



HAL
open science

EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS

Anne Boutet de Monvel, Lech Zielinski

► **To cite this version:**

Anne Boutet de Monvel, Lech Zielinski. EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS. 2008. hal-03663133

HAL Id: hal-03663133

<https://hal.science/hal-03663133>

Preprint submitted on 9 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS

ANNE BOUTET DE MONVEL* AND LECH ZIELINSKI†

ABSTRACT. We obtain eigenvalue asymptotics for Jacobi matrices of various Jaynes-Cummings type.

1. THE RESULTS

We consider a type of Jacobi matrices with unbounded entries related to some problems of quantum optics. See [1–4].

Let $\mathbb{N}^* = \{1, 2, \dots\}$ be the set of positive integers and let l^2 denote the Hilbert space of square summable complex sequences $x = (x_n)_{n \in \mathbb{N}^*}$. Let c_{00} be the subspace of sequences for which $\{n \in \mathbb{N}^* \mid x_n \neq 0\}$ is finite. We fix a real valued sequence $(\beta_n)_{n \in \mathbb{N}^*}$ and consider a linear operator J acting on $(x_n)_{n \in \mathbb{N}^*} \in c_{00}$ according to the formula

$$(1.1) \quad (Jx)_n = \begin{cases} nx_n + \beta_n x_{n+1} + \beta_{n-1} x_{n-1} & n \geq 2, \\ x_1 + \beta_1 x_2 & n = 1. \end{cases}$$

Then it is easy to establish the following elementary fact.

Proposition 1. *Assume that there exists $\rho > 0$ such that*

$$(1.2) \quad \beta_n = O(n^{1-\rho}).$$

Then the closure of the operator defined by (1.1) is a self-adjoint operator J , its spectrum is discrete and bounded from below. Let $(\lambda_n(J))_{n \in \mathbb{N}^}$ denote the sequence of eigenvalues of J repeated according to their multiplicities and ordered so that $\lambda_n(J) \leq \lambda_{n+1}(J)$ for all $n \in \mathbb{N}^*$. Then the following estimate*

$$(1.3) \quad \lambda_n(J) = n + O(n^{1-\rho})$$

holds as $n \rightarrow \infty$.

The aim of this paper is to obtain sharper estimates of the asymptotic behaviour of $(\lambda_n(J))_{n \in \mathbb{N}^*}$ which can be deduced from additional assumptions made on the sequence $(\beta_n)_{n \in \mathbb{N}^*}$. Our first result is

Theorem 1. *Assume that (1.2) holds with a certain $\rho > 0$ and*

$$(1.4) \quad \beta_{n+1} - \beta_n = O(n^{-\rho'})$$

holds with a certain $\rho' > 0$. Then one has the estimate

$$(1.5) \quad \lambda_n(J) = n + O(n^{1-\rho-\rho'})$$

Date: February 24, 2008.

Our second theorem depends on the behaviour of the sequence

$$(1.6) \quad \gamma_n = \begin{cases} \beta_{n-1}^2 - \beta_n^2 & n \geq 2, \\ -\beta_1^2 & n = 1. \end{cases}$$

Theorem 2. *Assume that $(\beta_n)_{n \in \mathbb{N}^*}$ satisfies the hypotheses of Theorem 1. Let $(\gamma_n)_{n \in \mathbb{N}^*}$ be the sequence given by (1.6). If*

$$(1.7) \quad \gamma_{n+1} - \gamma_n = O(n^{-\rho_1})$$

holds with a certain $\rho_1 > 0$, then one has

$$(1.8) \quad \lambda_n(J) = n + \gamma_n + O(n^{1-\rho-\rho_1}).$$

Remark. If it is possible to evaluate $\beta_n = b(n)$ by means of a function $b \in C^\infty((0, +\infty))$ satisfying the estimates

$$\begin{cases} b(\lambda) = O(\lambda^{1-\rho}), \\ b'(\lambda) = O(\lambda^{-\rho}), \end{cases}$$

then

$$\beta_{n+1} - \beta_n = \int_0^1 b'(n+s) ds = O(n^{-\rho}),$$

i.e., (1.4) holds with $\rho = \rho'$, and (1.5) takes the form

$$\lambda_n(J) = n + O(n^{1-2\rho}).$$

If moreover

$$b''(\lambda) = O(\lambda^{-1-\rho}),$$

then

$$\begin{aligned} \gamma_{n+1} - \gamma_n &= - \int_0^1 ds \int_0^1 b^{2''}(n+s-s') ds' \\ &\quad - 2 \int_0^1 ds \int_0^1 (bb'' + b'^2)(n+s-s') ds', \\ &= O(n^{-2\rho}), \end{aligned}$$

i.e., (1.7) holds with $\rho_1 = 2\rho$, and (1.8) takes the form

$$\lambda_n(J) = n + \gamma_n + O(n^{1-3\rho}).$$

2. PROOF OF PROPOSITION 1

Let $\mathcal{B}(l^2)$ denote the algebra of bounded operators in l^2 . Let $(e_k)_{k \in \mathbb{N}^*}$ be the canonical basis of l^2 , i.e. $e_k = (\delta_{k,n})_{n \in \mathbb{N}^*}$ where

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

We denote by Λ the self-adjoint operator on l^2 satisfying

$$(2.1) \quad \Lambda e_n = n e_n \text{ for } n \in \mathbb{N}^*.$$

Proof of Proposition 1. The estimate (1.2) allows us to find a constant $C > 0$ such that

$$(2.2) \quad -C\Lambda^{1-\rho} \leq J - \Lambda \leq C\Lambda^{1-\rho}$$

holds in the sense of quadratic forms and it follows straightforwardly that Λ and J are both bounded from below and essentially self-adjoint on c_{00} .

Next we choose $\lambda > 0$ large enough and we observe that the operator

$$Q_\lambda = (J + \lambda)^{-1} - (\Lambda + \lambda)^{-1} = -(J + \lambda)^{-1}(J - \Lambda)(\Lambda + \lambda)^{-1}$$

satisfies $Q_\lambda \Lambda^\rho \in \mathcal{B}(l^2)$. However $\Lambda^{-\rho}$ is compact on l^2 , hence Q_λ is compact as well and the essential spectrum $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(\Lambda) = \emptyset$. Moreover (2.2) gives

$$(2.3) \quad \Lambda - C\Lambda^{1-\rho} \leq J \leq \Lambda + C\Lambda^{1-\rho}$$

and the min-max principle ensures

$$(2.4) \quad \lambda_n(\Lambda - C\Lambda^{1-\rho}) \leq \lambda_n(J) \leq \lambda_n(\Lambda + C\Lambda^{1-\rho}),$$

where

$$(2.5) \quad \lambda_n(\Lambda \pm C\Lambda^{1-\rho}) = n \pm Cn^{1-\rho}$$

is the n -th eigenvalue of $\Lambda \pm C\Lambda^{1-\rho}$. This completes the proof of (1.3). \square

3. NOTATIONS AND CONVENTIONS

Below we describe further notations and conventions.

3.1. For any application $q: \mathbb{N}^* \rightarrow \mathbb{R}$ we denote by $q(\Lambda)$ the self-adjoint operator satisfying

$$(3.1) \quad q(\Lambda)e_n = q(n)e_n \text{ for } n \in \mathbb{N}^*,$$

i.e. the domain of $q(\Lambda)$ is $D(q(\Lambda)) = \{(x_n)_{n \in \mathbb{N}^*} \mid (q(n)x_n)_{n \in \mathbb{N}^*} \in l^2\}$.

3.2. Let B_1 and B_2 be operators acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$, i.e., the subspace of sequences satisfying $x_n = O(n^{-s})$ for every $s \in \mathbb{R}$. Then for $m \in \mathbb{R}$ we write

$$(3.2) \quad B_1 = B_2 + O(\Lambda^m)$$

if and only if $\Lambda^{s-m}(B_1 - B_2)\Lambda^{-s} \in \mathcal{B}(l^2)$ holds for any $s \in \mathbb{R}$.

3.3. We also observe property

$$(3.3) \quad \left. \begin{array}{l} B = O(\Lambda^m) \\ B' = O(\Lambda^{m'}) \end{array} \right\} \implies BB' = O(\Lambda^{m+m'}),$$

which follows immediately from the inequality

$$\|\Lambda^{s-m-m'}BB'\Lambda^{-s}\| \leq \|\Lambda^{(s-m')-m}B\Lambda^{-(s-m')}\| \cdot \|\Lambda^{s-m'}B'\Lambda^{-s}\|,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{B}(l^2)$.

3.4. Further on all operators are acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ and are assumed to be closable on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$. Moreover we often write $A + hc$ instead of $A + A^*$.

3.5. Let $S \in \mathcal{B}(l^2)$ denote the shift operator satisfying

$$(3.4) \quad Se_n = e_{n+1} \text{ for } n \in \mathbb{N}^*,$$

let $b: \mathbb{N}^* \rightarrow \mathbb{R}$ be given by the formula

$$(3.5) \quad b(n) = \beta_n \text{ for } n \in \mathbb{N}^*$$

and set

$$(3.6) \quad J_1 = Sb(\Lambda) + b(\Lambda)S^* = Sb(\Lambda) + \text{hc}.$$

Thus the operator J can be expressed

$$(3.7) \quad J = \Lambda + J_1.$$

3.6. We introduce the closed operator A defined on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ by

$$(3.8) \quad A = Sb(\Lambda) - b(\Lambda)S^* = Sb(\Lambda) - \text{hc}.$$

4. PROOF OF THEOREM 1

We deduce Theorem 1 from

Proposition 2. *The domain of A is the domain of the self-adjoint operator iA and*

$$(4.1) \quad \tilde{J} = e^{-A} J e^A = \Lambda + O(\Lambda^{1-\rho-\rho'})$$

holds under the hypotheses of Theorem 1.

Proof of Proposition 2. See Section 6. □

Proof of Theorem 1. It is easy to see that Theorem 1 follows from Proposition 2. Indeed, (4.1) implies

$$(4.2) \quad \Lambda - C\Lambda^{1-\rho-\rho'} \leq \tilde{J} \leq \Lambda + C\Lambda^{1-\rho-\rho'}$$

for a certain constant $C > 0$ and the min-max principle gives

$$(4.3) \quad \lambda_n(\Lambda - C\Lambda^{1-\rho-\rho'}) \leq \lambda_n(\tilde{J}) \leq \lambda_n(\Lambda + C\Lambda^{1-\rho-\rho'})$$

with $\lambda_n(\Lambda \pm C\Lambda^{1-\rho-\rho'}) = n \pm Cn^{1-\rho-\rho'}$. Hence (1.5) follows from the fact that J and \tilde{J} are unitary equivalent, which ensures $\lambda_n(J) = \lambda_n(\tilde{J})$ for all \mathbb{N}^* . □

5. PROOF OF THEOREM 2

Similarly we can deduce the assertion of Theorem 2 from

Proposition 3. *Under the hypotheses of Theorem 2 one has*

$$(5.1) \quad e^{-A} J e^A = \Lambda + g(\Lambda) + O(\Lambda^{1-\rho-\rho_1}),$$

where $g: \mathbb{N}^ \rightarrow \mathbb{R}$ satisfies $g(n) = \gamma_n$ for $n \in \mathbb{N}^*$.*

Proof of Proposition 3. See Section 7. □

Proof of Theorem 2. Theorem 2 follows from Proposition 3. Indeed, (5.1) ensures existence of a constant $C > 0$ such that

$$\lambda_n(\Lambda + g(\Lambda) - C\Lambda^{1-\rho-\rho_1}) \leq \lambda_n(\tilde{J}) \leq \lambda_n(\Lambda + g(\Lambda) + C\Lambda^{1-\rho-\rho_1}),$$

where

$$\lambda_n(\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}) = n + g(n) \pm Cn^{1-\rho-\rho_1}$$

is the n -th eigenvalue of $\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}$, and (1.8) follows from $\lambda_n(J) = \lambda_n(\tilde{J})$. \square

6. PROOF OF PROPOSITION 2

We begin by a few simple lemmas.

Lemma 1. *If $q: \mathbb{N}^* \rightarrow \mathbb{R}$ then the commutator of $q(\Lambda)$ with the shift operator S has the form*

$$(6.1) \quad [q(\Lambda), S] = (q(\Lambda + I) - q(\Lambda))S.$$

Proof. Indeed, the direct computation gives

$$\begin{aligned} Sq(\Lambda)e_n &= Sq(n)e_n = q(n)e_{n+1} = q(\Lambda)Se_n \\ q(\Lambda)Se_n &= q(\Lambda)e_{n+1} = q(n+1)e_{n+1} = q(\Lambda + I)Se_n \end{aligned}$$

for every $n \in \mathbb{N}^*$. \square

Lemma 2. *Let A and J_1 be as in Section 3. Then*

$$(6.2) \quad [\Lambda, A] = \Lambda A - A\Lambda = J_1.$$

Proof. Using Lemma 1 with $q(n) = n$ we find

$$(6.3) \quad [\Lambda, S] = S,$$

hence

$$[\Lambda, Sb(\Lambda)] = [\Lambda, S]b(\Lambda) = Sb(\Lambda)$$

and

$$[\Lambda, A] = [\Lambda, Sb(\Lambda)] + \text{hc} = Sb(\Lambda) + (Sb(\Lambda))^* = J_1. \quad \square$$

Lemma 3. *Let g be as in Proposition 3. Then*

$$(6.4) \quad [J_1, A] = -2g(\Lambda).$$

Proof. To begin we observe that

$$(6.5) \quad g(\Lambda) = Sb(\Lambda)^2 S^* - b(\Lambda)^2$$

follows from $Sb(\Lambda)S^*e_n = Sb(\Lambda)e_{n-1} = b(n-1)e_n$ if $n \geq 2$ and $S^*e_1 = 0$. Then

$$\begin{aligned} [J_1, A] &= [Sb(\Lambda) + b(\Lambda)S^*, Sb(\Lambda)] + \text{hc} \\ &= [b(\Lambda)S^*, Sb(\Lambda)] + \text{hc} \end{aligned}$$

and we complete the proof writing

$$[b(\Lambda)S^*, Sb(\Lambda)] = b(\Lambda)^2 - Sb(\Lambda)^2 S^* = -g(\Lambda),$$

where we used $S^*S = I$ and (6.5). \square

Proof of Proposition 2. The standard expansion formula gives

$$(6.6) \quad e^A \Lambda e^{-A} = \Lambda + [\Lambda, A] + \int_0^1 (1-s)e^{sA} [[\Lambda, A], A] e^{-sA} ds$$

and (6.2) allows us to rewrite (6.6) in the form

$$(6.7) \quad e^{-A} J e^A = \Lambda - \int_0^1 (1-s)e^{(s-1)A} [[\Lambda, A], A] e^{(1-s)A} ds.$$

However (6.2), (6.4) and (1.4) imply

$$[[\Lambda, A], A] = [J_1, A] = -2g(\Lambda) = O(\Lambda^{1-\rho-\rho'}),$$

which completes the proof due to □

Lemma 4. *For every $m \in \mathbb{R}$ one has*

$$(6.8) \quad \sup_{-1 \leq s \leq 1} \|\Lambda^m e^{sA} \Lambda^{-m}\| < \infty.$$

Proof. (a) To begin, we check that the estimate

$$(6.9) \quad [\Lambda^\varepsilon, A] = O(\Lambda^{\varepsilon-\rho})$$

holds for every $\varepsilon > 0$. Indeed, using Lemma 1 with $q(n) = n^\varepsilon$ we find

$$\begin{aligned} [\Lambda^\varepsilon, A] &= [\Lambda^\varepsilon, Sb(\Lambda)] + hc \\ &= [\Lambda^\varepsilon, S]b(\Lambda) + hc \\ &= ((\Lambda + I)^\varepsilon - \Lambda^\varepsilon)Sb(\Lambda) + hc. \end{aligned}$$

Hence using property (3.3) and

$$\begin{aligned} Sb(\Lambda) &= O(\Lambda^{1-\rho}), \\ (\Lambda + I)^\varepsilon - \Lambda^\varepsilon &= O(\Lambda^{\varepsilon-1}) \end{aligned}$$

we obtain (6.9).

(b) Further on we assume $0 < \varepsilon \leq \rho$ and we show that

$$(6.10) \quad \mathcal{M}_{k\varepsilon} = \sup_{-1 \leq s \leq 1} \|\Lambda^{k\varepsilon} e^{sA} \Lambda^{-k\varepsilon}\| < \infty$$

holds for every $k \in \mathbb{N}$. We introduce

$$R_{k\varepsilon}(s) = \Lambda^{(k+1)\varepsilon} e^{sA} \Lambda^{-(k+1)\varepsilon} - \Lambda^{k\varepsilon} e^{sA} \Lambda^{-k\varepsilon}$$

and observe that

$$R_\varepsilon(s) = [e^{s(1-t)A} \Lambda^\varepsilon e^{stA} \Lambda^{-\varepsilon}]_{t=0}^{t=1} = \int_0^1 e^{s(1-t)A} [A, \Lambda^\varepsilon] e^{stA} \Lambda^{-\varepsilon} dt$$

allows us to estimate (6.9) allows us to estimate

$$\begin{aligned} \|R_{k\varepsilon}(s)\| &= \|\Lambda^{k\varepsilon} R_\varepsilon(s) \Lambda^{-k\varepsilon}\| \\ &\leq \mathcal{M}_{k\varepsilon}^2 \|\Lambda^{k\varepsilon} [\Lambda^\varepsilon, A] \Lambda^{-k\varepsilon}\| < \infty \end{aligned}$$

if $\mathcal{M}_{k\varepsilon} < \infty$. □

7. PROOF OF PROPOSITION 3

(a) To begin, we observe that

$$(7.1) \quad [g(\Lambda), A] = O(\Lambda^{1-\rho-\rho_1})$$

follows from assumption (1.7). Indeed,

$$\begin{aligned} [g(\Lambda), A] &= [g(\Lambda), Sb(\Lambda)] + \text{hc} \\ &= [g(\Lambda), S]b(\Lambda) + \text{hc} \\ &= (g(\Lambda + I) - g(\Lambda))Sb(\Lambda) + \text{hc}, \end{aligned}$$

hence using property (3.2) and

$$\begin{aligned} Sb(\Lambda) &= O(\Lambda^{1-\rho}), \\ g(\Lambda + I) - g(\Lambda) &= O(\Lambda^{-\rho_1}) \end{aligned}$$

we obtain (7.1).

(b) Then the standard expansion formula gives

$$(7.2) \quad e^A \Lambda e^{-A} = \Lambda + [\Lambda, A] + \frac{1}{2} [[\Lambda, A], A] + \int_0^1 (1-s)^2 e^{sA} R e^{-sA} ds$$

with

$$R = \frac{1}{2} [[[\Lambda, A], A], A] = -[g(\Lambda), A].$$

(c) However we have $R = O(\Lambda^{1-\rho-\rho_1})$ due to (7.1) and Lemma 4 allows us to deduce

$$(7.3) \quad e^A \Lambda e^{-A} = J - g(\Lambda) + O(\Lambda^{1-\rho-\rho_1})$$

from (7.2). Applying Lemma 4 once more we obtain

$$(7.4) \quad e^{-A} J e^A = \Lambda + e^{-A} g(\Lambda) e^A + O(\Lambda^{1-\rho-\rho_1}).$$

(d) To complete the proof of Proposition 3 it remains to show

$$(7.5) \quad e^{-A} g(\Lambda) e^A = g(\Lambda) + O(\Lambda^{1-\rho-\rho_1}).$$

However

$$e^{-A} g(\Lambda) e^A - g(\Lambda) = \int_0^1 e^{-sA} [g(\Lambda), A] e^{sA} ds$$

and it is clear that (7.5) follows from (7.1) and Lemma 4.

REFERENCES

- [1] A. Boutet de Monvel, S. Naboko, and L. O. Silva, *Eigenvalue asymptotics of a modified Jaynes-Cummings model with periodic modulations*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 1, 103–107.
- [2] ———, *The asymptotic behavior of eigenvalues of a modified Jaynes-Cummings model*, Asymptot. Anal. **47** (2006), no. 3-4, 291–315.
- [3] J. Janas and S. Naboko, *Infinite Jacobi matrices with unbounded entries: asymptotics of eigenvalues and the transformation operator approach*, SIAM J. Math. Anal. **36** (2004), no. 2, 643–658.
- [4] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, vol. 72, American Mathematical Society, Providence, RI, 2000.

*IMJ, CASE 7012, UNIVERSITÉ PARIS DIDEROT, 175 RUE DU CHEVALERET, 75013 PARIS, FRANCE

†IMJ & UNIVERSITÉ DU LITTORAL, CALAIS, FRANCE