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Article

A Fisher Information-Based Incompatibility Criterion for Quantum Channels

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Abstract: We introduce a new incompatibility criterion for quantum channels based on the notion of (quantum) Fisher information. Our construction is based on a similar criterion for quantum measurements put forward by H. Zhu. We then study the power of the incompatibility criterion in different scenarios. First, we prove the first analytical conditions for the incompatibility of two Schur channels. Then, we study the incompatibility structure of a tuple of depolarizing channels, comparing the newly introduced criterion with the known results from asymmetric quantum cloning.

Keywords: incompatibility criterion; quantum channels; quantum measurements; Fisher information



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1. Introduction

The impossibility of simultaneous realizations of two quantum operations is one of the fundamental features of quantum theory [1,2]. Two of the most famous incarnations of this principle are the *Heisenberg uncertainty principle* (the position and momentum of a quantum particle can not be measured simultaneously [1]) and the *no-cloning theorem* (there is no physical operation producing two identical copies of an unknown, arbitrary, quantum state [3,4]). In general, two (or more) quantum operations, such as measurements, channels, or instruments, are called *compatible* if they can be seen as marginals of a common operation; if there is no physical operation having the original ones as marginals, they are called *incompatible*. As quantum theory is built on Hilbert space, general quantum measurements are considered the positive operator-valued measures (POVMs). In quantum information theory, there are many applications of the notion of incompatibility, such as the robustness of entanglement [5,6], the robustness of measurement incompatibility [7–9], quantum non-locality [10,11], quantum steering [7,12], quantum state discrimination [13–15], quantum resource theory [16], and quantum cryptography [17].

In the modern formalism of quantum theory, the most general description of physical transformations of quantum states is in terms of *quantum channels* [18,19]. The concept of incompatibility of quantum channels has been proposed in terms of the input–output devices [20,21]. In [21], the authors show that the definition of the *incompatibility of quantum channels* is a natural generalization of joint measurability of quantum observables. There exists a large body of work dealing with this notion from various points of view [15,22–24]. Generally speaking, deciding whether a given family of quantum operations is compatible can be formulated as a *semidefinite program* [25]. However, the size of the program grows *exponentially* with the number of operations considered. Hence, this method can be computationally prohibitive even for small system sizes (such as qubits) when the number of systems is moderately large. To cope with this dimensionality problem, (in-)compatibility criteria have been introduced; these are conditions that are only necessary, or sufficient, for the compatibility of the given tuple of channels. As is the case with quantum measurements [20], there exist much more compatibility criteria [26] than incompatibility criteria.

In this work, we introduce a new *incompatibility criterion for quantum channels* based on the notion of (quantum) Fisher information. Our criterion is based on a similar condition put forward by H. Zhu [27,28] in the case of quantum measurements.

After introducing the necessary background on Fisher information and quantum channel compatibility (Sections 2 and 3), we put forward the new incompatibility criterion in Section 4. The statement of the main result of the paper can be found in Theorem 1. We then apply this result to study, for the first time, the incompatibility of Schur channels, an important class of quantum operations with wide-ranging applications; see Section 5. In the final two sections of the paper, we introduce different compatibility structures for assemblages of quantum channels (Section 6), and we study them in the case of generalized depolarizing channels (Section 7).

2. Classical and Quantum Fisher Information

Consider a family of probability distributions $\{p(x|\theta), x \in R\}$ parametrized by θ . A central research direction in statistics is to estimate the accuracy of the value of parameter θ by observing x outcomes sampled from the distributions. Recall that the (classical) *Fisher information* of the model is defined as

$$I(\theta) := \sum_x p(x|\theta) \left(\frac{\partial \log p(x|\theta)}{\partial \theta} \right)^2.$$

when an estimator $\hat{\theta}(x)$ of the parameter θ is unbiased, the inverse of the classical Fisher information gives a lower bound on the *mean square error* (MSE) of the estimator, which is the well-known *Cramér-Rao bound* [29,30]. The notion of the classical Fisher information plays a significant role in the geometrical approach to statistics [31,32] and the information theory approach to physics [33].

In the multiple-parameter scenario, when θ is a vector, the classical Fisher information is considered as a matrix form, which is a real symmetric matrix with matrix elements [34–36]:

$$I_{ij}(\theta) := \sum_x p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial \theta_i} \frac{\partial \log p(x|\theta)}{\partial \theta_j}.$$

In a quantum parameter estimation scenario, we may perform the quantum positive operator-valued measurement (POVM) on a quantum state that depends on a parameter to extract the parameter information. Consider a quantum measurement $\mathbf{M} = \{M_x \geq 0, \sum_x M_x = I_d\}$ acting on the states $\rho(\theta) \in \mathcal{L}(H_d)$. The parameterized probability of outcomes x of the measurement is $p(x|\theta) = \text{Tr}[\rho(\theta)M_x]$. The corresponding measurement-induced Fisher information $I_{\mathbf{M}}(\theta)$ is then given by

$$I_{\mathbf{M}}(\theta) = \sum_x p(x|\theta) \text{Tr} \left(\frac{\partial \log \rho(\theta)}{\partial \theta} M_x \right)^2.$$

The *quantum Fisher information* of the model $\rho(\theta)$ is defined as [37]

$$J(\theta) := \text{Tr}[\rho(\theta)L(\theta)^2],$$

where the symmetric logarithmic derivative (SLD) operators $L(\theta)$ for the parameter θ are determined implicitly by

$$\frac{d\rho(\theta)}{d\theta} = \frac{1}{2}[\rho(\theta)L(\theta) + L(\theta)\rho(\theta)].$$

In contrast with the classical Cramér-Rao bound, the inverse of quantum Fisher information is also a lower bound for the MSE of an unbiased estimator, which is called the *quantum Cramér-Rao bound* [37].

In quantum multiple-parameter estimation scenarios, both the measure-induced Fisher information and quantum Fisher information are matrices

$$I_{\mathbf{M},ij}(\theta) = \sum_x p(x|\theta) \operatorname{Tr} \left(\frac{\partial \log \rho(\theta)}{\partial \theta_i} M_x \right) \operatorname{Tr} \left(\frac{\partial \log \rho(\theta)}{\partial \theta_j} M_x \right), \tag{1}$$

$$J_{ij}(\theta) = \frac{1}{2} \operatorname{Tr} \{ \rho(\theta) [L_i(\theta)L_j(\theta) + L_j(\theta)L_i(\theta)] \}, \tag{2}$$

where L_k is the SLD operator corresponding with θ_k . The measurement-induced Fisher information resembles the classical correlations, while the quantum Fisher information resembles the quantum mutual information. From the Braunstein–Caves theorem [36], the quantum Fisher information, independent of measurement, is an upper bound of the measurement-induced Fisher information in the positive semidefinite order for matrices:

$$I_{\mathbf{M}}(\theta) \leq J(\theta).$$

In this work, we shall consider another relationship proposed by Gill and Massar [38] for any d –dimensional quantum system:

$$\operatorname{Tr}[J^{-1}(\theta)I_{\mathbf{M}}(\theta)] \leq d - 1. \tag{3}$$

This inequality was the main ingredient in the incompatibility criterion invented by Zhu [27], which lies at the foundation of our incompatibility criterion for quantum channels.

3. Compatibility of Quantum Channels

In this section, we review the basic definitions of quantum channel compatibility.

Let H_d and H_D be Hilbert spaces, and $\mathcal{L}(H_d)$ denote the family of linear operators on H_d . In the Schrödinger picture, a *quantum channel* is defined as a linear map $\Phi : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_D)$ having the following two properties:

- *complete positivity*: for any dimension $k \geq 1$, the linear map $\operatorname{id}_k \otimes \Phi : \mathcal{L}(\mathbb{C}^k \otimes H_d) \rightarrow \mathcal{L}(\mathbb{C}^k \otimes H_D)$ is a positive operator;
- *trace-preservation*: for all operators $X \in \mathcal{L}(H_d)$, $\operatorname{Tr} \Phi(X) = \operatorname{Tr} X$.

We say that quantum channels are trace-preserving, completely positive (TPCP) maps. In this paper, we shall also consider the Heisenberg picture of quantum mechanics, where channels are seen as acting on observables instead of states. This amounts to considering the adjoint map $\Phi^* : \mathcal{L}(H_D) \rightarrow \mathcal{L}(H_d)$, where the adjoint is taken with respect to the Hilbert–Schmidt scalar product on the corresponding matrix spaces [21]:

$$\langle A, \Phi(\rho) \rangle = \langle \Phi^*(A), \rho \rangle,$$

where $\rho \in \mathcal{L}(H_d)$, $A \in \mathcal{L}(H_D)$, and $\langle X, Y \rangle := \operatorname{Tr}(X^*Y)$.

We now recall the definition of the compatibility of quantum channels and refer the reader to the review [20] for further properties.

Definition 1. Consider two quantum channels $\Phi_1 : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_{d_1})$ and $\Phi_2 : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_{d_2})$ having the same input space. The pair (Φ_1, Φ_2) is said to be compatible, if there exists a joint channel $\Lambda : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_{d_1} \otimes H_{d_2})$ such that $\Phi_{1,2}$ are the marginals of Λ :

$$\forall X \in \mathcal{L}(H_d), \quad \Phi_1(X) = \operatorname{Tr}_2 \Lambda(X) \quad \text{and} \quad \Phi_2(X) = \operatorname{Tr}_1 \Lambda(X),$$

where $\operatorname{Tr}_{1,2}$ denote the partial trace operations in $\mathcal{L}(H_{d_1} \otimes H_{d_2}) \cong \mathcal{L}(H_{d_1}) \otimes \mathcal{L}(H_{d_2})$.

In the Heisenberg (dual) picture, the condition above reads

$$\forall A \in \mathcal{L}(H_{d_1}), \quad \Phi_1^*(A) = \Lambda^*(A \otimes I_{d_2}), \quad \text{and} \quad \forall B \in \mathcal{L}(H_{d_2}), \quad \Phi_2^*(B) = \Lambda^*(I_{d_1} \otimes B).$$

The (in-)compatibility of more than two channels is defined in a similar manner.

As an example, let us consider the partially depolarizing channel, which is defined as:

$$\Phi_t = t \cdot \text{id} + (1 - t)\Delta, \quad 0 \leq t \leq 1, \tag{4}$$

with $\text{id}(A) = A$ and $\Delta(A) = \text{Tr}(A)I_d/d$ for any operator A ; these quantum channels will be discussed at length in Section 7. From the *no-cloning theorem* [3,4], it follows that two copies of the identity channel (id, id) , are incompatible. On the other hand, the completely depolarizing channel Δ is compatible with any other channel. A question is the self-incompatibility of Φ_t . It is well known that the channel Φ_t is self-compatible if $0 \leq t \leq \frac{d+2}{2(d+1)}$ [26,39]. The necessary and sufficient condition for the compatibility of two different depolarizing channels Φ_s and Φ_t were shown in [40–42]:

$$t + s - \frac{2}{d} \sqrt{(1-t)(1-s)} \leq 1. \tag{5}$$

As previously discussed, quantum channel incompatibility is a key phenomenon in quantum theory, being at the heart of fundamental results in quantum information, such as the *no-cloning theorem*. In order to measure the degree of incompatibility of a given set of quantum channels, several definitions of the *robustness of incompatibility* have been considered in the literature [8,26,43]. In this section, we introduce a new such measure for a tuple of channels, which has the merit of taking into consideration the asymmetry between the channels considered. A similar definition was considered in the case of POVMs in [44,45]. We shall consider only channels acting on $\mathcal{L}(H_d)$, and we recall that Δ denotes the fully depolarizing channel $\Delta(X) = (\text{Tr } X)I_d/d$.

Definition 2. Given an N -tuple of quantum channels $\Phi := (\Phi_1, \Phi_2, \dots, \Phi_N)$, define the compatibility region of Φ as

$$\Gamma_\Phi := \left\{ s \in [0, 1]^N : \text{the channels } \left[s_i \Phi_i + (1 - s_i)\Delta \right]_{i=1}^N \text{ are compatible} \right\}.$$

Note that the definition is a relevant event in the case where the channels Φ_i are identical: $\Phi_i = \Phi$ for all $i \in [N]$, in which case we call $\Gamma_\Phi := \Gamma_\Phi$ the *self-compatibility region* (note that the dependence in N is still present since we are consider N copies of the channel Φ).

The following result is a simple exercise.

Proposition 1. For any N -tuple of quantum channels $\Phi := (\Phi_1, \Phi_2, \dots, \Phi_N)$, the set Γ_Φ is convex and closed (i.e., a convex body). We have $0 \in \Gamma_\Phi$, and, for all $i \in [N]$,

$$e_i := (0, \dots, 0, \underbrace{1}_{i\text{-th position}}, 0, \dots, 0) \in \Gamma_\Phi.$$

4. Channel Incompatibility via POVM Incompatibility

This is the main section of our paper, where we put forward a new incompatibility criterion for quantum channels in Theorem 1. Our criterion is based on an incompatibility criterion for quantum measurements (POVMs) introduced by H. Zhu and his collaborators [27,28].

Let us start by recalling the definition of compatibility (or joint measurability) of quantum measurements. First, recall that a *quantum measurement* (or *POVM*) is a k -tuple of operators $\mathbf{A} = (A_1, A_2, \dots, A_k)$, having the following two properties:

- *positivity*: the operators $A_1, \dots, A_k \in \mathcal{L}(H_d)$ are positive semidefinite;
- *normalization*: $\sum_{i=1}^k A_i = I_d$.

A POVM gives the most general form of a physical process that produces the probabilities given by the *Born rule*: when measuring a quantum system described by a density matrix ρ , one obtains the result $i \in [k]$ with probability

$$\mathbb{P}[\text{outcome } i] = \text{Tr}(\rho A_i).$$

Naturally, one can see a POVM \mathbf{A} as a quantum-to-classical channel

$$\Phi_{\mathbf{A}}(X) = \sum_{i=1}^k \text{Tr}(X A_i) |i\rangle\langle i|,$$

where $\{|i\rangle\}_{i=1}^k$ denotes the canonical basis of \mathbb{C}^k corresponding to the pointer states of the measurement apparatus.

Whether two (or more) quantum measurements can be performed simultaneously is one of the crucial questions lying at the foundations of quantum theory [1,2]. Mathematically, we have the following important definition (compare with Definition 1).

Definition 3. Two POVMs $\mathbf{A} = \{A_i\}_{i \in [k]}$ and $\mathbf{B} = \{B_j\}_{j \in [l]}$ are said to be compatible (or jointly measurable) if there exists a third POVM $\mathbf{C} = \{C_{ij}\}_{(i,j) \in [k] \times [l]}$, called joint measurement, such that

$$\begin{aligned} \forall i \in [k], \quad A_i &= \sum_{j=1}^l C_{ij}, \\ \forall j \in [l], \quad B_j &= \sum_{i=1}^k C_{ij}. \end{aligned}$$

Otherwise, the measurements \mathbf{A} and \mathbf{B} are called incompatible [46]. The compatibility of more than two measurements is defined similarly.

Quantum measurement (in-)compatibility has received a lot of attention in the literature, see, e.g., the excellent reviews [20,47], or the recent perspective on the problem focusing on the post-processing partial order [48]. Importantly for us, in [27], H. Zhu proposed a family of universal POVM incompatibility criteria based on the classical Fisher information matrix. Assume a measurement \mathbf{C} is the joint measurement of \mathbf{A}_i . According to the Fisher information data-processing inequality, the measurement-induced Fisher information matrix of \mathbf{A}_i should not exceed that of \mathbf{C} , that is to say

$$I_{\mathbf{A}_i} \leq I_{\mathbf{C}}$$

for all quantum states θ (θ is omitted in the formula above for convenience); the Fisher information matrix I was defined in Equation (1). Define $\tilde{I}_{\mathbf{A}_i} := J^{-1/2} I_{\mathbf{A}_i} J^{-1/2}$ as the metric-adjusted Fisher information. The following inequality holds for compatible measurements based on the Gill–Massar inequality (3):

$$\min \left\{ \text{Tr } H : H \geq \tilde{I}_{\mathbf{A}_i} \quad \forall i \in [N] \right\} \leq d - 1. \tag{6}$$

Otherwise, the N -tuple of measurements $(\mathbf{A}_i)_{i \in [N]}$ is incompatible. When the parameter θ (the state around which we compute the Fisher information) corresponds to the maximally mixed state $\theta = I_d / d$, inequality (6) can be rephrased as the following proposition [27,28].

Proposition 2. For a set of N measurements $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N)$ on $\mathcal{L}(H_d)$, define the operators

$$\forall i \in [N] \quad G_{\mathbf{A}_i} := \sum_{s=1}^{k_i} |A_i(s)\rangle\langle A_i(s)| / [\text{Tr}(A_i(s)) \in \mathcal{L}(H_d^{\otimes 2})],$$

where $A_i(1), A_i(2), \dots, A_i(k_i)$ are the (non-zero) effects of the POVM \mathbf{A}_i , having k_i outcomes. Consider now the quantity

$$\tau(\mathbf{A}) := \min \text{Tr } H \tag{7}$$

$$\text{s.t. } H \geq G_{\mathbf{A}_i} \quad \forall i \in [N].$$

If $\tau(\mathbf{A}) > d$, then the N -tuple of POVMs $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N)$ is incompatible.

Remark 1. Note that the function $\tau(\mathbf{A})$ satisfies two basic requirements for a good measure of (in-)compatibility: monotonicity under coarse-graining and global unitary invariance.

Remark 2. For any POVM \mathbf{A} , the associated matrix $G_{\mathbf{A}}$ is larger, in the positive semidefinite order, than the maximally entangled state

$$\omega := \frac{1}{d} \sum_{i,j=1}^d |ii\rangle\langle jj|.$$

This fact is a consequence of the important observation that the $\mathbf{A} \mapsto G_{\mathbf{A}}$ is an order morphism for the post-processing order of quantum measurements [48], and $G_{\{\mathbb{I}\}} = \omega$.

A natural question is how to capture the incompatibility of quantum channels using measurements. Let $\{|e_j\rangle\}$ and $\{|f_k\rangle\}$ be any sets of the basis of Hilbert spaces H_{d_1} and H_{d_2} , respectively. Motivated by the definition of incompatibility of quantum channels, we dedicate to research properties of the induced sets $\{\Phi_1^*(|e_j\rangle\langle e_j|)\}$ and $\{\Phi_2^*(|f_k\rangle\langle f_k|)\}$. As we consider the quantum channel is trace-preserving, thus $\{\Phi_1^*(|e_j\rangle\langle e_j|)\}$ and $\{\Phi_2^*(|f_k\rangle\langle f_k|)\}$ can be regarded as POVMs [23], that is to say,

$$\sum_j \Phi_1^*(|e_j\rangle\langle e_j|) = \sum_k \Phi_2^*(|f_k\rangle\langle f_k|) = I_d. \tag{8}$$

Lemma 1. [21] If N quantum channels $\Phi_1, \Phi_2, \dots, \Phi_N$ are compatible, then, for all orthonormal bases $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(N)}$ of \mathbb{C}^d , the corresponding POVMs

$$\mathbf{A}_s := \left[\Phi_s^*(|\mathbf{e}_i^{(s)}\rangle\langle \mathbf{e}_i^{(s)}|) \right]_{i=1}^d, \quad \forall s \in [N]$$

are compatible.

Proof. Let Λ be a joint channel for the compatible N -tuple $(\Phi_1, \Phi_2, \dots, \Phi_N)$. Clearly, $\Lambda : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d^{\otimes N})$ thus its adjoint is an unital, completely positive map

$$\Lambda^* : \mathcal{L}(H_d^{\otimes N}) \rightarrow \mathcal{L}(H_d).$$

Define operators

$$\mathbf{B} := \left[\Lambda^* \left(\bigotimes_{s=1}^N |\mathbf{e}_{i_s}^{(s)}\rangle\langle \mathbf{e}_{i_s}^{(s)}| \right) \right]_{i_1, \dots, i_N \in [d]}.$$

From the fact that Λ^* is a completely positive, unital map, we infer that \mathbf{B} is a POVM (with d^N outcomes). Let us now compute the marginals of this POVM. For some fixed $s \in [N]$ and $i_s \in [d]$, we have

$$\sum_{i_1, \dots, i_{s-1}, i_{s+1}, \dots, i_N \in [d]} B_{i_1 \dots i_N} = \Lambda^*(I_d \otimes \dots \otimes |\mathbf{e}_{i_s}^{(s)}\rangle\langle \mathbf{e}_{i_s}^{(s)}| \otimes \dots \otimes I_d) = \Phi_s^*(|\mathbf{e}_{i_s}^{(s)}\rangle\langle \mathbf{e}_{i_s}^{(s)}|) = \mathbf{A}_s(i_s), \tag{9}$$

showing that the s -th marginal of \mathbf{B} is \mathbf{A}_s . Thus \mathbf{B} is a joint measurement of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_N$, proving the claim. \square

We leave open the reciprocal question, which we formulate as a conjecture (below for two channels, although the general version, for a N -tuple, can be easily stated).

Conjecture 1. Consider two quantum channels $\Phi, \Psi : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$ such that, for all orthonormal bases $\mathbf{e} = (e_1, \dots, e_d), \mathbf{f} = (f_1, \dots, f_d)$ of \mathbb{C}^d , the POVMs

$$\left[\Phi^*(|e_i\rangle\langle e_i|) \right]_{i=1}^d \quad \text{and} \quad \left[\Psi^*(|f_j\rangle\langle f_j|) \right]_{j=1}^d$$

are compatible. Then, Φ and Ψ are compatible channels.

We now turn to the main theoretical result of our paper: a criterion for quantum channel incompatibility. Informally, one can formulate it as follows: given an N -tuple of quantum channels, if one can find an N -tuple of orthonormal bases such that the corresponding quantum measurements are incompatible, then the original N -tuple of channels must also be incompatible. Our criterion is important since there are very few useful incompatibility criteria for channel incompatibility. On the other hand, there exist quite numerous incompatibility criteria for quantum measurements, so one can turn those into criteria for channels using Lemma 1. We introduce the following important notation: to a quantum channel $\Phi : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$ and an orthonormal basis $\mathbf{e} = (e_i)_{i=1}^d$ of \mathbb{C}^d , we associate the G matrix

$$G_{\Phi, \mathbf{e}} := \sum_{i=1}^d \frac{|\Phi^*(|e_i\rangle\langle e_i|)\rangle\langle\Phi^*(|e_i\rangle\langle e_i|)|}{\text{Tr } \Phi^*(|e_i\rangle\langle e_i|)}, \tag{10}$$

which corresponds to the G matrix associated to the POVM

$$\left[\Phi^*(|e_i\rangle\langle e_i|) \right]_{i=1}^d.$$

Theorem 1. Let $\Phi_1, \Phi_2, \dots, \Phi_N : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$ be N quantum channels. If there exists orthonormal bases $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(N)}$ of \mathbb{C}^d such that the value of the semidefinite program

$$\begin{aligned} \min \quad & \text{Tr } H \\ \text{s.t.} \quad & H \geq G_{\Phi_i, \mathbf{e}^{(i)}} \quad \forall i \in [N] \end{aligned} \tag{11}$$

is strictly larger than d , then the n -tuple of channels $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_N)$ is incompatible.

Proof. The theorem follows directly from Proposition 2 and Lemma 1. \square

Remark 3. If the quantum channel Φ is unital, that is to say, $\Phi(I_d) = I_d$, the formula (10) simplifies, in the sense that the denominator is trivial:

$$\text{Tr } \Phi^*(|e_i\rangle\langle e_i|) = \langle I_d, \Phi^*(|e_i\rangle\langle e_i|) \rangle = \langle \Phi(I_d), |e_i\rangle\langle e_i| \rangle = \langle I_d, |e_i\rangle\langle e_i| \rangle = \text{Tr}|e_i\rangle\langle e_i| = 1. \tag{12}$$

This will be the case for most of the examples we shall discuss in what follows.

It is important at this point to note that the incompatibility criterion we put forward in the result above is formulated as an SDP (semidefinite program). The usual way of formulating the compatibility of a tuple of quantum channels is also an SDP: one looks for a joint channel, a problem that can be formulated as an SDP thanks to the Choi formalism. However, let us compare the size of the SDPs:

- channel compatibility: the joint channel has a Choi matrix of size d^{N+1}
- incompatibility criterion from Theorem 1: the variable H has size d^2 .

Note also that one has, in both cases, N constraints of size d^2 . Therefore, we obtain a dramatic reduction in the size of the SDP at the price of having only a necessary compatibility condition (i.e., an incompatibility criterion).

There is, however, a situation when SDP (11) simplifies and can be analytically solved. This is when the matrices G corresponding to the channel are orthogonal (up to the maximally entangled state ω). We formalize this observation below.

Proposition 3. Consider N quantum channels $\Phi_1, \Phi_2, \dots, \Phi_N : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$ and orthonormal bases $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(N)}$ such that, for all $i, j \in [N], i \neq j$,

$$G_{\Phi_i, \mathbf{e}^{(i)}} - \omega \perp G_{\Phi_j, \mathbf{e}^{(j)}} - \omega.$$

Then, the value of SDP (11) is

$$1 - N + \sum_{i=1}^N \text{Tr} G_{\Phi_i, \mathbf{e}^{(i)}}.$$

Proof. Taking into consideration Remark 2, one can rewrite SDP (11) by subtracting ω everywhere:

$$1 + \min \quad \text{Tr} \tilde{H} \tag{13}$$

$$\text{s.t.} \quad \tilde{H} \geq G_{\Phi_i, \mathbf{e}^{(i)}} - \omega \quad \forall i \in [N]$$

where $\tilde{H} = H - \omega$. Using the hypothesis and noting that the matrices $G_{\Phi_i, \mathbf{e}^{(i)}} - \omega$ are all positive semidefinite, any feasible \tilde{H} must satisfy

$$\tilde{H} \geq \sum_{i=1}^N G_{\Phi_i, \mathbf{e}^{(i)}} - \omega.$$

Hence, the optimal \tilde{H} achieves equality above, and the conclusion follows. \square

This idea will be used in Sections 5 and 7 to obtain (analytical) incompatibility criteria for important classes of quantum channels.

As an example, let us work out the G matrix for the identity channel $\text{id}(X) = X$.

$$G_{\text{id}, \mathbf{e}} = \sum_{i=1}^d \left| |e_i\rangle\langle e_i| \right\rangle \left\langle |e_i\rangle\langle e_i| \right| = \sum_{i=1}^d |e_i \otimes \bar{e}_i\rangle\langle e_i \otimes \bar{e}_i| =: Z_{\mathbf{e}}. \tag{14}$$

The matrix $Z_{\mathbf{e}}$ will play an important role in what follows. We gather some useful facts about it below. Recall that two orthonormal bases \mathbf{e}, \mathbf{f} of \mathbb{C}^d are called *unbiased* if

$$\forall i, j \in [d], \quad |\langle e_i, f_j \rangle| = \frac{1}{\sqrt{d}}.$$

Lemma 2. For any orthonormal basis \mathbf{e} , we have

$$\langle Z_{\mathbf{e}}, \omega \rangle = 1.$$

Moreover, if \mathbf{e} and \mathbf{f} are unbiased orthonormal bases, then

$$\langle Z_{\mathbf{e}}, Z_{\mathbf{f}} \rangle = 1.$$

Let us close this section by mentioning how the matrices G behave when mixing noise into a quantum channel Φ . This property will be very useful in what follows when investigating the compatibility robustness of some classes of quantum channels.

Lemma 3. Given a quantum channel $\Phi : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$, consider its noisy version

$$\Phi_t := t\Phi + (1 - t)\Delta,$$

where $\Delta(X) = (\text{Tr } X)I/d$ is the completely depolarizing channel, and $t \in [0, 1]$ is some parameter. Then, for any orthonormal basis \mathbf{e} ,

$$G_{\Phi_t, \mathbf{e}} = t^2 G_{\Phi, \mathbf{e}} + (1 - t^2)\omega,$$

where ω is the maximally entangled state (note that $\omega = G_{\Delta, \mathbf{e}}$).

Proof. This can either be proven directly using formula (10) or by using the corresponding result for POVMs, see, e.g., ([48] Proposition 5.3). \square

5. Incompatibility of Two Schur Channels

As the first application of our newly introduced incompatibility criterion for quantum channels, we consider *Schur channels*. A Schur map is a linear map of the form

$$\Sigma_B(X) = B \circ X,$$

where B is a $d \times d$ complex matrix and \circ denotes the Hadamard product. The map Σ_B is completely positive if and only if matrix B is positive semidefinite, and it is trace-preserving if the diagonal of B is the identity: $B_{ii} = 1$ for all i . If both conditions are satisfied, we call map Σ_B a Schur channel (sometimes also called a Schur multiplier), see [49–52]. Schur channels have received a lot of attention in operator algebra and quantum information theory, and they contain the identity channel $\text{id} = \Sigma_I$, where I is the all 1s matrix, and the dephasing channel (the conditional expectation on the diagonal sub-algebra) $\text{diag} = \Sigma_I$ as examples.

For Schur channel Σ_B , we have

$$G_{\Sigma_B, \mathbf{e}} = |\bar{B}\rangle\langle \bar{B}| \circ Z_{\mathbf{e}},$$

for any orthonormal basis \mathbf{e} (recall the form of the matrix Z from (14)). If \mathbf{e} is the canonical basis, we have

$$G_{\Sigma_B, \text{can}} = Z_{\text{can}}.$$

Consider now a basis \mathbf{f} that is unbiased with respect to the canonical basis; in other words, the elements of \mathbf{f} form the columns of a Hadamard matrix U : $|f_j\rangle = U|j\rangle$ for all j . An important example of such a basis is the *Fourier basis*:

$$f_j(s) = \exp(2\pi i/d)^{js}, \quad \forall j, s \in [d].$$

Lemma 4. If B and C are two positive semidefinite matrices with unit diagonal, and can and \mathbf{f} are unbiased, then

$$G_{\Sigma_B, \text{can}} - \omega \perp G_{\Sigma_C, \mathbf{f}} - \omega.$$

Proof. Expanding the scalar product and using Lemma 2, we need to show that

$$\langle Z_{\text{can}}, |\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}} \rangle = \langle \omega, |\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}} \rangle.$$

Let us work out the left-hand-side:

$$\langle Z_{\text{can}}, |\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}} \rangle = \langle |C\rangle\langle C| \circ Z_{\text{can}}, Z_{\mathbf{f}} \rangle = \langle Z_{\text{can}}, Z_{\mathbf{f}} \rangle = 1,$$

where we have used Lemma 2 and the fact that $|C\rangle\langle C| \circ Z_{\text{can}} = Z_{\text{can}}$, which follows from the fact that C has unit diagonal. The right-hand-side can be dealt with in the same manner. \square

For a $d \times d$ matrix B with unit diagonal, define the real parameter $\beta(B)$ as follows:

$$\beta(B) := \frac{1}{d-1} \left(\frac{1}{d} \sum_{i,j=1}^d |B_{ij}|^2 - 1 \right) = \frac{1}{d-1} \sum_{i \neq j \in [d]} |B_{ij}|^2. \tag{15}$$

Recall that the torus \mathbb{T}^d is the set of vectors $b \in \mathbb{C}^d$ with $|b_i| = 1$ for all $i = 1, \dots, d$.

Lemma 5. *If B is a $d \times d$ positive semidefinite matrix with unit diagonal, then*

$$0 \leq \beta(B) \leq 1,$$

with $\beta(B) = 0$ iff $B = I$ and $\beta(B) = 1$ iff $B = |b\rangle\langle b|$ for a vector $b \in \mathbb{T}^d$.

Proof. The non-negativity of β , as well as the equality case, follows directly from definition (15). For the upper bound, use the ordering of the 1, 2-Schatten norms of B to write

$$\sum_{i,j=1}^d |B_{ij}|^2 = \|B\|_2^2 \leq \|B\|_1^2 = (\text{Tr } B)^2 = d^2,$$

proving the inequality. Equality holds if B is rank one, which, together with the condition on the diagonal, proves that $B = |b\rangle\langle b|$ for some vector $b \in \mathbb{T}^d$. \square

We can now, using Theorem 1, provide a new incompatibility criterion for Schur channels.

Theorem 2. *Consider two positive semidefinite matrices B and C with unit diagonal, and the corresponding depolarized Schur channels*

$$\begin{aligned} \Phi_s(X) &= s\Sigma_B(X) + (1-s)\Delta(X) = sB \circ X + (1-s)(\text{Tr } X) \frac{I}{d} \\ \Psi_t(X) &= t\Sigma_C(X) + (1-t)\Delta(X) = tC \circ X + (1-t)(\text{Tr } X) \frac{I}{d}. \end{aligned}$$

If $s^2 + \beta(C)t^2 > 1$, then the channels Φ_s and Ψ_t are incompatible. We have, thus, an upper bound for the compatibility region from Definition 2:

$$\Gamma_{\Phi, \Psi} \subseteq \{(s, t) \in [0, 1]^2 : s^2 + \beta(C)t^2 \leq 1 \text{ and } \beta(B)s^2 + t^2 \leq 1\}, \tag{16}$$

where $\Phi := \Phi_1$ and $\Psi := \Psi_1$.

Proof. The proof is an application of Theorem 1. To start, let us compute the G matrices associated with these channels, taking, respectively, the canonical basis can , and any unbiased base \mathbf{f} (e.g., the Fourier basis); this choice is inspired by Lemma 4 and Proposition 3. Applying these results, as well as the scaling Lemma 3, we have

$$\begin{aligned} G_{\Phi_s, \text{can}} &= s^2 G_{\Sigma_B, \text{can}} + (1-s^2)\omega = \omega + s^2(Z_{\text{can}} - \omega) \\ G_{\Psi_t, \mathbf{f}} &= t^2 G_{\Sigma_C, \mathbf{f}} + (1-t^2)\omega = \omega + t^2(|\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}} - \omega). \end{aligned}$$

Hence, the value of SDP (11) is given by (see Proposition 3)

$$1 - 2 + \text{Tr } G_{\Phi_s, \text{can}} + \text{Tr } G_{\Psi_t, \mathbf{f}} = s^2(d-1) + 1 - t^2 + t^2 \text{Tr}[|\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}}].$$

We can evaluate

$$\text{Tr}[|\bar{C}\rangle\langle \bar{C}| \circ Z_{\mathbf{f}}] = \frac{1}{d} \sum_{i,j=1}^d |C_{ij}|^2,$$

and, using parameter $\beta(C)$ from (15), the incompatibility criterion reads

$$s^2 + \beta(C)t^2 > 1,$$

which is the first claim. The second claim follows by swapping the roles of the unbiased bases **can** and **f**. \square

Remark 4. One can not easily generalize the result above to more than two Schur channels. This is due to the fact that one has to fix one of the bases in Theorem 1 to be the canonical basis. This is due to the fact that the Hadamard product used to define Schur channels is adapted to the canonical basis. We leave the generalization of the result (and method) above for three or more Schur channels open.

We compare, in Figure 1, the criterion from the theorem above with the actual incompatibility thresholds for some particular Schur channels, concluding that the incompatibility criterion is close to being exact.

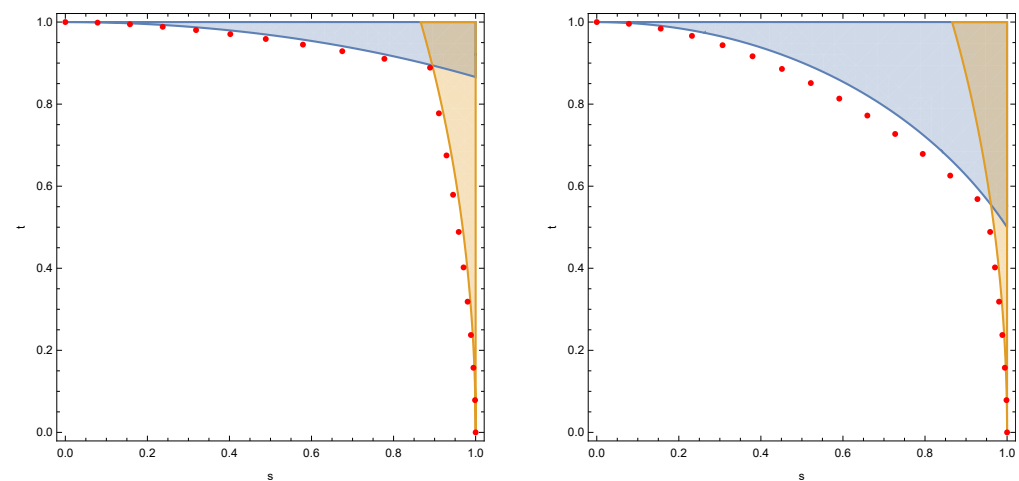


Figure 1. The Fisher information-based incompatibility criterion for Schur channels. In the left panel, we consider two noisy copies of the Schur channel corresponding to $B = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$. In the right panel, we consider noisy versions of Σ_B and Σ_C , where $C = \begin{bmatrix} 1 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1 \end{bmatrix}$. Shaded regions correspond to the conditions from (16), while the red dots correspond to the maximally compatible channels in the respective directions.

6. Channel Assemblages

The way in which several quantum measurement and quantum channels can be incompatible has been studied extensively in the literature [53–56]. The kind of (in-)compatibility structures that can be found in nature, and their relation to other important manifestations of non-locality (such as Bell inequality violations), is clearly a crucial question at the foundation of quantum theory.

Let $\{\Phi_i\}_{i=1}^N$ be a channel assemblage that is an N -tuple of quantum channels. If $\{\Phi_i\}_{i=1}^N$ are incompatible, there does not exist a joint quantum channel for *all* the N channels. However, a joint channel may exist when we consider a certain subset of $\{\Phi_i\}_{i=1}^N$. In other words, some subsets of $\{\Phi_i\}_{i=1}^N$ may be compatible, even though the whole set is incompatible. Obviously, if the whole set of N channels is compatible, then so is any subset: if Λ is a joint channel for the N -tuple. Then, for any subset $S \subseteq [N]$ of the channels, Λ_S , the marginal of Λ corresponding to the output indices in S

$$\Lambda_S : \mathcal{L}(H_d) \rightarrow \mathcal{L}\left(\bigotimes_{i \in S} H_d^{(i)}\right)$$

is a joint channel for $\{\Phi_i\}_{i \in S}$; above, we identify the different copies of the output space $\mathcal{L}(H_d)$ by a superscript. Therefore, it is significant to classify the incompatibility of subsets for a given quantum channel assemblage. A K -subset of $[N]$ is simply a subset $S \subseteq [N]$ of cardinality $|S| = K$.

Definition 4. Consider a quantum channel assemblage $\Phi = \{\Phi_i\}_{i=1}^N$ and $1 \leq K \leq N$ an integer. The N -tuple Φ is called:

- (N, K) -compatible if all K -subsets of Φ are compatible.
- (N, K) -incompatible if at least one K -subset of Φ is incompatible.
- (N, K) -strong incompatible if all K -subsets of Φ are incompatible.
- $(N, K + 1)$ -genuinely incompatible if it is (N, K) -compatible and $(N, K + 1)$ -incompatible.
- $(N, K + 1)$ -genuinely strong incompatible if it is (N, K) -compatible and $(N, K + 1)$ -strong incompatible.

Note that the assemblage $\{\Phi_i\}_{i=1}^N$ is compatible if and only if it is (N, N) -compatible. The previous definition is strongly inspired by the one from ([56] Section 2) in the case of POVMs. The incompatibility criterion from Theorem 1 can be readily adapted to the previous definition by considering subsets of the PSD constraints in (11). We restate it here for the convenience of the reader. We shall apply it in the next section for assemblages of depolarizing channels.

Theorem 3. Consider an assemblage $\Phi = \{\Phi_i\}_{i=1}^N$ of quantum channels acting on $\mathcal{L}(H_d)$. For a K -subset S of $[N]$, and K orthonormal bases $\mathbf{e} = (\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(K)})$ of \mathbb{C}^d , define the value of the following semidefinite program

$$\begin{aligned} \text{val}(\Phi, S, \mathbf{e}) &:= \min \quad \text{Tr } H \\ \text{s.t.} \quad &H \geq G_{\Phi_i, \mathbf{e}^{(i)}} \quad \forall i \in S. \end{aligned} \tag{17}$$

If there exists at least one $S \in [N]$ and a K -tuple of orthonormal bases \mathbf{e} such that $\text{val}(\Phi, S, \mathbf{e}) > d$, then the assemblage Φ is (N, K) -incompatible. Moreover, if for all K -subsets $S \subseteq [N]$, there exists a K -tuple of bases \mathbf{e}_S such that $\text{val}(\Phi, S, \mathbf{e}_S) > d$, the assemblage Φ is (N, K) -strong incompatible.

7. Assemblages of Depolarizing Channels

In this section, we address the (in-)compatibility properties of an N -tuple of partially depolarizing channels, using the Fisher information-based criterion from Theorem 1. Recall that the *partially depolarizing channel* is the linear map $\Phi_t : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)$ given by

$$\Phi_t = t \cdot \text{id} + (1 - t)\Delta, \tag{18}$$

where id is the identity channel $\text{id}(X) = X$ and Δ is the fully depolarizing channel $\Delta(X) = (\text{Tr } X)I/d$. The parameter $t \in [0, 1]$ interpolates between the identity channel and the fully depolarizing channel.

In this section, we shall study the incompatibility of N partially depolarizing channels $\{\Phi_{t_i}^j\}_{i=1}^N$, for some fixed parameters $t_1, t_2, \dots, t_N \in [0, 1]$, with the help of the criterion from Theorem 1. To do so, let us first compute the G matrices of depolarizing channels, which are just noisy versions of the identity channel. Recall from Equation (14) that, for the identity channel, we have, for an arbitrary basis \mathbf{e} ,

$$Z_{\mathbf{e}} = G_{\text{id}, \mathbf{e}} = \sum_{i=1}^d |e_i \otimes \bar{e}_i\rangle\langle e_i \otimes \bar{e}_i|,$$

where \bar{e}_i denotes the (entrywise) complex conjugate of the vector e_i . Hence, by Lemma 3, we have

$$G_{\Phi_t, \mathbf{e}} = t^2 Z_{\mathbf{e}} + (1 - t^2)\omega.$$

As in Section 5, we are going to use the orthogonality of the G matrices in order to put forward analytical incompatibility criteria for depolarizing channels (see Proposition 3). To do so, recall that the Z_e matrices have tractable scalar products for unbiased bases. As it turns out, mutually unbiased bases [57] will play an important role in what follows. Let D_d be the maximal cardinality of a set of mutually unbiased bases of \mathbb{C}^d . It is known that $3 \leq D_d \leq d + 1$ [58–60]. The upper bounds are attained for all dimensions d , which are prime powers; whether it is always attained is an important open problem in quantum information theory, even the case $d = 6$ being undecided.

We now state the main result of this section, an incompatibility criterion for depolarizing channels.

Proposition 4. *Let N be an integer such that $N \leq D_d$, the maximal number of mutually unbiased bases of \mathbb{C}^d . Consider N depolarizing channels $\Phi_{t_1}, \dots, \Phi_{t_N}$, where $t_1, \dots, t_N \in [0, 1]$ are noise parameters. If*

$$t_1^2 + t_2^2 + \dots + t_N^2 > 1 \tag{19}$$

then the N depolarizing channels Φ_{t_i} are incompatible.

Proof. Since the number of channels we consider is smaller than D_d , we can choose N mutually unbiased bases $e^{(1)}, \dots, e^{(N)}$. SDP (11) reads

$$\begin{aligned} \min \quad & \text{Tr } H \\ \text{s.t.} \quad & H \geq t_i^2 Z_{e^{(i)}} + (1 - t_i^2)\omega \quad \forall i \in [N]. \end{aligned}$$

Proposition 3 applies, so the value of the SDP above is

$$\min \text{Tr } H = 1 + (d - 1) \sum_{i=1}^N t_i^2.$$

Hence, if condition (19) holds, by Theorem 1, the N quantum depolarizing channels $\Phi_{t_1}, \dots, \Phi_{t_N}$ are incompatible. \square

As mentioned in the introduction, the compatibility of depolarizing channels is equivalent to approximate quantum cloning: how much noise one needs to add to N copies of the identity channel to render them compatible. In Figure 2, we present the relative performance of the criterion from Proposition 4, with the true values of the noise parameters for $1 \rightarrow 2$ asymmetric approximate quantum clonings from Equation (5).

We can specialize the result above to assemblages of depolarizing channels in the spirit of Definition 4.

Corollary 1. *Consider N partially depolarizing channels $\{\Phi_{t_i}\}_{i=1}^N$ acting on $\mathcal{L}(H_d)$ and let $K \leq \min(N, D_d)$ be an integer. If there exists a subset $S \subseteq [N]$ of cardinality K such that*

$$\sum_{i \in S} t_i^2 > 1,$$

then the channels are (N, K) -incompatible. Moreover, if for every subset $S \subseteq [N]$ of cardinality K the condition above holds, the channels are (N, K) -strongly incompatible.

Note that in the statement above, we do not require that the number N of channels must be smaller than the number of mutually unbiased bases in the corresponding Hilbert space; this is required only of the parameter K . This criterion might thus be useful in situations where one has a large number of channels.

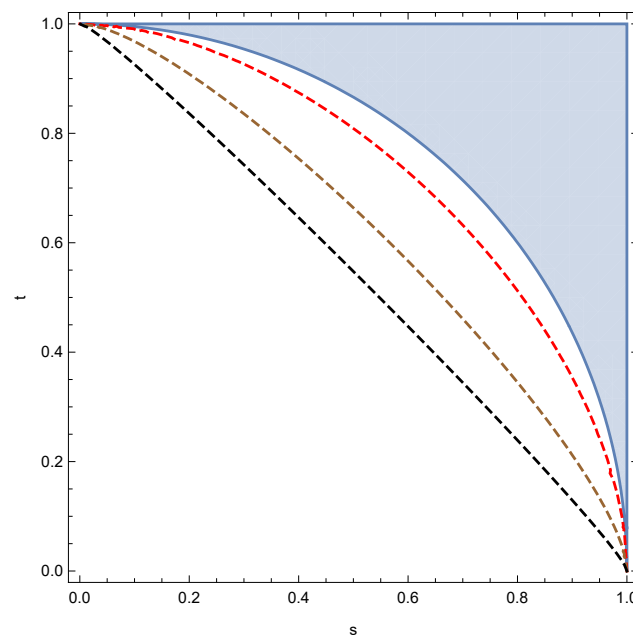


Figure 2. Comparing the incompatibility criterion from Proposition 4 (filled region) with the incompatibility thresholds from Equation (5) (dashed curves) for different values of d : $d = 2$ (red curve), $d = 5$ (brown curve), $d = 20$ (black curve).

We end this section by a similar corollary, in the setting where the channels are identical.

Corollary 2. *If N, K are integers such that $K \leq \min(N, D_d)$, then the partially depolarizing channel Φ_t from Equation (18) is (N, K) -self-(strong) incompatible as soon as $t > 1/\sqrt{K}$.*

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