) - g(\Lambda) &= \mathcal{O}(\Lambda^{-\rho_1}) \end{split}$$

we obtain (7.1).

(b) Then the standard expansion formula gives

(7.2)
$$e^{A}\Lambda e^{-A} = \Lambda + [\Lambda, A] + \frac{1}{2}[[\Lambda, A], A] + \int_{0}^{1} (1-s)^{2} e^{sA} R e^{-sA} ds$$

with

$$R = \frac{1}{2} \big[[[\Lambda, A], A], A] = -[g(\Lambda), A].$$

(c) However we have $R = O(\Lambda^{1-\rho-\rho_1})$ due to (7.1) and Lemma 4 allows us to deduce

(7.3)
$$e^{A}\Lambda e^{-A} = J - g(\Lambda) + O(\Lambda^{1-\rho-\rho_1})$$

from (7.2). Applying Lemma 4 once more we obtain

(7.4)
$$e^{-A}Je^{A} = \Lambda + e^{-A}g(\Lambda)e^{A} + O(\Lambda^{1-\rho-\rho_{1}}).$$

(d) To complete the proof of Proposition 3 it remains to show

(7.5)
$$e^{-A}g(\Lambda)e^{A} = g(\Lambda) + O(\Lambda^{1-\rho-\rho_1}).$$

However

$$e^{-A}g(\Lambda)e^{A} - g(\Lambda) = \int_{0}^{1} e^{-sA}[g(\Lambda), A]e^{sA}ds$$

and it is clear that (7.5) follows from (7.1) and Lemma 4.

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A. BOUTET DE MONVEL AND L. ZIELINSKI

*IMJ, case 7012, Université Paris Diderot, 175 rue du Chevaleret, 75013 Paris, France
 † IMJ & Université du Littoral, Calais, France