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ON BANG–BANG CONSTRAINED SOLUTIONS OF A CONTROL SYSTEM

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Abstract. Given \( \phi_1, \phi_2 \in L^1([0,T]) \) and a function \( x \in W^{2,1}([0,T]) \) solving the control problem (P)

\[
x'' + a_1(t)x' + a_0(t)x \in [\phi_1(t), \phi_2(t)] \quad \text{a.e.}
\]

\[
x(0) = x_0, \quad x(T) = x_1, \quad x'(0) = v_0, \quad x'(T) = v_1
\]

there exists a bang–bang solution \( y \) to (P) satisfying \( y \leq x \); moreover there exists a finite union of intervals \( E \) such that 

\[
y'' + a_1y' + a_0y = \phi_1 \chi_E + \phi_2 \chi_{[0,T]\setminus E}.
\]

The reachable set of bang–bang constrained solutions is convex; an application to the calculus of variations.

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Key words and phrases. bang–bang, linear control system, range of a vector measure, reachable set, calculus of variations.

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1. Introduction

We consider the family of bidimensional linear control systems \((P)\) described by a generic second order equation subject to a scalar control:

\[ x'' + a_1(t)x' + a_0(t)x \in \Phi(t) = [\phi_1(t), \phi_2(t)] \text{ a.e., } (x(0), x'(0), x(T), x'(T)) = (x_0, v_0, x_1, v_1) \]

where \( \phi_1 \leq \phi_2 \in L^1([0, T]) \) and \( a_1, a_0 \in C([0, T]) \), \( x_0, v_0, x_1, v_1 \in \mathbb{R} \), \( x \in W^{2,1}([0, T]) \).

The function \( y \) is said to be a bang–bang solution to \((P)\) if it solves \((P)\) and moreover

\[ y'' + a_1(t)y' + a_0(t)y \in \text{extr } \Phi(t) = \{ \phi_1(t), \phi_2(t) \} \text{ a.e.} \tag{1} \]

Existence of bang–bang solutions has been proved for instance by Cesari [4, Theorem 16.3].

The purpose of this paper is to prove that given an arbitrary solution \( x \) to \((P)\), there exists a bang–bang solution \( y \) such that

\[ \forall t \in [0, T] \quad y(t) \leq x(t) \tag{2} \]

and in addition \( y'' + a_1y' + a_0y \) steers from \( \phi_1 \) to \( \phi_2 \) only a finite number of times.

Motivation of such a problem was to study the reachable set

\[ \mathcal{Y}_T = \{(y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \text{extr } \Phi(t) \text{ a.e., } (y(0), y'(0)) = (x_0, v_0)\} \]

where \( c \) is an arbitrary function. A consequence of Theorem 3.1 is that \( \mathcal{Y}_T \) coincides with

\[ \mathcal{X}_T = \{(y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \Phi(t) \text{ a.e., } (y(0), y'(0)) = (x_0, v_0)\} \]

Notice that \( \mathcal{X}_T \) is convex so that the above assumption implies that \( \mathcal{Y}_T \) is convex too.

Another motivation arises from non–convex problems of the calculus of variations (see [1]).

A possible approach in order to find bang–bang solutions is to use Lyapunov Theorem on the range of a vector measure [4, §16.1].

Here, the solution of \( x'' + a_1(t)x' + a_0(t)x = \rho(t), x(0) = x'(0) = 0 \) is given by

\[ x(t) = \int_0^t h(t, s)\rho(s) \, ds \]

where \( h \in C^1([0, T] \times [0, T]) \) and for each \( s \in [0, T] \) the function \( h_s(.) = h(., s) \in C^2([0, T]) \)

is the solution to the associated homogeneous differential equation satisfying the initial conditions \( h_s(0) = 0, h_s'(0) = 1 \); Lyapunov Theorem yields the existence of a measurable subset \( E \) of \([0, T]\) such that

\[ \int_0^T h(T, s)\rho(s) \, ds = \int_0^T h(T, s)\chi_E(s) + \phi_2(s)\chi_{[0, T] \setminus E}(s) \, ds, \tag{3} \]

\[ \int_0^T \frac{\partial h}{\partial t}(T, s)\rho(s) \, ds = \int_0^T \frac{\partial}{\partial t}(T, s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0, T] \setminus E}(s)) \, ds. \tag{4} \]
Clearly, by differentiating under the integral sign, the function $y$ defined by

$$y(t) = \int_0^t h(t, s) \phi_1(s) \chi_E(s) + \phi_2(s) \chi_{[0,T] \setminus E}(s) \, ds$$

(5)

is a bang–bang solution. However, this approach does not give any information on the behaviour of $y$ with respect to $x$ on $[0, T]$. Here we prove a new Lyapunov’s type theorem concerning the range of a two–dimensional vector measure whose densities are such that their quotient is monotone: in this case, the set $E$ can be chosen in the form $[\alpha, \beta]$. Remark that this is not true in general; for instance there are no $\alpha, \beta \in [0, 3\pi]$ satisfying

$$\int_\alpha^\beta \sin t \, dt = \int_0^{3\pi} \sin t \chi_{[\alpha, \beta]}(t) \, dt.$$ 

In our application the equalities $h(s, s) = 0$ and $\frac{\partial h}{\partial t}(s, s) = 1$ imply that the monotonicity condition is locally fulfilled; this allows us to build a set $E$ satisfying (3)–(4) as a finite union of intervals and, in the case where $\phi_1 < \rho < \phi_2$ are continuous, to choose $E$ in such a way that neither 0 nor $T$ belong to its closure.

These facts, together with a decomposition of the kernel $h(t, s)$ into a linear combination of linearly independent functions are the main tools that we use in order to show that the bang–bang solution $y$ defined by (5) satisfies the inequality $y \leq x$.

As an application we consider the problem of minimizing the integral functionals

$$I(x, u) = \int_0^T f(t, x(t), u(t)) \, dt$$

where $x : [0, T] \to \mathbb{R}^2$ is such that $x(0), x'(0), x(T), x'(T)$ are fixed and $u$ is a control belonging to $U(t, x) \subset \mathbb{R}^2$. The classical approach to obtain existence of a minimum is to impose conditions in order to have the lower semicontinuity of $I$ with respect to $u$ (for instance convexity of $u \mapsto f(t, x, u)$).

Recently in an effort to provide existence criteria other than convexity in $u$ some sufficient conditions have been given: for problems of the calculus of variations ($x' = u$ in the above setting) and for maps of the form $f(t, x, x') = g(t, x) + h(t, x')$, existence of solutions has been obtained by requiring that the real map $x \mapsto g(t, x)$ be monotone [5] or, for $x$ in $\mathbb{R}^n$, that the same function be concave [2]. Optimal control problems escaping to convexity conditions have been handled in [6].

It has been proved further in [3] that there exists a dense subset $D$ of $C(\mathbb{R})$ such that, for $g$ in it, the problem

$$\text{minimize } \int_0^T g(x(t)) \, dt + \int_0^T h(x'(t)) \, dt : \quad x(0) = x_0, x(T) = x_1$$

admits a solution for every lower semicontinuous $h$ satisfying growth conditions.

Our theorem gives a straightforward generalization of the above result.
2. Assumptions and Preliminary Results

Let $\phi_1, \phi_2 \in L^1[0,T]$, $\phi_1 \leq \phi_2$ and put $\Phi(t) = [\phi_1(t), \phi_2(t)] \subset \mathbb{R}$. We are interested in the solutions of the following control problem.

**Problem P.** $a_1, a_0 \in C([0,T])$, $x_0, x_1, v_0, v_1 \in \mathbb{R}$, $x \in W^{2,1}[0,T]

\begin{align*}
  x'' + a_1(t)x' + a_0(t)x & \in \Phi(t) \text{ a.e. (P)} \\
  x(0) = x_0, & \quad x'(0) = v_0, \quad x(T) = x_1, & \quad x'(T) = v_1.
\end{align*}

By extr $\Phi$ we mean the extreme points of $\Phi$ i.e. extr $\Phi(t) = \{\phi_1(t), \phi_2(t)\}$.

**Definition 2.1.** A function $y \in W^{2,1}[0,T]$ is said to be a bang–bang solution to (P) if $y$ solves (P) and, moreover,

\begin{align*}
  y'' + a_1(t)y' + a_0(t)y & \in \text{extr } \Phi(t) \text{ a.e.}
\end{align*}

The following representation formula of the solutions to (P) will be used later.

**Proposition 2.1.** There exists a function $h \in C^1([0,T] \times [0,T])$ satisfying Property (S) below such that, for each function $\rho \in L^1([0,T])$, the solution of

\begin{align*}
  x'' + a_1(t)x' + a_0(t)x &= \rho(t), \quad x(0) = x'(0) = 0 \quad (P_{\rho})
\end{align*}

is given by the formula

\begin{align*}
  x(t) &= \int_0^t h(t,s)\rho(s) \, ds. \tag{2.1}
\end{align*}

Moreover for each $s \in [0,T]$ the function $h(.,s)$ is of class $C^2([0,T])$.

**Property S.**

1) There exist $w_1, w_2 \in C^2([0,T])$, $z_1, z_2 \in C^1([0,T])$ such that

\begin{align*}
  \forall s, t \in [0,T] \quad h(t, s) = w_1(t)z_1(s) + w_2(t)z_2(s) \tag{2.2}
\end{align*}

and $W(w_1, w_2, t) = \det \begin{vmatrix}
  w_1(t) & w_2(t) \\
  w_1'(t) & w_2'(t)
\end{vmatrix} \neq 0$.

For each $t_0$ in $[0,T]$ there exists $\delta > 0$ such that if we set $I_\delta = [t_0 - \delta, t_0 + \delta] \cap [0,T]$ then:

2) $\forall s, t \in I_\delta \quad h(t, s) > 0$ if $s < t$, $h(t, s) < 0$ if $t < s$ (whence $h(s, s) = 0$);

3) $\forall s, t \in I_\delta \quad \frac{\partial h}{\partial t}(t, s) > 0$;

4) $\forall t \in I_\delta \quad s \mapsto h(t, s)/\frac{\partial h}{\partial t}(t, s)$ is decreasing on $I_\delta$.  


Proof of Proposition 2.1. For each \( s \in [0,T] \), let \( h_s(\cdot) = h(\cdot, s) \in C^2([0,T]) \) be the solution to
\[
  h''_s(t) + a_1(t) h'_s(t) + a_0(t) h_s(t) = 0 \quad h_s(s) = 0, \quad h'_s(s) = 1.
\]
Set \( z(t) = \int_0^t h(t,s) \rho(s) \, ds \). Differentiation under the integral sign shows that \( z \) is a solution to \( (P_\rho) \) whence, by uniqueness, \( z = x \).

In order to prove the second part of the claim, let \( w_1, w_2 \in C^2([0,T]) \) be two solutions of the differential equation
\[
x'' + a_1(t) x' + a_0(t) x = 0 \quad (2.3)
\]
such that their wronskian
\[
W(w_1, w_2, t) = \det \begin{vmatrix} w_1(t) & w_2(t) \\ w'_1(t) & w'_2(t) \end{vmatrix}
\]
is non zero for every \( t \). Such functions exist since the set of the solutions of a second order linear differential equation is a two–dimensional vector space. Since for each \( s \in [0,T] \), the function \( h_s \) is a solution to (2.3) then there exist \( z_1, z_2 \) defined on \([0,T]\) such that
\[
\forall s, t \in [0,T] \quad h_s(t) = w_1(t) z_1(s) + w_2(t) z_2(s). \quad (2.4)
\]
Conditions on \( h_s \) at \( s \) and equation (2.4) yield
\[
\begin{align*}
  h_s(s) &= 0 = w_1(s) z_1(s) + w_2(s) z_2(s) \\
  h'_s(s) &= 1 = w'_1(s) z_1(s) + w'_2(s) z_2(s)
\end{align*}
\]
Since \( W(w_1, w_2, s) \neq 0 \) for each \( s \), we find
\[
  z_1(s) = -\frac{w_2(s)}{W(w_1, w_2, s)} , \quad z_2(s) = \frac{w_1(s)}{W(w_1, w_2, s)}
\]
so that \( z_1, z_2 \in C^1([0,T]) \) hence \( h(t,s) = h_s(t) \) belongs to \( C^1([0,T] \times [0,T]) \).

By construction
\[
\forall s \in [0,T] \quad h(s, s) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(s, s) = 1
\]
implying
\[
\forall s \in [0,T] \quad \frac{d}{ds} h(s, s) = 0 \iff \forall s \in [0,T] \quad \frac{\partial h}{\partial t}(s, s) + \frac{\partial h}{\partial s}(s, s) = 0
\]
\[
\iff \forall s \in [0,T] \quad \frac{\partial h}{\partial s}(s, s) = -1.
\]
As a consequence

\[ \forall s \in [0, T] \quad \frac{\partial}{\partial s} \left( \frac{h}{\partial t} \right) (s, s) = -1. \]

By continuity for a fixed \( t_0 \) in \([0, T]\), there exists \( \delta > 0 \) such that

\[ \forall s, t \in [t_0 - \delta, t_0 + \delta] \cap [0, T] \quad \frac{\partial h}{\partial t} (t, s) > 0 \quad \text{and} \quad \frac{\partial}{\partial s} \left( \frac{h}{\partial t} \right) (t, s) < 0; \]

for this \( \delta \) properties S 2)(3)4) are satisfied.

Assume for instance \( \Phi(t) = [0, \phi(t)] \) and let \( \mu \in L^1[0, T] \) be such that \( 0 \leq \rho \leq \phi \). For a solution \( x \) to \((P_{\mu})\) formula (2.1) yields, in particular,

\[ x(T) = \int_0^T h(T, s) \rho(s) \, ds, \quad (2.5) \]
\[ x'(T) = \int_0^T \frac{\partial h}{\partial t}(T, s) \rho(s) \, ds. \quad (2.6) \]

Let us point out that the classical Lyapunov Theorem on the range of a vector measure [4, §16.1] allows to find a bang–bang solution. In fact its application yields the existence of a measurable subset \( E \) of \([0, T]\) such that

\[ \int_0^T h(T, s) \rho(s) \, ds = \int_0^T h(T, s) \phi(s) \chi_E(s) \, ds, \quad (2.7) \]
\[ \int_0^T \frac{\partial h}{\partial t}(T, s) \rho(s) \, ds = \int_0^T \frac{\partial h}{\partial t}(T, s) \phi(s) \chi_E(s) \, ds, \quad (2.8) \]

so that the function \( \bar{x} \) defined by

\[ \bar{x}(t) = \int_0^t h(t, s) \phi(s) \chi_E(s) \, ds \]

is, by Proposition 2.1, a bang–bang solution to \((P)\) (with \( \phi_1 = 0, \phi_2 = \phi, x_0 = v_0 = 0 \)). However, for \( 0 < t < T \), the Lyapunov Theorem does not give any information on the relative positions of \( \bar{x} \) and the original solution \( x \).

The purpose of Proposition 2.2 below is to show that if \( s \mapsto \left( \frac{h}{\partial t} \right) (t, s) \) is monotone on \([0, T]\) then the measurable subset \( E \) can be chosen to be an interval \([\alpha, \beta]\) with \( 0 \leq \alpha \leq \beta \leq T \). This will allow us, taking into account property S 4), to define in §3 a bang–bang solution \( y \) satisfying \( y(t) \leq x(t) \) for each \( t \).

In what follows \([a, b]\) is an interval of \( \mathbb{R} \), \( \rho \) and \( \phi \) are two functions belonging to \( L^1([a, b]) \) satisfying \( 0 \leq \rho \leq \phi \). We say that \( r \in \mathbb{R} \) is positive (resp. negative) if \( r \geq 0 \) (resp. \( r \leq 0 \)). We consider the following hypothesis.

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**Hypothesis H.** The functions \( f, g \) belong to \( L^\infty([a,b]) \) and are positive almost everywhere. Moreover there exists a strictly monotone positive function \( k \) such that
\[
g(t) = k(t)f(t) \quad \text{a.e.}
\]

We have the following Lyapunov’s type result.

**Proposition 2.2.** Let \( f, g \) satisfy hypothesis H. Then there exist \( \alpha, \beta \in \mathbb{R} \) such that, if we put \( E = [\alpha, \beta] \), we have:
\[
\int_a^b \rho(s)f(s)\,ds = \int_\alpha^\beta \phi(s)f(s)\,ds = \int_a^b \phi(s)f(s)\chi_E(s)\,ds; \quad (2.9)
\]
\[
\int_a^b \rho(s)g(s)\,ds = \int_\alpha^\beta \phi(s)g(s)\,ds = \int_a^b \phi(s)g(s)\chi_E(s)\,ds. \quad (2.10)
\]
Moreover, \( \alpha \) and \( \beta \) are unique if \( \rho, \phi, f, g \) are continuous, and \( 0 < \rho < \phi, f > 0, g > 0 \).

In order to prove Proposition 2.2, we need the following fundamental Lemma.

**Lemma 2.1.** Assume that \( f, g \) satisfy hypothesis H and let \( \alpha, \beta \in [a,b] \) be such that
\[
\int_a^\beta \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds \quad (2.11)
\]
\[
\int_a^\alpha \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds. \quad (2.12)
\]
Then, if \( k \) is increasing, we have
\[
\int_a^\beta \phi(s)g(s)\,ds \geq \int_a^b \rho(s)g(s)\,ds, \quad (2.13)
\]
\[
\int_a^\alpha \phi(s)g(s)\,ds \leq \int_a^b \rho(s)g(s)\,ds. \quad (2.14)
\]
If \( k \) is decreasing on \([a,b]\), inequalities (2.13) and (2.14) are reversed. Moreover, inequalities (2.13)–(2.14) are strict if \( 0 < \rho < \phi \) and \( f > 0, g > 0 \) a.e.

**Proof of Lemma 2.1.** Assume for instance that \( k \) is increasing. To prove (2.14) let \( f_{\phi}, f_{\rho} \) be the monotone functions defined by
\[
f_{\phi}(t) = \int_a^t \phi(s)f(s)\,ds \quad f_{\rho}(t) = \int_a^t \rho(s)f(s)\,ds.
\]
The Lebesgue–Stieltjes formula for integration by parts yields
\[
\int_a^b \rho(s)g(s) \, ds = \int_a^b \rho(s)k(s)f(s) \, ds = \int_a^b k(s) \, df_\rho(s) = k(b)f(b) - k(a)f(a) - \int_a^b f_\rho(s) \, dk(s);
\]
analogously we have
\[
\int_a^\beta \phi(s)g(s) \, ds = k(\beta)f_\phi(\beta) - k(a)f_\phi(a) - \int_a^\beta f_\phi(s) \, dk(s).
\]
Taking into account that \(f_\phi(a) = f_\rho(a) = 0\) and that by (2.12) \(f_\rho(b) = f_\phi(\beta)\), we are thus led to show that
\[
\int_a^b f_\rho(s) \, dk(s) - \int_a^\beta f_\phi(s) \, dk(s) \leq (k(b) - k(\beta))f_\rho(b). \quad (2.15)
\]
By our assumptions we have
\[
\forall t \in [a, b] \quad f_\phi(t) \geq f_\rho(t); \quad (2.16)
\]
therefore
\[
\int_a^b f_\rho(s) \, dk(s) - \int_a^\beta f_\phi(s) \, dk(s) \leq \int_a^\beta f_\phi(s) \, dk(s). \quad (2.17)
\]
Furthermore the functions \(f_\rho\) and \(k\) being increasing we have
\[
\int_a^b f_\rho(s) \, dk(s) \leq (k(b) - k(\beta))f_\rho(b)
\]
which, together with (2.17), gives (2.15).

To prove the final part of the lemma, it is enough to remark that if \(f > 0\) and \(\rho > 0\) then, by (2.12), \(\beta \neq a\); if moreover \(0 < \rho < \phi\) a.e. then inequality (2.16) is strict for every \(t > a\) so that (2.17) is strict too (\(k\) being increasing). Similar arguments prove (2.13). □
Proof of Proposition 2.2.

i) Existence.

a) Assume first $0 < \rho < \phi$ and $f > 0$, $g > 0$ a.e. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [a,b]$ be such that

\[
\int_{\alpha_1}^{b} \phi(s)f(s)\,ds = \int_{a}^{b} \rho(s)f(s)\,ds, \tag{2.18}
\]
\[
\int_{\alpha_2}^{b} \phi(s)g(s)\,ds = \int_{a}^{b} \rho(s)g(s)\,ds, \tag{2.19}
\]
\[
\int_{\beta_1}^{b} \phi(s)f(s)\,ds = \int_{a}^{b} \rho(s)f(s)\,ds, \tag{2.20}
\]
\[
\int_{\beta_2}^{b} \phi(s)g(s)\,ds = \int_{a}^{b} \rho(s)g(s)\,ds. \tag{2.21}
\]

Assume for instance that $k$ is decreasing on $[a,b]$. In this situation Lemma 2.1 yields

\[\beta_2 \leq \beta_1 \quad \alpha_2 \leq \alpha_1. \tag{2.22}\]

The function $v$ defined by

\[v(x) = \int_{a}^{x} \phi(s)f(s)\,ds\]

is continuous and increasing with values in $[0, v(b)]$: let $v^{-1}$ denote its inverse function. Set $m$ to be

\[m = \int_{a}^{b} \rho(s)f(s)\,ds.\]

Since, by (2.18), $v(b) = v(\alpha_1) + m$ then $v(\alpha) + m \in [0, v(b)]$ if and only if $a \leq \alpha \leq \alpha_1$; this allows us to introduce the continuous function $\xi_1$ defined by the formula

\[\forall \alpha \in [a, \alpha_1] \quad \xi_1(\alpha) = v^{-1}(v(\alpha) + m).\]

By definition, we have

\[\forall \alpha \in [a, \alpha_1] \quad \int_{\alpha}^{\xi_1(\alpha)} \phi(s)f(s)\,ds = v(\xi_1(\alpha)) - v(\alpha) = m = \int_{a}^{b} \rho(s)f(s)\,ds \tag{2.23}\]

so that, by (2.20) and (2.22), we deduce

\[\forall \alpha \in [a, \alpha_1] \quad \xi_1(\alpha) \geq \beta_1 \geq \beta_2. \tag{2.24}\]
Similarly, equality (2.21) allows to define a continuous function \( \xi_2 : [\beta_2, b] \to \mathbb{R} \) such that we have
\[
\forall \beta \geq \beta_2 \quad \int_{\xi_2(\beta)}^{\beta} \phi(s)g(s) \, ds = \int_a^b \rho(s)g(s) \, ds \tag{2.25}
\]
from which joint with (2.19) and (2.22) we deduce
\[
\forall \beta \geq \beta_2 \quad \xi_2(\beta) \leq \alpha_2 \leq \alpha_1. \tag{2.26}
\]
We deduce from (2.24) and (2.26) that the composed application
\[
\xi_2 \circ \xi_1 : [a, \alpha_1] \xrightarrow{\xi_1} [\beta_2, b] \xrightarrow{\xi_2} [a, \alpha_1]
\]
is defined and continuous from \([a, \alpha_1]\) into itself and therefore admits a fixed point \(\bar{\alpha}\). Thus, if we set \(\bar{\beta} = \xi_1(\bar{\alpha})\) we have \(\bar{\alpha} = \xi_2(\bar{\beta})\). Equalities (2.23) and (2.25) with \(\alpha, \beta\) replaced by \(\bar{\alpha}, \bar{\beta}\) yield the conclusion.

b) let \(\rho_n = \rho + \frac{1}{n}\), \(\phi_n = \phi + \frac{2}{n}\), \(f_n = f + \frac{1}{n}\) so that \(0 < \rho_n < \phi_n\) and \(f_n > 0\) a.e. and set \(g_n = k f_n\) so that the monotonicity of \(k\) implies that \(g_n > 0\) a.e. and \(f_n, g_n\) satisfy H. By a) there exist \(\alpha_n, \beta_n\) such that
\[
\int_a^b \rho_n(s)f_n(s) \, ds = \int_{\alpha_n}^{\beta_n} \phi_n(s)f_n(s) \, ds; \tag{2.27}
\]
\[
\int_a^b \rho_n(s)g_n(s) \, ds = \int_{\alpha_n}^{\beta_n} \phi_n(s)g_n(s) \, ds. \tag{2.28}
\]
By compactness we may assume \(\alpha_n \to \alpha, \beta_n \to \beta\). The conclusion follows by passing through the limit in (2.27) and (2.28).

ii) Uniqueness.
Assume that \(0 < \rho < \phi, f > 0, g > 0\) are continuous and that, for instance, \(k\) is decreasing. By i) a) the points \(\alpha\) such that there exists \(\beta\) satisfying (2.11) and (2.12) are the fixed points of the composed map \(\xi_2 \circ \xi_1\). By definition the functions \(\xi_1, \xi_2\) are differentiable and we have
\[
\forall \alpha \in [a, \alpha_1] \quad \xi'_1(\alpha) = \frac{\psi'(\alpha)}{\phi'(\xi_1(\alpha))} = \frac{\phi(\alpha)f(\alpha)}{\phi(\xi_1(\alpha))f(\xi_1(\alpha))};
\]
\[
\forall \beta \in [\beta_2, b] \quad \xi'_2(\beta) = \frac{\phi(\beta)g(\beta)}{\phi(\xi_2(\beta))g(\xi_2(\beta))}.\]
In order to prove the claim we notice that if \( \alpha \) satisfies \( \xi_2 \circ \xi_1(\alpha) = \alpha \) then

\[
(\xi_2 \circ \xi_1)'(\alpha) = \xi_2'(\xi_1(\alpha))\xi_1'(\alpha) = \frac{k(\xi_1(\alpha))}{k(\alpha)}.
\]  

(2.29)

By (2.23) we have \( \xi_1(\alpha) > \alpha \) so that the strict monotonicity of \( k \) implies \( k(\xi_1(\alpha)) < k(\alpha) \) and thus \( (\xi_2 \circ \xi_1)'(\alpha) < 1 \) whenever \( \xi_2 \circ \xi_1(\alpha) = \alpha \). Let \( S = \{ \alpha \in [a, b] : \xi_2 \circ \xi_1(\alpha) = \alpha \} \). Clearly, \( S \) is compact and non-empty by 1); moreover, taking (2.29) into account, for each \( \alpha \in S \) there exists \( \eta \) such that

\[
\forall t \in [\alpha - \eta, \alpha] \quad \xi_2 \circ \xi_1(t) > t
\]
\[
\forall t \in [\alpha, \alpha + \eta] \quad \xi_2 \circ \xi_1(t) < t.
\]  

(2.30)

As a consequence, the set \( S \) has no accumulation points and is therefore finite.

Let \( \alpha_1 = \min S \) and assume \( S \neq \{ \alpha_1 \} \); let \( \alpha_2 = \min S \setminus \{ \alpha_1 \} \). Then by (2.30) there exist \( t_1 < t_2 \in [\alpha_1, \alpha_2] \) such that \( \xi_2 \circ \xi_1(t_1) < t_1 \) and \( \xi_2 \circ \xi_1(t_2) > t_2 \). Therefore there exists \( \bar{t} \in [t_1, t_2] \) such that \( \xi_2 \circ \xi_1(\bar{t}) = \bar{t} \), a contradiction. \( \square \)
3. Main Result

**Theorem 3.1.** Let $x \in W^{2,1}([0,T])$ be a solution to $(P)$. Then there exists a bang–bang solution $y$ to $(P)$ satisfying

$$\forall t \in [0,T] \quad y(t) \leq x(t).$$

Moreover there exists a set $E$ which is a finite union of intervals such that

$$y'' + a_1(t)y' + a_0(t)y = \phi_1(t)\chi_E(t) + \phi_2(t)\chi_{[0,T]\setminus E}(t) \text{ a.e.}$$

**Corollary 1.** Under the above assumption, there exists a bang–bang solution $y$ satisfying

$$\forall t \in [0,T] \quad y(t) \geq x(t).$$

**Proof of Corollary 1.** Let $-\Phi$ be defined by the equality $(-\Phi)(t) = -\Phi(t)$. Clearly, $\tilde{x} = -x$ solves

$$\tilde{x}'' + a_1(t)\tilde{x}' + a_0(t)\tilde{x} \in -\Phi(t) \text{ a.e.}$$

By Theorem 3.1 there exists a bang–bang solution $\tilde{y}$ satisfying the same boundary conditions as $\tilde{x}$ and satisfying

$$\forall t \in [0,T] \quad \tilde{y}(t) \leq \tilde{x}(t).$$

Then the function $y$ defined by

$$\forall t \in [0,T] \quad y(t) = -\tilde{y}(t)$$

is a solution to our problem. □

**Proof of Theorem 3.1.** Let $h$ be the function defined in Proposition 2.1.

i) We show that it is not restrictive to assume

$$\Phi(t) = [0, \phi(t)] \quad (\phi \in L^1([0,T]), \phi > 0 \text{ a.e.}) \quad \text{and} \quad x_0 = v_0 = 0.$$ 

In fact, let $\Phi(t) = [\phi_1(t), \phi_2(t)]$ and $x$ satisfy

$$x'' + a_1(t)x' + a_0(t)x \in \Phi(t) \text{ a.e.}$$

Then the function $\tilde{x}$ defined by

$$\tilde{x}(t) = x(t) - x'(0)t - x(0)$$
satisfies \( \ddot{x}(0) = \dot{x}'(0) = 0 \) and

\[
\dddot{x} + a_1(t)\ddot{x} + a_0(t)\dot{x} \in [\psi_1(t), \psi_2(t)] \text{ a.e.}
\]

where

\[
\psi_1(t) = \phi_1(t) - a_0(t)x''(0)t - a_1(t)x'(0) - a_0(t)x(0),
\]

\[
\psi_2(t) = \phi_2(t) - a_0(t)x''(0)t - a_1(t)x'(0) - a_0(t)x(0).
\]

Moreover, by Proposition 2.1, the function \( \bar{x} \) defined by

\[
\bar{x}(t) = \ddot{x}(t) - \int_0^t h(t,s)\psi_1(s) \, ds
\]

satisfies \( \bar{x}(0) = 0, \bar{x}'(0) = 0 \) and

\[
\dddot{x} + a_1(t)\ddot{x} + a_0(t)\dot{x} \in [0, \psi_2(t) - \psi_1(t)] \text{ a.e.}
\]

If we assume that Theorem 3.1 holds for such an interval and initial boundary conditions, there exists a function \( \bar{y} \) satisfying

\[
\bar{y}(0) = \bar{x}(0), \quad \bar{y}'(0) = \bar{x}'(0), \quad \bar{y}(T) = \bar{x}(T), \quad \bar{y}'(T) = \bar{x}'(T),
\]

\[
\bar{y}'' + a_1(t)\bar{y}' + a_0(t)\bar{y} \in \{0, \psi_2(t) - \psi_1(t)\} \text{ a.e.,}
\]

\[
\forall t \in [0,T] \quad \bar{y}(t) \leq \bar{x}(t).
\]

It is now easy to check that the function \( y \) defined by

\[
y(t) = \bar{y}(t) + \int_0^t h(t,s)\psi_1(s) \, ds + x'(0)t + x(0)
\]

is a solution to our problem.

ii) Assume first that \( \delta \) of property (S) can be chosen in such a way that \( I_\delta = [0,T] \). In this case, if we set

\[
\rho = \dddot{x} + a_1\dot{x}' + a_0x
\]

then by Proposition 2.1 we can write

\[
x(t) = \int_0^t h(t,s)\rho(s) \, ds,
\]
where $h$ satisfies property S 1) and in addition:

$$\forall s, t \in [0, T] \quad h(t, s) > 0 \text{ if } s < t, \quad h(t, s) < 0 \text{ if } t < s \tag{3.2}$$

$$\forall s, t \in [0, T] \quad \frac{\partial h}{\partial t}(t, s) > 0, \tag{3.3}$$

$$\forall t \in [0, T] \quad s \mapsto h(t, s)/\frac{\partial h}{\partial t}(t, s) \text{ is decreasing on } [0, t]. \tag{3.4}$$

In particular the functions $f$ and $g$ defined on $[0, T]$ by

$$g(s) = h(T, s) \quad f(s) = \frac{\partial h}{\partial t}(T, s)$$

verify hypothesis H with $k(.) = h(T, .)/\frac{\partial h}{\partial t}(T, .)$.

By Proposition 2.1, each bang–bang solution $y$ such that $x(0) = x'(0) = 0$ is given by the formula $y(t) = \int_0^t h(t, s) \nu(s) \, ds$ for some measurable function $\nu$ with values in $\{0, \phi(t)\}$.

We are thus led to show that there exists such a $\nu$ satisfying

$$\int_0^T h(T, s) \rho(s) \, ds = \int_0^T h(T, s) \nu(s) \, ds, \tag{3.5}$$

$$\int_0^T \frac{\partial h}{\partial t}(T, s) \rho(s) \, ds = \int_0^T \frac{\partial h}{\partial t}(T, s) \nu(s) \, ds \tag{3.6}$$

and for each $t$ in $[0, T]$,

$$\int_0^t h(t, s) \rho(s) \, ds \geq \int_0^t h(t, s) \nu(s) \, ds. \tag{3.7}$$

a) Assume $0 < \rho < \phi$ a.e.

By Proposition 2.2 there exist $\alpha, \beta \in [0, T]$ such that

$$\int_0^T h(T, s) \rho(s) \, ds = \int_0^\beta h(T, s) \phi(s) \, ds, \tag{3.8}$$

$$\int_0^T \frac{\partial h}{\partial t}(T, s) \rho(s) \, ds = \int_\alpha^\beta \frac{\partial h}{\partial t}(T, s) \phi(s) \, ds. \tag{3.9}$$

It is clear that if we set

$$\nu(s) = \phi(s) \chi_{[\alpha, \beta]}(s) \tag{3.10}$$

then (3.5) and (3.6) are satisfied. In order to prove (3.7) we first show that under our assumptions on $\rho$ and $\phi$ we have

$$0 < \alpha < \beta < T. \tag{3.11}$$
Notice first that the equalities \((\alpha, \beta) = (0, T)\) or \(\alpha = \beta\) cannot hold otherwise by (3.8) \(\rho = \phi\) or \(\rho = 0\) a.e., a contradiction. Assume, for instance, \(\alpha = 0\) and \(\beta < T\), the case \(\alpha > 0\) and \(\beta = T\) being similar. Under this assumption, equalities (3.8) and (3.9) become

\[
\int_0^T h(T, s)\rho(s)\, ds = \int_0^\beta h(T, s)\phi(s)\, ds,
\]

(3.12)

\[
\int_0^T \frac{\partial h}{\partial t}(T, s)\rho(s)\, ds = \int_0^\beta \frac{\partial h}{\partial t}(T, s)\phi(s)\, ds.
\]

(3.13)

Property (3.4) and the assumption \(0 < \rho < \phi\) a.e. allow us to apply Lemma 2.1 from which we deduce

\[
\int_0^T h(T, s)\rho(s)\, ds < \int_0^\beta h(T, s)\phi(s)\, ds,
\]

contradicting (3.12).

Set \(y(t) = \int_0^t h(t, s)\nu(s)\, ds\) so that (3.8) and (3.9) become \(y(T) = x(T)\) and \(y'(T) = x'(T)\). Purpose of what follows is to show (3.7), i.e. that \(y(t) \leq x(t)\) for each \(t\). We consider the cases \(t \in [0, \alpha]\), \(t \in [\beta, T]\), \(t \in [\alpha, \beta]\) separately.

Inequality (3.7) is trivial if \(t \leq \alpha\); in fact we have

\[
y(t) = 0 \leq \int_0^t h(t, s)\rho(s)\, ds = x(t),
\]

the inequality being strict for \(t \in [0, \alpha]\). In particular

\[
y(\alpha) < x(\alpha).
\]

(3.14)

Assume \(t \in [\beta, T]\).

Since, taking (3.2) into account, \(h(t, s) \leq 0\) whenever \(s \geq t\), we have

\[
\forall t \geq \beta \quad \int_t^T h(t, s)\rho(s)\, ds \leq 0 = \int_t^T h(t, s)\nu(s)\, ds
\]

(3.15)

or equivalently

\[
\forall t \geq \beta \quad \int_0^T h(t, s)\rho(s)\, ds - \int_0^t h(t, s)\rho(s)\, ds \leq \int_0^T h(t, s)\nu(s)\, ds - \int_0^t h(t, s)\nu(s)\, ds.
\]

(3.16)

Therefore, in order to prove that \(y(t) \leq x(t)\) for \(t \in [\beta, T]\) it is enough to show that

\[
\forall t \in [\beta, T] \quad \int_0^T h(t, s)\rho(s)\, ds = \int_0^T h(t, s)\nu(s)\, ds.
\]

(3.17)
For this purpose, we use property S 1). Equalities (3.8) and (3.9) become

\[
\begin{align*}
&\begin{cases}
w_1(T) \int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds + w_2(T) \int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0 \\
w_1'(T) \int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds + w_2'(T) \int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0
\end{cases}
\end{align*}
\]

The condition on the wronskian of \(w_1, w_2\) at \(T\) implies

\[
\begin{align*}
&\int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds = 0, \quad (3.18) \\
&\int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0. \quad (3.19)
\end{align*}
\]

Multiplying (3.18) by \(w_1(t)\), (3.19) by \(w_2(t)\) and adding the two equations we obtain

\[
\int_0^T (w_1(t)z_1(s) + w_2(t)z_2(s))\rho(s) \, ds = \int_0^T (w_1(t)z_1(s) + w_2(t)z_2(s))\nu(s) \, ds
\]

which, together with property S 1), gives (3.17). Moreover remark that since inequality (3.15) is strict for \(t \neq T\), then

\[
y(\beta) < x(\beta). \quad (3.20)
\]

At this stage, we only need to prove that (3.7) holds for \(t \in [\alpha, \beta]\).

Assume by contradiction that there exists \(t \in [\alpha, \beta]\) such that \(x(t) = y(t)\). Let

\[
\bar{t} = \sup\{t \in [\alpha, \beta] : x(t) = y(t)\}.
\]

Then \(\alpha < \bar{t} < \beta\) and by the very definition of \(\bar{t}\), \(x(\bar{t}) = y(\bar{t})\) so that

\[
y'(\bar{t}) - x'(\bar{t}) = \lim_{t \to \bar{t}^+} \frac{y(t) - x(t)}{t - \bar{t}} \leq 0.
\]

It follows that

\[
\begin{align*}
&\int_\alpha^\bar{t} h(\bar{t}, s)\phi(s) \, ds = \int_0^\bar{t} h(\bar{t}, s)\rho(s) \, ds, \quad (3.21) \\
&\int_\alpha^\bar{t} \frac{\partial h}{\partial t}(\bar{t}, s)\phi(s) \, ds \leq \int_0^\bar{t} \frac{\partial h}{\partial t}(\bar{t}, s)\rho(s) \, ds. \quad (3.22)
\end{align*}
\]
For each \( s \in [0, \tilde{t}] \) let \( f(s) = h(\tilde{t}, s) \), \( g(s) = \frac{\partial h}{\partial t}(\tilde{t}, s) \) and \( k = f/g \) so that by (3.2)-(3.4) the function \( k \) is increasing and \( f > 0, g > 0 \). If we replace \((a, b)\) by \((0, \tilde{t})\), Lemma 2.1 together with (3.21) imply that

\[
\int_0^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s) \phi(s) \, ds > \int_0^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s) \rho(s) \, ds
\]

thus contradicting (3.22).

b) Assume, in general, \( 0 \leq \rho \leq \phi \) a.e. and let \( \phi_n, \rho_n \in L^1([0, T]) \) be such that

\[
0 < \rho_n < \phi_n \text{ a.e. and } \rho_n \to \rho, \phi_n \to \phi \text{ in } L^1([0, T])
\]

(for instance \( \rho_n = \rho + \frac{1}{n}, \phi_n = \phi + \frac{1}{n} \)). Corresponding to each \( n \), there exist \( \alpha_n, \beta_n \in [0, T] \) such that, if we set \( \nu_n = \phi_n \chi_{[\alpha_n, \beta_n]} \) then we have

\[
\int_0^T h(T, s) \rho_n(s) \, ds = \int_0^T h(T, s) \nu_n(s) \, ds,
\]

\[
\int_0^T \frac{\partial h}{\partial t}(T, s) \rho_n(s) \, ds = \int_0^T \frac{\partial h}{\partial t}(T, s) \nu_n(s) \, ds
\]

and, for each \( t \in [0, T] \),

\[
\int_0^t h(t, s) \rho_n(s) \, ds > \int_0^t h(t, s) \nu_n(s) \, ds.
\]

The interval \([0, T]\) being compact, we may assume \( \alpha_n \to \alpha, \beta_n \to \beta \) for some \( \alpha \leq \beta \in [0, T] \). Clearly \( \nu_n = \phi_n \chi_{[\alpha_n, \beta_n]} \) converges to \( \phi \chi_{[\alpha, \beta]} \) in \( L^1([0, T]) \), therefore if we pass through the limit in (3.23), (3.24), (3.25) and we set \( \nu = \phi \chi_{[\alpha, \beta]} \) we obtain (3.5), (3.6) and (3.7).

iii) In the general case, using property \( S \) and the compactness of \([a, b]\), there exists a subdivision \( a_0 = 0 < a_1 < \cdots < a_l = T = a_{l+1} \) of \([0, T]\) such that, if we put \( I_j = [a_j, a_{j+1}] \), we have

- \( \forall s, t \in I_j \) \( h(t, s) > 0 \) if \( s < t \), \( h(t, s) < 0 \) if \( t < s \);
- \( \forall s, t \in I_j \) \( \frac{\partial h}{\partial t}(t, s) > 0 \);
- \( \forall t \in I_j \) \( s \mapsto h(t, s) \frac{\partial h}{\partial t}(t, s) \) is decreasing on \( I_j \).

By ii), on each interval \( I_j \) there exist \( \alpha_j, \beta_j \) such that the solution \( y_j \) to the problem

\[
y'' + a_1(t)y' + a_0(t)y = \phi_1(t)\chi_{[\alpha_j, \alpha_j]}(t) + \phi_2(t)\chi_{[\beta_j, \beta_j]}(t) \text{ a.e. on } I_j
\]

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with the initial conditions

\[ y_j(a_j) = x(a_j), \quad y'_j(a_j) = x'(a_j) \]

satisfies the equalities

\[ y_j(a_{j+1}) = x(a_{j+1}), \quad y'_j(a_{j+1}) = x'(a_{j+1}) \]

and moreover \( y_j(t) \leq x(t) \) for each \( t \in I_j \).

Clearly the function \( y \in W^{2,1}([0, T]) \) obtained by glueing together the functions \( y_j \) is a solution to our problem. \( \square \)

**Remark 3.1.** The proof of Theorem 3.1, part ii)a) shows in fact that when \( 0 < \rho < \phi \), we have \( y(t) < x(t) \) on \( ]0, T[ \).

**Remark 3.2.** With the notations introduced in Proposition 2.1, the proof of Theorem 3.1 (part ii)) shows that if \( T = \delta \) then, given a solution \( x \) to (P), there exists a bang–bang solution \( y \leq x \) satisfying

\[
\begin{align*}
y'' + a_1(t)y' + a_0(t)y &= \min \Phi(t) \text{ on } [0, \alpha] \cup [\beta, T], \\
y'' + a_1(t)y' + a_0(t)y &= \max \Phi(t) \text{ on } [\alpha, \beta].
\end{align*}
\]

The number \( \delta \) depending only on the function \( h \), it can happen that \( \delta = +\infty \).

This is the case when \( a_1 \) and \( a_0 \) are constant and the equation \( \lambda^2 + a_1 \lambda + a_0 = 0 \) admits two real roots \( \lambda_1, \lambda_2 \). In fact, under this assumption we have either

\[
\begin{align*}
h(t, s) &= \frac{1}{\lambda_2 - \lambda_1} (e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}) \text{ if } \lambda_1 \neq \lambda_2, \text{ or} \\
h(t, s) &= (t-s)e^{\lambda(t-s)} \text{ if } \lambda_1 = \lambda_2 = \lambda.
\end{align*}
\]
4. Applications

Our first application concerns the reachable set of bang–bang constrained solutions. Let \( c \) be an arbitrary function defined on \([0, T]\) and consider the reachable sets \( X^c_T \) and \( Y^c_T \) associated to \((P)\) defined by

\[
X^c_T = \{ (y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \Phi(t) \text{ a.e.}, \quad (y(0), y'(0)) = (x_0, v_0) \}\]

\[
Y^c_T = \{ (y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \text{extr} \Phi(t) \text{ a.e.}, \quad (y(0), y'(0)) = (x_0, v_0) \}\.
\]

Then Theorem 3.1 claims \( X^c_T = Y^c_T \) whence \( Y^c_T \) is convex.

Finally, we give an application to the calculus of variations.

**Theorem 4.1.** Let \( a_0, a_1 \in C([0, T]) \), \( \phi_1, \phi_2 \in L^1([0, T]) \) verify \( \phi_1(t) \leq \phi_2(t) \) a.e. Let \( x_0, v_0, x_1, v_1 \) be 4 fixed real numbers. Then there exists a dense subset \( D \) of \( C(\mathbb{R}) \) for the uniform convergence such that for \( g \) in \( D \) the problem

\[
\text{minimize} \quad \left\{ \int_0^T g(x(t)) \, dt + \int_0^T h(\rho(t)) \, dt \right\}
\]

on the subset of \( W^{2,1}([0, T]) \times L^1([0, T]) \) of those functions \( (x, \rho) \) satisfying

\[
(x(0), x'(0), x(T), x'(T)) = (x_0, v_0, x_1 v_1), \quad x'' + a_1(t)x' + a_0(t) x = \rho(t) \in [\phi_1(t), \phi_2(t)] \text{ a.e.}
\]

admits at least one solution for every lower semicontinuous function \( h \) satisfying the growth condition \( h(u) \geq c \psi(|u|) \), \( \psi \) being l.s.c. and convex, \( \lim_{r \to +\infty} \psi(r)/r = +\infty \).

**Proof.** With our theorem 3.1 and the preceding application, the proof is a direct adaptation of the proof given in [3]. □

References


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