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EIGENVALUE ASYMPTOTICS FOR JAYNES-CUMMINGS TYPE MODELS WITHOUT MODULATIONS

ANNE BOUTET DE MONVEL^{*} AND LECH ZIELINSKI[†]

ABSTRACT. We obtain eigenvalue asymptotics for Jacobi matrices of various Jaynes-Cummings type.

1. THE RESULTS

We consider a type of Jacobi matrices with unbounded entries related to some problems of quantum optics. See [1–4].

Let $\mathbb{N}^* = \{1, 2, \dots\}$ be the set of positive integers and let l^2 denote the Hilbert space of square summable complex sequences $x = (x_n)_{n \in \mathbb{N}^*}$. Let c_{00} be the subspace of sequences for which $\{n \in \mathbb{N}^* \mid x_n \neq 0\}$ is finite. We fix a real valued sequence $(\beta_n)_{n \in \mathbb{N}^*}$ and consider a linear operator J acting on $(x_n)_{n \in \mathbb{N}^*} \in c_{00}$ according to the formula

$$(1.1) \quad (Jx)_n = \begin{cases} nx_n + \beta_n x_{n+1} + \beta_{n-1} x_{n-1} & n \geq 2, \\ x_1 + \beta_1 x_2 & n = 1. \end{cases}$$

Then it is easy to establish the following elementary fact.

Proposition 1. *Assume that there exists $\rho > 0$ such that*

$$(1.2) \quad \beta_n = O(n^{1-\rho}).$$

Then the closure of the operator defined by (1.1) is a self-adjoint operator J , its spectrum is discrete and bounded from below. Let $(\lambda_n(J))_{n \in \mathbb{N}^}$ denote the sequence of eigenvalues of J repeated according to their multiplicities and ordered so that $\lambda_n(J) \leq \lambda_{n+1}(J)$ for all $n \in \mathbb{N}^*$. Then the following estimate*

$$(1.3) \quad \lambda_n(J) = n + O(n^{1-\rho})$$

holds as $n \rightarrow \infty$.

The aim of this paper is to obtain sharper estimates of the asymptotic behaviour of $(\lambda_n(J))_{n \in \mathbb{N}^*}$ which can be deduced from additional assumptions made on the sequence $(\beta_n)_{n \in \mathbb{N}^*}$. Our first result is

Theorem 1. *Assume that (1.2) holds with a certain $\rho > 0$ and*

$$(1.4) \quad \beta_{n+1} - \beta_n = O(n^{-\rho'})$$

holds with a certain $\rho' > 0$. Then one has the estimate

$$(1.5) \quad \lambda_n(J) = n + O(n^{1-\rho-\rho'})$$

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Our second theorem depends on the behaviour of the sequence

$$(1.6) \quad \gamma_n = \begin{cases} \beta_{n-1}^2 - \beta_n^2 & n \geq 2, \\ -\beta_1^2 & n = 1. \end{cases}$$

Theorem 2. *Assume that $(\beta_n)_{n \in \mathbb{N}^*}$ satisfies the hypotheses of Theorem 1. Let $(\gamma_n)_{n \in \mathbb{N}^*}$ be the sequence given by (1.6). If*

$$(1.7) \quad \gamma_{n+1} - \gamma_n = O(n^{-\rho_1})$$

holds with a certain $\rho_1 > 0$, then one has

$$(1.8) \quad \lambda_n(J) = n + \gamma_n + O(n^{1-\rho-\rho_1}).$$

Remark. If it is possible to evaluate $\beta_n = b(n)$ by means of a function $b \in C^\infty((0, +\infty))$ satisfying the estimates

$$\begin{cases} b(\lambda) = O(\lambda^{1-\rho}), \\ b'(\lambda) = O(\lambda^{-\rho}), \end{cases}$$

then

$$\beta_{n+1} - \beta_n = \int_0^1 b'(n+s) ds = O(n^{-\rho}),$$

i.e., (1.4) holds with $\rho = \rho'$, and (1.5) takes the form

$$\lambda_n(J) = n + O(n^{1-2\rho}).$$

If moreover

$$b''(\lambda) = O(\lambda^{-1-\rho}),$$

then

$$\begin{aligned} \gamma_{n+1} - \gamma_n &= - \int_0^1 ds \int_0^1 b^{2''}(n+s-s') ds' \\ &\quad - 2 \int_0^1 ds \int_0^1 (bb'' + b'^2)(n+s-s') ds', \\ &= O(n^{-2\rho}), \end{aligned}$$

i.e., (1.7) holds with $\rho_1 = 2\rho$, and (1.8) takes the form

$$\lambda_n(J) = n + \gamma_n + O(n^{1-3\rho}).$$

2. PROOF OF PROPOSITION 1

Let $\mathcal{B}(l^2)$ denote the algebra of bounded operators in l^2 . Let $(e_k)_{k \in \mathbb{N}^*}$ be the canonical basis of l^2 , i.e. $e_k = (\delta_{k,n})_{n \in \mathbb{N}^*}$ where

$$\delta_{k,n} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

We denote by Λ the self-adjoint operator on l^2 satisfying

$$(2.1) \quad \Lambda e_n = n e_n \text{ for } n \in \mathbb{N}^*.$$

Proof of Proposition 1. The estimate (1.2) allows us to find a constant $C > 0$ such that

$$(2.2) \quad -C\Lambda^{1-\rho} \leq J - \Lambda \leq C\Lambda^{1-\rho}$$

holds in the sense of quadratic forms and it follows straightforwardly that Λ and J are both bounded from below and essentially self-adjoint on c_{00} .

Next we choose $\lambda > 0$ large enough and we observe that the operator

$$Q_\lambda = (J + \lambda)^{-1} - (\Lambda + \lambda)^{-1} = -(J + \lambda)^{-1}(J - \Lambda)(\Lambda + \lambda)^{-1}$$

satisfies $Q_\lambda \Lambda^\rho \in \mathcal{B}(l^2)$. However $\Lambda^{-\rho}$ is compact on l^2 , hence Q_λ is compact as well and the essential spectrum $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(\Lambda) = \emptyset$. Moreover (2.2) gives

$$(2.3) \quad \Lambda - C\Lambda^{1-\rho} \leq J \leq \Lambda + C\Lambda^{1-\rho}$$

and the min-max principle ensures

$$(2.4) \quad \lambda_n(\Lambda - C\Lambda^{1-\rho}) \leq \lambda_n(J) \leq \lambda_n(\Lambda + C\Lambda^{1-\rho}),$$

where

$$(2.5) \quad \lambda_n(\Lambda \pm C\Lambda^{1-\rho}) = n \pm Cn^{1-\rho}$$

is the n -th eigenvalue of $\Lambda \pm C\Lambda^{1-\rho}$. This completes the proof of (1.3). \square

3. NOTATIONS AND CONVENTIONS

Below we describe further notations and conventions.

3.1. For any application $q: \mathbb{N}^* \rightarrow \mathbb{R}$ we denote by $q(\Lambda)$ the self-adjoint operator satisfying

$$(3.1) \quad q(\Lambda)e_n = q(n)e_n \text{ for } n \in \mathbb{N}^*,$$

i.e. the domain of $q(\Lambda)$ is $D(q(\Lambda)) = \{(x_n)_{n \in \mathbb{N}^*} \mid (q(n)x_n)_{n \in \mathbb{N}^*} \in l^2\}$.

3.2. Let B_1 and B_2 be operators acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$, i.e., the subspace of sequences satisfying $x_n = O(n^{-s})$ for every $s \in \mathbb{R}$. Then for $m \in \mathbb{R}$ we write

$$(3.2) \quad B_1 = B_2 + O(\Lambda^m)$$

if and only if $\Lambda^{s-m}(B_1 - B_2)\Lambda^{-s} \in \mathcal{B}(l^2)$ holds for any $s \in \mathbb{R}$.

3.3. We also observe property

$$(3.3) \quad \left. \begin{array}{l} B = O(\Lambda^m) \\ B' = O(\Lambda^{m'}) \end{array} \right\} \implies BB' = O(\Lambda^{m+m'}),$$

which follows immediately from the inequality

$$\|\Lambda^{s-m-m'}BB'\Lambda^{-s}\| \leq \|\Lambda^{(s-m')-m}B\Lambda^{-(s-m')}\| \cdot \|\Lambda^{s-m'}B'\Lambda^{-s}\|,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{B}(l^2)$.

3.4. Further on all operators are acting on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ and are assumed to be closable on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$. Moreover we often write $A + \text{hc}$ instead of $A + A^*$.

3.5. Let $S \in \mathcal{B}(l^2)$ denote the shift operator satisfying

$$(3.4) \quad Se_n = e_{n+1} \text{ for } n \in \mathbb{N}^*,$$

let $b: \mathbb{N}^* \rightarrow \mathbb{R}$ be given by the formula

$$(3.5) \quad b(n) = \beta_n \text{ for } n \in \mathbb{N}^*$$

and set

$$(3.6) \quad J_1 = Sb(\Lambda) + b(\Lambda)S^* = Sb(\Lambda) + \text{hc}.$$

Thus the operator J can be expressed

$$(3.7) \quad J = \Lambda + J_1.$$

3.6. We introduce the closed operator A defined on $\cap_{k \in \mathbb{N}} D(\Lambda^k)$ by

$$(3.8) \quad A = Sb(\Lambda) - b(\Lambda)S^* = Sb(\Lambda) - \text{hc}.$$

4. PROOF OF THEOREM 1

We deduce Theorem 1 from

Proposition 2. *The domain of A is the domain of the self-adjoint operator iA and*

$$(4.1) \quad \tilde{J} = e^{-A} J e^A = \Lambda + O(\Lambda^{1-\rho-\rho'})$$

holds under the hypotheses of Theorem 1.

Proof of Proposition 2. See Section 6. □

Proof of Theorem 1. It is easy to see that Theorem 1 follows from Proposition 2. Indeed, (4.1) implies

$$(4.2) \quad \Lambda - C\Lambda^{1-\rho-\rho'} \leq \tilde{J} \leq \Lambda + C\Lambda^{1-\rho-\rho'}$$

for a certain constant $C > 0$ and the min-max principle gives

$$(4.3) \quad \lambda_n(\Lambda - C\Lambda^{1-\rho-\rho'}) \leq \lambda_n(\tilde{J}) \leq \lambda_n(\Lambda + C\Lambda^{1-\rho-\rho'})$$

with $\lambda_n(\Lambda \pm C\Lambda^{1-\rho-\rho'}) = n \pm Cn^{1-\rho-\rho'}$. Hence (1.5) follows from the fact that J and \tilde{J} are unitary equivalent, which ensures $\lambda_n(J) = \lambda_n(\tilde{J})$ for all \mathbb{N}^* . □

5. PROOF OF THEOREM 2

Similarly we can deduce the assertion of Theorem 2 from

Proposition 3. *Under the hypotheses of Theorem 2 one has*

$$(5.1) \quad e^{-A} J e^A = \Lambda + g(\Lambda) + O(\Lambda^{1-\rho-\rho_1}),$$

where $g: \mathbb{N}^ \rightarrow \mathbb{R}$ satisfies $g(n) = \gamma_n$ for $n \in \mathbb{N}^*$.*

Proof of Proposition 3. See Section 7. □

Proof of Theorem 2. Theorem 2 follows from Proposition 3. Indeed, (5.1) ensures existence of a constant $C > 0$ such that

$$\lambda_n(\Lambda + g(\Lambda) - C\Lambda^{1-\rho-\rho_1}) \leq \lambda_n(\tilde{J}) \leq \lambda_n(\Lambda + g(\Lambda) + C\Lambda^{1-\rho-\rho_1}),$$

where

$$\lambda_n(\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}) = n + g(n) \pm Cn^{1-\rho-\rho_1}$$

is the n -th eigenvalue of $\Lambda + g(\Lambda) \pm C\Lambda^{1-\rho-\rho_1}$, and (1.8) follows from $\lambda_n(J) = \lambda_n(\tilde{J})$. \square

6. PROOF OF PROPOSITION 2

We begin by a few simple lemmas.

Lemma 1. *If $q: \mathbb{N}^* \rightarrow \mathbb{R}$ then the commutator of $q(\Lambda)$ with the shift operator S has the form*

$$(6.1) \quad [q(\Lambda), S] = (q(\Lambda + I) - q(\Lambda))S.$$

Proof. Indeed, the direct computation gives

$$\begin{aligned} Sq(\Lambda)e_n &= Sq(n)e_n = q(n)e_{n+1} = q(\Lambda)Se_n \\ q(\Lambda)Se_n &= q(\Lambda)e_{n+1} = q(n+1)e_{n+1} = q(\Lambda + I)Se_n \end{aligned}$$

for every $n \in \mathbb{N}^*$. \square

Lemma 2. *Let A and J_1 be as in Section 3. Then*

$$(6.2) \quad [\Lambda, A] = \Lambda A - A\Lambda = J_1.$$

Proof. Using Lemma 1 with $q(n) = n$ we find

$$(6.3) \quad [\Lambda, S] = S,$$

hence

$$[\Lambda, Sb(\Lambda)] = [\Lambda, S]b(\Lambda) = Sb(\Lambda)$$

and

$$[\Lambda, A] = [\Lambda, Sb(\Lambda)] + \text{hc} = Sb(\Lambda) + (Sb(\Lambda))^* = J_1. \quad \square$$

Lemma 3. *Let g be as in Proposition 3. Then*

$$(6.4) \quad [J_1, A] = -2g(\Lambda).$$

Proof. To begin we observe that

$$(6.5) \quad g(\Lambda) = Sb(\Lambda)^2 S^* - b(\Lambda)^2$$

follows from $Sb(\Lambda)S^*e_n = Sb(\Lambda)e_{n-1} = b(n-1)e_n$ if $n \geq 2$ and $S^*e_1 = 0$. Then

$$\begin{aligned} [J_1, A] &= [Sb(\Lambda) + b(\Lambda)S^*, Sb(\Lambda)] + \text{hc} \\ &= [b(\Lambda)S^*, Sb(\Lambda)] + \text{hc} \end{aligned}$$

and we complete the proof writing

$$[b(\Lambda)S^*, Sb(\Lambda)] = b(\Lambda)^2 - Sb(\Lambda)^2 S^* = -g(\Lambda),$$

where we used $S^*S = I$ and (6.5). \square

Proof of Proposition 2. The standard expansion formula gives

$$(6.6) \quad e^A \Lambda e^{-A} = \Lambda + [\Lambda, A] + \int_0^1 (1-s) e^{sA} [[\Lambda, A], A] e^{-sA} ds$$

and (6.2) allows us to rewrite (6.6) in the form

$$(6.7) \quad e^{-A} J e^A = \Lambda - \int_0^1 (1-s) e^{(s-1)A} [[\Lambda, A], A] e^{(1-s)A} ds.$$

However (6.2), (6.4) and (1.4) imply

$$[[\Lambda, A], A] = [J_1, A] = -2g(\Lambda) = O(\Lambda^{1-\rho-\rho'}),$$

which completes the proof due to □

Lemma 4. *For every $m \in \mathbb{R}$ one has*

$$(6.8) \quad \sup_{-1 \leq s \leq 1} \|\Lambda^m e^{sA} \Lambda^{-m}\| < \infty.$$

Proof. (a) To begin, we check that the estimate

$$(6.9) \quad [\Lambda^\varepsilon, A] = O(\Lambda^{\varepsilon-\rho})$$

holds for every $\varepsilon > 0$. Indeed, using Lemma 1 with $q(n) = n^\varepsilon$ we find

$$\begin{aligned} [\Lambda^\varepsilon, A] &= [\Lambda^\varepsilon, Sb(\Lambda)] + hc \\ &= [\Lambda^\varepsilon, S]b(\Lambda) + hc \\ &= ((\Lambda + I)^\varepsilon - \Lambda^\varepsilon)Sb(\Lambda) + hc. \end{aligned}$$

Hence using property (3.3) and

$$\begin{aligned} Sb(\Lambda) &= O(\Lambda^{1-\rho}), \\ (\Lambda + I)^\varepsilon - \Lambda^\varepsilon &= O(\Lambda^{\varepsilon-1}) \end{aligned}$$

we obtain (6.9).

(b) Further on we assume $0 < \varepsilon \leq \rho$ and we show that

$$(6.10) \quad \mathcal{M}_{k\varepsilon} = \sup_{-1 \leq s \leq 1} \|\Lambda^{k\varepsilon} e^{sA} \Lambda^{-k\varepsilon}\| < \infty$$

holds for every $k \in \mathbb{N}$. We introduce

$$R_{k\varepsilon}(s) = \Lambda^{(k+1)\varepsilon} e^{sA} \Lambda^{-(k+1)\varepsilon} - \Lambda^{k\varepsilon} e^{sA} \Lambda^{-k\varepsilon}$$

and observe that

$$R_\varepsilon(s) = [e^{s(1-t)A} \Lambda^\varepsilon e^{stA} \Lambda^{-\varepsilon}]_{t=0}^{t=1} = \int_0^1 e^{s(1-t)A} [A, \Lambda^\varepsilon] e^{stA} \Lambda^{-\varepsilon} dt$$

allows us to estimate (6.9) allows us to estimate

$$\begin{aligned} \|R_{k\varepsilon}(s)\| &= \|\Lambda^{k\varepsilon} R_\varepsilon(s) \Lambda^{-k\varepsilon}\| \\ &\leq \mathcal{M}_{k\varepsilon}^2 \|\Lambda^{k\varepsilon} [\Lambda^\varepsilon, A] \Lambda^{-k\varepsilon}\| < \infty \end{aligned}$$

if $\mathcal{M}_{k\varepsilon} < \infty$. □

7. PROOF OF PROPOSITION 3

(a) To begin, we observe that

$$(7.1) \quad [g(\Lambda), A] = O(\Lambda^{1-\rho-\rho_1})$$

follows from assumption (1.7). Indeed,

$$\begin{aligned} [g(\Lambda), A] &= [g(\Lambda), Sb(\Lambda)] + \text{hc} \\ &= [g(\Lambda), S]b(\Lambda) + \text{hc} \\ &= (g(\Lambda + I) - g(\Lambda))Sb(\Lambda) + \text{hc}, \end{aligned}$$

hence using property (3.2) and

$$\begin{aligned} Sb(\Lambda) &= O(\Lambda^{1-\rho}), \\ g(\Lambda + I) - g(\Lambda) &= O(\Lambda^{-\rho_1}) \end{aligned}$$

we obtain (7.1).

(b) Then the standard expansion formula gives

$$(7.2) \quad e^A \Lambda e^{-A} = \Lambda + [\Lambda, A] + \frac{1}{2}[[\Lambda, A], A] + \int_0^1 (1-s)^2 e^{sA} R e^{-sA} ds$$

with

$$R = \frac{1}{2}[[[\Lambda, A], A], A] = -[g(\Lambda), A].$$

(c) However we have $R = O(\Lambda^{1-\rho-\rho_1})$ due to (7.1) and Lemma 4 allows us to deduce

$$(7.3) \quad e^A \Lambda e^{-A} = J - g(\Lambda) + O(\Lambda^{1-\rho-\rho_1})$$

from (7.2). Applying Lemma 4 once more we obtain

$$(7.4) \quad e^{-A} J e^A = \Lambda + e^{-A} g(\Lambda) e^A + O(\Lambda^{1-\rho-\rho_1}).$$

(d) To complete the proof of Proposition 3 it remains to show

$$(7.5) \quad e^{-A} g(\Lambda) e^A = g(\Lambda) + O(\Lambda^{1-\rho-\rho_1}).$$

However

$$e^{-A} g(\Lambda) e^A - g(\Lambda) = \int_0^1 e^{-sA} [g(\Lambda), A] e^{sA} ds$$

and it is clear that (7.5) follows from (7.1) and Lemma 4.

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