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Central Submonads and Notions of Computation

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Abstract

The notion of "centre" has been introduced for many algebraic structures in mathematics. A notable example is the centre of a monoid which always determines a commutative submonoid. Monads (in category theory) can be seen as generalisations of monoids and in this paper we show how the notion of centre may be extended to strong monads acting on symmetric monoidal categories. We show that the centre of a strong monad \mathcal{T} , if it exists, determines a commutative submonad \mathcal{Z} of \mathcal{T} , such that the Kleisli category of \mathcal{Z} is isomorphic to the premonoidal centre (in the sense of Power and Robinson) of the Kleisli category of \mathcal{T} . We provide three equivalent conditions which characterise the existence of the centre of \mathcal{T} . While not every strong monad admits a centre, we show that every strong monad on well-known naturally occurring categories does admit a centre, thereby showing that this new notion is ubiquitous. We also provide a computational interpretation of our ideas which consists in giving a refinement of Moggi's monadic metalanguage. The added benefit is that this allows us to immediately establish a large class of contextually equivalent terms for monads that admit a non-trivial centre by simply looking at the richer syntactic structure provided by the refinement.

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1 Introduction

The importance of monads in programming semantics has been demonstrated in seminal work by Moggi [5, 6]. The main idea is that monads allow us to introduce computational effects (e.g., state, input/output, recursion, probability, continuations) into pure type systems in a controlled way. This idea has been very successful and monads have been incorporated as a programming abstraction into the design of modern programming languages such as Haskell and others (see [1]).

In most cases, especially when considering lambda calculi, we are interested in *strong* monads, which are monads that satisfy additional coherence conditions with respect to the monoidal structure of the base category. A subclass of these monads are the *commutative* monads, which satisfy even stronger coherence conditions with respect to the monoidal structure. In general, the Kleisli category of a strong monad has a canonical *premonoidal* structure [7], which is slightly weaker than a monoidal one, as the name suggests. In fact, this premonoidal structure is a monoidal one iff the monad in question is commutative [7]. Furthermore, this difference has an important computational interpretation: given a commutative monad that describes some computational effect, if two monadic operations may be performed in any order, then the overall computational result will be the same regardless of the order of monadic sequencing. However, this is *not* true, in general, for strong monads: doing the monadic operation f followed by g is not generally the same as doing the monadic operation g followed by f , even when both orders of applications are admissible. Indeed,

many monads of interest in computer science are strong, but not commutative, so one must be aware of this fact which prevents us from incorporating certain program transformations.

Nevertheless, just because a monad is strong, but not commutative, it does not mean that *every* monadic operation does not satisfy the above commutative property with respect to any other monadic operation. In fact, it is very useful to determine which monadic operations do satisfy these additional commutative properties, because this allows us to establish a greater class of contextually equivalent programs (even when the monad is not commutative) and guarantee the safety of certain program transformations. In this paper, we approach this problem by taking inspiration from the algebraic notion of "centre", which we extend to strong monads.

Our Contributions. The notion of "centre" is a simple algebraic idea that has been formulated for many algebraic structures in mathematics. For example, the centre of a monoid is always a *commutative* submonoid of the original one. Strong monads in category theory can be seen as a generalisation of monoids (on sets): if M is a monoid, then the functor $\mathcal{M} \stackrel{\text{def}}{=} (M \times -): \mathbf{Set} \rightarrow \mathbf{Set}$ can be equipped with a canonical monad structure that makes it the *free monad* induced by the monoid M . In fact, \mathcal{M} is always a strong monad on \mathbf{Set} and it is commutative iff M is a commutative monoid. Therefore, a natural question to ask is how to extend the notion of centre from monoids to strong monads¹.

In this paper we show that for every monad on \mathbf{Set} (which is necessarily strong), we can form its *central submonad* which is always commutative and we show that this generalises the centre of monoids. More generally, we introduce the notion of centre for strong monads \mathcal{T} acting on a symmetric monoidal category \mathbf{C} . We provide three equivalent conditions for the existence of this centre and we show that, if the centre exists, then this gives a canonical commutative submonad \mathcal{Z} of \mathcal{T} . We say that \mathcal{Z} is the *central submonad* of \mathcal{T} , which we justify by showing that the Kleisli category of \mathcal{Z} is isomorphic (in a canonical sense) to the premonoidal centre (in the sense of Power and Robinson [7]) of the Kleisli category of \mathcal{T} . The central submonad \mathcal{Z} has interesting computational properties that address the problem we described above: monadic operations that are identified by \mathcal{Z} commute with any other monadic operation of the larger monad \mathcal{T} .

It is possible to describe a strong monad that does not admit a centre. We constructed one such monad by defining a category specifically for this purpose. However, for most naturally occurring categories that we are aware of, every strong monad acting on them does admit a centre and we give many examples in this paper. Furthermore, we are not aware of any other naturally occurring monad that has been described in the literature and that does not admit a centre. Because of this, we believe the new notion is widely applicable.

We also provide a computational interpretation of our ideas. This consists in giving a refinement of Moggi's monadic metalanguage. The new system has an additional type that keeps track of central terms. In particular, the unit of the monad is central, some of the monadic constants may also be central (depending on the choice of monad), and central terms are closed under monadic sequencing. We describe a mathematical semantics for this system that we conjecture is sound and adequate (proofs are work-in-progress and will be finished soon). This allows us to easily establish a large class of contextually equivalent terms for monads that are strong and which admit a non-trivial centre. We give several examples of such monads in this paper and we believe that it should be possible to discover more in the future.

¹ We do not see any satisfactory way to introduce the notion of centre to monads which are *not* strong and for which we do not presuppose any monoidal structure on the base category.

2 Background

We start by introducing some background on strong and commutative monads and their premonoidal structure. We also use this as opportunity to fix notation.

2.1 Strong and Commutative Monads

► **Definition 1** (Monad). *A monad over a category \mathbf{C} is an endofunctor \mathcal{T} equipped with two natural transformations $\eta : \text{Id} \Rightarrow \mathcal{T}$ and $\mu : \mathcal{T}^2 \Rightarrow \mathcal{T}$ such that the following diagrams:*

$$\begin{array}{ccc}
 \mathcal{T}^3 X & \xrightarrow{\mu_{\mathcal{T}X}} & \mathcal{T}^2 X \\
 \mathcal{T}\mu_X \downarrow & & \downarrow \mu_X \\
 \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{T}X & \xrightarrow{\mathcal{T}\eta_X} & \mathcal{T}^2 X \\
 \eta_{\mathcal{T}X} \downarrow & \searrow & \downarrow \mu_X \\
 \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X
 \end{array}$$

commute.

► **Definition 2** (Strong Monad). *A strong monad over a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda)$ is a monad (\mathcal{T}, η, μ) equipped with a natural transformation $\tau_{X,Y} : X \otimes \mathcal{T}Y \rightarrow \mathcal{T}(X \otimes Y)$, called strength, such that the following diagrams:*

$$\begin{array}{ccc}
 I \otimes \mathcal{T}X & \xrightarrow{\tau_{I,X}} & \mathcal{T}(I \otimes X) \\
 \lambda_{\mathcal{T}X} \searrow & & \downarrow \mathcal{T}\lambda_X \\
 & & \mathcal{T}X
 \end{array}
 \qquad
 \begin{array}{ccc}
 (W \otimes X) \otimes \mathcal{T}Y & \xrightarrow{\tau_{W \otimes X, Y}} & \mathcal{T}((W \otimes X) \otimes Y) \\
 \alpha_{W,X,\mathcal{T}Y} \downarrow & & \downarrow \mathcal{T}\alpha_{W,X,Y} \\
 W \otimes (X \otimes \mathcal{T}Y) & \xrightarrow{W \otimes \tau_{X,Y}} & W \otimes \mathcal{T}(X \otimes Y) \xrightarrow{\tau_{W,X \otimes Y}} & \mathcal{T}(W \otimes (X \otimes Y))
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{X \otimes \eta_Y} & X \otimes \mathcal{T}Y \\
 \eta_{X \otimes Y} \searrow & & \downarrow \tau_{X,Y} \\
 & & \mathcal{T}(X \otimes Y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes \mathcal{T}^2 Y & \xrightarrow{\tau_{X,\mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \xrightarrow{\mathcal{T}\tau_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
 X \otimes \mu_Y \downarrow & & \downarrow \mu_{X \otimes Y} \\
 X \otimes \mathcal{T}Y & \xrightarrow{\tau_{X,Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

commute.

Next, we recall the definition of a commutative monad which is of central importance in this paper.

► **Definition 3** (Commutative Monad). *Given a strong monad \mathcal{T} on a symmetric monoidal category, the costrength $\tau'_{X,Y} : \mathcal{T}X \otimes Y \rightarrow \mathcal{T}(X \otimes Y)$ is given by $\tau'_{X,Y} \stackrel{\text{def}}{=} \mathcal{T}(\gamma_{Y,X}) \circ_{\mathcal{T}Y,X} \circ \gamma_{\mathcal{T}X,Y}$.*

Then, \mathcal{T} is said to be commutative if the following diagram:

$$\begin{array}{ccccc}
 \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
 \tau'_{X,\mathcal{T}Y} \downarrow & & & & \downarrow \mu_{X \otimes Y} \\
 \mathcal{T}(X \otimes \mathcal{T}Y) & \xrightarrow{\mathcal{T}\tau_{X,Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y)
 \end{array} \tag{1}$$

commutes for every choice of objects X and Y .

The appropriate notion of morphism between two strong monads is given by our next definition.

► **Definition 4** (Morphism of Strong Monads [2]). *Given two strong monads $(\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}}, \tau^{\mathcal{T}})$ and $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}, \tau^{\mathcal{P}})$ over a category \mathbf{C} , we say that a morphism of strong monads is a natural transformation $\iota : \mathcal{T} \Rightarrow \mathcal{P}$ that makes the following diagrams:*

$$\begin{array}{ccccc}
 \begin{array}{ccc} X & & \\ \eta_X^{\mathcal{T}} \swarrow & & \searrow \eta_X^{\mathcal{P}} \\ \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X \end{array} & \begin{array}{ccc} \mathcal{T}^2 X & \xrightarrow{\iota_{\mathcal{T}X}} \mathcal{P}\mathcal{T}X & \xrightarrow{\mathcal{P}\iota_X} \mathcal{P}^2 X \\ \mu_X^{\mathcal{T}} \downarrow & & \downarrow \mu_X^{\mathcal{P}} \\ \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X \end{array} & \begin{array}{ccc} A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}^{\mathcal{T}}} & \mathcal{T}(A \otimes B) \\ A \otimes \iota_B \downarrow & & \downarrow \iota_{A \otimes B} \\ A \otimes \mathcal{P}B & \xrightarrow{\tau_{A,B}^{\mathcal{P}}} & \mathcal{P}(A \otimes B) \end{array}
 \end{array}$$

commute.

It is easy to see that strong monads over a symmetric monoidal category \mathbf{C} and strong monad morphisms between them form a category which we denote by writing $\mathbf{StrMnd}(\mathbf{C})$.

► **Definition 5** (Submonad). *In the situation of Definition 4, if ι is a monomorphism in $\mathbf{StrMnd}(\mathbf{C})$, then \mathcal{T} is said to be a strong submonad of \mathcal{P} and ι is said to be a submonad morphism.*

The submonad relation induces an embedding between the Kleisli categories of the corresponding monads that we recall below.

► **Definition 6** (Kleisli category). *Given a monad (\mathcal{T}, η, μ) over a category \mathbf{C} , the Kleisli category $\mathbf{C}_{\mathcal{T}}$ of \mathcal{T} is the category whose objects are the same as those of \mathbf{C} , but whose morphisms are given by $\mathbf{C}_{\mathcal{T}}[X, Y] = \mathbf{C}[X, \mathcal{T}Y]$. Composition in $\mathbf{C}_{\mathcal{T}}$ is given by $g \odot f \stackrel{\text{def}}{=} \mu_Z \circ \mathcal{T}g \circ f$ where $f : X \rightarrow \mathcal{T}Y$ and $g : Y \rightarrow \mathcal{T}Z$. The identity at X is given by the monadic unit $\eta_X : X \rightarrow \mathcal{T}X$.*

► **Proposition 7.** *If $\iota : \mathcal{T} \Rightarrow \mathcal{P}$ is a submonad morphism, the functor $\mathcal{I} : \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{P}}$ which is defined by $\mathcal{I}(X) = X$ on objects and on morphisms by $\mathcal{I}(f : X \rightarrow \mathcal{T}Y) = \iota_Y \circ f : X \rightarrow \mathcal{P}Y$ is an embedding of categories.*

The functor \mathcal{I} described in the above proposition is the canonical embedding of $\mathbf{C}_{\mathcal{T}}$ into $\mathbf{C}_{\mathcal{P}}$ induced by the submonad morphism ι .

2.2 Premonoidal Structure of Strong Monads

Let \mathcal{T} be a strong monad on a symmetric monoidal category (\mathbf{C}, I, \otimes) . Then, the Kleisli category $\mathbf{C}_{\mathcal{T}}$ does *not* necessarily have a canonical monoidal structure. However, it does have a canonical *premonoidal structure* as shown by Power and Robinson [7]. In fact, they show that this premonoidal structure is monoidal iff the monad \mathcal{T} is commutative. Next, we briefly recall the premonoidal structure of \mathcal{T} as outlined by Power and Robinson.

For every objects X and Y of $\mathbf{C}_{\mathcal{T}}$, their tensor product $X \otimes Y$ is also an object of $\mathbf{C}_{\mathcal{T}}$. But, the monoidal product \otimes of \mathbf{C} does not necessarily induce a monoidal functor on $\mathbf{C}_{\mathcal{T}}$. However, by using the strength and the costrength of \mathcal{T} , we can define two families of functors as follows:

- for any object X , a functor $(- \otimes_l X): \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{T}}$ whose action on objects sends Y to $Y \otimes X$, and sends $f: Y \rightarrow \mathcal{T}Z$ to $\tau'_{Z,X} \circ (f \otimes X): Y \otimes X \rightarrow \mathcal{T}(Z \otimes X)$;
- for any object X , a functor $(X \otimes_r -): \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{T}}$ whose action on objects sends Y to $X \otimes Y$, and sends $f: Y \rightarrow \mathcal{T}Z$ to $\tau_{X,Z} \circ (X \otimes f): X \otimes Y \rightarrow \mathcal{T}(X \otimes Z)$.

This categorical data satisfies the axioms and coherence properties of *premonoidal categories* as outlined by Power and Robinson [7], but which we omit here for brevity. We wish to note that in a premonoidal category, $f \otimes_l X'$ and $X \otimes_r g$ do not always commute. This leads us to the next definition, which plays a crucial role in the theory of premonoidal categories.

► **Definition 8** (Central morphism [7]). *Given a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on a symmetric monoidal category (\mathbf{C}, I, \otimes) , we say that a morphism $f: X \rightarrow Y$ in $\mathbf{C}_{\mathcal{T}}$ is central if for any morphism $f': X' \rightarrow Y'$ in $\mathbf{C}_{\mathcal{T}}$, the following diagram:*

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{f \otimes_l X'} & Y \otimes X' \\ X \otimes_r f' \downarrow & & \downarrow Y \otimes_r f' \\ X \otimes Y' & \xrightarrow{f \otimes_l X'} & Y \otimes Y' \end{array}$$

commutes in $\mathbf{C}_{\mathcal{T}}$; or equivalently, the following diagram:

$$\begin{array}{ccccccc} X \otimes X' & \xrightarrow{f \otimes X'} & \mathcal{T}Y \otimes X' & \xrightarrow{\tau'_{Y,X'}} & \mathcal{T}(Y \otimes X') & \xrightarrow{\mathcal{T}(Y \otimes f')} & \mathcal{T}(Y \otimes \mathcal{T}Y') \\ X \otimes f' \downarrow & & & & & & \downarrow \mathcal{T}\tau_{Y,Y'} \\ X \otimes \mathcal{T}Y' & & & & & & \mathcal{T}^2(Y \otimes Y') \\ \tau_{X,Y'} \downarrow & & & & & & \downarrow \mu_{Y \otimes Y'} \\ \mathcal{T}(X \otimes Y') & \xrightarrow{\mathcal{T}(f \otimes Y')} & \mathcal{T}(\mathcal{T}Y \otimes Y') & \xrightarrow{\mathcal{T}\tau'_{Y,Y'}} & \mathcal{T}^2(Y \otimes Y') & \xrightarrow{\mu_{Y \otimes Y'}} & \mathcal{T}(Y \otimes Y') \end{array}$$

commutes in \mathbf{C} .

In [7], Power and Robinson have proven that the wide subcategory of $\mathbf{C}_{\mathcal{T}}$ consisting of the central morphisms, and denoted by $Z(\mathbf{C}_{\mathcal{T}})$, is a symmetric *monoidal* subcategory

of $\mathbf{C}_{\mathcal{T}}$. This subcategory is called the *centre* of $\mathbf{C}_{\mathcal{T}}$. In particular, this means that Kleisli composition and the tensor functors $(- \otimes_l X)$ and $(X \otimes_r -)$ preserve central morphisms. However, it does not necessarily hold that the subcategory $Z(\mathbf{C}_{\mathcal{T}})$ determines a monad over \mathbf{C} . Nevertheless, in this situation, the left adjoint of the Kleisli adjunction $\mathcal{J} : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ always corestricts to $Z(\mathbf{C}_{\mathcal{T}})$ and we write $\hat{\mathcal{J}} : \mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$ to indicate this corestriction.

We refer to this notion of centre, as introduced by Power and Robinson, as the *premonoidal centre* of a premonoidal category. We do this in order to avoid confusion with the new notion of centre of a monad that we introduce next.

3 Central Submonads on Set

The inspiration for the construction of the central submonad (if it exists) comes from the category **Set** where it can always be defined for any monad acting on it. Next, we outline how this works.

► **Notation 9.** Throughout the remainder of the section, we write $(\mathcal{T}, \eta, \mu, \tau)$ to indicate an arbitrary strong monad on the category **Set**. We also write τ' to indicate the costrength of \mathcal{T} , which is induced by the strength τ and the symmetry of **Set** in the usual way.

► **Definition 10 (Centre).** Given a set X , the centre of \mathcal{T} at X , written $\mathcal{Z}X$, is defined to be the set

$$\mathcal{Z}X \stackrel{\text{def}}{=} \{t \in \mathcal{T}X \mid \forall Y \in \text{Ob}(\mathbf{Set}), \forall s \in \mathcal{T}Y. \mu(\mathcal{T}\tau'(\tau(t, s))) = \mu(\mathcal{T}\tau(\tau'(t, s)))\}.$$

We write $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ for the indicated subset inclusion.

In other words, the centre of \mathcal{T} at X is the subset of $\mathcal{T}X$ which contains all monadic elements for which (1) holds.

► **Remark 11.** Notice that $\mathcal{Z}X \supseteq \eta_X(X)$, i.e., the centre of \mathcal{T} at X always contains all monadic elements which are in the image of the monadic unit. This follows easily from the axioms of strong monads.

In fact, the assignment $\mathcal{Z}(-)$ extends to a *commutative submonad* of \mathcal{T} . This is made precise by the following lemmas and theorems.

► **Lemma 12.** The assignment $\mathcal{Z}(-)$ extends to a functor $\mathcal{Z} : \mathbf{Set} \rightarrow \mathbf{Set}$ when we define

$$\mathcal{Z}f \stackrel{\text{def}}{=} \mathcal{T}f|_{\mathcal{Z}X} : \mathcal{Z}X \rightarrow \mathcal{Z}Y,$$

for any function $f : X \rightarrow Y$, where $\mathcal{T}f|_{\mathcal{Z}X}$ indicates the restriction of $\mathcal{T}f : \mathcal{T}X \rightarrow \mathcal{T}Y$ to the subset $\mathcal{Z}X$.

Proof. The validity of this definition is equivalent to showing that $\mathcal{T}f(\mathcal{Z}X) \subseteq \mathcal{Z}Y$. This follows as a special case of Theorem 25. ◀

► **Lemma 13.** For every sets X, Y , the monadic unit $\eta_X : X \rightarrow \mathcal{T}X$, the monadic multiplication $\mu_X : \mathcal{T}^2X \rightarrow \mathcal{T}X$, and the monadic strength $\tau_{X,Y} : X \times \mathcal{T}Y \rightarrow \mathcal{T}(X \times Y)$ (co)restrict respectively to functions $\eta_X^{\mathcal{Z}} : X \rightarrow \mathcal{Z}X$, $\mu_X^{\mathcal{Z}} : \mathcal{Z}^2X \rightarrow \mathcal{Z}X$, $\tau_{X,Y}^{\mathcal{Z}} : X \times \mathcal{Z}Y \rightarrow \mathcal{Z}(X \times Y)$.

Proof. Special case of Theorem 25. ◀

The above lemmas now immediately imply the main theorem of this section.

► **Theorem 14.** *The assignment $\mathcal{Z}(-)$ extends to a commutative submonad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ of \mathcal{T} with $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ the required submonad morphism. Furthermore, there exists a canonical isomorphism $\mathbf{Set}_{\mathcal{Z}} \cong \mathcal{Z}(\mathbf{Set}_{\mathcal{T}})^2$.*

Proof. Special case of Theorem 27. ◀

We refer to the commutative monad determined by this theorem as *the central submonad* of \mathcal{T} . This theorem, together with our next example, shows that we have successfully generalised the notion of centre for monoids to monads on \mathbf{Set} .

► **Example 15.** Given a monoid (M, e, m) , the free monad induced by M is the monad $\mathcal{T} = (M \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$ with unit $\eta_X :: x \mapsto (e, x)$, monad multiplication $\mu_X :: (z, (z', x)) \mapsto (m(z, z'), x)$ and strength given by $\tau_{X,Y} :: (x, (t, y)) \mapsto (t, (x, y))$. Then, the central submonad \mathcal{Z} of \mathcal{T} is given by the commutative monad $(\mathcal{Z}(M) \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$, where $\mathcal{Z}(M)$ is the centre of the monoid M and where the monad data is given by the (co)restrictions of the monad data of \mathcal{T} .

► **Example 16.** Let \mathcal{T} be a commutative monad. Then its central submonad is \mathcal{T} itself.

► **Example 17.** Let S be a set. The continuation monad is $\mathcal{T} = [[-, S], S] : \mathbf{Set} \rightarrow \mathbf{Set}$, equipped with $\eta_X = x \mapsto \lambda f.f(x)$, $\mu_X : F \mapsto \lambda g.F(\lambda h.h(g))$ and $\tau_{X,Y} = (x, f) \mapsto \lambda g.f(\lambda y.g(x, y))$. Note that, if S is the empty set or a singleton set, then \mathcal{T} is commutative, so we are in the situation of Example 16. Otherwise, the central submonad of \mathcal{T} maps X to the set of $\varphi \in \mathcal{T}X$, such that: $\forall Y, \forall \psi \in \mathcal{T}Y, \forall g \in [X \times Y, S]$:

$$\psi(\lambda y.\varphi(\lambda x.g(x, y))) = \varphi(\lambda x.\psi(\lambda y.g(x, y)))$$

Suppose that $Y = \{*\}$. Then ψ is a function $S \rightarrow S$ and the condition becomes:

$$\psi(\varphi(\lambda x.g(x))) = \varphi(\lambda x.\psi(g(x)))$$

If φ is constant, this does not hold for any ψ . φ is necessarily some $\lambda f.\sigma(f(z))$ with $\sigma : S \rightarrow S$ and $z \in X$, which has to verify $\forall \psi : S \rightarrow S, \forall g : X \rightarrow S$:

$$\psi(\sigma(g(z))) = \sigma(\psi(g(z)))$$

σ can then only be the identity, so φ can only be $\eta_X(z) = \lambda f.f(z)$ for some $z \in X$. Besides, those functions are solutions of the problem. Thus, when S is not trivial, $\mathcal{Z}X = \eta_X(X) \simeq X$ and the central submonad of \mathcal{T} is naturally isomorphic to the identity monad.

Example 17 shows that the centre of a monad may be trivial in the sense that it is precisely the image of the monadic unit, and by Remark 11, this is the least it can be. Therefore, the central submonad of such a monad is not very useful, because it does not contain any additional information about the nature of the specific monadic effect. At the other extreme, Example 16 shows that the centre of a commutative monad coincides with itself (as one would expect) and therefore we also do not get anything new. Therefore, the monads that have interesting central submonads are those monads which are strong, but not commutative, and which have non-trivial centres, such as the one in Example 15.

² We explain later (see Theorem 27) in what sense this isomorphism is canonical.

4 Centralisable Monads

In this section we show how to define the central submonad of a strong monad on a symmetric monoidal category. This submonad does not always exist (but it *usually* does) and we present three equivalent conditions that characterise its existence. In Subsection 4.1 we present the first such characterisation in terms of *central cones*. Then, in Subsection 4.2 we present the remaining ones that allow us to establish a link to the theory of premonoidal categories of Power and Robinson. In Subsection 4.3 we construct a category and a monad that does not admit a centre. Despite this, in Subsection 4.4 we discuss some illustrative examples of central submonads and we show that in most naturally occurring categories, all strong monads have centres.

4.1 Central cones

In this subsection we show how the construction of the central submonad can be generalised to many categories other than **Set**.

► **Notation 18.** *Throughout the remainder of the section, we assume we are given a symmetric monoidal category (\mathbf{C}, \otimes, I) . We also assume that $(\mathcal{T}, \eta, \mu, \tau)$ is an arbitrary strong monad on the category \mathbf{C} . We write τ' to indicate the costrength of \mathcal{T} , which is induced by the strength τ and the symmetry of \mathbf{C} in the usual way. All theorems and definitions in this section are stated with respect to this monad structure.*

In **Set**, the centre is defined pointwise through subsets of $\mathcal{T}X$ which only contain elements that satisfy the coherence condition for a commutative monad. However, \mathbf{C} is an arbitrary symmetric monoidal category, so we cannot easily form subobjects in the required way. This leads us to the definition of a *central cone* which allows us to overcome this problem.

► **Definition 19 (Central Cone).** *Let X be an object of \mathbf{C} . A central cone of \mathcal{T} at X is given by a pair (Z, ι) of an object Z and a morphism $\iota : Z \rightarrow \mathcal{T}X$, such that for any object Y , we have that the diagram:*

$$\begin{array}{ccccc}
 Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X, \mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\
 \downarrow \iota \otimes \mathcal{T}Y & & & & \downarrow \mathcal{T}\tau_{X, Y} \\
 \mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\
 \downarrow \tau_{\mathcal{T}X, Y} & & & & \downarrow \mu_{X \otimes Y} \\
 \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X, Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

commutes. If (Z, ι) and (Z', ι') are two central cones of \mathcal{T} at X , then a morphism of central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$ is a morphism $\varphi : Z' \rightarrow Z$, such that $\iota \circ \varphi = \iota'$. A terminal central cone of \mathcal{T} at X is a central cone (Z, ι) for \mathcal{T} at X , such that for any central cone (Z', ι') of \mathcal{T} at X , there exists a unique morphism of central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$.

The above diagram is very similar to the one in Definition 8. The names "central morphism" and "central cone" also hint that there should be a relation between them. This is indeed the case and we show that the two definitions are equivalent.

► **Proposition 20.** *Let $f : X \rightarrow \mathcal{T}Y$ be a morphism in \mathbf{C} . The pair (X, f) is a central cone of \mathcal{T} at Y if and only if f is central in $\mathbf{C}_{\mathcal{T}}$ in the premonoidal sense (see Def. 8).*

Proof. Let (X, f) be a central cone and let $f' : X' \rightarrow \mathcal{T}Y'$ be a morphism. The following diagram:

$$\begin{array}{ccccccc}
 X \otimes X' & \xrightarrow{f \otimes X'} & \mathcal{T}Y \otimes X' & \xrightarrow{\tau'_{Y, X'}} & \mathcal{T}(Y \otimes X') & \xrightarrow{\mathcal{T}(Y \otimes f')} & \mathcal{T}(Y \otimes \mathcal{T}Y') \\
 \downarrow X \otimes f' & & \downarrow \mathcal{T}Y \otimes f' & & & & \downarrow \mathcal{T}\tau_{Y, Y'} \\
 & \text{(1)} & & \text{(2)} & & \nearrow \tau'_{Y, \mathcal{T}Y'} & \\
 X \otimes \mathcal{T}Y' & \xrightarrow{f \otimes \mathcal{T}Y'} & \mathcal{T}Y \otimes \mathcal{T}Y' & & & & \mathcal{T}^2(Y \otimes Y') \\
 \downarrow \tau_{X, Y'} & \text{(3)} & \downarrow \tau_{\mathcal{T}Y, Y'} & & \text{(4)} & & \downarrow \mu_{Y \otimes Y'} \\
 \mathcal{T}(X \otimes Y') & \xrightarrow{\mathcal{T}(f \otimes Y')} & \mathcal{T}(\mathcal{T}Y \otimes Y') & \xrightarrow{\mathcal{T}\tau'_{Y, Y'}} & \mathcal{T}^2(Y \otimes Y') & \xrightarrow{\mu_{Y \otimes Y'}} & \mathcal{T}(Y \otimes Y')
 \end{array}$$

commutes because: (1) \mathbf{C} is monoidal; (2) τ' is natural; (3) τ is natural; and (4) the pair (X, f) is a central cone. Therefore, the morphism f is central in the premonoidal sense.

For the other direction, if f is central in $\mathbf{C}_{\mathcal{T}}$, the following diagram:

$$\begin{array}{ccccccc}
 Z \otimes \mathcal{T}Y & \xrightarrow{f \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X, \mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) & & \\
 \downarrow f \otimes \mathcal{T}Y & \searrow f \otimes \mathcal{T}Y & \downarrow & \searrow & \downarrow \mathcal{T}\tau_{X, Y} & & \\
 \mathcal{T}X \otimes \mathcal{T}Y & \xleftarrow{f \otimes \mathcal{T}Y} & Z \otimes \mathcal{T}Y & & \mathcal{T}^2(X \otimes Y) & & \\
 \downarrow \tau_{\mathcal{T}X, Y} & \text{(1)} & \downarrow \tau_{Z, Y} & \text{(2)} & \downarrow \mu_{X \otimes Y} & & \\
 \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X, Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) & &
 \end{array}$$

commutes because: (1) τ is natural; (2) f is a central morphism; all remaining subdiagrams commute trivially. This shows the pair (X, f) is a central cone. ◀

From now on, we rely heavily on the fact that central cones and central morphisms are equivalent notions and we use Proposition 20 implicitly in the sequel.

Central cones – or central morphisms – have several nice properties. We focus on the ones necessary for our purposes. To begin, we show that central cones are stable by left and right composition.

► **Lemma 21.** *If $(X, f : X \rightarrow \mathcal{T}Y)$ is a central cone of \mathcal{T} at Y , then for any $g : Z \rightarrow X$, it follows that $(Z, f \circ g)$ is a central cone of \mathcal{T} at Y .*

Proof.

$$\begin{array}{ccc}
 Z \otimes \mathcal{T}X' & \xrightarrow{g \otimes \mathcal{T}X'} & X \otimes \mathcal{T}X' \xrightarrow{f \otimes \mathcal{T}X'} \mathcal{T}Y \otimes \mathcal{T}X' \xrightarrow{\tau'_{Y, \mathcal{T}X'}} \mathcal{T}(Y \otimes \mathcal{T}X') \\
 & & \downarrow f \otimes \mathcal{T}X' \qquad \qquad \qquad \downarrow \mathcal{T}\tau_{Y, X'} \\
 & & \mathcal{T}Y \otimes \mathcal{T}X' \qquad \qquad \qquad \mathcal{T}^2(Y \otimes X') \\
 & & \downarrow \tau_{\mathcal{T}Y, X'} \qquad \qquad \qquad \downarrow \mu_{Y \otimes X'} \\
 & & \mathcal{T}(\mathcal{T}Y \otimes X') \xrightarrow{\tau_{\mathcal{T}Y, X'}} \mathcal{T}^2(Y \otimes X') \xrightarrow{\mu_{Y \otimes X'}} \mathcal{T}(Y \otimes X')
 \end{array}$$

commutes directly from the definition of central cone for f . ◀

► **Lemma 22.** *If $(X, f : X \rightarrow \mathcal{T}Y)$ is a central cone of \mathcal{T} at Y then for any $g : Y \rightarrow Z$, it follows that $(X, \mathcal{T}g \circ f)$ is a central cone of \mathcal{T} at Z .*

Proof. The following diagram:

$$\begin{array}{ccccccc}
 X \otimes \mathcal{T}X' & \xrightarrow{f \otimes \mathcal{T}X'} & \mathcal{T}Y \otimes \mathcal{T}X' & \xrightarrow{\mathcal{T}g \otimes \mathcal{T}X'} & \mathcal{T}Z \otimes \mathcal{T}X' & & \\
 & & \downarrow \tau'_{Y, \mathcal{T}X'} & \text{(2)} & \downarrow \tau'_{Z, \mathcal{T}X'} & & \\
 & & \mathcal{T}(Y \otimes \mathcal{T}X') & \xrightarrow{\mathcal{T}(g \otimes \mathcal{T}X')} & \mathcal{T}(Z \otimes \mathcal{T}X') & & \\
 & & \downarrow \mathcal{T}\tau_{Y, X'} & \text{(3)} & \downarrow \mathcal{T}\tau_{Z, X'} & & \\
 & & \mathcal{T}^2(Y \otimes X') & \xrightarrow{\mathcal{T}^2(g \otimes X')} & \mathcal{T}^2(Z \otimes X') & & \\
 & & \downarrow \mu_{Y \otimes X'} & \text{(4)} & \downarrow \mu_{Z \otimes X'} & & \\
 & & \mathcal{T}(Y \otimes X') & \xrightarrow{\mathcal{T}(g \otimes X')} & \mathcal{T}(Z \otimes X') & & \\
 & & \downarrow \mathcal{T}(g \otimes X') & \text{(7)} & \downarrow \mu_{Z \otimes X'} & & \\
 & & \mathcal{T}Z \otimes \mathcal{T}X' & \xrightarrow{\tau_{\mathcal{T}Z, X'}} & \mathcal{T}(\mathcal{T}Z \otimes X') & \xrightarrow{\tau_{\mathcal{T}Z, X'}} & \mathcal{T}^2(Z \otimes X') \xrightarrow{\mu_{Z \otimes X'}} \mathcal{T}(Z \otimes X') \\
 & & \downarrow \mathcal{T}g \otimes \mathcal{T}X' & \text{(5)} & \downarrow \mathcal{T}(\mathcal{T}g \otimes X') & \text{(6)} & \downarrow \mathcal{T}^2(g \otimes X') \\
 & & \mathcal{T}Y \otimes \mathcal{T}X' & \xrightarrow{\tau_{\mathcal{T}Y, X'}} & \mathcal{T}(\mathcal{T}Y \otimes X') & \xrightarrow{\tau_{\mathcal{T}Y, X'}} & \mathcal{T}^2(Y \otimes X') \xrightarrow{\mu_{Y \otimes X'}} \mathcal{T}(Y \otimes X')
 \end{array}$$

commutes, because: (1) f is a central cone, (2) τ' is natural, (3) τ is natural, (4) μ is natural (5) τ is natural, (6) τ' is natural, (7) μ is natural. ◀

► **Lemma 23.** *If (Z, ι) is a terminal central cone of \mathcal{T} at X , then ι is a monomorphism.*

Proof. Let us consider $f, g : Y \rightarrow Z$ such that $\iota \circ f = \iota \circ g$; this morphism is a central cone at X (Lemma 21), and since (Z, ι) is a terminal central cone, it factors uniquely through ι . Thus $f = g$ and therefore ι is monic. ◀

It is easy to see that if a terminal central cone of \mathcal{T} at X exists, then it is unique up to a unique isomorphism of central cones. We also note that Lemma 23 is crucial for defining the centre of \mathcal{T} through terminal central cones, because the morphisms ι would be the components of a submonad morphism. The main definition of this subsection follows next and gives the foundation for the construction of the central submonad.

► **Definition 24** (Centralisable Monad). *We say that the monad \mathcal{T} is centralisable if for any object X , a terminal central cone of \mathcal{T} at X exists. In this situation, we write $(\mathcal{Z}X, \iota_X)$ for the terminal central cone of \mathcal{T} at X . If (Y, φ) is another central cone of \mathcal{T} at X , we write $\varphi^{\mathcal{Z}} : Y \rightarrow \mathcal{Z}X$ for the unique central cone morphism $\varphi^{\mathcal{Z}} : (Y, \varphi) \rightarrow (\mathcal{Z}X, \iota_X)$.*

In fact, for a centralisable monad \mathcal{T} , its terminal central cones induce a commutative submonad \mathcal{Z} of \mathcal{T} . This is the main theorem of this subsection, which is stated next, and its proof reveals constructively how the monad structure arises from the terminal central cones.

► **Theorem 25.** *If the monad \mathcal{T} is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on \mathbf{C} . Moreover, \mathcal{Z} is a commutative submonad of \mathcal{T} in the sense that the morphisms $\iota_X : \mathcal{Z}X \rightarrow \mathcal{T}X$ constitute a monomorphism of strong monads $\iota : \mathcal{Z} \Rightarrow \mathcal{T}$.*

Proof. First let us describe the functorial structure of \mathcal{Z} . Recall that \mathcal{Z} maps every object X to its terminal central cone at X . Let $f : X \rightarrow Y$ be a morphism. We know that $\mathcal{T}f \circ \iota_X : \mathcal{Z}X \rightarrow \mathcal{T}Y$ is a central cone according to Lemma 22. Therefore, we define $\mathcal{Z}f$ as the unique map such that the following diagram:

$$\begin{array}{ccc} & \mathcal{Z}f & \\ \mathcal{Z}X & \dashrightarrow & \mathcal{Z}Y \\ \iota_X \downarrow & & \downarrow \iota_Y \\ \mathcal{T}X & \xrightarrow{\mathcal{T}f} & \mathcal{T}Y \end{array}$$

commutes. It follows directly that \mathcal{Z} maps the identity to the identity, and that ι is natural. \mathcal{Z} also preserves composition, which follows by the commutative diagram below.

$$\begin{array}{ccccc} & & \mathcal{Z}(f \circ g) & & \\ & & \text{---} & & \\ & \mathcal{Z}A & \xrightarrow{\mathcal{Z}g} & \mathcal{Z}B & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}C \\ & \downarrow \iota_A & & \downarrow \iota_B & & \downarrow \iota_C \\ \mathcal{T}A & \xrightarrow{\mathcal{T}g} & \mathcal{T}B & \xrightarrow{\mathcal{T}f} & \mathcal{T}C \\ & & \text{---} & & \\ & & \mathcal{T}(f \circ g) & & \end{array}$$

This proves that \mathcal{Z} is a functor. Next, we describe its monad structure and after that we show that it is commutative.

The monadic unit η_X is central, because it is the identity morphism in $Z(\mathbf{C}_{\mathcal{T}})$, thus it factors

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through ι_X to define $\eta_X^{\mathcal{Z}}$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{T}X \\ \eta_X^{\mathcal{Z}} \searrow & & \nearrow \iota_X \\ & \mathcal{Z}X & \end{array}$$

Next, observe that, by definition, $\mu_X \circ \mathcal{T}\iota_X \circ \iota_{\mathcal{Z}X} = \iota_X \odot \iota_{\mathcal{Z}X}$, where $(- \odot -)$ indicates Kleisli composition. Since ι is central and Kleisli composition preserves central morphisms, it follows that this morphism factors through ι_X and we use this to define $\mu_X^{\mathcal{Z}}$ as in the diagram below.

$$\begin{array}{ccc} \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{Z}X \\ \iota_{\mathcal{Z}X} \downarrow & & \downarrow \iota_X \\ \mathcal{T}\mathcal{Z}X & \xrightarrow{\mathcal{T}\iota_X} \mathcal{T}^2 X \xrightarrow{\mu_X} & \mathcal{T}X \end{array}$$

Again, by definition, $\tau_{A,B} \circ (A \otimes \iota_B) = A \otimes_r \iota_B$. Central morphisms are preserved by the premonoidal products (as we noted in Section 2) and therefore, this morphism factors through $\iota_{A \otimes B}$ which we use to define $\tau_{A,B}^{\mathcal{Z}}$ as in the diagram below.

$$\begin{array}{ccc} A \otimes \mathcal{Z}B & \xrightarrow{\tau_{A,B}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes B) \\ A \otimes \iota_B \downarrow & & \downarrow \iota_{A \otimes B} \\ A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}} & \mathcal{T}(A \otimes B) \end{array}$$

Note that the last three diagrams are exactly those of a morphism of strong monads (see Definition 4).

Using the fact that ι is monic (see Lemma 23) we show that the following commutative diagram shows that $\eta^{\mathcal{Z}}$ is natural.

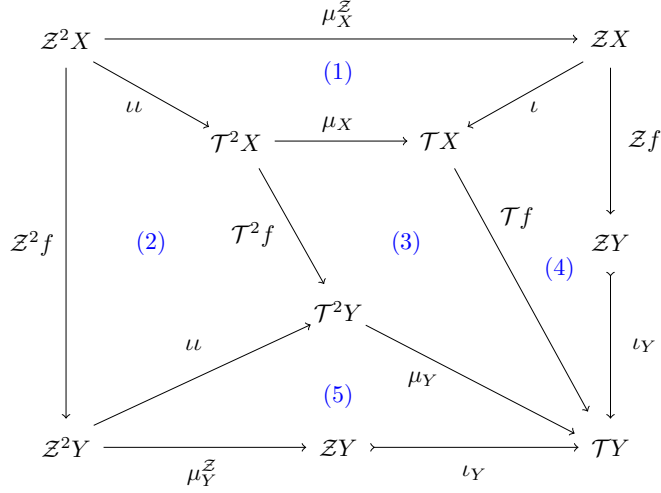
$$\begin{array}{ccccc} X & \xrightarrow{\eta_X^{\mathcal{Z}}} & \mathcal{Z}X & & \\ \downarrow f & \searrow \eta_X & \mathcal{T}X & \xleftarrow{\iota_X} & \mathcal{Z}X \\ & & & & \downarrow \mathcal{Z}f \\ & & & & \mathcal{Z}Y \\ & & & & \downarrow \iota_Y \\ & & & & \mathcal{T}Y \\ & \nearrow \eta_Y & \mathcal{Z}Y & \xrightarrow{\iota_Y} & \mathcal{T}Y \\ Y & \xrightarrow{\eta_Y^{\mathcal{Z}}} & \mathcal{Z}Y & & \end{array}$$

(1) $\eta_X^{\mathcal{Z}} \circ \eta_X = \iota_X \circ \mathcal{Z}f$
(2) $\iota_X \circ \mathcal{Z}f = \iota_Y \circ \mathcal{Z}f$
(3) $\eta_X \circ f = \eta_Y \circ f$
(4) $\eta_Y^{\mathcal{Z}} \circ \eta_Y = \iota_Y \circ \mathcal{Z}f$

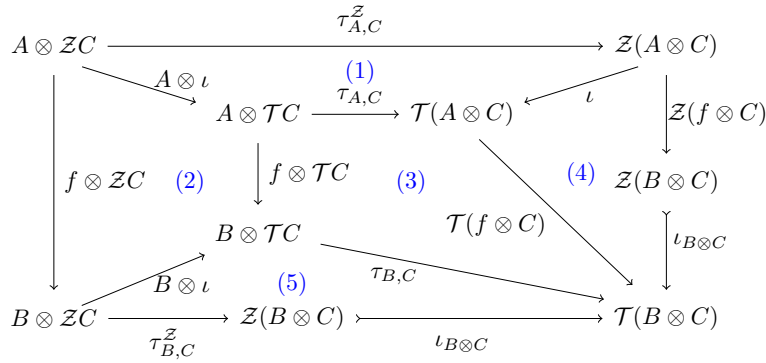
(1) definition of $\eta^{\mathcal{Z}}$, (2) ι is natural, (3) η is natural and (4) definition of $\eta^{\mathcal{Z}}$. Thus we have proven that for any $f : X \rightarrow Y$, $\iota_Y \circ \mathcal{Z}f \circ \eta_X^{\mathcal{Z}} = \iota_Y \circ \eta_Y^{\mathcal{Z}} \circ f$. Besides, ι is monic, thus

$Zf \circ \eta_X^Z = \eta_Y^Z \circ f$ which proves that η^Z is natural. We will prove all the remaining diagrams will the same reasoning.

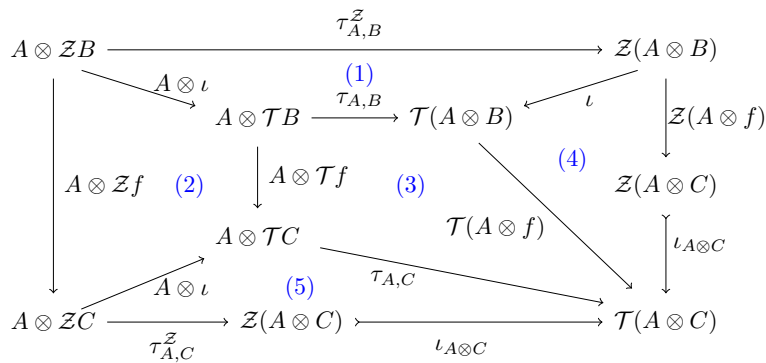
The following commutative diagram shows that μ^Z is natural.



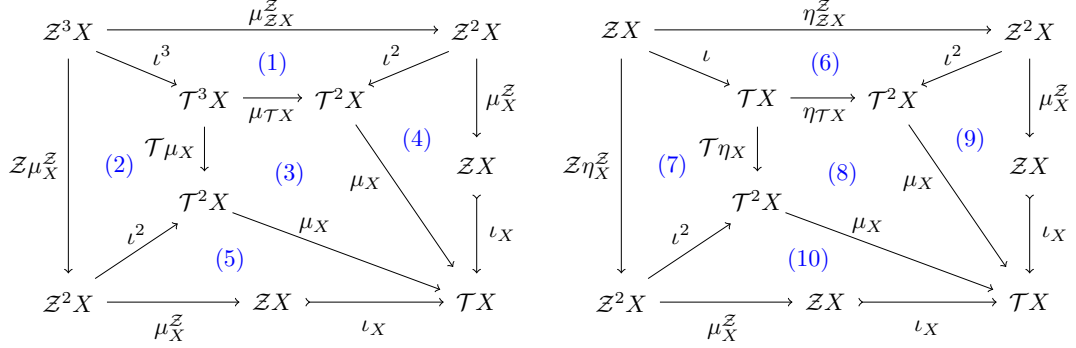
(1) definition of μ^Z , (2) ι is natural, (3) μ is natural, (4) ι is natural and (5) definition of μ^Z . The following commutative diagrams shows that τ^Z is natural.



(1) definition of τ^Z , (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of τ^Z .

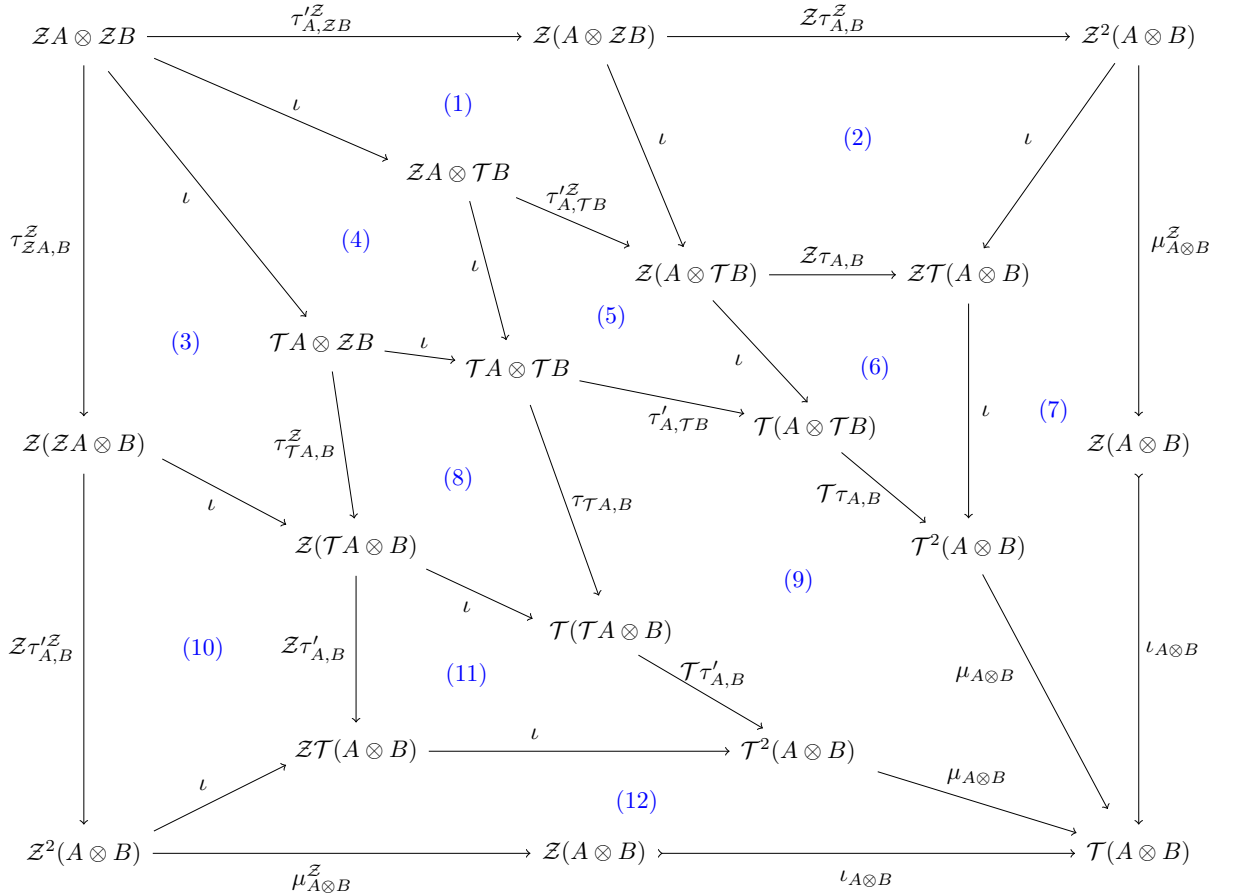


(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$. The following commutative diagrams prove that \mathcal{Z} is a monad.



(1) and (2) involve the definition of $\mu^{\mathcal{Z}}$ and the naturality of ι and $\mu^{\mathcal{Z}}$, (3) is Def. 1, (4) definition of $\mu^{\mathcal{Z}}$ and (5) also. (6) and (7) involve the definition of $\eta^{\mathcal{Z}}$ and the naturality of ι and $\eta^{\mathcal{Z}}$, (8) is Def. 1, (9) definition of $\mu^{\mathcal{Z}}$ and (10) also.

\mathcal{Z} is proven strong with very similar diagrams. The commutative diagram:



proves that \mathcal{Z} is a commutative monad, with (1) $\tau'^{\mathcal{Z}}$ is natural, (2) definition of $\tau^{\mathcal{Z}}$, (3) $\tau^{\mathcal{Z}}$ is natural, (4) \mathbf{C} is monoidal, (5) definition of $\tau'^{\mathcal{Z}}$, (6) ι is natural, (7) definition of $\mu^{\mathcal{Z}}$, (8)

definition of $\tau^{\mathcal{Z}}$, (9) ι is central, (10) definition of $\tau'^{\mathcal{Z}}$, (11) ι is natural and (12) definition of $\mu^{\mathcal{Z}}$. ◀

This theorem shows that centralisable monads always induce a canonical commutative submonad. However, we still have not precisely explained in what sense this submonad is "central". We justify this next. Note that, since \mathcal{Z} is a commutative monad, its Kleisli category $\mathbf{C}_{\mathcal{Z}}$ has a canonical symmetric monoidal structure. Furthermore, since \mathcal{Z} is a submonad of \mathcal{T} , we know that $\mathbf{C}_{\mathcal{Z}}$ canonically embeds into $\mathbf{C}_{\mathcal{T}}$. The next theorem shows that this embedding factors through the premonoidal centre of $\mathbf{C}_{\mathcal{T}}$, and moreover, the two categories are isomorphic.

► **Theorem 26.** *In the situation of Theorem 25, the canonical embedding functor $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.*

Proof.

- Let us note the corestriction $\hat{\mathcal{I}}$. For any f , Lemma 21 ensures that $\hat{\mathcal{I}}f$ is central.
- We define the inverse functor $G : Z(\mathbf{C}_{\mathcal{T}}) \rightarrow \mathbf{C}_{\mathcal{Z}}$.
 G maps objects to themselves. If $f : X \rightarrow \mathcal{T}Y$ is a central morphism in the premonoidal sense, (X, f) is a central cone of \mathcal{T} at Y ; then there exists a unique $f^{\mathcal{Z}} : X \rightarrow \mathcal{Z}Y$ such that $\iota_Y \circ f^{\mathcal{Z}} = f$. We define $Gf = f^{\mathcal{Z}}$.
 The proof that it is a functor is direct considering that any $f^{\mathcal{Z}}$ is a morphism of central cones and that components of ι are monic.
- Let $f : X \rightarrow \mathcal{T}Y$ be a morphism of $Z(\mathbf{C}_{\mathcal{T}})$, *i.e.* a central morphism. $\hat{\mathcal{I}}Gf = \iota_Y \circ f^{\mathcal{Z}} = f$ by definition of morphism of central cones (see Def. 19). Let $g : X \rightarrow \mathcal{Z}Y$ in \mathbf{C} . $\iota_Y \circ G\hat{\mathcal{I}}g = \iota_Y \circ (\iota_Y \circ g)^{\mathcal{Z}} = \iota_Y \circ g$ by Definition 19 and thus $G\hat{\mathcal{I}}g = g$ since ι_Y is monic. ◀

Because of the previous two theorems, we are justified in naming the submonad \mathcal{Z} as *the* central submonad of \mathcal{T} .

4.2 Characterising the Centre of Strong Monads

In the previous subsection we showed that the existence of terminal central cones is sufficient to construct the central submonad and we provided a constructive proof of this fact. Next, we show that the existence of these central cones is also necessary for this. Furthermore, we provide another equivalent characterisation in terms of the premonoidal structure of the monad. This is precisely formulated in the main theorem of this paper which is presented next.

► **Theorem 27 (Centralisability).** *Let \mathbf{C} be a symmetric monoidal category and \mathcal{T} a strong monad on it. The following are equivalent:*

1. *For any object X of \mathbf{C} , \mathcal{T} admits a terminal central cone at X ;*
2. *There exists a commutative submonad \mathcal{Z} of \mathcal{T} such that the canonical embedding functor $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$;*
3. *The corestriction of the Kleisli left adjoint $\mathcal{J} : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ to the premonoidal centre $\hat{\mathcal{J}} : \mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$ also is a left adjoint.*

Proof.

- (1 \Rightarrow 2) By Theorem 25 and Theorem 26.

- (2 \Rightarrow 3) Let us consider the Kleisli left adjoint $\mathcal{J}^{\mathcal{Z}}$ associated to the monad \mathcal{Z} . All our hypotheses can be summarised in the following commutative diagram, because \mathcal{Z} is a submonad of \mathcal{T} :

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{\mathcal{J}} & \mathbf{C}_{\mathcal{T}} \\
 \mathcal{J}^{\mathcal{Z}} \downarrow & \searrow \hat{\mathcal{J}} & \uparrow \\
 \mathbf{C}_{\mathcal{Z}} & \xrightarrow[\hat{\mathcal{I}}]{\sim} & Z(\mathbf{C}_{\mathcal{T}})
 \end{array}$$

Since $\hat{\mathcal{I}}$ is an isomorphism, $\hat{\mathcal{J}} = \hat{\mathcal{I}}\mathcal{J}^{\mathcal{Z}}$ is also a left adjoint.

- (3 \Rightarrow 1) Let $\mathcal{R} : Z(\mathbf{C}_{\mathcal{T}}) \rightarrow \mathbf{C}$ be the right adjoint of $\hat{\mathcal{J}}$ and let ε be the counit of the adjunction. We will show that the pair $(\mathcal{R}X, \varepsilon_X)$ is the terminal central cone of \mathcal{T} at X . First, since ε_X is a morphism in $Z(\mathbf{C}_{\mathcal{T}})$, it follows that it is central. Thus the pair $(\mathcal{R}X, \varepsilon_X)$ is a central cone of \mathcal{T} at X . Next, let $\Phi : Z(\mathbf{C}_{\mathcal{T}})[\hat{\mathcal{J}}Y, X] \cong \mathbf{C}(Y, \mathcal{R}X)$ be the natural bijection induced by the adjunction. If $f : Y \rightarrow \mathcal{T}X$ is central, meaning a morphism of $Z(\mathbf{C}_{\mathcal{T}})$, the following diagram:

$$\begin{array}{ccc}
 \hat{\mathcal{J}}Y & & \\
 \hat{\mathcal{J}}\Phi(f) \downarrow \text{---} & \searrow f & \\
 \hat{\mathcal{J}}\mathcal{R}X & \xrightarrow{\varepsilon_X} & X
 \end{array}$$

commutes in $Z(\mathbf{C}_{\mathcal{T}})$. Equivalently, the following diagram in \mathbf{C} :

$$\begin{array}{ccc}
 Y & & \\
 \Phi(f) \downarrow \text{---} & \searrow f & \\
 \mathcal{R}X & \xrightarrow{\varepsilon_X} & \mathcal{T}X
 \end{array}$$

commutes. Note that the pair (Y, f) is equivalently a central cone for \mathcal{T} at X (by Proposition 20). Thus f uniquely factors through the counit $\varepsilon_X : \mathcal{R}X \rightarrow \mathcal{T}X$ and therefore $(\mathcal{R}X, \varepsilon_X)$ is the terminal central cone of \mathcal{T} at X . ◀

This theorem shows that Definition 24 may be stated by choosing any one of the above equivalent criteria. We note that the first condition is the easiest to verify in practice. The second condition is the most useful for providing a computational interpretation, as we discuss in the sequel. The third condition is the easiest to state categorically and provides an important link to premonoidal categories.

4.3 A non Centralisable Monad

In **Set**, we heavily relied on the notion of subset to define the central submonad. One may wonder what happens if not all subsets are objects in the category. The following example describes such a situation, which gives rise to a non-centralisable strong monad.

► **Example 28.** Consider the Dihedral group \mathbb{D}_4 which has 8 elements. Its center is non-trivial and has 2 elements.

Let \mathbf{C} be the full subcategory of **Set** with objects that are finite products of the set \mathbb{D}_4 with itself. This category has a cartesian structure and the terminal object is the singleton set (which is the empty product). Notice that every object in this category has cardinality which is a power of 8. Therefore the cardinality of every homset of \mathbf{C} is also a power of 8.

Since \mathbf{C} has a cartesian structure and since \mathbb{D}_4 is a monoid, we can consider the action monad \mathcal{M} on \mathbf{C} induced by \mathbb{D}_4 , which can be defined in exactly the same way as in Example 15. It follows that \mathcal{M} is a strong monad on \mathbf{C} . However, it is easy to show that this monad is not centralisable.

Let us assume, for the purpose of reaching a contradiction, that there is a monad $\mathcal{T} : \mathbf{C} \rightarrow \mathbf{C}$ such that $\mathbf{C}_{\mathcal{T}} \cong Z(\mathbf{C}_{\mathcal{M}})$. Next, observe that the homset $Z(\mathbf{C}_{\mathcal{M}})[1, 1]$ has the same cardinality as the centre of the monoid \mathbb{D}_4 , i.e., its cardinality is 2. However, $\mathbf{C}_{\mathcal{T}}$ cannot have such a homset since $\mathbf{C}_{\mathcal{T}}[X, Y] = \mathbf{C}[X, \mathcal{T}Y]$ which must have cardinality a power of 8. Therefore there exists no such monad \mathcal{T} and \mathcal{M} is not centralisable.

4.4 Examples of Centralisable Monads

Despite the fact that not every monad is centralisable, in most naturally occurring categories every strong monad is indeed centralisable. For example, in many categories of interest, the objects of the category have a suitable notion of subobject (e.g., subsets in **Set**, subspaces in **Vect**) and for such categories it is not hard to show that every monad is centralisable.

► **Example 29.** Any strong monad on **Set** is centralisable (see Section 3). The same is true for any strong monad on the categories **DCPO**, **Meas**, **Top**, **Hilb**, **Vect** and many others.

► **Example 30.** The valuations monad $\mathcal{V} : \mathbf{DCPO} \rightarrow \mathbf{DCPO}$ is strong, but its commutativity is an open problem [4]. The central submonad of \mathcal{V} is precisely the "central valuations monad" described in [3]. In fact, the latter work inspired the present paper, which may be seen as a categorical generalisation of the ideas presented in [3].

► **Example 31.** The *unbounded* Giry monad $\mathcal{G} : \mathbf{Meas} \rightarrow \mathbf{Meas}$, which assigns the space of all (possibly unbounded) measures to a measurable space, is a strong monad which is *not* commutative. This monad is centralisable and its central submonad \mathcal{Z} is such that $\mathcal{Z}X$ contains all discrete measures on the measurable space X (and possibly others).

5 Computational interpretation (Work in Progress)

5.1 Syntax

Finally, we provide a computational interpretation of our ideas by presenting a refinement of Moggi's metalanguage [6]. The types are extended by simply adding the \mathcal{Z} unary connective that represents the central submonad of \mathcal{T} :

$$\begin{aligned}
 (\text{Types}) \quad a, b &::= 1 \mid a \rightarrow b \mid a \times b \mid \mathcal{Z}a \mid \mathcal{T}a \\
 (\text{Terms}) \quad M, N &::= x \mid * \mid \lambda x^a. M \mid MN \mid \mathbf{ret}_{\mathcal{Z}} M \mid \iota M \\
 &\quad \mid \mathbf{do} \ x \leftarrow_{\mathcal{Z}} M ; N \mid \mathbf{do} \ x \leftarrow_{\mathcal{T}} M ; N
 \end{aligned}$$

The typing rules are the following:

$$\begin{array}{c}
\frac{}{\Gamma, x : a \vdash x : a} \quad \frac{}{\Gamma \vdash * : 1} \quad \frac{\Gamma \vdash M : a \rightarrow b \quad \Gamma \vdash N : a}{\Gamma \vdash MN : b} \quad \frac{\Gamma \vdash M : a \quad \Gamma \vdash N : b}{\Gamma \vdash \langle M, N \rangle : a \times b} \\
\frac{\Gamma \vdash M : a_1 \times a_2}{\Gamma \vdash \pi_i M : a_i} \quad \frac{\Gamma \vdash M : a}{\Gamma \vdash \text{ret}_{\mathcal{Z}} M : \mathcal{Z}a} \quad \frac{\Gamma \vdash M : \mathcal{Z}a \quad \Gamma, x : a \vdash N : \mathcal{Z}b}{\Gamma \vdash \text{do } x \leftarrow_{\mathcal{Z}} M ; N : \mathcal{Z}b} \\
\frac{\Gamma, x : a \vdash M : b}{\Gamma \vdash \lambda x^a. M : a \rightarrow b} \quad \frac{\Gamma \vdash M : \mathcal{Z}a}{\Gamma \vdash \iota M : \mathcal{T}a} \quad \frac{\Gamma \vdash M : \mathcal{T}a \quad \Gamma, x : a \vdash N : \mathcal{T}b}{\Gamma \vdash \text{do } x \leftarrow_{\mathcal{T}} M ; N : \mathcal{T}b}
\end{array}$$

As part of future work, we will describe a sound and adequate semantics of this system (which is fairly straightforward) and we will also consider the issue of full abstraction for specific monads with non-trivial centres.

5.2 Operational semantics

The operational semantics is defined as follows:

$$\begin{array}{c}
\frac{M \rightsquigarrow M'}{\text{do } x \leftarrow_{\mathcal{Z}} M ; N \rightsquigarrow \text{do } x \leftarrow_{\mathcal{Z}} M' ; N} \quad \frac{M \rightsquigarrow M'}{\text{do } x \leftarrow_{\mathcal{T}} M ; N \rightsquigarrow \text{do } x \leftarrow_{\mathcal{T}} M' ; N} \\
\frac{M \rightsquigarrow M'}{\text{ret}_{\mathcal{Z}} M \rightsquigarrow \text{ret}_{\mathcal{Z}} M'} \quad \frac{M \rightsquigarrow M'}{\iota M \rightsquigarrow \iota M'} \quad \frac{}{\text{do } x \leftarrow_{\mathcal{Z}} \text{ret}_{\mathcal{Z}} M ; N \rightsquigarrow N[M/x]} \quad \frac{}{(\lambda x^a. M)N \rightsquigarrow M[N/x]} \\
\frac{M \rightsquigarrow M'}{\pi_i M \rightsquigarrow \pi_i M'} \quad \frac{M \rightsquigarrow M'}{\langle M, N \rangle \rightsquigarrow \langle M', N \rangle} \quad \frac{M \rightsquigarrow M'}{\langle N, M \rangle \rightsquigarrow \langle N, M' \rangle}
\end{array}$$

5.3 Denotational semantics

Let us consider a cartesian closed category \mathbf{C} equipped with a centralisable monad \mathcal{T} , whose central submonad is noted \mathcal{Z} . The types are denoted as objects in \mathbf{C} , as following:

$$\begin{aligned}
[[1]] &= I \\
[[a \rightarrow b]] &= [[a]] \rightarrow [[b]] \\
[[\mathcal{Z}a]] &= \mathcal{Z} [[a]] \\
[[\mathcal{T}a]] &= \mathcal{T} [[a]]
\end{aligned}$$

When $\Gamma = x_1 : a_1 \dots x_n : a_n$, $[[\Gamma]] = [[a]]_1 \times \dots \times [[a]]_n$. We use morphisms of \mathbf{C} for terms. ev is the evaluation map and ϕ is the natural isomorphism $\mathbf{C}[X \times Y, Z] \cong \mathbf{C}[X, Z^Y]$. Their semantics is defined recursively on the typing derivation:

$$\begin{aligned}
[[\Gamma, x : a \vdash x : a]] &= \pi_a \\
[[\Gamma \vdash \pi_i M : a]] &= \pi_i \circ [[\Gamma \vdash M : a \times b]] \\
[[\Gamma \vdash \langle M, N \rangle : a \times b]] &= \langle [[\Gamma \vdash M : a]], [[\Gamma \vdash N : b]] \rangle \\
[[\Gamma \vdash MN : b]] &= ev_{[[a],[b]]} \circ \langle [[\Gamma \vdash M : a \rightarrow b]], [[\Gamma \vdash N : a]] \rangle \\
[[\Gamma \vdash \lambda x^a. M : a \rightarrow b]] &= \phi_{[[\Gamma],[a],[b]]} ([[\Gamma, x : a \vdash M : b]) \\
[[\Gamma \vdash \text{ret}_{\mathcal{Z}} M : \mathcal{Z}a]] &= \eta_a^{\mathcal{Z}} \circ [[\Gamma \vdash M : a]] \\
[[\Gamma \vdash \iota M : \mathcal{T}a]] &= \iota_{[[a]]} \circ [[\Gamma \vdash M : \mathcal{Z}a]] \\
[[\Gamma \vdash \text{do } x \leftarrow_{\mathcal{Z}} M ; N : \mathcal{Z}b]] &= \mu_{[[b]]}^{\mathcal{Z}} \circ \mathcal{Z} [[N]] \circ \tau_{[[\Gamma],[a]]}^{\mathcal{Z}} \circ ([[\Gamma] \times [[M]]) \circ \Delta \\
[[\Gamma \vdash \text{do } x \leftarrow_{\mathcal{T}} M ; N : \mathcal{T}b]] &= \mu_{[[b]]} \circ \mathcal{T} [[N]] \circ \tau_{[[\Gamma],[a]]} \circ ([[\Gamma] \times [[M]]) \circ \Delta
\end{aligned}$$

5.4 Properties

- ▶ **Conjecture 32** (Soundness). *If $M \rightsquigarrow M'$, then $\llbracket M \rrbracket = \llbracket M' \rrbracket$.*
- ▶ **Conjecture 33** (Adequacy). *If η is a monomorphism, then $\llbracket M \rrbracket = \llbracket M' \rrbracket$ implies that there exists a value V such that $M \rightsquigarrow^* V$ and $M' \rightsquigarrow^* V$.*
- ▶ **Conjecture 34.**

$$\llbracket \text{do } x \leftarrow_{\mathcal{T}} \iota M ; \text{do } y \leftarrow_{\mathcal{T}} N ; P \rrbracket = \llbracket \text{do } y \leftarrow_{\mathcal{T}} N ; \text{do } x \leftarrow_{\mathcal{T}} \iota M ; P \rrbracket$$

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