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Central Submonads and Notions of Computation

Titouan Carette

Université Paris-Saclay, Inria, CNRS, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Louis Lemonnier

Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Vladimir Zamdzhiev

Université de Lorraine, CNRS, Inria, LORIA, F 54000 Nancy, France

Abstract

The notion of "centre" has been introduced for many algebraic structures in mathematics. A notable example is the centre of a monoid which always determines a commutative submonoid. Monads (in category theory) can be seen as generalisations of monoids and in this paper we show how the notion of centre may be extended to strong monads acting on symmetric monoidal categories. We show that the centre of a strong monad \mathcal{T} , if it exists, determines a commutative submonad \mathcal{Z} of \mathcal{T} , such that the Kleisli category of \mathcal{Z} is isomorphic to the premonoidal centre (in the sense of Power and Robinson) of the Kleisli category of \mathcal{T} . We provide three equivalent conditions which characterise the existence of the centre of \mathcal{T} . While not every strong monad admits a centre, we show that every strong monad on well-known naturally occurring categories does admit a centre, thereby showing that this new notion is ubiquitous. We also provide a computational interpretation of our ideas which consists in giving a refinement of Moggi's monadic metalanguage. The added benefit is that this allows us to immediately establish a large class of contextually equivalent terms for monads that admit a non-trivial centre by simply looking at the richer syntactic structure provided by the refinement.

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1 Introduction

Commutative submonads might be constructed through a variety of different methods, see [2, 3].

2 Background

► **Definition 1** (Monad). *A monad over a category \mathbf{C} is an endofunctor \mathcal{T} equipped with two natural transformations $\eta : Id \Rightarrow \mathcal{T}$ and $\mu : \mathcal{T}^2 \Rightarrow \mathcal{T}$ such that the following diagrams:*

$$\begin{array}{ccc} \mathcal{T}^3 X & \xrightarrow{\mu_{\mathcal{T}X}} & \mathcal{T}^2 X \\ \mathcal{T}\mu_X \downarrow & & \downarrow \mu_X \\ \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X \end{array} \qquad \begin{array}{ccc} \mathcal{T}X & \xrightarrow{\mathcal{T}\eta_X} & \mathcal{T}^2 X \\ \eta_{\mathcal{T}X} \downarrow & \searrow & \downarrow \mu_X \\ \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X \end{array}$$

commute.

► **Definition 2** (Strong Monad). A strong monad over a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda)$ is a monad (\mathcal{T}, η, μ) equipped with a natural transformation $\tau_{X,Y} : X \otimes TY \rightarrow \mathcal{T}(X \otimes Y)$ (called strength), such that the following diagrams:

$$\begin{array}{ccc}
 I \otimes TX & \xrightarrow{\tau_{I,X}} & \mathcal{T}(I \otimes X) \\
 \searrow \lambda_{TX} & & \downarrow \mathcal{T}\lambda_X \\
 & & TX \\
 \\
 (W \otimes X) \otimes TY & \xrightarrow{\tau_{W \otimes X, Y}} & \mathcal{T}((W \otimes X) \otimes Y) \\
 \downarrow \alpha_{W,X, TY} & & \downarrow \mathcal{T}\alpha_{W,X,Y} \\
 W \otimes (X \otimes TY) & \xrightarrow{W \otimes \tau_{X,Y}} W \otimes \mathcal{T}(X \otimes Y) \xrightarrow{\tau_{W, X \otimes Y}} & \mathcal{T}(W \otimes (X \otimes Y))
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{X \otimes \eta_Y} & X \otimes TY \\
 \searrow \eta_{X \otimes Y} & & \downarrow \tau_{X,Y} \\
 & & \mathcal{T}(X \otimes Y) \\
 \\
 X \otimes \mathcal{T}^2 Y & \xrightarrow{\tau_{X, TY}} & \mathcal{T}(X \otimes TY) \xrightarrow{\mathcal{T}\tau_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
 \downarrow X \otimes \mu_Y & & \downarrow \mu_{X \otimes Y} \\
 X \otimes TY & \xrightarrow{\tau_{X,Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

commute.

Next, we recall the definition of a commutative monad.

► **Definition 3** (Commutative Monad). Given a strong monad \mathcal{T} on a symmetric monoidal category, the costrength $\tau'_{X,Y} : \mathcal{T}X \otimes Y \rightarrow \mathcal{T}(X \otimes Y)$ is given by $\tau'_{X,Y} \stackrel{\text{def}}{=} \mathcal{T}(\gamma_{Y,X}) \circ \tau_{Y,X} \circ \gamma_{\mathcal{T}X,Y}$. Then, \mathcal{T} is said to be commutative if the following diagram:

$$\begin{array}{ccc}
 \mathcal{T}X \otimes TY & \xrightarrow{\tau_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
 \downarrow \tau'_{X, TY} & & & \downarrow \mu_{X \otimes Y} \\
 \mathcal{T}(X \otimes TY) & \xrightarrow{\mathcal{T}\tau_{X,Y}} & \mathcal{T}^2(X \otimes Y) \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y)
 \end{array} \tag{1}$$

commutes for every choice of objects X and Y .

The appropriate notion of morphism between two strong monads is given by our next definition.

► **Definition 4** (Morphism of Strong Monads [1]). Given two strong monads $(\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}}, \tau^{\mathcal{T}})$ and $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}, \tau^{\mathcal{P}})$ over a category \mathbf{C} , we say that a morphism of strong monads is a natural transformation $\iota : \mathcal{T} \Rightarrow \mathcal{P}$ that makes the following diagrams:

$$\begin{array}{ccc}
 \begin{array}{ccc} X & & \\ \eta_X^{\mathcal{T}} \swarrow & & \searrow \eta_X^{\mathcal{P}} \\ \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X \end{array} & \begin{array}{ccc} \mathcal{T}^2 X & \xrightarrow{\iota_{\mathcal{T}X}} \mathcal{P}\mathcal{T}X & \xrightarrow{\mathcal{P}\iota_X} \mathcal{P}^2 X \\ \mu_X^{\mathcal{T}} \downarrow & & \downarrow \mu_X^{\mathcal{P}} \\ \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X \end{array} & \begin{array}{ccc} A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}^{\mathcal{T}}} & \mathcal{T}(A \otimes B) \\ A \otimes \iota_B \downarrow & & \downarrow \iota_{A \otimes B} \\ A \otimes \mathcal{P}B & \xrightarrow{\tau_{A,B}^{\mathcal{P}}} & \mathcal{P}(A \otimes B) \end{array}
 \end{array}$$

commute.

It is easy to see that strong monads over a symmetric monoidal category \mathbf{C} and strong monad morphisms between them form a category which we denote by writing $\mathbf{StrMnd}(\mathbf{C})$.

► **Definition 5** (Submonad). *In the situation of Definition 4, if ι is a monomorphism in the category of strong monads over \mathbf{C} , then \mathcal{T} is said to be a strong submonad of \mathcal{P} and ι is said to be a submonad morphism.*

The submonad relation induces an embedding between the Kleisli categories of the corresponding monads that we recall below.

► **Definition 6** (Kleisli category). *Given a monad (\mathcal{T}, η, μ) over a category \mathbf{C} , the Kleisli category $\mathbf{C}_{\mathcal{T}}$ of \mathcal{T} is the category whose objects are the same as those of \mathbf{C} , but whose morphisms are given by $\mathbf{C}_{\mathcal{T}}[X, Y] = \mathbf{C}[X, \mathcal{T}Y]$. Composition in $\mathbf{C}_{\mathcal{T}}$ is given by $g \odot f = \mu_Z \circ \mathcal{T}g \circ f$ where $f : X \rightarrow \mathcal{T}Y$ and $g : Y \rightarrow \mathcal{T}Z$. The identity at X is given by the monadic unit $\eta_X : X \rightarrow \mathcal{T}X$.*

► **Proposition 7.** *If $\iota : \mathcal{T} \Rightarrow \mathcal{P}$ is a submonad morphism, the functor $\mathcal{I} : \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{P}}$ which is defined by $\mathcal{I}(X) = X$ on objects and on morphisms by $\mathcal{I}(f : X \rightarrow \mathcal{T}Y) = \iota_Y \circ f : X \rightarrow \mathcal{P}Y$ is an embedding of categories.*

This functor \mathcal{I} is the canonical embedding of $\mathbf{C}_{\mathcal{T}}$ into $\mathbf{C}_{\mathcal{P}}$ induced by the submonad morphism ι .

2.1 Premonoidal Structure of Strong Monads

Let \mathcal{T} be a strong monad on a symmetric monoidal category (\mathbf{C}, I, \otimes) . Then, the Kleisli category $\mathbf{C}_{\mathcal{T}}$ does *not* necessarily have a canonical monoidal structure. However, it does have a canonical *premonoidal structure* as shown by Power and Robinson [5]. In fact, they show that this premonoidal structure is monoidal iff the monad \mathcal{T} is commutative. Next, we briefly recall the premonoidal structure of \mathcal{T} as outlined by Power and Robinson.

For every objects X and Y of $\mathbf{C}_{\mathcal{T}}$, their tensor product $X \otimes Y$ is also an object of $\mathbf{C}_{\mathcal{T}}$. But, the monoidal product \otimes of \mathbf{C} does not necessarily induce a monoidal functor on $\mathbf{C}_{\mathcal{T}}$. However, by using the strength and the costrength of \mathcal{T} , we can define two families of functors as follows:

- for any object X , a functor $(- \otimes_l X) : \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{T}}$ whose action on objects sends Y to $Y \otimes X$, and sends $f : Y \rightarrow \mathcal{T}Z$ to $\tau'_{Z,X} \circ (f \otimes X) : Y \otimes X \rightarrow \mathcal{T}(Z \otimes X)$;
- for any object X , a functor $(X \otimes_r -) : \mathbf{C}_{\mathcal{T}} \rightarrow \mathbf{C}_{\mathcal{T}}$ whose action on objects sends Y to $X \otimes Y$, and sends $f : Y \rightarrow \mathcal{T}Z$ to $\tau_{X,Z} \circ (X \otimes f) : X \otimes Y \rightarrow \mathcal{T}(X \otimes Z)$.

This categorical data satisfies the axioms and coherence properties of *premonoidal categories* as outlined by Power and Robinson [5], but which we omit here for brevity. We wish to note that in a premonoidal category, $f \otimes_l X'$ and $X \otimes_r g$ do not always commute. This leads us to the next definition, which plays a crucial role in the theory of premonoidal categories.

► **Definition 8** (Central morphism [5]). *Given a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on a symmetric monoidal category (\mathbf{C}, I, \otimes) , we say that a morphism $f : X \rightarrow Y$ in $\mathbf{C}_{\mathcal{T}}$ is central if for any*

$f' : X' \rightarrow Y'$, the following diagram:

$$\begin{array}{ccc} X \otimes X' & \xrightarrow{f \otimes_l X'} & Y \otimes X' \\ \downarrow X \otimes_r f' & & \downarrow Y \otimes_r f' \\ X \otimes Y' & \xrightarrow{f \otimes_l X'} & Y \otimes Y' \end{array}$$

commutes in $\mathbf{C}_{\mathcal{T}}$; or equivalently, the following diagram:

$$\begin{array}{ccccccc} X \otimes X' & \xrightarrow{f \otimes X'} & \mathcal{T}Y \otimes X' & \xrightarrow{\tau'_{Y,X'}} & \mathcal{T}(Y \otimes X') & \xrightarrow{\mathcal{T}(Y \otimes f')} & \mathcal{T}(Y \otimes \mathcal{T}Y') \\ \downarrow X \otimes f' & & & & & & \downarrow \mathcal{T}\tau_{Y,Y'} \\ X \otimes \mathcal{T}Y' & & & & & & \mathcal{T}^2(Y \otimes Y') \\ \downarrow \tau_{X,Y'} & & & & & & \downarrow \mu_{Y \otimes Y'} \\ \mathcal{T}(X \otimes Y') & \xrightarrow{\mathcal{T}(f \otimes Y')} & \mathcal{T}(\mathcal{T}Y \otimes Y') & \xrightarrow{\mathcal{T}\tau'_{Y,Y'}} & \mathcal{T}^2(Y \otimes Y') & \xrightarrow{\mu_{Y \otimes Y'}} & \mathcal{T}(Y \otimes Y') \end{array}$$

commutes in \mathbf{C} .

In [5], Power and Robinson have proven that the wide subcategory of $\mathbf{C}_{\mathcal{T}}$ consisting of the central morphisms, and denoted by $Z(\mathbf{C}_{\mathcal{T}})$, is a symmetric monoidal subcategory of $\mathbf{C}_{\mathcal{T}}$. This subcategory is called the *centre* of $\mathbf{C}_{\mathcal{T}}$. In particular, this means that Kleisli composition and the tensor functors $(- \otimes_l X)$ and $(X \otimes_r -)$ preserve central morphisms. However, it does not necessarily hold that the subcategory $Z(\mathbf{C}_{\mathcal{T}})$ determines a monad over \mathbf{C} . Nevertheless, in this situation, the left adjoint of the Kleisli adjunction $\mathcal{J} : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ always corestricts to $Z(\mathbf{C}_{\mathcal{T}})$ and we write $\hat{\mathcal{J}} : \mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$ to indicate this corestriction.

3 Central Submonads on Set

The inspiration for the construction of the central submonad (if it exists) comes from the category **Set** where it can always be defined for any monad acting on it. Next, we outline how this works.

► **Notation 9.** Throughout the remainder of the section, we write $(\mathcal{T}, \eta, \mu, \tau)$ to indicate an arbitrary strong monad on the category **Set**. We also write τ' to indicate the costrength of \mathcal{T} , which is induced by the strength τ and the symmetry of **Set** in the usual way.

► **Definition 10 (Centre).** Given a set X , the centre of \mathcal{T} at X , written $\mathcal{Z}X$, is defined to be the set

$$\mathcal{Z}X \stackrel{\text{def}}{=} \{t \in \mathcal{T}X \mid \forall Y \in \text{Ob}(\mathbf{Set}). \forall s \in \mathcal{T}Y. \mu(\mathcal{T}\tau'(\tau(t, s))) = \mu(\mathcal{T}\tau(\tau'(t, s)))\}.$$

We write $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ for the indicated subset inclusion.

In other words, the centre of \mathcal{T} at X is the subset of $\mathcal{T}X$ which contains all monadic elements for which (1) holds. In fact, the assignment $\mathcal{Z}(-)$ extends to a *commutative submonad* of \mathcal{T} . This is made precise by the following lemmas and theorems.

► **Lemma 11.** *The assignment $\mathcal{Z}(-)$ extends to a functor $\mathcal{Z} : \mathbf{Set} \rightarrow \mathbf{Set}$ when we define*

$$\mathcal{Z}f \stackrel{\text{def}}{=} \mathcal{T}f|_{\mathcal{Z}X} : \mathcal{Z}X \rightarrow \mathcal{Z}Y,$$

for any function $f : X \rightarrow Y$, where $\mathcal{T}f|_{\mathcal{Z}X}$ indicates the restriction of $\mathcal{T}f : \mathcal{T}X \rightarrow \mathcal{T}Y$ to the subset $\mathcal{Z}X$.

Proof. The validity of this definition is equivalent to showing that $\mathcal{T}f(\mathcal{Z}X) \subseteq \mathcal{Z}Y$. This follows by Theorem 26. ◀

► **Lemma 12.** *For every sets X, Y , the monadic unit $\eta_X : X \rightarrow \mathcal{T}X$, the monadic multiplication $\mu_X : \mathcal{T}^2X \rightarrow \mathcal{T}X$, and the monadic strength $\tau_{X,Y} : X \times \mathcal{T}Y \rightarrow \mathcal{T}(X \times Y)$ (co)restrict respectively to functions $\eta_X^{\mathcal{Z}} : X \rightarrow \mathcal{Z}X$, $\mu_X^{\mathcal{Z}} : \mathcal{Z}^2X \rightarrow \mathcal{Z}X$, $\tau_{X,Y}^{\mathcal{Z}} : X \times \mathcal{Z}Y \rightarrow \mathcal{Z}(X \times Y)$.*

Proof. Special case of Theorem 26. ◀

The above lemmas now immediately imply the main theorem of this section.

► **Theorem 13.** *The assignment $\mathcal{Z}(-)$ extends to a commutative submonad of \mathcal{T} with $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ the required submonad morphism. Furthermore, there exists a canonical isomorphism $\mathbf{Set}_{\mathcal{Z}} \cong \mathcal{Z}(\mathbf{Set}_{\mathcal{T}})^1$.*

Proof. Special case of Theorem 28. ◀

We refer to the commutative monad determined by this theorem as *the central submonad* of \mathcal{T} . This theorem, together with our next example, shows that we have successfully generalised the notion of centre for monoids to monads on \mathbf{Set} .

3.1 Examples

In this subsection we present specific examples of central submonads in \mathbf{Set} .

► **Example 14.** Let W be a set. Then we consider the state monad: $\mathcal{T} = [W, W \times -] : \mathbf{Set} \rightarrow \mathbf{Set}$.

$$\eta_X : \begin{cases} X \rightarrow \mathcal{T}X \\ x \mapsto \lambda w. (w, x) \end{cases}$$

$$\mu_X : \begin{cases} \mathcal{T}^2X \rightarrow \mathcal{T}X \\ F \mapsto \lambda w. F(w)_2(w) \end{cases}$$

$\mathcal{Z}X$ is the set of $f \in \mathcal{T}X$ st.

$$\forall Y, \forall g \in \mathcal{T}Y, \lambda w. (x(y(w)_1)_1, (x(y(w)_1)_2, y(w)_2)) = \lambda w. (y(x(w)_1)_1, (x(w)_2, y(x(w)_1)_2))$$

Or st. $\forall Y, \forall g \in \mathcal{T}Y, \forall w \in W$:

$$\begin{cases} x(y(w)_1)_1 = y(x(w)_1)_1 \\ x(y(w)_1)_2 = x(w)_2 \\ y(w)_2 = y(x(w)_1)_2 \end{cases}$$

We have that $\mathcal{Z}X = \{\lambda w. (w, x) | x \in X\} \simeq X$.

¹ We explain later (see Theorem 28) in what sense this isomorphism is canonical.

► **Example 15.** Let S be a set. The continuation monad is $\mathcal{T} = [[-, S], S] : \mathbf{Set} \rightarrow \mathbf{Set}$, equipped with $\eta_X = x \mapsto \lambda f.f(x)$, $\mu_X : F \mapsto \lambda g.F(\lambda h.h(g))$ and $\tau_{X,Y} = (x, f) \mapsto \lambda g.f(\lambda y.g(x, y))$.

The central submonad of \mathcal{T} then maps X to the set of $\varphi \in \mathcal{T}X$, such that: $\forall Y, \forall \psi \in \mathcal{T}Y, \forall g \in [X \times Y, S]$:

$$\psi(\lambda y.\varphi(\lambda x.g(x, y))) = \varphi(\lambda x.\psi(\lambda y.g(x, y)))$$

Thus, when S is not trivial, $\mathcal{Z}X \simeq X$.

► **Example 16.** Let (M, e, m) be a monoid. We show in this example that our construction of central submonad is a direct generalisation of the centre for a monoid. We consider the action monad, which is the following: $\mathcal{T} = M \times - : \mathbf{Set} \rightarrow \mathbf{Set}$, equipped with $\eta_X : x \mapsto (e, x)$ and $\mu_X : (z, (z', x)) \mapsto (m(z, z'), x)$. The strength simply changes the order in the tensor products: $- \times (M \times -) \Rightarrow M \times (- \times -)$.

The central submonad maps X to the set of $(z, x) \in \mathcal{T}X$ such that for all set Y , for all $(z', y) \in \mathcal{T}Y$:

$$(m(z, z'), (x, y)) = (m(z', z), (x, y))$$

This ends up with the following conditions:

$$\begin{cases} m(z, z') = m(z', z) \\ (x, y) = (x, y) \end{cases}$$

Thus, $\mathcal{Z}X = \{(z, x) | z \in Z(M), x \in X\} = Z(M) \times X$. We have proven that the central submonad of the action monad induced by M , is the action monad induced by its centre $Z(M)$.

4 Centralisable Monads

4.1 Central cones

In this section we show how the construction of the central submonad can be generalised to many categories other than \mathbf{Set} .

► **Notation 17.** Throughout the remainder of the section, we assume we are given a symmetric monoidal category (\mathbf{C}, \otimes, I) . We also assume that $(\mathcal{T}, \eta, \mu, \tau)$ is an arbitrary strong monad on the category \mathbf{C} . We write τ' to indicate the costrength of \mathcal{T} , which is induced by the strength τ and the symmetry of \mathbf{C} in the usual way. All theorems and definitions in this section are stated with respect to this monad structure.

► **Definition 18 (Central Cone).** Let X be an object of \mathbf{C} . A central cone of \mathcal{T} at X is given by a pair (Z, ι) of an object Z and a morphism $\iota : Z \rightarrow \mathcal{T}X$, such that for any object Y , we

have that the diagram:

$$\begin{array}{ccccc}
 Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X,\mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\
 \downarrow \iota \otimes \mathcal{T}Y & & & & \downarrow \mathcal{T}\tau_{X,Y} \\
 \mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\
 \downarrow \tau_{\mathcal{T}X,Y} & & & & \downarrow \mu_{X \otimes Y} \\
 \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

commutes. If (Z, ι) and (Z', ι') are two central cones of \mathcal{T} at X , then a morphism of central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$ is a morphism $\varphi : Z' \rightarrow Z$, such that $\iota \circ \varphi = \iota'$. A terminal central cone of \mathcal{T} at X is a central cone (Z, ι) for \mathcal{T} at X , such that for any central cone (Z', ι') of \mathcal{T} at X , there exists a unique morphism of central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$.

► **Proposition 19.** Let $f : X \rightarrow \mathcal{T}Y$ be a morphism in \mathbf{C} . (X, f) is a central cone of \mathcal{T} at Y if and only if f is central in $\mathbf{C}_{\mathcal{T}}$, i.e. in the premonoidal sense (see Def. 8).

Proof. Consider $f' : X' \rightarrow \mathcal{T}Y'$ a morphism part of a central cone. The following diagram:

$$\begin{array}{ccccccc}
 X \otimes X' & \xrightarrow{f \otimes X'} & \mathcal{T}Y \otimes X' & \xrightarrow{\tau'_{Y,X'}} & \mathcal{T}(Y \otimes X') & \xrightarrow{\mathcal{T}(Y \otimes f')} & \mathcal{T}(Y \otimes \mathcal{T}Y') \\
 \downarrow X \otimes f' & (1) & \downarrow \mathcal{T}Y \otimes f' & (2) & \nearrow \tau'_{Y,\mathcal{T}Y'} & & \downarrow \mathcal{T}\tau_{Y,Y'} \\
 X \otimes \mathcal{T}Y' & \xrightarrow{f \otimes \mathcal{T}Y'} & \mathcal{T}Y \otimes \mathcal{T}Y' & & & & \mathcal{T}^2(Y \otimes Y') \\
 \downarrow \tau_{X,Y'} & (3) & \downarrow \tau_{\mathcal{T}Y,Y'} & (4) & & & \downarrow \mu_{Y \otimes Y'} \\
 \mathcal{T}(X \otimes Y') & \xrightarrow{\mathcal{T}(f \otimes Y')} & \mathcal{T}(\mathcal{T}Y \otimes Y') & \xrightarrow{\mathcal{T}\tau'_{Y,Y'}} & \mathcal{T}^2(Y \otimes Y') & \xrightarrow{\mu_{Y \otimes Y'}} & \mathcal{T}(Y \otimes Y')
 \end{array}$$

commutes with : (1) \mathbf{C} is monoidal, (2) τ' is natural, (3) τ is natural and (4) f is a central cone.

Now if f is central in $\mathbf{C}_{\mathcal{T}}$, this diagram:

$$\begin{array}{ccccc}
 Z \otimes TY & \xrightarrow{f \otimes TY} & TX \otimes TY & \xrightarrow{\tau'_{X,TY}} & T(X \otimes TY) \\
 \downarrow f \otimes TY & \searrow f \otimes TY & \parallel & \searrow \tau'_{X,TY} & \parallel \downarrow T\tau_{X,Y} \\
 & & TX \otimes TY & \xrightarrow{\tau'_{X,TY}} & T(X \otimes TY) \\
 & & \parallel & & \parallel \\
 TX \otimes TY & \xrightarrow{\tau_{TX,Y}} & T(TX \otimes Y) & & T^2(X \otimes Y) \\
 \downarrow \tau_{TX,Y} & & \parallel & & \downarrow \mu_{X \otimes Y} \\
 T(TX \otimes Y) & \xrightarrow{\tau_{TX,Y}} & T(TX \otimes Y) & \xrightarrow{\tau'_{TX,Y}} & T^2(X \otimes Y) \\
 & & & \parallel & \parallel \\
 & & & T^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & T(X \otimes Y)
 \end{array} \quad (1)$$

commutes with : (1) f is a central morphism. ◀

From now on, we rely heavily on the fact that central cone and central morphism are equivalent notions.

► **Lemma 20.** *If $(X, f : X \rightarrow TY)$ is a central cone of \mathcal{T} at Y then for any $g : Z \rightarrow X$, $(Z, f \circ g)$ is a central cone of \mathcal{T} at Y .*

Proof.

$$\begin{array}{ccc}
 Z \otimes TX' & \xrightarrow{g \otimes TX'} & X \otimes TX' \xrightarrow{f \otimes TX'} TY \otimes TX' \xrightarrow{\tau'_{Y,TX'}} T(Y \otimes TX') \\
 \downarrow f \otimes TX' & & \downarrow T\tau_{Y,X'} \\
 TX' \otimes TX' & & T^2(Y \otimes X') \\
 \downarrow \tau_{TX',X'} & & \downarrow \mu_{Y \otimes X'} \\
 T(TX' \otimes X') & \xrightarrow{\tau'_{TX',X'}} & T^2(Y \otimes X') \xrightarrow{\mu_{Y \otimes X'}} T(Y \otimes X')
 \end{array}$$

commutes directly from the definition of central cone for f . ◀

► **Lemma 21.** *If $(X, f : X \rightarrow TY)$ is a central cone of \mathcal{T} at Y then for any $g : Y \rightarrow Z$, $(X, Tg \circ f)$ is a central cone of \mathcal{T} at Z .*

Proof. The following diagram:

$$\begin{array}{ccccc}
 X \otimes \mathcal{T}X' & \xrightarrow{f \otimes \mathcal{T}X'} & \mathcal{T}Y \otimes \mathcal{T}X' & \xrightarrow{\mathcal{T}g \otimes \mathcal{T}X'} & \mathcal{T}Z \otimes \mathcal{T}X' \\
 \downarrow f \otimes \mathcal{T}X' & & \downarrow \tau'_{Y, \mathcal{T}X'} & \text{(2)} & \downarrow \tau'_{Z, \mathcal{T}X'} \\
 & & \mathcal{T}(Y \otimes \mathcal{T}X') & \xrightarrow{\mathcal{T}(g \otimes \mathcal{T}X')} & \mathcal{T}(Z \otimes \mathcal{T}X') \\
 & & \downarrow \mathcal{T}\tau_{Y, X'} & \text{(3)} & \downarrow \mathcal{T}\tau_{Z, X'} \\
 & & \mathcal{T}^2(Y \otimes X') & \xrightarrow{\mathcal{T}^2(g \otimes X')} & \mathcal{T}^2(Z \otimes X') \\
 & & \downarrow \mu_{Y \otimes X'} & \text{(4)} & \downarrow \mu_{Z \otimes X'} \\
 \mathcal{T}Y \otimes \mathcal{T}X' & \xrightarrow{\tau_{\mathcal{T}Y, X'}} & \mathcal{T}(\mathcal{T}Y \otimes X') & \xrightarrow{\mathcal{T}\tau'_{Y, X'}} & \mathcal{T}^2(Y \otimes X') & \xrightarrow{\mu_{Y \otimes X'}} & \mathcal{T}(Y \otimes X') \\
 \downarrow \mathcal{T}g \otimes \mathcal{T}X' & \text{(5)} & \downarrow \mathcal{T}(\mathcal{T}g \otimes X') & \text{(6)} & \downarrow \mathcal{T}^2(g \otimes X') & \text{(7)} & \searrow \mathcal{T}(g \otimes X') \\
 \mathcal{T}Z \otimes \mathcal{T}X' & \xrightarrow{\tau_{\mathcal{T}Z, X'}} & \mathcal{T}(\mathcal{T}Z \otimes X') & \xrightarrow{\mathcal{T}\tau'_{Z, X'}} & \mathcal{T}^2(Z \otimes X') & \xrightarrow{\mu_{Z \otimes X'}} & \mathcal{T}(Z \otimes X')
 \end{array}$$

commutes with : (1) f is a central cone, (2) τ' is natural, (3) τ is natural, (4) μ is natural
 (5) τ is natural, (6) τ' is natural, (7) μ is natural. \blacktriangleleft

► **Lemma 22.** *If (Z, ι) is a terminal central cone of \mathcal{T} at X , then ι is monic.*

Proof. Let us consider $f, g : Y \rightarrow Z$ such that $\iota \circ f = \iota \circ g$; this morphism is a central cone (Lemma 20) at X , and since (Z, ι) is a terminal central cone, it factors uniquely through ι . Thus ι is monic. \blacktriangleleft

It is easy to see that if a terminal central cone for \mathcal{T} at X exists, then it is unique up to a unique isomorphism of central cones.

► **Definition 23** (Centralisable Monad). *We say that the monad \mathcal{T} is centralisable if for any object X , a terminal central cone of \mathcal{T} at X exists.*

► **Notation 24.** *If the monad \mathcal{T} is centralisable, we write (ZX, ι_X) for the terminal central cone of \mathcal{T} at X . If (Y, φ) is another central cone of \mathcal{T} at X , we write $\varphi^Z : Y \rightarrow ZX$ for the unique central cone morphism $\varphi^Z : (Y, \varphi) \rightarrow (ZX, \iota_X)$.*

► **Example 25.** Any strong monad on **Set** is centralisable (see previous section). The same is true for any strong monad on the categories **DCPO**, **Meas**, **Top**, **Hilb**, **Vect**, etc.

► **Theorem 26.** *If the monad \mathcal{T} is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative monad $(\mathcal{Z}, \eta^Z, \mu^Z, \tau^Z)$ on **C**. Moreover, \mathcal{Z} is a commutative submonad of \mathcal{T} in the sense that the morphisms $\iota_X : ZX \rightarrow \mathcal{T}X$ constitute a monomorphism of strong monads $\iota : \mathcal{Z} \Rightarrow \mathcal{T}$.*

Proof. First let us prove that \mathcal{Z} is a functor. It maps every object to their terminal central cone. Let $f : X \rightarrow Y$ be a map, we know that $\mathcal{T}f \circ \iota_X : \mathcal{Z}X \rightarrow \mathcal{T}Y$ is central according to Lemma 21; We define $\mathcal{Z}f$ as the unique map such that the following diagram:

$$\begin{array}{ccc} \mathcal{Z}X & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}Y \\ \iota_X \downarrow & & \downarrow \iota_Y \\ \mathcal{T}X & \xrightarrow{\mathcal{T}f} & \mathcal{T}Y \end{array}$$

commutes. It falls then directly that \mathcal{Z} maps the identity to the identity, and that ι is natural. \mathcal{Z} also preserves composition, proven with the commutative diagram below.

$$\begin{array}{ccccc} & & \mathcal{Z}(f \circ g) & & \\ & \text{---} & \text{---} & \text{---} & \\ \mathcal{Z}A & \xrightarrow{\mathcal{Z}g} & \mathcal{Z}B & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}C \\ \iota_A \downarrow & & \downarrow \iota_B & & \downarrow \iota_C \\ \mathcal{T}A & \xrightarrow{\mathcal{T}g} & \mathcal{T}B & \xrightarrow{\mathcal{T}f} & \mathcal{T}C \\ & \text{---} & \text{---} & \text{---} & \\ & & \mathcal{T}(f \circ g) & & \end{array}$$

This proves that \mathcal{Z} is a functor. Next we prove it is a monad, then that it is commutative. η_X is central as unit of $Z(\mathbf{C}_{\mathcal{T}})$, thus it factors through ι_X to define $\eta_X^{\mathcal{Z}}$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{T}X \\ \eta_X^{\mathcal{Z}} \searrow & & \nearrow \iota_X \\ & \mathcal{Z}X & \end{array}$$

$\mu_X \circ \mathcal{T}\iota_X \circ \iota_{\mathcal{Z}X} = \iota_X \odot \iota_{\mathcal{Z}X}$ is central, thus it factors through ι_X to define $\mu_X^{\mathcal{Z}}$.

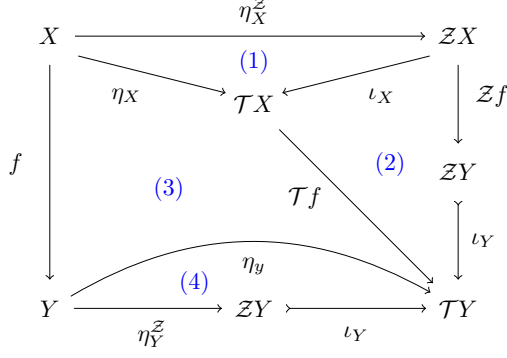
$$\begin{array}{ccc} \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{Z}X \\ \iota_{\mathcal{Z}X} \downarrow & & \downarrow \iota_X \\ \mathcal{T}\mathcal{Z}X & \xrightarrow{\mathcal{T}\iota_X} \mathcal{T}^2 X & \xrightarrow{\mu_X} \mathcal{T}X \end{array}$$

$\tau_{A,B} \circ (A \otimes \iota_B) = A \otimes_r \iota_B$ is central, thus it factors through $\iota_{A \otimes B}$ to define $\tau_{A,B}^{\mathcal{Z}}$.

$$\begin{array}{ccc} A \otimes \mathcal{Z}B & \xrightarrow{\tau_{A,B}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes B) \\ A \otimes \iota_B \downarrow & & \downarrow \iota_{A \otimes B} \\ A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}} & \mathcal{T}(A \otimes B) \end{array}$$

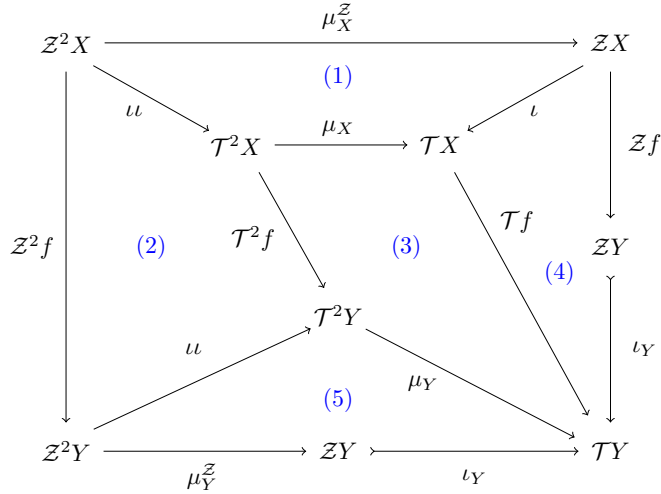
Note that those diagrams are exactly those of a morphism of strong monads.

Using the fact that ι is monic (see Lemma 22) we show that the following commutative diagram shows that $\eta^{\mathcal{Z}}$ is natural.



(1) definition of $\eta^{\mathcal{Z}}$, (2) ι is natural, (3) η is natural and (4) definition of $\eta^{\mathcal{Z}}$. Thus we have proven that for any $f : X \rightarrow Y$, $\iota_Y \circ \mathcal{Z}f \circ \eta_X^{\mathcal{Z}} = \iota_Y \circ \eta_Y^{\mathcal{Z}} \circ f$. Besides, ι is monic, thus $\mathcal{Z}f \circ \eta_X^{\mathcal{Z}} = \eta_Y^{\mathcal{Z}} \circ f$ which proves that $\eta^{\mathcal{Z}}$ is natural. We will prove all the remaining diagrams will the same reasoning.

The following commutative diagram shows that $\mu^{\mathcal{Z}}$ is natural.



(1) definition of $\mu^{\mathcal{Z}}$, (2) ι is natural, (3) μ is natural, (4) ι is natural and (5) definition of $\mu^{\mathcal{Z}}$.

The following commutative diagrams shows that $\tau^{\mathcal{Z}}$ is natural.

$$\begin{array}{ccccc}
 A \otimes \mathcal{Z}C & \xrightarrow{\tau_{A,C}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes C) & & \\
 \downarrow f \otimes \mathcal{Z}C & \searrow A \otimes \iota & \downarrow \mathcal{Z}(f \otimes C) & \swarrow \iota & \\
 & A \otimes \mathcal{T}C & \xrightarrow{\tau_{A,C}} & \mathcal{T}(A \otimes C) & \\
 & \downarrow f \otimes \mathcal{T}C & & \downarrow \mathcal{T}(f \otimes C) & \\
 B \otimes \mathcal{Z}C & \xrightarrow{\tau_{B,C}^{\mathcal{Z}}} & \mathcal{Z}(B \otimes C) & \xrightarrow{\iota_{B \otimes C}} & \mathcal{T}(B \otimes C) \\
 \uparrow B \otimes \iota & \swarrow B \otimes \mathcal{T}C & \downarrow \mathcal{T}(f \otimes C) & \swarrow \tau_{B,C} & \\
 & B \otimes \mathcal{T}C & & &
 \end{array}$$

(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.

$$\begin{array}{ccccc}
 A \otimes \mathcal{Z}B & \xrightarrow{\tau_{A,B}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes B) & & \\
 \downarrow A \otimes \mathcal{Z}f & \searrow A \otimes \iota & \downarrow \mathcal{Z}(A \otimes f) & \swarrow \iota & \\
 & A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}} & \mathcal{T}(A \otimes B) & \\
 & \downarrow A \otimes \mathcal{T}f & & \downarrow \mathcal{T}(A \otimes f) & \\
 A \otimes \mathcal{Z}C & \xrightarrow{\tau_{A,C}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes C) & \xrightarrow{\iota_{A \otimes C}} & \mathcal{T}(A \otimes C) \\
 \uparrow A \otimes \iota & \swarrow A \otimes \mathcal{T}C & \downarrow \mathcal{T}(A \otimes f) & \swarrow \tau_{A,C} & \\
 & A \otimes \mathcal{T}C & & &
 \end{array}$$

(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.

The following commutative diagrams prove that \mathcal{Z} is a monad.

$$\begin{array}{ccc}
 \mathcal{Z}^3 X & \xrightarrow{\mu_{\mathcal{Z}^2 X}^{\mathcal{Z}}} & \mathcal{Z}^2 X \\
 \downarrow \mathcal{Z}\mu_X^{\mathcal{Z}} & \searrow \iota^3 & \downarrow \mu_X^{\mathcal{Z}} \\
 & \mathcal{T}^3 X & \xrightarrow{\mu_{\mathcal{T}^2 X}} \mathcal{T}^2 X \\
 & \downarrow \mathcal{T}\mu_X & \downarrow \mu_X \\
 & \mathcal{T}^2 X & \xrightarrow{\mu_X} \mathcal{T}X \\
 \downarrow \iota^2 & \swarrow \iota^2 & \downarrow \iota_X \\
 \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} \mathcal{Z}X & \xrightarrow{\iota_X} \mathcal{T}X
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{Z}X & \xrightarrow{\eta_{\mathcal{Z}^2 X}^{\mathcal{Z}}} & \mathcal{Z}^2 X \\
 \downarrow \mathcal{Z}\eta_X^{\mathcal{Z}} & \searrow \iota & \downarrow \mu_X^{\mathcal{Z}} \\
 & \mathcal{T}X & \xrightarrow{\eta_{\mathcal{T}^2 X}} \mathcal{T}^2 X \\
 & \downarrow \mathcal{T}\eta_X & \downarrow \mu_X \\
 & \mathcal{T}^2 X & \xrightarrow{\mu_X} \mathcal{T}X \\
 \downarrow \iota^2 & \swarrow \iota^2 & \downarrow \iota_X \\
 \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} \mathcal{Z}X & \xrightarrow{\iota_X} \mathcal{T}X
 \end{array}$$

(1) and (2) involve the definition of $\mu^{\mathcal{Z}}$ and the naturality of ι and $\mu^{\mathcal{Z}}$, (3) is Def. 1, (4) definition of $\mu^{\mathcal{Z}}$ and (5) also. (6) and (7) involve the definition of $\eta^{\mathcal{Z}}$ and the naturality of ι and $\eta^{\mathcal{Z}}$, (8) is Def. 1, (9) definition of $\mu^{\mathcal{Z}}$ and (10) also.

\mathcal{Z} is proven strong with very similar diagrams. The commutative diagram:

The diagram illustrates the commutativity of various transformations between tensor products of objects in a monoidal category. The nodes represent different ways of associating the functors \mathcal{Z} and \mathcal{T} with the tensor product. The arrows represent natural transformations, and the blue numbers (1) through (12) indicate the specific properties or definitions used to prove the commutativity of each part of the diagram.

proves that \mathcal{Z} is a commutative monad, with (1) $\tau'^{\mathcal{Z}}$ is natural, (2) definition of $\tau^{\mathcal{Z}}$, (3) $\tau^{\mathcal{Z}}$ is natural, (4) \mathbf{C} is monoidal, (5) definition of $\tau'^{\mathcal{Z}}$, (6) ι is natural, (7) definition of $\mu^{\mathcal{Z}}$, (8) definition of $\tau^{\mathcal{Z}}$, (9) ι is central, (10) definition of $\tau'^{\mathcal{Z}}$, (11) ι is natural and (12) definition of $\mu^{\mathcal{Z}}$. \blacktriangleleft

► **Theorem 27.** *In the same situation as the previous theorem, the canonical embedding functor $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong \mathcal{Z}(\mathbf{C}_{\mathcal{T}})$.*

Proof.

- Let us note the corestriction $\hat{\mathcal{I}}$. For any f , Lemma 20 ensures that $\hat{\mathcal{I}}f$ is central.
- We define the inverse functor $G : \mathcal{Z}(\mathbf{C}_{\mathcal{T}}) \rightarrow \mathbf{C}_{\mathcal{Z}}$.
 G maps objects to themselves. If $f : X \rightarrow \mathcal{T}Y$ is a central morphism in the premonoidal sense, (X, f) is a central cone of \mathcal{T} at Y ; then there exists a unique $f^{\mathcal{Z}} : X \rightarrow \mathcal{Z}Y$ such that $\iota_Y \circ f^{\mathcal{Z}} = f$. We define $Gf = f^{\mathcal{Z}}$.
 The proof that it is a functor is direct considering that any $f^{\mathcal{Z}}$ is a morphism of central cones and that components of ι are monic.
- Let $f : X \rightarrow \mathcal{T}Y$ be a morphism of $\mathcal{Z}(\mathbf{C}_{\mathcal{T}})$, i.e. a central morphism. $\hat{\mathcal{I}}Gf = \iota_Y \circ f^{\mathcal{Z}} = f$ by definition of morphism of central cones (see Def. 18). Let $g : X \rightarrow \mathcal{Z}Y$ in \mathbf{C} . $\iota_Y \circ G\hat{\mathcal{I}}g = \iota_Y \circ (\iota_Y \circ g)^{\mathcal{Z}} = \iota_Y \circ g$ by Definition 18 and thus $G\hat{\mathcal{I}}g = g$ since ι_Y is monic.



Because of the previous two theorems, we are justified in naming the submonad \mathcal{Z} as *the* central submonad of \mathcal{T} .

4.2 The central theorem

► **Theorem 28** (Centralisability). *Let us consider a symmetric monoidal category (\mathbf{C}, \otimes, I) . A strong monad \mathcal{T} on \mathbf{C} is centralisable if any of the following equivalent criteria holds:*

1. *For any object X of \mathbf{C} , \mathcal{T} admits a terminal central cone at X ;*
2. *There exists a commutative submonad \mathcal{Z} of \mathcal{T} such that the canonical embedding functor $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$;*
3. *The corestriction of the left adjoint $\mathcal{J} : \mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ to $\hat{\mathcal{J}} : \mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$ is also a left adjoint.*

Proof.

- (1 \Rightarrow 2) Theorem 26 proves that the terminal central cones define a commutative submonad \mathcal{Z} of \mathcal{T} . Theorem 27 proves the isomorphism of categories.
- (2 \Rightarrow 3) Let us consider the left adjoint $\mathcal{J}^{\mathcal{Z}}$ associated to the monad \mathcal{Z} . All our hypotheses can be summed up in the following commutative diagram, because \mathcal{Z} is a submonad of \mathcal{T} :

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{\mathcal{J}} & \mathbf{C}_{\mathcal{T}} \\
 \mathcal{J}^{\mathcal{Z}} \downarrow & \searrow \hat{\mathcal{J}} & \uparrow \\
 \mathbf{C}_{\mathcal{Z}} & \xrightarrow[\hat{\mathcal{I}}]{\sim} & Z(\mathbf{C}_{\mathcal{T}})
 \end{array}$$

Since $\hat{\mathcal{I}}$ is isomorphic, $\hat{\mathcal{J}} = \hat{\mathcal{I}}\mathcal{J}^{\mathcal{Z}}$ is also a left adjoint.

- (3 \Rightarrow 1) Let $R : Z(\mathbf{C}_{\mathcal{T}}) \rightarrow \mathbf{C}$ be the right adjoint. We note Φ the natural isomorphism $Z(\mathbf{C}_{\mathcal{T}}) \cong \mathbf{C}_{\mathcal{Z}}$. If $f : Y \rightarrow \mathcal{T}X$ is central, meaning a morphism of $Z(\mathbf{C}_{\mathcal{T}})$, the following diagram:

$$\begin{array}{ccc}
 \hat{\mathcal{J}}Y & & \\
 \hat{\mathcal{J}}\Phi(f) \downarrow \quad \searrow f & & \\
 \hat{\mathcal{J}}RX & \xrightarrow{\varepsilon_X} & X
 \end{array}$$

commutes in $Z(\mathbf{C}_{\mathcal{T}})$. The corresponding diagram in \mathbf{C} :

$$\begin{array}{ccc}
 Y & & \\
 \Phi(f) \downarrow \quad \searrow f & & \\
 RX & \xrightarrow{\varepsilon_X} & X
 \end{array}$$

commutes. Thus f uniquely factors through the counit $\varepsilon_X : RX \rightarrow \mathcal{T}X$. It follows that (RX, ε_X) is a terminal central cone of \mathcal{T} at X .



4.3 A non Centralisable Monad

► **Example 29.** Consider the Dihedral group \mathbb{D}_4 which has 8 elements. Its center is non-trivial and has 2 elements.

Let \mathbf{C} be the full subcategory of \mathbf{Set} with objects that are finite products of the set \mathbb{D}_4 with itself. This category has a cartesian structure and the terminal object is the singleton set which is the empty product. The morphisms are the same as in \mathbf{Set} . Notice that every object in this category has cardinality which is a power of 8. Therefore the cardinality of every homset of \mathbf{C} is also a power of 8.

Since \mathbf{C} has a cartesian structure and since \mathbb{D}_4 is a monoid, we can consider the action monad \mathcal{M} on \mathbf{C} induced by \mathbb{D}_4 , meaning $\mathbb{D}_4 \times -$. This is a strong monad on \mathbf{C} . We will show that this monad is not centralisable.

Let us assume that there is a monad \mathcal{T} such that $\mathbf{C}_{\mathcal{T}} \cong Z(\mathbf{C}_{\mathcal{M}})$. There are homsets of $Z(\mathbf{C}_{\mathcal{M}})$ with cardinality 2, such as $Z(\mathbf{C}_{\mathcal{M}})[1, 1]$. $\mathbf{C}_{\mathcal{T}}$ cannot have such a homset since $\mathbf{C}_{\mathcal{T}}[X, Y] = \mathbf{C}[X, \mathcal{T}Y]$ which must have cardinality a power of 8. Therefore there exists no such monad and \mathcal{M} is not centralisable.

5 Computational interpretation (Work in Progress)

5.1 Syntax

Finally, we provide a computational interpretation of our ideas by presenting a refinement of Moggi's metalanguage [4]. The types are extended by simply adding the \mathcal{Z} unary connective that represents the central submonad of \mathcal{T} :

$$\begin{aligned} \text{(Types)} \quad a, b &::= 1 \mid a \rightarrow b \mid a \times b \mid \mathcal{Z}a \mid \mathcal{T}a \\ \text{(Terms)} \quad M, N &::= x \mid * \mid \lambda x^a. M \mid MN \mid \mathbf{ret}_{\mathcal{Z}} M \mid \iota M \\ &\quad \mid \mathbf{do} \ x \leftarrow_{\mathcal{Z}} M ; N \mid \mathbf{do} \ x \leftarrow_{\mathcal{T}} M ; N \end{aligned}$$

The typing rules are the following:

$$\begin{array}{c} \frac{}{\Gamma, x : a \vdash x : a} \quad \frac{}{\Gamma \vdash * : 1} \quad \frac{\Gamma \vdash M : a \rightarrow b \quad \Gamma \vdash N : a}{\Gamma \vdash MN : b} \quad \frac{\Gamma \vdash M : a \quad \Gamma \vdash N : b}{\Gamma \vdash \langle M, N \rangle : a \times b} \\[10pt] \frac{\Gamma \vdash M : a_1 \times a_2}{\Gamma \vdash \pi_i M : a_i} \quad \frac{\Gamma \vdash M : a}{\Gamma \vdash \mathbf{ret}_{\mathcal{Z}} M : \mathcal{Z}a} \quad \frac{\Gamma \vdash M : \mathcal{Z}a \quad \Gamma, x : a \vdash N : \mathcal{Z}b}{\Gamma \vdash \mathbf{do} \ x \leftarrow_{\mathcal{Z}} M ; N : \mathcal{Z}b} \\[10pt] \frac{\Gamma, x : a \vdash M : b}{\Gamma \vdash \lambda x^a. M : a \rightarrow b} \quad \frac{\Gamma \vdash M : \mathcal{Z}a}{\Gamma \vdash \iota M : \mathcal{T}a} \quad \frac{\Gamma \vdash M : \mathcal{T}a \quad \Gamma, x : a \vdash N : \mathcal{T}b}{\Gamma \vdash \mathbf{do} \ x \leftarrow_{\mathcal{T}} M ; N : \mathcal{T}b} \end{array}$$

As part of future work, we will describe a sound and adequate semantics of this system (which is fairly straightforward) and we will also consider the issue of full abstraction for specific monads with non-trivial centres.

5.2 Operational semantics

The operational semantics is defined as follows:

$$\begin{array}{c}
\frac{M \rightsquigarrow M'}{\text{do } x \leftarrow_{\mathcal{Z}} M ; N \rightsquigarrow \text{do } x \leftarrow_{\mathcal{Z}} M' ; N} \quad \frac{M \rightsquigarrow M'}{\text{do } x \leftarrow_{\mathcal{T}} M ; N \rightsquigarrow \text{do } x \leftarrow_{\mathcal{T}} M' ; N} \\
\frac{M \rightsquigarrow M'}{\text{ret}_{\mathcal{Z}} M \rightsquigarrow \text{ret}_{\mathcal{Z}} M'} \quad \frac{M \rightsquigarrow M'}{\iota M \rightsquigarrow \iota M'} \quad \frac{}{\text{do } x \leftarrow_{\mathcal{Z}} \text{ret}_{\mathcal{Z}} M ; N \rightsquigarrow N[M/x]} \\
\frac{}{\text{do } x \leftarrow_{\mathcal{T}} \iota(\text{ret}_{\mathcal{Z}} M) ; N \rightsquigarrow N[M/x]} \quad \frac{}{(\lambda x^a.M)N \rightsquigarrow M[N/x]} \\
\frac{M \rightsquigarrow M'}{\pi_i M \rightsquigarrow \pi_i M'} \quad \frac{M \rightsquigarrow M'}{\langle M, N \rangle \rightsquigarrow \langle M', N \rangle} \quad \frac{M \rightsquigarrow M'}{\langle N, M \rangle \rightsquigarrow \langle N, M' \rangle}
\end{array}$$

5.3 Denotational semantics

Let us consider a cartesian closed category \mathbf{C} equipped with a centralisable monad \mathcal{T} , whose central submonad is noted \mathcal{Z} . The types are denoted as objects in \mathbf{C} , as following:

$$\begin{aligned}
\llbracket 1 \rrbracket &= I \\
\llbracket a \rightarrow b \rrbracket &= \llbracket a \rrbracket \rightarrow \llbracket b \rrbracket \\
\llbracket \mathcal{Z}a \rrbracket &= \mathcal{Z} \llbracket a \rrbracket \\
\llbracket \mathcal{T}a \rrbracket &= \mathcal{T} \llbracket a \rrbracket
\end{aligned}$$

When $\Gamma = x_1 : a_1 \dots x_n : a_n$, $\llbracket \Gamma \rrbracket = \llbracket a \rrbracket_1 \times \dots \times \llbracket a \rrbracket_n$. We use morphisms of \mathbf{C} for terms. ev is the evaluation map and ϕ is the natural isomorphism $\mathbf{C}[X \times Y, Z] \cong \mathbf{C}[X, Z^Y]$. Their semantics is defined recursively on the typing derivation:

$$\begin{aligned}
\llbracket \Gamma, x : a \vdash x : a \rrbracket &= \pi_a \\
\llbracket \Gamma \vdash \pi_i M : a \rrbracket &= \pi_a \circ \llbracket \Gamma \vdash M : a \times b \rrbracket \\
\llbracket \Gamma \vdash \langle M, N \rangle : a \times b \rrbracket &= \langle \llbracket \Gamma \vdash M : a \rrbracket, \llbracket \Gamma \vdash N : b \rrbracket \rangle \\
\llbracket \Gamma \vdash MN : b \rrbracket &= ev_{\llbracket a \rrbracket, \llbracket b \rrbracket} \circ \langle \llbracket \Gamma \vdash M : a \rightarrow b \rrbracket, \llbracket \Gamma \vdash N : a \rrbracket \rangle \\
\llbracket \Gamma \vdash \lambda x^a.M : a \rightarrow b \rrbracket &= \phi_{\llbracket \Gamma \rrbracket, \llbracket a \rrbracket, \llbracket b \rrbracket}(\llbracket \Gamma, x : a \vdash M : b \rrbracket) \\
\llbracket \Gamma \vdash \text{ret}_{\mathcal{Z}} M : \mathcal{Z}a \rrbracket &= \eta_a^{\mathcal{Z}} \circ \llbracket \Gamma \vdash M : a \rrbracket \\
\llbracket \Gamma \vdash \iota M : \mathcal{T}a \rrbracket &= \iota_{\llbracket a \rrbracket} \circ \llbracket \Gamma \vdash M : \mathcal{Z}a \rrbracket \\
\llbracket \Gamma \vdash \text{do } x \leftarrow_{\mathcal{Z}} M ; N : \mathcal{Z}b \rrbracket &= \mu_{\llbracket b \rrbracket}^{\mathcal{Z}} \circ \mathcal{Z} \llbracket N \rrbracket \circ \tau_{\llbracket \Gamma \rrbracket, \llbracket a \rrbracket}^{\mathcal{Z}} \circ (\llbracket \Gamma \rrbracket \times \llbracket M \rrbracket) \circ \Delta \\
\llbracket \Gamma \vdash \text{do } x \leftarrow_{\mathcal{T}} M ; N : \mathcal{T}b \rrbracket &= \mu_{\llbracket b \rrbracket} \circ \mathcal{T} \llbracket N \rrbracket \circ \tau_{\llbracket \Gamma \rrbracket, \llbracket a \rrbracket} \circ (\llbracket \Gamma \rrbracket \times \llbracket M \rrbracket) \circ \Delta
\end{aligned}$$

5.4 Properties

- **Theorem 30** (Soundness). *If $M \rightsquigarrow M'$, then $\llbracket M \rrbracket = \llbracket M' \rrbracket$.*
- **Theorem 31** (Adequacy). *If $\llbracket M \rrbracket = \llbracket M' \rrbracket$, then there exists a value V such that $M \rightsquigarrow^* V$ and $M' \rightsquigarrow^* V$.*
- **Theorem 32.**

$$\llbracket \text{do } x \leftarrow_{\mathcal{T}} \iota M ; \text{do } y \leftarrow_{\mathcal{T}} N ; P \rrbracket = \llbracket \text{do } y \leftarrow_{\mathcal{T}} N ; \text{do } x \leftarrow_{\mathcal{T}} \iota M ; P \rrbracket$$

6 Conclusion

We defined the central submonad of a strong monad.

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