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Abstract. We consider the Weyl formula for the asymptotic number of eigenvalues of self-adjoint elliptic differential operators with coefficients which have Hölder continuous first order derivatives. Our aim is to prove that the Weyl formula holds with a remainder usually considered in the case of operators with smooth coefficients.

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1 Introduction

This paper presents a generalization of our earlier results described in [14], [16], [17] and motivated by the well-known result of L. Hörmander [5] on spectral asymptotics of a self-adjoint elliptic operator A with smooth coefficients on a smooth (boundaryless) manifold M. If M is a compact manifold of dimension d and $2m$ is the order of A, then the associated counting function $\mathcal{N}(A,\lambda)$, defined as the number of eigenvalues (counted with their multiplicities) smaller than λ, satisfies the Weyl formula

$$\mathcal{N}(A,\lambda) = c\lambda^{\frac{d}{2m}} + O(\lambda^{\frac{d-\mu}{2m}}), \tag{1.1}$$

where $c = c_{A,M}$ is a constant and $\mu = 1$.

It is natural to ask if (1.1) still holds for the operators with non-smooth coefficients. In [16] we proved that for coefficients which are Hölder continuous of exponent $r \in]0;\ 1]$, the formula (1.1) holds if $\mu < r$. In [17] we proved (1.1) with $\mu = 1$ assuming that the first order derivatives of coefficients are Lipschitz continuous and in this paper we show a way of replacing the hypothesis of Lipschitz continuity by Hölder continuity. Moreover we use the semiclassical framework, which allows us to obtain the classical Weyl formula (1.1) as a corollary.

Let $\mathcal{B}^r$ denote the set of bounded, Hölder continuous functions on $\mathbb{R}^d$, i.e. $a \in \mathcal{B}^r$ means that $a \in L^\infty(\mathbb{R}^d)$ and there is $C > 0$ such that

$$|a(x) - a(y)| \leq C|x - y|^r \quad (x, y \in \mathbb{R}^d).$$

Further on $m \in \mathbb{N} \setminus \{0\}$ and we fix $r \in]0;\ 1]$. For $\nu, \bar{\nu} \in \mathbb{N}^d$, $|\nu|, |\bar{\nu}| \leq m$, let $a_{\nu,\bar{\nu}}$ be real valued coefficients satisfying $a_{\nu,\bar{\nu}} = a_{\bar{\nu},\nu}$ and

$$\partial_{x_j} a_{\nu,\bar{\nu}} \in \mathcal{B}^r \ \ (j = 1, ..., d). \tag{1.2}$$

We assume moreover the existence of a constant $c > 0$ such that

$$\sum_{|\nu|=|\bar{\nu}|=m} a_{\nu,\bar{\nu}}(x)\xi^{\nu+\bar{\nu}} \geq c|\xi|^{2m} \quad (x, \xi \in \mathbb{R}^d) \tag{1.3}$$

and for $h > 0$ we introduce the quadratic form $\mathcal{A}_h$ defined for $\varphi, \psi \in C_0^m(\mathbb{R}^d)$ by the formula

$$\mathcal{A}_h[\varphi, \psi] = \sum_{|\nu|,|\bar{\nu}|\leq m} (a_{\nu,\bar{\nu}}(hD)^\nu \varphi, (hD)^{\bar{\nu}} \psi), \tag{1.4}$$

where $(\cdot, \cdot)$ is the scalar product of $L^2(\mathbb{R}^d)$ and $(hD)^\nu = (-ih)^{|\nu|}\partial^\nu / \partial x^\nu$.

The ellipticity hypothesis (1.3) ensures the fact that $\mathcal{A}_h$ is bounded from below and its closure defines a self-adjoint operator A_h. We introduce

$$a(x, \xi) = \sum_{|\nu|,|\bar{\nu}|\leq m} a_{\nu,\bar{\nu}}(x)\xi^{\nu+\bar{\nu}} \tag{1.5}$$

and for $E \in \mathbb{R}$ we denote

$$\Gamma_E = a^{-1}(]-\infty;\ E[) = \{(x, \xi) \in \mathbb{R}^{2d} :\ a(x, \xi) < E\}. \tag{1.6}$$

We will present a proof of

Theorem 1.1 *Let $E_0, E \in \mathbb{R}$ be such that $E_0 < E$ and Γ_E is bounded. Then the operators A_h described above have discrete spectrum in $]-\infty;\ E_0]$ for $h \in]0;\ h_0]$, if $h_0 > 0$ is fixed small enough. We assume moreover*

$$a(x,\xi) = E_0 \ \ \Rightarrow \ \ \nabla_\xi a(x,\xi) \neq 0. \tag{1.7}$$

Then, for $h \in]0;\ h_0]$, the counting function $\mathcal{N}(A_h, E_0)$ satisfies the asymptotic formula

$$\mathcal{N}(A_h, E_0) = \ |\Gamma_{E_0}|(2\pi h)^{-d} + O(h^{1-d}), \tag{1.8}$$

where $|\Gamma_{E_0}|$ denotes the Lebesgue measure of Γ_{E_0}.

2 Regularization of coefficients

Let $\gamma_1 \in C_0^\infty(\mathbb{R}^d)$ be such that $\gamma_1(-x) = \gamma_1(x)$, $\int \gamma_1(x)\,\mathrm{d}x = 1$ and for $\varepsilon > 0$ set $\gamma_\varepsilon(x) = \varepsilon^{-d}\gamma_1(x/\varepsilon)$. We fix $\delta \in]0;\ 1[$ and introduce

$$a_{\nu,\bar\nu,h}(x) = (a_{\nu,\bar\nu} * \gamma_{h^\delta})(x) = \int a_{\nu,\bar\nu}(y)\gamma_1(h^{-\delta}(x-y))\,h^{-\delta d}\,\mathrm{d}y. \tag{2.1}$$

It is easy to see (cf. Section 7(A)) that (1.2) ensures

$$|a_{\nu,\bar\nu}(x) - a_{\nu,\bar\nu,h}(x)| \leq Ch^{\delta(1+r)}, \tag{2.2}$$

$$|\partial_x^\alpha a_{\nu,\bar\nu,h}(x)| \leq C_\alpha h^{-\delta(|\alpha|-1-r)_+}, \tag{2.3}$$

where $s_+ = (s+|s|)/2$ is the positive part of $s \in \mathbb{R}$. We define

$$p_h(x,\xi) = \sum_{|\nu|,|\bar\nu|\leq m} a_{\nu,\bar\nu,h}(x)\xi^{\nu+\bar\nu} \tag{2.4}$$

and assume further on $(1+r)\delta \geq 1$, hence (2.2) ensures

$$|\partial_\xi^\alpha(a-p_h)(x,\xi)| \leq C_\alpha h\langle\xi\rangle^{2m-|\alpha|}, \tag{2.5}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Moreover the operator

$$P_h = \sum_{|\nu|,|\bar\nu|\leq m} (hD)^\nu a_{\nu,\bar\nu,h}(x)(hD)^{\bar\nu} \tag{2.6}$$

is an approximation of A_h satisfying

$$\pm(A_h - P_h) \le Ch\langle hD\rangle^{2m} = Ch(I - h^2\Delta)^m$$

and $P_h^- \le A_h \le P_h^+$ holds (in the sens of quadratic forms) with

$$P_h^\pm = P_h \pm Ch\langle hD\rangle^{2m}. \tag{2.7}$$

Our principal aim is to show

Theorem 2.1 *The assertion of Theorem 1.1 holds if A_h is replaced by $P_h^\pm$.*

First of all we remark that the discreteness of the spectrum of $P_h^\pm$ is a standard result and analogical reasoning can be applied to A_h (cf. Section 7(B) for details). Then the asymptotic formula (1.8) follows from the estimates of the counting functions for $P_h^\pm$ due to the min-max principle (cf. [12]), which ensures

$$P_h^- \le A_h \le P_h^+ \Rightarrow \mathcal{N}(P_h^+, E) \le \mathcal{N}(A_h, E) \le \mathcal{N}(P_h^-, E). \tag{2.8}$$

In the remaining part of this section we describe a version of Theorem 2.1 formulated suitably for the proof that will follow.

Let $\mathbb{1}_Z : \mathbb{R} \to \{0, 1\}$ be the characteristic function of $Z \subset \mathbb{R}$ and set

$$\mathcal{N}_h(p_h, E) = \int_{p_h(x,\xi)\le E} \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d} = \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, \mathbb{1}_{]-\infty;\, E]} \circ p_h. \tag{2.9}$$

Further on we also write $\mathbb{1}_Z(P_h^\pm)$ to denote the spectral projector of $P_h^\pm$ on a Borel set $Z \subset \mathbb{R}$ and due to (2.8) and (2.11), to obtain the asymptotic formula (1.8) it suffices to show

$$\operatorname{tr} \mathbb{1}_{]-\infty;\, E_0]}(P_h^\pm) = \mathcal{N}(P_h^\pm, E_0) \,=\, \mathcal{N}_h(p_h, E_0) + O(h^{1-d}). \tag{2.15}$$

To show (2.15) we introduce

$$\begin{cases} g \in C_0^\infty(]E_0 - 2c_0;\ E_0 + 2c_0[),\ \ g = 1 \text{ on } [E_0 - c_0;\ E_0 + c_0], \\ \tilde{g} \in C_0^\infty(]-\infty;\ E_0 - c_0[) \end{cases} \tag{2.16}$$

and assume $\tilde{g} + g^2 = 1$ on $[-E_0';\ E_0 + c_0]$ where $E_0' > 0$ is chosen sufficiently large to ensure $-E_0' \le \inf\, p_h^\pm$ and $-E_0' \le \inf\, \sigma(P_h^\pm)$ for $h \in]0;\ h_0]$.
Then for $E \in [E_0 - c_0;\ E_0 + c_0]$ we can write the decompositions

$$\mathcal{N}_h(p_h, E) = \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, (\tilde{g} + g^2\mathbb{1}_Z)(p_h(x,\xi)),$$

$$\mathbb{1}_{]-\infty;\, E]}(P_h^\pm) = (\tilde{g} + g^2\mathbb{1}_Z)(P_h^\pm),$$

where $Z = [E_0 - 2c_0;\ E]$. Now it is clear that (2.15) is a consequence of

Theorem 2.2 *One has*

$$\operatorname{tr} \tilde{g}(P_h^{\pm}) \;=\; \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\;\; \tilde{g}\circ p_h \;\; + O(h^{1-d}), \tag{2.17}$$

$$\operatorname{tr} (g^2 \mathbf{1}_Z)(P_h^{\pm}) \;=\; \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\;\; (g^2\mathbf{1}_Z)\circ p_h \;\; + O(h^{1-d}). \tag{2.18}$$

Further on $\bar{c} > 0$ is small enough and $f_0 \in C_0^\infty(]\bar{c}/2;\ \bar{c}/2[)$. Then we define $f_1 = f_0 * f_0 \in C_0^\infty(]\bar{c};\ \bar{c}[)$ and

$$\tilde{f}_h(\lambda) = \mathcal{F}_h^{-1} f_1(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda t/h} f_1(t) \frac{\mathrm{d}t}{2\pi h}. \tag{2.19}$$

We note that $\tilde{f}_h(\lambda) = \tilde{f}_1(\lambda/h)/h$, $\tilde{f}_1$ is rapidly decreasing and $\tilde{f}_1 > 0$. Moreover we can choose f_0 such that $\int_{-\infty}^{\infty} \tilde{f}_h = f_1(0) = 1$. Then we use

$$\tilde{f}_h^Z(\lambda) = (\mathbf{1}_Z * \tilde{f}_h)(\lambda) = \int_Z \mathrm{d}\lambda'\, \tilde{f}_h(\lambda - \lambda') \tag{2.20}$$

as an approximation of $\mathbf{1}_Z$. In Section 7(C) we recall a proof of

Proposition 2.3 *The estimate* (2.18) *follows if*

$$\operatorname{tr} (g^2 \tilde{f}_h^Z)(P_h^{\pm}) \;=\; \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, (g^2 \tilde{f}_h^Z)\circ p_h \;+\; O(h^{1-d}) \tag{2.21}$$

holds uniformly with respect to $Z = [E';\ E]$.

3 Pseudodifferential approximation

In this section we show the estimate (2.17). We use pseudodifferential operators in a reasoning which will be developed in Section 4 to treat (2.18). We denote by $||\cdot||$ the norm of bounded operators $B(L^2(\mathbb{R}^d))$ and $||B||_{\mathrm{tr}} = \operatorname{tr}(B^*B)^{1/2}$ is the trace class norm. If $b_h \in C_0^\infty(\mathbb{R}^{2d})$ then we denote by $b_h(x, hD)$ the pseudodifferential operator acting on $\varphi \in \mathcal{S}(\mathbb{R}^d)$ according to the formula

$$(b_h(x,hD)\varphi)(x) = \int \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi h)^d}\, \mathrm{e}^{i(x-y)\xi/h} b_h(x,\xi)\varphi(y). \tag{3.1}$$

If $b_h, l_h \in C_0^\infty(\mathbb{R}^{2d})$, then $b_h(x,hD)l_h(x,hD)^*$ has the integral kernel

$$\mathcal{K}_h(x,y) = \int \frac{\mathrm{d}\xi}{(2\pi h)^d}\, \mathrm{e}^{i(x-y)\xi/h} b_h(x,\xi)\overline{l_h(y,\xi)},$$

hence

$$\operatorname{tr} b_h(x,hD)l_h(x,hD)^* = \int \mathrm{d}x\, \mathcal{K}_h(x,x) = \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, b_h\overline{l}_h. \qquad (3.2)$$

In this section we consider $\tilde{g} \in C_0^\infty(]-\infty;\ E_1[)$ with E_1 such that $\Gamma_{E_1+2c_1}$ is bounded for a certain $c_1 > 0$. Then using (2.5) we can find $h_0 > 0$ small enough to ensure

$$0 < h < h_0 \ \Rightarrow\ \operatorname{supp}(\tilde{g}\circ p_h) \ \subset \Gamma_{E_1+c_1}. \qquad (3.3)$$

Next we fix an auxiliary h-independent cut-off function $l \in C_0^\infty(\mathbb{R}^{2d})$ such that $l = 1$ on a neigbourhood of $\overline{\Gamma_{E_1+c_1}}$ and $l \geq 0$. We denote $L_h = l(x,hD)$ and recall standard estimates of pseudodifferential operators:

$$||L_h|| = O(1), \quad ||L_h||_{\mathrm{tr}} = O(h^{-d}). \qquad (3.4)$$

Moreover (3.2) allows us to write

$$\int \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi h)^d}\, \tilde{g}\circ p_h \ = \int \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi h)^d}\, l(\tilde{g}\circ p_h) \ = \operatorname{tr}(\tilde{g}\circ p_h)(x,hD)L_h^* \qquad (3.5)$$

and since $|\operatorname{tr} B| \leq ||B||_{\mathrm{tr}}$, it is clear that (2.17) follows from

$$||\tilde{g}(P_h^\pm) - L_h^*(\tilde{g}\circ p_h)(x,hD)||_{\mathrm{tr}} = O(h^{1-d}). \qquad (3.6)$$

Our proof of (3.6) is based on an auxiliary lemma (cf. Section 7(D)):

Lemma 3.1 *If $\tilde{g}$ and L_h are as above, then for every $N \in \mathbb{N}$ one has*

$$||\tilde{g}(P_h^\pm)(I - L_h)||_{\mathrm{tr}} = O(h^N). \qquad (3.7)$$

Since $||\tilde{g}(P_h^\pm)L_h||_{\mathrm{tr}} \leq ||\tilde{g}(P_h^\pm)||\, ||L_h||_{\mathrm{tr}} = O(h^{-d})$ due to (3.4), Lemma 3.1 ensures $||\tilde{g}(P_h^\pm)||_{\mathrm{tr}} = O(h^{-d})$. Another immediate corollary of (3.7) is

$$||\tilde{g}(P_h^\pm) - L_h^*\,\tilde{g}(P_h^\pm)L_h||_{\mathrm{tr}} = O(h^N)$$

and we can conclude that instead of (3.6) it suffices to show

$$||L_h^*\tilde{g}(P_h^\pm)L_h - L_h^*(\tilde{g}\circ p_h)(x,hD)||_{\mathrm{tr}} = O(h^{1-d}). \qquad (3.6')$$

Further on $g_1 \in \mathcal{S}(\mathbb{R})$ denotes the Fourier transform of $\tilde{g}$, i.e. we have $\tilde{g}(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, g_1(t)\, \mathrm{e}^{it\lambda}$ and

$$(\tilde{g}\circ p_h)(x,\xi) = \tilde{g}(p_h(x,\xi))l(x,\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, g_1(t)\, (\mathrm{e}^{itp_h}l)(x,\xi), \qquad (3.8)$$

$$\tilde{g}(P_h^{\pm}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, g_1(t)\, \mathrm{e}^{itP_h^{\pm}}. \tag{3.9}$$

Therefore we can write

$$\tilde{g}(P_h^{\pm})L_h - (\tilde{g}\circ p_h)(x,hD) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, g_1(t)\, R_{h,t}$$

with

$$R_{h,t} = \mathrm{e}^{itP_h^{\pm}} L_h - (\mathrm{e}^{itp_h} l)(x,hD) \tag{3.10}$$

and the left hand side of (3.6′) can be estimated by

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi}\, |g_1(t)|\, ||L_h^*\, R_{h,t}||_{\mathrm{tr}}. \tag{3.11}$$

However $||L_h^*\, R_{h,t}||_{\mathrm{tr}} \leq ||L_h^*||_{\mathrm{tr}}\, ||R_{h,t}|| \leq Ch^{-d}||R_{h,t}||$ and g_1 is rapidly decaying, hence in order to estimate the quantity (3.11) by $O(h^{1-d})$ it suffices to show

Lemma 3.2 *There is $C>0$ such that $||R_{h,t}|| \leq Ch\langle t\rangle^C$.*

Before starting the proof of Lemma 3.2 it is useful to introduce some notations. We write the differential operators $P_h^{\pm}$ in the standard form

$$P_h^{\pm} = \sum_{|\alpha|\leq 2m} a_{\nu,\bar{\nu},h}^{\pm}(x)(hD)^{\alpha} = p_h^{\pm}(x,hD), \tag{3.12}$$

$$p_h^{\pm}(x,\xi) = \sum_{|\alpha|\leq 2m} a_{\nu,\bar{\nu},h}^{\pm}(x)\xi^{\alpha} \tag{3.13}$$

and it is easy to check that (2.5) still holds with $p_h^{\pm}$ instead of p_h.
Let $\tilde{m}\in\mathbb{R}$ and $\delta_1 \geq \delta_0 \geq 0$ be such that $\delta_0+\delta_1<1$. If $b=\{b_h\}_{0<h\leq h_0}$ is a family of smooth functions satisfying the estimates

$$|\partial_{\xi}^{\alpha}\partial_x^{\beta} b_h(x,\xi)| \leq C_{n,\alpha,\beta} h^{-\tilde{m}-|\alpha|\delta_0-|\beta|\delta_1}\langle\xi\rangle^{-n} \tag{3.14}$$

(for every $\alpha,\beta\in\mathbb{N}^d$ and $n\in\mathbb{N}$), then we write $b\in S_{\delta_0,\delta_1}^{\tilde{m}}$.
If $l\in C_0^{\infty}(\mathbb{R}^{2d})$ is h-independent, then the families lp_h, $l\tilde{p}_h^{\pm}$, $l\mathrm{e}^{itp_h}$ belong to $S_{0,\delta}^0$ and well-known L^2-estimates of pseudodifferential operators (cf. [2],[6]) give

$$b\in S_{0,\delta}^{\tilde{m}} \;\Rightarrow\; \exists C>0,\;\; ||(\mathrm{e}^{itp_h} l b_h)(x,hD)|| \leq Ch^{-\tilde{m}}\langle t\rangle^C. \tag{3.15}$$

If $s \in \mathbb{R}$ then we write $B_h = L_h + O(h^s)$ if and only if there is a constant $C > 0$ such that $||B_h - L_h|| \leq Ch^s$ holds for all $h \in]0;\ h_0]$. Moreover further on we adopt the convention to drop the index h whenever there is no confusion. Instead of $P_h^{\pm}$, L_h, $R_{h,t}$, p_h, $p_h^{\pm}$, we are going to write simply P, L, R_t, p, $p^{\pm}$.

Proof of Lemma 3.2 We are going to obtain suitable estimates of

$$\tilde{R}_t = (i\partial_t + P)(\mathrm{e}^{itp}l)(x, hD) \tag{3.16}$$

and the assertion of Lemma 3.2 will follow from

$$R_t = -[\mathrm{e}^{i(t-\tau)P}(\mathrm{e}^{i\tau p}l)(x, hD)]_{\tau=0}^{\tau=t} = i\int_0^t \mathrm{d}\tau\, \mathrm{e}^{i(t-\tau)P}\tilde{R}_\tau. \tag{3.17}$$

The standard formula of the composition with a differential operator gives

$$p^{\pm}(x, hD)(\mathrm{e}^{itp}l)(x, hD) = \tilde{q}_t(x, hD), \tag{3.18}$$

with

$$\tilde{q}_t = \sum_{|\alpha|\leq 2m} \frac{(-ih)^{|\alpha|}}{\alpha!}\partial_x^\alpha(\mathrm{e}^{itp}l)\partial_\xi^\alpha p^{\pm} = \mathrm{e}^{itp}\sum_{k=0}^{2m} t^k q_k, \tag{3.19}$$

$$q_k = \sum_{k\leq|\alpha|\leq 2m}\ \sum_{\substack{\alpha_0+\ldots+\alpha_k=\alpha \\ \alpha_j\neq 0 \text{ if } j\neq 0}} c_{\alpha_0,\ldots,\alpha_k}\, \partial_x^{\alpha_0} l\, \partial_x^{\alpha_1} p \ldots \partial_x^{\alpha_k} p\, \partial_\xi^\alpha p^{\pm} \tag{3.20}$$

and $q_k \in S_{0,\delta}^{-k}$ follows from $-|\alpha| + \sum_{j=1}^k \delta(|\alpha_j| - 1) \leq -k - (1-\delta)(|\alpha| - k)$. However (3.15) ensures

$$q_k \in S_{0,\delta}^{-k} \ \Rightarrow\ (\mathrm{e}^{itp}q_k)(x, hD) = O(h^k\langle t\rangle^{C_k}),$$

hence

$$P(\mathrm{e}^{itp}l)(x, hD) = (\mathrm{e}^{itp}q_0)(x, hD) + O(h\langle t\rangle^C).$$

To complete the proof we note that $l(p^{\pm} - p) \in S_{0,\delta}^{-1}$ and

$$q_0 - lp^{\pm} = \sum_{1\leq|\alpha|\leq 2m} \frac{(-ih)^{|\alpha|}}{\alpha!}\partial_x^\alpha l\, \partial_\xi^\alpha p^{\pm} \ \in S_{0,\delta}^{-1} \tag{3.21}$$

ensure $q_0 - lp \in S_{0,\delta}^{-1}$, hence

$$i\partial_t(\mathrm{e}^{itp}l)(x, hD) + (\mathrm{e}^{itp}q_0)(x, hD) = (\mathrm{e}^{itp}(q_0 - lp))(x, hD) = O(h\langle t\rangle^C).$$

4 Refinement

Due to Proposition 2.3, to complete the proof of Theorem 2.2 it suffices to show (2.21). We will use a reasoning of Section 3 in a refined form. We keep notations introduced before and in particular we drop the index h.

To begin we remark that replacing $\tilde{g}$ by $g \in C_0^\infty(]E_0 - c_0;\ E_0 + c_0[)$ in (3.6), we can introduce $l_0 = g \circ p$, $L_0 = l_0(x, hD)$ and write

$$\operatorname{tr}(g^2 \tilde{f}_h^Z)(P) \ = \ \operatorname{tr} L_0^* \tilde{f}_h^Z(P) L_0 \ + O(h^{1-d}). \tag{4.1}$$

As before $Z = [E';\ ]$ and we use the Fourier transform

$$f_h^Z(t) := \ \mathcal{F}_h \tilde{f}_h^Z(t) = \textstyle\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-it\lambda/h} \tilde{f}_h^Z(\lambda)$$
$$= \ f_1(t)\ \textstyle\int_{E'}^{E} \mathrm{d}\lambda\, \mathrm{e}^{-it\lambda/h} \ = \ f_1(t)\, \frac{\mathrm{e}^{-it(E+E')/(2h)}}{t/(2h)} \sin \frac{(E'-E)t}{2h}$$

to write

$$\operatorname{tr} L_0^* \tilde{f}_h^Z(P) L_0 \ = \ \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t) \operatorname{tr} L_0^* \mathrm{e}^{itP/h} L_0, \tag{4.2}$$

$$\int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, l_0^2 (\tilde{f}_h^Z \circ p) \ = \int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t)\, l_0^2 \mathrm{e}^{itp/h}. \tag{4.3}$$

However (3.2) allows us to express (4.3) as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t) \operatorname{tr} L_0^* (l_0 \mathrm{e}^{itp/h})(x, hD) \tag{4.3$'$}$$

and introducing

$$R_{t/h}^0 = \mathrm{e}^{itP/h} L_0 - (\mathrm{e}^{itp/h} l_0)(x, hD) \tag{4.4}$$

we find that (2.21) is equivalent to the estimate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t) \operatorname{tr} L_0^* R_{t/h}^0 \ = O(h^{1-d}). \tag{4.5}$$

Next we observe that (2.5) and (2.10) ensure

$$E_0 - 2c_0 \le p(x,\xi) \le E_0 + 2c_0 \Rightarrow |\nabla_\xi p(x,\xi)| \ge c_0 \tag{4.6}$$

for $h \in]0;\ h_0]$ if $h_0 > 0$ is fixed small enough. The condition (4.6) ensures

Lemma 4.1 *Assume that $b \in S^0_{0,\delta}$. Then for every $n \in \mathbb{N}$ one can find $b_n \in S^0_{0,\delta}$ such that*

$$t^n \int \mathrm{d}\xi \;\; b\,\mathrm{e}^{itp/h} \;=\; h^n \int \mathrm{d}\xi \;\; b_n\,\mathrm{e}^{itp/h}. \tag{4.7(n)}$$

Proof. Using (4.6) in a standard reasoning we obtain

$$\forall\, b \in S^0_{0,\delta} \;\; \exists \tilde{b}_j \in S^0_{0,\delta}, \quad b = \sum\nolimits_{j=1}^{d} \tilde{b}_j \partial_{\xi_j} p. \tag{4.8}$$

Therefore performing the integration by parts

$$t \int \mathrm{d}\xi \;\; \sum\nolimits_{j=1}^{d} b_j \partial_{\xi_j} p\, \mathrm{e}^{itp/h} \;=\; ih \int \mathrm{d}\xi \sum\nolimits_{j=1}^{d} \partial_{\xi_j} \tilde{b}_j\, \mathrm{e}^{itp/h}$$

we obtain the assertion of Lemma 4.1 for $n = 1$ with $b_1 = \sum_{j=1}^{d} i\partial_{\xi_j} \tilde{b}_j$. Obviously the general case follows by induction with respect to n. $\triangle$.

We will use Lemma 4.1 to replace $R^0_{t/h}$ in (4.5) by

$$R_{t/h} = \mathrm{e}^{itP/h} L_0 - (\mathrm{e}^{itp/h} l_t)(x, hD), \tag{4.4$'$}$$

where $l_t = l_0 + t l'_0 + \frac{1}{2} t^2 l''_0$ with certain $l'_0 \in S^0_{0,\delta}$ and $l''_0 \in S^1_{0,\delta}$.
In fact $\operatorname{tr} L_0^*(R_{t/h} - R^0_{t/h})$ can be expressed as

$$\int \frac{\mathrm{d}x\mathrm{d}\xi}{(2\pi h)^d}\, l_0(l_t - l_0)\mathrm{e}^{itp/h} = h^{1-d} \int \mathrm{d}x\mathrm{d}\xi\, \tilde{l}\, \mathrm{e}^{itp/h},$$

with a ceratin $\tilde{l} \in S^0_{0,\delta}$ (due to Lemma 4.1 with $n = 1$ and 2). Thus

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t) \operatorname{tr} L_0^*(R_{t/h} - R^0_{t/h}) \;= h^{1-d} \int \mathrm{d}x\mathrm{d}\xi\, \tilde{l}(\tilde{f}_h^Z \circ p)$$

is clearly $O(h^{1-d})$, hence (4.5) is equivalent to

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi h}\, f_h^Z(t) \operatorname{tr} L_0^* R_{t/h} \;= O(h^{1-d}). \tag{4.5$'$}$$

obtain (4.5$'$) it suffices to show that

$$\sup_{|t| \le \bar{c}} |\operatorname{tr} L_0^* R_{t/h}| \;\le C h^{1-d+\varepsilon} \tag{4.9}$$

holds if $\bar{c}, \varepsilon > 0$ are small enough. The main result of this section is

5 Construction of the approximation

$$[\mathrm{Op}_t(b),\, x_j/h] = t/overh\mu\mathrm{Op}_t(\partial_{\xi_j} p\, b) + \mathrm{Op}_t(i\partial_{\xi_j} b),$$
$$[\mathrm{Op}_t(b),\, D_j] = t/overh\mu\mathrm{Op}_t(\partial_{x_j} p\, b) + \mathrm{Op}_t(i(\partial_{x_j} + \partial_{y_j})b),$$

In this section $\bar{N} \in \mathbf{N}$ is fixed and we describe the construction of

$$Q_{\bar{N},n}(t) = \Big(\mathrm{e}^{-itp} q_{\bar{N},n}(t)\Big)(x, D) \tag{4.1}$$

such that (3.15) holds. The proof of (3.14) will be given in Sections 5-7.
For $N = 0, ..., \bar{N}$ we are going to consider

$$q_{N,n}(t)(x,\xi) = \sum_{0\le k\le N} t^k q^0_{k,n}(x,\xi) \tag{4.2(N)}$$

where $q_{N,n}(0) = q^0_{0,n} = l_n$ and for $k \ge 1$ we have

$$q^0_{k,n} \in S(hp(h^{\sigma/2}\tilde{p})^{k-1}, g), \quad \mathrm{supp}\, q^0_{k,n} \subset \mathcal{B}_n(\bar{c}/2). \tag{4.3(k)}$$

We construct $q_{N,n}(t)$ by induction with respect to N and its final step corresponding to $N = \bar{N}$ gives $q_{\bar{N},n}(t)$ to be used in (4.1).
For a smooth function $(t,x,\xi) \to b(t,x,\xi) \in \mathbf{C}$ we denote

$$\mathcal{P}_{\bar{N}} b(t) = \sum_{|\alpha|\le\bar{N}} (-i)^{|\alpha|}\partial^a_\xi \left(b(t)\overline{\partial^\alpha_x p_0}\right)/\alpha!,$$

where p_0 is given by (3.24).

Proposition 5.1 *Let $N = 0, ..., \bar{N}$. Then we can find $q_{N,n}(t)$ satisfying (4.2(N)) with $q_{N,n}(0) = l_n$, $q^0_{k,n}$ satisfying (4.3(k)) for $k = 1, ..., N$ and*

$$(\partial_t + i\mathcal{P}_{\bar{N}})\Big(\mathrm{e}^{-itp} q_{N,n}(t)\Big) = \mathrm{e}^{-itp}\tilde{q}^0_{N,n}(t) \tag{4.4(N)}$$

holds with

$$\tilde{q}^0_{N,n}(t)(x,\xi) = \sum_{N\le k\le N+\bar{N}} t^k \tilde{q}^0_{N,k,n}(x,\xi), \tag{4.5(N)}$$

$$\tilde{q}^0_{N,k,n} \in S(hp(h^{\sigma/2}\tilde{p})^k, g) \ \ for \ \ k = N, ..., N+\bar{N} \tag{4.6(N)}$$

and $\mathrm{supp}\, \tilde{q}^0_{N,k,n} \subset \mathcal{B}_n(\bar{c}/2)$.

Proof. We introduce the notation

$$\tilde{\mathcal{P}}_{\bar{N}}q(t)=\mathrm{e}^{itp}(\partial_t+i\mathcal{P}_{\bar{N}})\Big(\mathrm{e}^{-itp}q(t)\Big).$$

If $q(t,x,\xi)=q_0(x,\xi)$ is independent of t, then

$$\tilde{\mathcal{P}}_{\bar{N}}q(t)=\sum_{0\leq k\leq\bar{N}}t^k\tilde{q}_k^0 \tag{4.7}$$

with

$$\tilde{q}_0^0=i(\bar{p}_0-p)q_0-\sum_{1\leq|\alpha|\leq\bar{N}}(-i)^{|\alpha|+1}\partial_\xi^\alpha(q_0\,\overline{\partial_x^\alpha p_0})/\alpha!,$$

$$\tilde{q}_k^0=\sum_{\substack{|\alpha_0+...+\alpha_k|\leq\bar{N}\\ \alpha_j\neq 0\ \text{if}\ \ j\neq 0}}c_{\alpha_0,...,\alpha_k}\partial_\xi^{\alpha_0}(q_0\,\overline{\partial_x^{\alpha_0+...+\alpha_k}p_0})\;\partial_\xi^{\alpha_1}p...\partial_\xi^{\alpha_k}p\ \ (k=1,...,\bar{N}).$$

Using the fact that the estimates (2.8) still hold with p_0 instead of p we find

$$q_0\in S(m,g)\implies\partial_\xi^{\alpha_0}(q_0\,\partial_x^\alpha\bar{p}_0)\in S(mph^{1+\sigma|\alpha|}\langle\xi\rangle^{|\alpha|-|\alpha_0|},g) \tag{4.8}$$

if $\alpha\neq 0$. Moreover the estimates $(2.10')$ imply

$$\partial_\xi^{\alpha_1}p...\partial_\xi^{\alpha_k}p\in S(\tilde{p}^k\langle\xi\rangle^{-|\alpha_1|-...-|\alpha_k|},g) \tag{4.9}$$

if $\alpha_1,...,\alpha_k\neq 0$. Combining (4.8), (4.9) with $\alpha=\alpha_0+...+\alpha_k\Longrightarrow|\alpha|\geq\max\{k,\,1\}$ in the expression of $\tilde{q}_k^0$ we find

$$q_0\in S(m,g)\implies\tilde{q}_k^0\in S(mph^{1+\sigma\max\{k,\,1\}}\tilde{p}^k,g) \tag{4.10}$$

(where in the case $k=0$ we use moreover (3.24)).
Therefore in the case $N=0$ when we take $q(t)=l_n\in S(1,g)$ in (4.7), we obtain (4.4(0)) with $\tilde{q}_{0,n}^0$ of the form (4.5(0)) and (4.10) with $m=1$ imply (4.6(0)), i.e. Proposition 4.1 holds for $N=0$.

Further on we assume that the statement of Proposition 4.1 holds for a given $N\leq\bar{N}-1$ and we prove that it still holds for $N+1$ instead of N.

Using the induction hypothesis to express $\tilde{\mathcal{P}}_{\bar{N}}q_{N,n}(t)$ we find

$$\tilde{\mathcal{P}}_{\bar{N}}q_{N+1,n}(t)=\tilde{\mathcal{P}}_{\bar{N}}(t^{N+1}q_{N+1,n}^0)+\tilde{\mathcal{P}}_{\bar{N}}q_{N,n}(t)=$$

$$t^N\Big((N+1)q_{N+1,n}^0+\tilde{q}_{N,N,n}^0\Big)+t^{N+1}\tilde{\mathcal{P}}_{\bar{N}}q_{N,n+1}^0+\sum_{N+1\leq k\leq N+\bar{N}}t^k\,\tilde{q}_{N,k,n}^0.$$

In order to obtain $(4.5(N+1))$ it suffices to cancel the term with t^N taking

$$q^0_{N+1,n} = -\tilde{q}^0_{N,N,n}/(N+1) \tag{4.11}$$

and we have $\tilde{q}^0_{N,N,n} \in S(hp(h^{\sigma/2}\tilde{p})^N, g)$ by the induction hypothesis.
Let us introduce the following notation :

$$a_n(t) \in \sum_{k\in\mathcal{K}} t^k S(m(k), g) \iff a_n(t) = \sum_{k\in\mathcal{K}} t^k a^0_{k,n} \text{ with } a^0_{k,n} \in S(m(k), g).$$

Then using $ph \le \tilde{p}$ and $\max\{k,\, 1\} \ge (k+1)/2$ we can write (4.7), (4.10) in the following form

$$q(t) = q_0 \in S(m, g) \Longrightarrow \tilde{\mathcal{P}}_{\bar{N}} q(t) \in \sum_{0\le k\le \bar{N}} t^k\, S(m(h^{\sigma/2}\tilde{p})^{k+1}, g). \tag{4.12}$$

Since (4.11) gives $(4.3(N+1))$, applying (4.12) with $m = ph(h^{\sigma/2}\tilde{p})^N$ we find

$$t^{N+1}\tilde{\mathcal{P}}_{\bar{N}} q^0_{N,n+1} \in \sum_{0\le k\le \bar{N}} t^{N+1+k}\, S(ph(h^{\sigma/2}\tilde{p})^{N+k+1}, g)$$

and $(4.6(N+1))$ follows. $\triangle$

Proposition 5.2 *Let $Q_{\bar{N},n}(t)$ be defined by (4.1) with $q_{\bar{N},n}$ given by $(4.2(\bar{N}))$ where $q^0_{n,0} = l_n$ and $q^0_{k,n}$ satisfy $(4.3(k))$ for $k = 1, ..., \bar{N}$. Then (3.15) holds.*

Proof. For $k \in \mathbf{N}$, $n \in \mathbf{N}^*$, $q \in C^\infty_0(\mathbf{R}^d \times \mathbf{R}^d)$ and $\lambda \in \mathbf{R}$ we denote

$$\mathcal{N}_{k,n}(q, \lambda) = \int_{-\infty}^{\infty} dt\, \check{f}^{\pm}_{n,\lambda}(t)\, t^k\, J_t(q), \tag{4.13}$$

where

$$J_t(q) = (2\pi)^{-d} \int \mathrm{e}^{-itp(x,\xi)} q(x, \xi)\, dx d\xi. \tag{4.14}$$

Since $t^k\, \check{f}^{\pm}_{n,\lambda}(t)$ is the Fourier inverse of $i^k (f^{\pm}_{n,\lambda})^{(k)}$, changing the order of integrals (4.13) and (4.14) we find

$$\mathcal{N}_{k,n}(q, \lambda) = (2\pi)^{-d} \int i^k (f^{\pm}_{n,\lambda})^{(k)}\Big(p(x, \xi)\Big) q(x, \xi)\, dx d\xi. \tag{4.15}$$

Since $\Big(\mathrm{e}^{-itp} q\Big)(x, D) L^*_n$ has the integral kernel

$$(x, y) \to (2\pi)^{-d} \int \mathrm{e}^{-itp(x,\xi)} q(x, \xi) l_n(y, \xi)\, d\xi,$$

we have $\operatorname{tr} L_n^*\Big(\mathrm{e}^{-itp}q\Big)(x,D) = J_t(ql_n)$ and

$$\int_{-\infty}^{\infty} dt\, \check{f}_{n,\lambda}^{\pm}(t) \operatorname{tr} Q_{\bar{N},n}(t) L_n^* = \sum_{0\le k\le \bar{N}} \mathcal{N}_{k,n}(q_{k,n}^0 l_n, \lambda). \tag{4.16}$$

For $k=0$ we have $q_{0,n}^0 = l_n$, hence (4.15) and (3.8) give

$$\begin{aligned}
&\Big|\mathcal{N}_{0,n}(q_{0,n}^0 l_n,\lambda) - (2\pi)^{-d}\int_{p<\lambda} l_n(x,\xi)^2\, dxd\xi\Big| \\
&= \Big|(2\pi)^{-d}\int \Big(f_{n,\lambda}^{\pm} - \mathbf{1}_{]-\infty,\lambda[}\Big)\Big(p(x,\xi)\Big) l_n(x,\xi)^2\, dxd\xi\Big| \\
&\le \int_{\lambda - h_n p_n < p < \lambda + h_n p_n} \mathbf{1}_{\mathcal{B}_n(\bar{c})}(x,\xi)\, dxd\xi.
\end{aligned}$$

It remains to show that the estimate

$$|\mathcal{N}_{k,n}(q_n,\lambda)| \le \bar{C}\int_{\lambda - h_n p_n < p < \lambda + h_n p_n} \mathbf{1}_{\mathcal{B}_n(\bar{c})}(x,\xi)\, dxd\xi \tag{4.17(k)}$$

holds with $q_n = q_{k,n}^0 l_n,\ k = 1, ..., \bar{N}$.
The expression (4.15) allows to write the obvious inequality

$$|\mathcal{N}_{k,n}(q_n,\lambda)| \le \int \Big|(f_{n,\lambda}^{\pm})^{(k)}\Big(p(x,\xi)\Big)\Big| |q_n(x,\xi)|\, dxd\xi. \tag{4.18(k)}$$

In the case $k=1$ the above inequality leads to the following :

Corollary 5.3 *If $q_n \in S(ph,g)$ are such that $\operatorname{supp} q_n \subset \mathcal{B}_n(\bar{c})$, then $|q_n| \le Ch_n p_n$ and using (3.9(1)) in (4.18(1)) we obtain (4.17(1)). In particular (4.17(1)) holds with $q_n = q_{1,n}^0 l_n$.*

In order to prove (4.17(k)) for $q_n = q_{k,n}^0 l_n$, $k \ge 2$, let $\chi \in C_0^\infty(]-2;\ 2[)$ be such that $\chi = 1$ on $[-1;\ 1]$ and for $s > 0$ let

$$\chi_s(x,\xi) = \chi(m(x,\xi)/s^2) \quad \text{ with } m = \langle\xi\rangle^2 |\nabla_\xi p|^2 h^{-2} p^{-2}. \tag{4.19}$$

We consider also $\tilde{\chi}_s := 1 - \chi_s$ and remark that using $1 + m \in S(1+m,g)$ it is easy to check that $\chi_s,\ \tilde{\chi}_s \in S(1,g)$. Moreover

$$(x,\xi) \in \operatorname{supp}\chi_s \implies \tilde{p}(x,\xi) = ((1+m^{1/2})hp)(x,\xi) \le (1+2s)(hp)(x,\xi). \tag{4.20}$$

Therefore $|q_{k,n}^0 l_n \chi_s| \le C_k h_n^k p_n^k$ and using (3.9(k)) in (4.18(k)) we find that (4.17(k)) holds with $q_n = q_{k,n}^0 l_n \chi_s$, $k \ge 2$.
Thus it remains to show that (4.17(k)) holds with $q_n = q_{k,n}^0 l_n \tilde{\chi}_s$, $k \ge 2$.
To obtain this result it suffices to show

Lemma 5.4 *If $q_n \in S(m,g)$ then we can find $\tilde{q}_n \in S(m/\tilde{p},g)$ such that* $\operatorname{supp}\tilde{q}_n \subset \operatorname{supp} q_n$ *and*

$$t\, J_t(q_n\tilde{\chi}_s) = J_t(\tilde{q}_n\tilde{\chi}_{s/2}). \tag{4.21}$$

Indeed, iterating the assertion of Lemma 4.4 we can write

$$t^k J_t(q^0_{k,n} l_n \tilde{\chi}_s) = t^{k-1} J_t(q_{1,k,n}\tilde{\chi}_{s/2}) = ... = t^{k-\bar{k}} J_t(q_{\bar{k},k,n}\tilde{\chi}_{s/2^{\bar{k}}}). \tag{4.22}$$

for some $q_{\bar{k},k,n} \in S(ph\tilde{p}^{k-1-\bar{k}},g)$ with $\operatorname{supp} q_{\bar{k},k,n} \subset \mathcal{B}_n(\bar{c}/2)$.
Thus using (4.22) with $\bar{k} = k-1$ we obtain

$$\mathcal{N}_{k,n}(q^0_{k,n} l_n \tilde{\chi}_s, \lambda) = \mathcal{N}_{1,n}(q_n,\lambda)$$

with $q_n \in S(hp,g)$, $\operatorname{supp} q_n \subset \mathcal{B}_n(\bar{c}/2)$, and (4.17(1)) holds due to Corollary 4.3.

Proof of Lemma 4.4. Using $\langle\xi\rangle^2|\nabla_\xi p|^2 \in S(\tilde{p}^2,g)$ and

$$s^2\tilde{p}(x,\xi)^2 \le 2s^2(hp)(x,\xi)^2 + 2\langle\xi\rangle^2|\nabla_\xi p(x,\xi)|^2 \le 2(s^2+1)\langle\xi\rangle^2|\nabla_\xi p(x,\xi)|^2$$

for $(x,\xi) \in \operatorname{supp}\tilde{\chi}_s$, we obtain $\tilde{\chi}_s\langle\xi\rangle^{-2}|\nabla_\xi p|^{-2} \in S(\tilde{p}^{-2},g)$ due to (3.20) and combining with $\langle\xi\rangle^2\partial_{\xi_j}p \in S(\langle\xi\rangle\tilde{p},g)$ we can define

$$\tilde{\chi}_{j,s} := \tilde{\chi}_s\, \partial_{\xi_j} p|\nabla_\xi p|^{-2} \in S(\langle\xi\rangle/\tilde{p},g). \tag{4.23}$$

Writing $q_n\tilde{\chi}_s = \sum_{j=1}^d q_n\tilde{\chi}_{j,s}\partial_{\xi_j}p$ and integrating by parts we find

$$t\, J_t(q_n\tilde{\chi}_{j,s}\partial_{\xi_j}p) = (2\pi)^{-d}\int i\partial_{\xi_j}\left(\mathrm{e}^{-itp(x,\xi)}\right)(q_n\tilde{\chi}_{j,s})(x,\xi)\,dxd\xi$$

$$= -iJ_t(\partial_{\xi_j}(q_n\tilde{\chi}_{j,s})),$$

which completes the proof due to $\partial_{\xi_j}(q_n\tilde{\chi}_{j,s}) \in S(m/\tilde{p},g)$ and $\operatorname{supp}\tilde{\chi}_{j,s} \cap \operatorname{supp}\chi_{s/2} = \emptyset$. $\triangle$

6 Preliminary remarks about the approximation error

In this section we begin a study of $\operatorname{tr}\tilde{Q}_{\bar{N},n}(t-\tau)\mathrm{e}^{-i\tau P}L_n^*$ with the purpose to establish (3.18), which implies (3.14).
To abbreviate notations we denote the elements of $\mathcal{V}$ by the letter ν, adopting the following convention : if the index ν appears in a formula simultaneously

with a letter n, t or τ, then $\nu = (n, t, \tau)$. Moreover the notation $s_\nu = \tilde{s}_\nu + O(m_{n,t})$ means that $|s_{n,t,\tau} - \tilde{s}_{n,t,\tau}| \leq C m_{n,t}$ holds with a constant $C > 0$ independent of $\nu = (n, t, \tau) \in \mathcal{V}$.

Assume that $Q_\nu = \left(\mathrm{e}^{-itp} q_\nu\right)(x, D)$ with $q_\nu \in S(m, g)$, $\operatorname{supp} q_\nu \subset \mathcal{B}_n(\bar{c})$ and $Y_\nu \in B(L^2(\mathbf{R}^d))$ for $\nu \in \mathcal{V}$. Then Lemma 9.1 allows to estimate

$$|t^k \operatorname{tr} Q_\nu \mathrm{e}^{-i\tau P} Y_\nu| \leq |t|^k \, ||Q_\nu||_{\mathrm{tr}} \, ||Y_\nu|| \leq$$

$$\leq C m_n h_n^{-C_0} \langle h_n p_n t \rangle^{k + C_0} ||Y_\nu||, \tag{5.1}$$

where m_n are as in (3.6).
The family $\{Y_\nu\}_{\nu \in \mathcal{V}} \subset B(L^2(\mathbf{R}^d))$ will be called negligible if for every $N \in \mathbf{N}$ we can find $C(N) > 0$ such that

$$||Y_\nu|| = O\left(h_n^N \langle h_n p_n t \rangle^{C(N)}\right). \tag{5.2}$$

If $N', k \in \mathbf{N}$ and $\{Y_\nu\}_{\nu \in \mathcal{V}}$ is negligible, then (5.1) allows to find $C(k, N') > 0$ such that

$$t^k \operatorname{tr} Q_\nu \mathrm{e}^{-i\tau P} Y_\nu = O\left(h_n^{N'} \langle h_n p_n t \rangle^{C(k, N')}\right). \tag{5.3}$$

Proposition 6.1 *Let $c > \bar{c}/2$ and consider $b_\nu \in S(m, g)$ satisfying the condition $\operatorname{supp} b_\nu \cap \mathcal{B}_n(c) = \emptyset$. Then the family*

$$R_\nu = b_\nu^{\mathcal{W}}(x, D) \mathrm{e}^{-i\tau P} L_n^* \tag{5.4}$$

is negligible.

Before giving the proof of this result we describe its consequences.

We assume $\bar{c} > c > c' > \bar{c}/2$ and note that the method of ch. 18.4 [16] (cf. the beginning of Section 3) allows to find $\tilde{\theta}_k^- \in C_0^\infty(\mathcal{B}_{\bar{x}(k)}^-(c))$ such that $\tilde{\theta}_k^- = 1$ on $\mathcal{B}_{\bar{x}(k)}^-(c')$ and

$$|\partial^\alpha \tilde{\theta}_n^-(x)| \leq C_\alpha v(x)^{-\rho|\alpha|}.$$

If $\tilde{\Theta}_k^-$ denotes the operator of multiplication by $\tilde{\theta}_k^-$, then Proposition 5.1 ensures that $(I - \tilde{\Theta}_{k_n^-}^-) \mathrm{e}^{-i\tau P} L_n^*$ and $P(I - \tilde{\Theta}_{k_n^-}^-) \mathrm{e}^{-i\tau P} L_n^*$ are negligible, hence for every $N \in \mathbf{N}$ we can find $C(N) > 0$ such that

$$\operatorname{tr} \tilde{Q}_{\bar{N}, n}(t - \tau) \mathrm{e}^{-i\tau P} L_n^* =$$

$$\operatorname{tr} \tilde{Q}_{\bar{N}, n}(t - \tau) \tilde{\Theta}_{k_n^-}^- \mathrm{e}^{-i\tau P} L_n^* + O\left(h_n^N \langle h_n p_n t \rangle^{C(N)}\right). \tag{5.5}$$

Further on we prefer using the operators $\tilde{Q}_{\bar{N},n}(t-\tau)\tilde{\Theta}^-_{k_n^-}$ instead of $\tilde{Q}_{\bar{N},n}(t-\tau)$ and for this purpose we introduce new notations.

Symbols depending on $(x,\xi,y)\in\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d$.

If $q_\nu\in C_0^\infty(\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d)$, then $\mathrm{Op}(\mathrm{e}^{i(t-\tau)p}q_\nu)$ will denote the integral operator with the kernel

$$(x,y)\to\mathcal{K}_\nu(q,x,y)=(2\pi)^{-d}\int\mathrm{e}^{i(x-y)\xi-i(t-\tau)p(x,\xi)}q_\nu(x,\xi,y)\,d\xi. \tag{5.6}$$

For $c>0$ and $n\in\mathbf{N}^*$ we will denote

$$\tilde{\mathcal{B}}_n(c):=\mathcal{B}^-_{\bar{x}(k_n^-)}(c)\times\mathcal{B}^+_{\bar{\xi}(k_n^+)}\times\mathcal{B}^-_{\bar{x}(k_n^-)}(c)$$

and for $m\in\mathcal{M}_{\bar{c}}(\tilde{g})$ we will write $q_\nu\in\tilde{S}(m)$ if and only if there is $c<\bar{c}$ such that $q_\nu\in C_0^\infty(\tilde{\mathcal{B}}_n(c))$ and

$$|\partial^\beta_{x,y}\partial^\alpha_\xi q_\nu(x,\xi,y)|\le C_{\alpha,\beta}\,m_n h_n^{\sigma|\beta|}\langle\xi\rangle^{|\beta|-|\alpha|}$$

holds for every $\alpha\in\mathbf{N}^d$, $\beta\in\mathbf{N}^{2d}$, where m_n are as in (3.6).

These notations will be used below to express the approximation error in a particular form, similar to (4.13), (4.14), (4.16).

Expressions of $\mathrm{tr}\,\tilde{Q}_{\bar{N},n}(t-\tau)\tilde{\Theta}^-_{k_n^-}\mathrm{e}^{-i\tau P}L_n^*$.

We have $p^{\mathcal{W}}(x,D)=p_0(x,D)=p_0(x,D)^*$ and

$$Q_{\bar{N},n}(t)p_0(x,D)^*=\mathrm{Op}\left(\mathrm{e}^{-itp}\tilde{q}_{\bar{N},n,t}\right)$$

with

$$\tilde{q}_{\bar{N},n,t}(x,\xi,y)=q_{\bar{N},n}(t)(x,\xi)\overline{p_0(y,\xi)}.$$

Writing the Taylor's development of $\tilde{q}_{\bar{N},n,t}(x,\xi,\cdot)$ in x and applying standard integrations by parts in the integrals of the form (5.6) based on the equality $(x-y)^\alpha\mathrm{e}^{i(x-y)\xi}=i^{|\alpha|}\partial^\alpha_{\xi_j}(\mathrm{e}^{i(x-y)\xi})$, we find

$$\tilde{Q}_{\bar{N},n}(t)=\mathrm{Op}\left(\mathrm{e}^{-itp}(\tilde{q}^0_{\bar{N},n}+\tilde{q}^1_{\bar{N},n})(t)\right),$$

where $\tilde{q}^0_{\bar{N},n}(t)$ are as in Proposition 4.1 and the remainder term of the Taylor's development of order $\bar{N}$ gives

$$\tilde{q}^1_{\bar{N},n}(t)(x,\xi,y)=\mathrm{e}^{itp(x,\xi)}(\bar{N}+1)\int_0^1 ds\,(1-s)^{\bar{N}}\tilde{q}_{\bar{N},n}(t,s)(x,\xi,y) \tag{5.7}$$

with

$$\tilde{q}_{\bar{N},n}(t,s)(x,\xi,y) =$$

$$\sum_{|\alpha|=\bar{N}+1} (-i)^{|\alpha|}\Big(\partial_\xi^\alpha \left(q_{\bar{N},n}(t)\mathrm{e}^{-itp}\right)(x,\xi)\,\overline{\partial_x^\alpha p_0(x+s(y-x),\xi)}\Big)/\alpha!.$$

We can write

$$\tilde{q}^0_{\bar{N},n}(t-\tau)(x,\xi)\tilde{\theta}^-_{k_n^-}(y) = \sum_{\bar{N}\le k\le 2\bar{N}} (t-\tau)^k\ \tilde{q}^0_{\bar{N},k,n}(x,\xi)\tilde{\theta}^-_{k_n^-}(y)$$

$$= \sum_{\bar{N}\le k\le 2\bar{N}} t^k\ \tilde{q}^0_{\bar{N},k,\nu}(x,\xi,y),$$

where $\nu=(n,t,\tau)$ according to our convention and

$$\tilde{q}^0_{\bar{N},k,\nu}(x,\xi,y) = \left(1-\frac{\tau}{t}\right)^k\ \tilde{q}^0_{\bar{N},k,n}(x,\xi)\tilde{\theta}^-_{k_n^-}(y).$$

It is clear that $\tilde{q}^0_{\bar{N},k,\nu}\in\tilde{S}(ph^{\bar{N}\sigma/2}\tilde{p}^k)$ and using the form of $q^0_{\bar{N},n}(t)$ in (5.7) we find a similar expression

$$\tilde{q}^1_{\bar{N},n}(t-\tau)(x,\xi,y)\tilde{\theta}^-_{k_n^-}(y) = \sum_{0\le k\le 2\bar{N}} t^k\ \tilde{q}^1_{\bar{N},k,\nu}(x,\xi,y)$$

with $\tilde{q}^1_{\bar{N},k,\nu}\in\tilde{S}(ph^{\bar{N}\sigma/2}\tilde{p}^k)$.

If $Y_\nu\in B(L^2(\mathbf{R}^d))$ and $q_\nu\in\tilde{S}(m)$ then the quantity

$$J_\nu(q,Y) := \mathrm{tr}\,\left(\mathrm{Op}(\mathrm{e}^{i(\tau-t)p}q_\nu)\mathrm{e}^{-i\tau P}Y_\nu\right), \tag{5.8}$$

is well defined due to Lemma 9.1 and analogically to (5.1) we have

$$|t^k\,J_\nu(q,Y)| \le Cm_nh_n^{-C_0}\langle h_np_nt\rangle^{k+C_0}||Y_\nu||, \tag{5.9}$$

where m_n are as in (3.6).

Using the above notation we can state the following conclusion

Corollary 6.2 *There exist $\tilde{q}_{\bar{N},k,\nu}\in\tilde{S}(ph^{\bar{N}\sigma/2}\tilde{p}^k)$ such that*

$$\mathrm{tr}\,\tilde{Q}_{\bar{N},n}(t-\tau)\tilde{\Theta}^-_{k_n^-}\mathrm{e}^{-i\tau P}L_n^* = \sum_{0\le k\le 2\bar{N}} t^k\,J_\nu(\tilde{q}_{\bar{N},k},Y) \tag{5.10}$$

holds with $Y_\nu = L_n^*$.

We will complete the proof of Theorem 2.1 by showing

Proposition 6.3 *Assume $N_0, k \in \mathbf{N}^*$, $Y_\nu = L_n^*$ and $q_\nu \in \tilde{S}(m)$. Then one can find $k_0 \in \mathbf{N}^*$ and $C(N_0) > 0$ such that*

$$t^k J_\nu(q, Y) = \sum_{1 \le \bar{k} \le k_0} J_\nu(q_{\bar{k}}, Y_{\bar{k}}) + O\left(h_n^{N_0} \langle h_n p_n t\rangle^{C(N_0)}\right). \tag{5.11}$$

holds for certain families of symbols $\{q_{\bar{k},\nu}\}_{\nu\in\mathcal{V}}$ and operators $\{Y_{\bar{k},\nu}\}_{\nu\in\mathcal{V}}$ satisfying

$$q_{\bar{k},\nu} \in \tilde{S}(m\tilde{p}^{-k}), \qquad ||Y_{\bar{k},\nu}|| = O\left(\langle h_n p_n t\rangle^{C(N_0)}\right) \tag{5.12}$$

for $\bar{k} = 1, ..., k_0$.

Indeed, using (5.10) and Proposition 5.3 with $q_\nu = \tilde{q}_{\bar{N},k,\nu}$, $m = ph^{\bar{N}\sigma/2}\tilde{p}^k$, we can write

$$\operatorname{tr} \tilde{Q}_{\bar{N},n}(t-\tau)\tilde{\Theta}^-_{k_n^-} \mathrm{e}^{-i\tau P} L_n^* = \sum_{1 \le \bar{k} \le k(\bar{N})} J_\nu(q_{\bar{N},\bar{k}}, Y_{\bar{N},\bar{k}}) + O\left(h_n^{N_0} \langle h_n p_n t\rangle^{C(\bar{N},N_0)}\right),$$

with

$$q_{\bar{N},\bar{k},\nu} \in \tilde{S}(ph^{\bar{N}\sigma/2}), \qquad ||Y_{\bar{N},\bar{k},\nu}|| = O\left(\langle h_n p_n t\rangle^{C(\bar{N},N_0)}\right),$$

hence choosing $\bar{N} = \bar{N}(N_0)$ large enough we can ensure

$$J_\nu(q_{\bar{N},\bar{k}}, Y_{\bar{N},\bar{k}}) = O\left(h_n^{N_0} \langle h_n p_n t\rangle^{\bar{C}(\bar{N},N_0)}\right)$$

due to (5.9). This proves (3.18) completing the proof of Theorem 3.2 and 2.1.

Proof of Proposition 5.1. The method of ch. 18.4 [16] allows to find symbols $l_n^0(x,\xi) = \theta^{0-}_{k_n^-}(x)\theta^{0+}_{k_n^+}(\xi)$ satisfying

$$\begin{gathered} l_n^0 \in S(1,g), \quad \partial_{x_j} l_n^0 \in S(v(x)^{-\rho}, g), \quad \operatorname{supp} l_n^0 \subset \mathcal{B}_n(c), \\ l_n^0 = 1 \text{ on } \mathcal{B}_n(c') \text{ with } c > c' > \bar{c}/2. \end{gathered} \tag{5.13}$$

We set $L_n^0 = l_n^0(x,D)$ and note that $||b_\nu^{\mathcal{W}}(x,D)L_n^0|| = O(h_n^N)$ for every $N \in \mathbf{N}$ holds due to $\operatorname{supp} b_\nu \cap \operatorname{supp} l_n^0 = \emptyset$ (cf. Theorem 18.5.4 and 18.6.3 [16]). Therefore it suffices to show that

$$||(I - L_n^0)\mathrm{e}^{-i\tau P} L_n^*|| = O\left(h_n^{k\sigma} \langle h_n p_n \tau\rangle^k\right) \tag{5.14(k)}$$

holds for every $k \in \mathbf{N}$ if l_n^0 satisfies (5.13). Obviously (5.14(k)) holds for $k = 0$ and we will prove the general statement by induction with respect to $k \in \mathbf{N}$.

Further on we assume that $\sigma > 0$ is small enough to ensure $\rho'(1+\sigma) \leq \rho$ and $r'(1+\sigma) \leq 1$. This assumption implies $v(x)^{-\rho}\langle\xi\rangle^{-1} \leq h(x,\xi)^{1+\sigma}$ and $\partial_{x_j} l_n^0 \partial_{\xi_j} p$, $\partial_{x_j} p \partial_{\xi_j} l_n^0 \in S(ph^{1+\sigma}, g)$ if l_n^0 satisfies (5.13). Thus (3.23) ensures

$$[I - L_n^0,\, P] = -[L_n^0,\, P] = \tilde{l}_n^{\mathcal{W}}(x,D) \;\text{ with }\; \tilde{l}_n \in S(ph^{1+\sigma}, g). \tag{5.15}$$

Moreover we can find

$$\begin{gathered} l_n^1 \in S(1,g), \quad \partial_{x_j} l_n^1 \in S(v(x)^{-\rho}, g), \quad \operatorname{supp} l_n^1 \subset \mathcal{B}_n(c'), \\ l_n^1 = 1 \;\text{ on }\; \mathcal{B}_n(c'') \;\text{ with }\; c' > c'' > \bar{c}/2 \end{gathered} \tag{5.16}$$

and setting $L_n^1 = l_n^1(x,D)$ we have $\operatorname{supp} \tilde{l}_n \cap \operatorname{supp} l_n^1 = \emptyset$, hence

$$||\tilde{l}_n^{\mathcal{W}}(x,D) L_n^1|| = O(h_n^N) \;\text{ for every }\; N \in \mathbf{N}. \tag{5.17}$$

Since $||(I - L_n^0)L_n^*|| = O(h_n^N)$ for every $N \in \mathbf{N}$ and

$$(I - L_n^0)\mathrm{e}^{-i\tau P} L_n^* = \mathrm{e}^{-i\tau P}(I - L_n^0)L_n^* + \left[I - L_n^0,\, \mathrm{e}^{-i\tau P}\right] L_n^*, \tag{5.18}$$

it remains to estimate the norm of

$$\left[I - L_n^0,\, \mathrm{e}^{-i\tau P}\right] L_n^* = \int_0^1 ds\, \tau \mathrm{e}^{i(s-1)\tau P} [I - L_n^0,\, P] \mathrm{e}^{-is\tau P} L_n^*. \tag{5.19}$$

and due to (5.17) it suffices to estimate

$$|\tau| ||\tilde{l}_n^{\mathcal{W}}(x,D)(I - L_n^1)\mathrm{e}^{-is\tau P} L_n^*||. \tag{5.20}$$

However we have $||\tilde{l}_n^{\mathcal{W}}(x,D)|| \leq C p_n h_n^{1+\sigma}$ (cf. Theorem 18.6.3 [16]) and using the induction hypothesis we can assume that (5.14(k)) holds with L_n^1 instead of L_n^0 and c'' instead of c'. Thus the quantity (5.20) can be estimated by

$$C|\tau| p_n h_n^{1+\sigma} h_n^{k\sigma} \langle h_n p_n \tau\rangle^k \leq C h_n^{(k+1)\sigma} \langle h_n p_n \tau\rangle^{k+1},$$

completing the proof of (5.14($k+1$)). $\triangle$.

7 Auxiliary commutator formulas

Notations. We will write $b_\nu \in S_n(m,g)$ if and only if there exist $c_0 < \bar{c}$, $l_n^0 \in S(1,g)$ satisfying supp $l_n^0 \subset \mathcal{B}_n(c_0)$ and $(1-l_n^0)b_\nu \in S(h^N,g)$ for every $N \in \mathbf{N}$.
Then Theorem 18.5.4 [16] ensures $b\tilde{b}$, $b \odot \tilde{b} \in S_n(m\tilde{m},g)$ if $b_\nu \in S_n(m,g)$, $\tilde{b}_\nu \in S(\tilde{m},g)$ and (3.23) still holds with $b, \tilde{b}, S(m',g)$ replaced by $b_\nu, \tilde{b}_\nu, S_n(m',g)$. Moreover $b_\nu \in S_n(m,g)$ implies

$$|\partial_x^\beta \partial_\xi^\alpha b_\nu(x,\xi)| \leq C_{\alpha,\beta}\, m_n h_n^{\sigma|\beta|} \langle \xi \rangle^{|\beta|-|\alpha|} \tag{6.1}$$

where m_n are as in (3.6) and Theorem 8.6.3 [16] ensures

$$||b_\nu^{\mathcal{W}}(x,D)|| \leq C m_n, \qquad ||(I - l_n^{0\,\mathcal{W}}(x,D))b_\nu^{\mathcal{W}}(x,D)|| = O(h_n^N) \tag{6.2}$$

for every $N \in \mathbf{N}$.
We introduce the following formal notation

$$Y(\tau,B) := \mathrm{e}^{-i\tau P} B \mathrm{e}^{i\tau P}. \tag{6.3}$$

Let $\{Y_\nu\}_{\nu\in\mathcal{V}}$ be a family of bounded operators and let $m \in \mathcal{M}_{\bar{c}}(\tilde{g})$. We write $Y_\nu \in \mathcal{Y}(m)$ if and only if there exist $N \in \mathbf{N}$, $C_0 > 0$, the weights $m(k,k') \in \mathcal{M}_{\bar{c}}(\tilde{g})$, the symbols $b_{k,k',\nu} \in S_n(m(k,k'),g)$ and functions $s_{k,k'} : [0;\,1]^N \to \mathbf{R}$, $s_{k,\nu} : [0;\,1]^N \to \mathbf{C}$, satisfying

$$\prod_{k'=1}^N m(k,k') \leq m, \qquad |s_{k,k'}(w)| \leq C_0, \qquad |s_{k,\nu}(w)| \leq C_0 \langle h_n p_n t \rangle^{C_0},$$

for $k,k' = 1,...,N$ and

$$Y_\nu = \sum_{k=1}^N \int_{[0;\,1]^N} dw\, s_{k,\nu}(w) Y(s_{k,1}(w)\tau, B_{k,1,\nu})...Y(s_{k,N}(w)\tau, B_{k,N,\nu}) + R_\nu \tag{6.4}$$

where $B_{k,k',\nu} = b_{k,k',\nu}^{\mathcal{W}}(x,D)$ and the family $\{R_\nu\}_{\nu\in\mathcal{V}}$ is negligible.

Taking $s_{k,\nu}(w) = s_{k,k'}(w) = 1$ we can forget the integration with respect to w, hence $Y(\tau,B_{k,1,\nu})...Y(\tau,B_{k,N,\nu}) \in \mathcal{Y}(m)$ and more generally

$$Y_\nu \in \mathcal{Y}(m),\ \ \tilde{Y}_\nu \in \mathcal{Y}(\tilde{m}) \implies Y_\nu \tilde{Y}_\nu \in \mathcal{Y}(m\tilde{m}), \tag{6.5}$$

$$Y_\nu \in \mathcal{Y}(m) \Longrightarrow ||Y_\nu|| \leq C m_n \langle h_n p_n t \rangle^C. \tag{6.6}$$

Reformulation of Proposition 5.3. In Section 7 we will prove

Proposition 7.1 *Assume $q_{0,\nu} \in \tilde{S}(m)$, $Y_{0,\nu} \in \mathcal{Y}(1)$ and $N_0 \in \mathbf{N}$. Then one can find $k_0 \in \mathbf{N}^*$ and $C(N_0) > 0$ such that*

$$t\, J_\nu(q_0, Y_0) = \sum_{1 \le \bar{k} \le k_0} (J_\nu(q_{\bar{k}}, Y_{\bar{k}}) + tJ_\nu(q_{-\bar{k}}, Y_{-\bar{k}})) + O\left(h_n^{N_0} \langle h_n p_n t \rangle^{C(N_0)}\right) \tag{6.7}$$

holds with certain symbols $q_{\pm\bar{k},\nu}$ and operators $Y_{\pm\bar{k},\nu}$ satisfying

$$q_{\bar{k},\nu} \in \tilde{S}(m/\tilde{p}), \quad q_{-\bar{k},\nu} \in \tilde{S}(mh^\sigma), \quad Y_{\pm\bar{k},\nu} \in \mathcal{Y}(1) \tag{6.8}$$

for $\bar{k} = 1, ..., k_0$.

It is easy to see that Proposition 6.1 implies Proposition 5.3. Indeed, first of all we note that the assertion of Proposition 6.1 can be applied to express $tJ_\nu(q_{-\bar{k}}, Y_{-\bar{k}})$, $\bar{k} = 1, ..., k_0$ and iterating this procedure N times we find the expression of $tJ_\nu(q_0, Y_0)$ in the form (6.7) with new symbols $q_{-\bar{k},\nu} \in \tilde{S}(mh^{N\sigma})$, $\bar{k} = 1, ..., k_N$. Thus for $N = N(N_0)$ large enough all terms $tJ_\nu(q_{-\bar{k}}, Y_{-\bar{k}})$, $\bar{k} = 1, ..., k_N$, become $O\left(h_n^{N_0} \langle h_n p_n t \rangle^{C(N_0)}\right)$, i.e. the assertion of Proposition 6.1 holds with $q_{-\bar{k}} = 0$ for $\bar{k} \ge 1$. This proves Proposition 5.3 in the case $k = 1$ and it is clear that the general case follows after k iterations.

In the remaining part of this section we describe the properties of $Y_\nu \in \mathcal{Y}(m)$ needed in the proof of Proposition 6.1. More precisely we consider the commutator of Y_ν with the operator of multiplication by the j-th coordinate, denoted by x_j.

Lemma 7.2 *Assume $Y_\nu \in \mathcal{Y}(m)$. Then there exist*

$$Y_\nu^+ \in \mathcal{Y}(\langle \xi \rangle^{-1} m), \quad Y_\nu^- \in \mathcal{Y}(\langle \xi \rangle^{-1} \tilde{p} h^\sigma m) \tag{6.9}$$

such that $[Y_\nu,\ x_j] = Y_\nu^+ + \tau Y_\nu^-$.

Proof. Let $B_{k,k',\nu} = b^{\mathcal{W}}_{k,k',\nu}(x, D)$ with $b_{k,k',\nu} \in S_n(m(k, k'), g)$.
If we know that

$$[Y(\tau, B_{k,k',\nu}),\, x_j] = Y^+_{k,k',\nu} + \tau Y^-_{k,k',\nu} \tag{6.10}$$

holds with

$$Y^+_{k,k',\nu} \in \mathcal{Y}(\langle \xi \rangle^{-1} m(k, k')), \qquad Y^-_{k,k',\nu} \in \mathcal{Y}(\langle \xi \rangle^{-1} \tilde{p} h^\sigma m(k, k')),$$

then commuting succesively x_j with $Y(s_{k,k'}(w)\tau, B_{k,k',\nu})$, $k' = 1, ..., N$, we obtain easily the general statement of Lemma 6.2.

To begin we write

$$[Y(\tau, B_{k,k',\nu}),\, x_j] = \mathrm{e}^{-i\tau P}[B_{k,k',\nu},\, Y(-\tau, x_j)]\mathrm{e}^{i\tau P} \tag{6.11}$$

and denote

$$P_j := [iP,\, x_j] = \partial_{\xi_j} p^{\mathcal{W}}(x, D) = \partial_{\xi_j} p_0(x, D). \tag{6.12}$$

Then we can write the Taylor formula

$$\begin{aligned} Y(-\tau, x_j) &= x_j - \tau\partial_\tau Y(0, x_j) + \tau^2 \int_0^1 ds\,(1-s)\partial_\tau^2 Y(-s\tau, x_j) \\ &= x_j + \tau P_j + \tau \mathrm{e}^{-i\tau P}\overline{Y}(\tau, [i\tau P, P_j])\mathrm{e}^{i\tau P}, \end{aligned} \tag{6.13}$$

where

$$\overline{Y}(\tau, B) = \tau \int_0^1 ds\,(1-s) Y((-s-1)\tau, B).$$

Using (6.13) we can express the commutator (6.11) in the form

$$Y(\tau, [B_{k,k',\nu},\, x_j]) + \tau Y(\tau, [B_{k,k',\nu},\, P_j]) + \tau[Y(\tau, B_{k,k',\nu}),\, \overline{Y}(\tau, [i\tau P, P_j]) \tag{6.14}$$

and since $[B_{k,k',\nu},\, x_j] = \partial_{\xi_j} b^{\mathcal{W}}_{k,k',\nu}(x, D)$, it is clear that the first term of (6.13) is in $\mathcal{Y}(\langle\xi\rangle^{-1} m(k, k'))$. Next we check that

$$\partial_{x_{j'}} b_{k,k',\nu} \partial_{\xi_{j'}} \partial_{\xi_j} p \in S_n(\langle\xi\rangle^{-1} \tilde{p} h^\sigma m(k, k'), g)$$

$$\partial_{\xi_{j'}} b_{k,k',\nu} \partial_{x_{j'}} \partial_{\xi_j} p S_n(\langle\xi\rangle^{-1} p h^{1+\sigma} m(k, k'), g) \subset S_n(\langle\xi\rangle^{-1} \tilde{p} h^\sigma m(k, k'), g)$$

due to $hp \leq \tilde{p}$, hence (3.23) implies

$$b_{k,k',\nu} \odot \partial_{\xi_j} p - \partial_{\xi_j} p \odot b_{k,k',\nu} \in S_n(\langle\xi\rangle^{-1} \tilde{p} h^\sigma m(k, k'), g),$$

and consequently $Y(\tau, [B_{k,k',\nu},\, P_j]) \in \mathcal{Y}(\langle\xi\rangle^{-1} \tilde{p} h^\sigma m(k, k'))$.
Moreover we have

$$\partial_{x_{j'}} p \partial_{\xi_{j'}} \partial_{\xi_j} p,\, \partial_{\xi_{j'}} p \partial_{x_{j'}} \partial_{\xi_j} p \in S(\langle\xi\rangle^{-1} \tilde{p} h^{1+\sigma} p, g),$$

hence (3.23) ensures

$$p \odot \partial_{\xi_j} p - \partial_{\xi_j} p \odot p \in S(\langle\xi\rangle^{-1} \tilde{p} h^{1+\sigma} p, g). \tag{6.15}$$

Introducing $l_n^0 \in S(1,g)$ such that $(I - L_n^0)B_{k,k',\nu}$ is negligible with $L_n^0 = l_n^0(x,D)$ and supp $l_n^0 \subset \mathcal{B}_n(c_0)$ with $c_0 < \bar{c}$, we can write

$$Y(\tau, B_{k,k',\nu})\overline{Y}(\tau,[P_j, i\tau P]) = Y(\tau, B_{k,k',\nu})\overline{Y}(\tau, L_n^0[P_j, i\tau P]) + R_{k,k',\nu} \quad (6.16)$$

with $\{R_{k,k',\nu}\}_{\nu\in\mathcal{V}}$ negligible. However using

$$\langle h_n p_n \tau\rangle^{-1}|\tau| l_n^0 \in S_n(h^{-1}p^{-1}, g)$$

and (6.15) we find $p_{j,\nu} \in S_n(\langle\xi\rangle^{-1}\tilde{p}h^\sigma, g)$ such that

$$\langle h_n p_n \tau\rangle^{-1}\tau L_n^0[P_j, iP] = p_{j,\nu}^{\mathcal{W}}(x,D),$$

hence $\overline{Y}(\tau, L_n^0[P_j, i\tau P]) \in \mathcal{Y}(\langle\xi\rangle^{-1}\tilde{p}h^\sigma)$. Therefore the right hand side of (6.16) belongs to $\mathcal{Y}(\langle\xi\rangle^{-1}\tilde{p}h^\sigma m(k,k'))$ and $\overline{Y}(\tau,[P_j, i\tau P])Y(\tau, B_{k,k',\nu})$ belongs to the same class, i.e. (6.14) gives the desired decomposition (6.10). $\triangle$

Corollary 7.3 *Let $Y_{0,\nu} \in \mathcal{Y}(1)$. If $P_j = [iP, x_j]$, then one has*

$$\tau P_j \mathrm{e}^{-i\tau P} Y_{0,\nu} = [\mathrm{e}^{-i\tau P} Y_{0,\nu},\ \ x_j] + \mathrm{e}^{-i\tau P}(Y_{0,\nu}^+ + \tau Y_{0,\nu}^-) \quad (6.17)$$

with some $Y_{0,\nu}^+ \in \mathcal{Y}(\langle\xi\rangle^{-1})$ and $Y_{0,\nu}^- \in \mathcal{Y}(\langle\xi\rangle^{-1}\tilde{p}h^\sigma)$.

Indeed, using (6.13) to express $[\mathrm{e}^{-i\tau P},\ \ x_j]$ and applying Lemma 6.2 with $Y_\nu = Y_{0,\nu}$ we can write

$$[\mathrm{e}^{-i\tau P} Y_{0,\nu},\ \ x_j] = [\mathrm{e}^{-i\tau P},\ \ x_j] Y_{0,\nu} + \mathrm{e}^{-i\tau P}[Y_{0,\nu},\ \ x_j]$$
$$= -\tau P_j \mathrm{e}^{-i\tau P} Y_{0,\nu} - \tau \mathrm{e}^{-i\tau P}\overline{Y}(\tau,[P_j, i\tau P]) Y_{0,\nu} + \mathrm{e}^{-i\tau P}(Y_\nu^+ + \tau Y_\nu^-).$$

It remains to remark that the reasoning of the proof of Lemma 6.2 ensures the fact that $Y_{0,\nu}^+ = Y_\nu^+$ and $Y_{0,\nu}^- = -\overline{Y}(\tau,[P_j, i\tau P])Y_{0,\nu} + Y_\nu^-$ belong to the indicated classes. $\triangle$

8 End of the proof of Theorem 2.1

Throughout this section we use the following notation

$$Q_\nu = \mathrm{Op}(\mathrm{e}^{i(\tau-t)p} q_\nu) \quad \text{with} \quad q_\nu \in \tilde{S}(\langle\xi\rangle m/\tilde{p}). \quad (7.1)$$

We adopt the convention that the symbol $(x,\xi) \to p(x,\xi)$ can be considered as $(x,\xi,y) \to p(x,\xi)$, allowing to define $q_\nu \partial_{\xi_j} p \in \tilde{S}(m)$ by the formula

$$(q_\nu \partial_{\xi_j} p)(x,\xi,y) = q_\nu(x,\xi,y)\partial_{\xi_j} p(x,\xi). \quad (7.2)$$

Lemma 8.1 *If Q_ν and $q_\nu\partial_{\xi_j}p$ are as in* (7.1), (7.2), *then*

$$[Q_\nu,\, x_j] = (t-\tau)\mathrm{Op}(\mathrm{e}^{i(\tau-t)p}q_\nu\partial_{\xi_j}p) + \mathrm{Op}(\mathrm{e}^{i(\tau-t)p}i\partial_{\xi_j}q_\nu). \tag{7.3}$$

Proof. Since the integral kernel of $[Q_\nu,\, x_j]$ is

$$(x,y) \to (2\pi)^{-d}\int (y_j - x_j)\mathrm{e}^{i(x-y)\xi + i(\tau-t)p(x,\xi)}q_\nu(x,\xi,y)\,d\xi \tag{7.4}$$

and $(y_j - x_j)\mathrm{e}^{i(x-y)\xi} = -i\partial_{\xi_j}\mathrm{e}^{i(x-y)\xi}$, the integration by parts allows to write (7.4) in the form

$$(x,y) \to (2\pi)^{-d}\int \mathrm{e}^{i(x-y)\xi + i(\tau-t)p(x,\xi)}((t-\tau)\partial_{\xi_j}pq_\nu + i\partial_{\xi_j}q_\nu)(x,\xi,y)\,d\xi,$$

which gives (7.3). $\triangle$

The computation of the composition kernel gives

$$\mathrm{Op}(\mathrm{e}^{i(\tau-t)p}q_\nu)b_\nu(x,D)^* = \mathrm{Op}(\mathrm{e}^{i(\tau-t)p}(q_\nu \bullet b_\nu)) \tag{7.5}$$

with

$$(q_\nu \bullet b_\nu)(x,\xi,y) = (2\pi)^{-d}\int \mathrm{e}^{i(\tilde{y}-y)(\xi-\tilde{\xi})}q_\nu(x,\xi,\tilde{y})\overline{b_\nu(y,\tilde{\xi})}$$

and the usual Taylor development with integrations by parts give

$$q_\nu \bullet b_\nu = q_\nu \bullet_{N'} b_\nu + r_{N',\nu}, \tag{7.6}$$

with

$$(q_\nu \bullet_{N'} b_\nu)(x,\xi,y) = \sum_{|\alpha|<N'}(-i)^{|\alpha|}\partial_y^\alpha q_\nu(x,\xi,y)\overline{\partial_\xi^\alpha b_\nu(y,\xi)}/\alpha!, \tag{7.7}$$

$$r_{N',\nu} = \sum_{|\alpha|=N'}(q_{\alpha,\nu}(z) \bullet \partial_\xi^\alpha b_\nu)|_{z=y},$$

$$q_{\alpha,\nu}(z)(x,\xi,y) = \int_0^1 \partial_y^\alpha q_\nu(x,\xi,z+s(y-z))\frac{i^{N'}N'ds}{\alpha!}.$$

Then for an arbitrary $N \in \mathbf{N}$ we can find $N' = N'(N)$ such that the family of operators $R_{N',\nu} = \mathrm{Op}(\mathrm{e}^{i(\tau-t)p}r_{N',\nu})$ satisfies

$$||R_{N',\nu}|| \le C_N h_n^N \langle h_n p_n t\rangle^{C(N)}. \tag{7.8}$$

Denoting $(q_\nu \bar{b}_\nu^\sharp)(x,\xi,y) = q_\nu(x,\xi,y)\overline{b_\nu(y,\xi)}$ we can write

$$q_\nu \in \tilde{S}(\tilde{m}), b_\nu \in S(m',g) \Rightarrow q_\nu \bullet_{N'} b_\nu \in \tilde{S}(\tilde{m}m'), q_\nu \bullet_{N'} b_\nu - q_\nu \bar{b}_\nu^\sharp \in \tilde{S}(\tilde{m}m'h^\sigma).$$

If in particular $q_\nu \in \tilde{S}(\langle\xi\rangle m/\tilde{p})$, $\partial_{\xi_j} p_0 \in S(\langle\xi\rangle^{-1}\tilde{p}, g)$, then

$$Q_\nu P_j = \mathrm{Op}(\mathrm{e}^{i(\tau-t)p}(q_\nu \bullet \partial_{\xi_j} p_0)), \qquad q_\nu \bullet_{N'} \partial_{\xi_j} p_0 \in \tilde{S}(m),$$

$$q_\nu \bullet_{N'} \partial_{\xi_j} p_0 - q_\nu \overline{\partial_{\xi_j} p_0}^\sharp \in \tilde{S}(h^\sigma m).$$

Since (3.20) ensures $\partial_{\xi_j} p - \overline{\partial_{\xi_j} p_0} \in S(\langle\xi\rangle^{-1} h^{1+\sigma} p, g) \subset S(\langle\xi\rangle^{-1} h^\sigma \tilde{p}, g)$,

$$q^-_{N',\nu} := q_\nu \bullet_{N'} \partial_{\xi_j} p_0 - q_\nu \partial_{\xi_j} p^\sharp \in \tilde{S}(h^\sigma m). \tag{7.9}$$

For $j = 1, ..., d$ we introduce

$$p_{(j),\nu}(x,\xi,y) := \left(1 - \frac{\tau}{t}\right) \partial_{\xi_j} p(x,\xi) + \frac{\tau}{t} \partial_{\xi_j} p(y,\xi). \tag{7.10}$$

Proposition 8.2 *Let $q_\nu \in \tilde{S}(\langle\xi\rangle m/\tilde{p})$, $Y_{0,\nu} \in \mathcal{Y}(1)$ and $N_0 \in \mathbf{N}$. Then we can find $C(N_0) > 0$ such that*

$$t\, J_\nu(q p_{(j)}, Y_0) = \sum_{1 \le k \le 2} (J_\nu(q_k, Y_k) + t J_\nu(q_{-k}, Y_{-k})) + O\left(h_n^{N_0} \langle h_n p_n t\rangle^{C(N_0)}\right) \tag{7.11}$$

where for $k = 1$ and 2 we have

$$q_{k,\nu} \in \tilde{S}(m/\tilde{p}), \quad q_{-k,\nu} \in \tilde{S}(mh^\sigma), \quad Y_{k,\nu} \in \mathcal{Y}(1), \quad Y_{-k,\nu} \in \mathcal{Y}(1). \tag{7.12}$$

Proof. By definition of $p_{(j),\nu}$ we have

$$t\, \mathrm{Op}(\mathrm{e}^{i(\tau-t)p} q_\nu p_{(j),\nu}) = (t-\tau)\mathrm{Op}(\mathrm{e}^{i(\tau-t)p} q_\nu \partial_{\xi_j} p) + \tau \mathrm{Op}(\mathrm{e}^{i(\tau-t)p} q_\nu \partial_{\xi_j} p^\sharp) \tag{7.13}$$

and applying (7.3), (7.9), we can write the above expression as

$$[Q_\nu,\, x_j] + Q_\nu \tau P_j + \mathrm{Op}(\mathrm{e}^{i(\tau-t)p}(\tau q^-_{N',\nu} - i\partial_{\xi_j} q_\nu)) + \tau R_{N',\nu}$$

with $R_{N',\nu}$ satisfying (7.8). Thus $t\, J_\nu(q p_{(j)}, Y_0)$ can be written as

$$\mathrm{tr}\,[Q_\nu,\, x_j]\mathrm{e}^{-i\tau P} Y_{0,\nu} \;+\; \mathrm{tr}\, Q_\nu \tau P_j \mathrm{e}^{-i\tau P} Y_{0,\nu} \;+$$

$$J_\nu(-i\partial_{\xi_j} q, Y_0) + \tau J_\nu(q^-_{N'}, Y_0) + O(h_n^{N_0} \langle h_n p_n t\rangle^{C(N_0)}). \tag{7.14}$$

Due to Corollary 6.3, the sum of two first terms in (7.14) equals to

$$\begin{aligned}\operatorname{tr}[Q_\nu,\, x_j]\mathrm{e}^{-i\tau P}Y_{0,\nu} \;+\; \operatorname{tr} Q_\nu[\mathrm{e}^{-i\tau P}Y_{0,\nu},\, x_j] \;+ J_\nu(q,Y_0^+)+\tau J_\nu(q,Y_0^-)\\ = \operatorname{tr}[Q_\nu \mathrm{e}^{-i\tau P}Y_{0,\nu},\, x_j] \;+ J_\nu(q,Y_0^+)+\tau J_\nu(q,Y_0^-)\\ = J_\nu(q,Y_0^+)+\tau J_\nu(q,Y_0^-). \qquad (7.15)\end{aligned}$$

Thus we obtain (7.11) with

$$q_{1,\nu}=i\partial_{\xi_j}q_\nu, \qquad q_{-1,\nu}=\frac{\tau}{t}q^-_{N',\nu}, \qquad Y_{1,\nu}=Y_{-1,\nu}=Y_{0,\nu},$$

$$q_{2,\nu}=\langle\bar\xi(k_n^+)\rangle^{-1}q_\nu, \qquad Y_{2,\nu}=\langle\bar\xi(k_n^+)\rangle Y^+_{0,\nu},$$

$$q_{-2,\nu}=\langle\bar\xi(k_n^+)\rangle^{-1}\tilde p_n h_n^\sigma q_\nu, \qquad Y_{-2,\nu}=\langle\bar\xi(k_n^+)\rangle\tilde p_n^{-1}h_n^{-\sigma}Y^-_{0,\nu},$$

where according to our convention $\tilde p_n=\tilde p_n(\bar x(k_n^-),\bar\xi(k_n^+))$. $\triangle$.

Proof of Proposition 6.1. Let χ_s, $\tilde\chi_s$ be as in the proof of Proposition 4.2. Then $q_{0,\nu}\in\tilde S(m,g)\Longrightarrow \tilde q_\nu:=q_{0,\nu}\chi_s h_n^{-1}p_n^{-1}\in\tilde S(m/\tilde p,g)$ and

$$tJ_\nu(q_0\chi_s,Y_0)=J_\nu(\tilde q,\tilde Y) \text{ with } \tilde Y_\nu=h_np_ntY_{0,\nu}\in\mathcal{Y}(1).$$

Therefore it sufficesto prove the statement of Proposition 6.1 with $q_{0,\nu}\tilde\chi_s$ instead of $q_{0,\nu}$.

Further on we assume $s=1$ and note that for $(x,\xi)\in\operatorname{supp}\tilde\chi_1$ we have

$$\langle\xi\rangle^{-1}\tilde p(x,\xi)/2\le|\nabla_\xi p(x,\xi)|\le 2\langle\xi\rangle^{-1}\tilde p(x,\xi) \qquad (7.16)$$

and due to (2.10), for $(x,\xi,y)\in\operatorname{supp} q_{0,\nu}\tilde\chi_1$ we have

$$\begin{aligned}|\nabla_\xi p(x,\xi)|-\left(\sum\nolimits_{j=1}^d p^2_{(j),\nu}\right)^{1/2}\le|\nabla_\xi p(x,\xi)-\nabla_\xi p(y,\xi)|\\ \le\langle\xi\rangle^{-1}\tilde p(x,\xi)/4\le|\nabla_\xi p(x,\xi)|/2,\end{aligned}$$

which implies

$$\left(\sum\nolimits_{j=1}^d p^2_{(j),\nu}\right)^{1/2}\ge|\nabla_\xi p(x,\xi)|/2\ge\langle\xi\rangle^{-1}\tilde p(x,\xi)/4. \qquad (7.17)$$

Using (7.16), (7.17), we can write $q_{0,\nu}\tilde\chi_1=\sum_{j=1}^d q_{(j),\nu}p_{(j),\nu}$ with

$$q_{(j),\nu}=q_{0,\nu}\tilde\chi_1 p_{(j),\nu}\left(\sum\nolimits_{j=1}^d p^2_{(j),\nu}\right)^{-1}\in\tilde S(\langle\xi\rangle m/\tilde p,g)$$

and we complete the proof applying Proposition 7.2 with $q_\nu=q_{(j),\nu}$.

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