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A REMARK ABOUT THE PERIODIC HOMOGENIZATION OF CERTAIN COMPOSITE FIBERED MEDIA

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Abstract. **We explain in this paper the similarity arising in the homogenization process of some composite fibered media with the problem of the reduction of dimension $3d-1d$. More precisely, we highlight the fact that when the homogenization process leads to a local reduction of dimension, studying the homogenization problem in the reference configuration domain of the composite amounts to the study of the corresponding reduction of dimension in the reference cell. We give two examples in the framework of the thermal conduction problem: the first one concerns the reduction of dimension in a thin parallelepiped of size ε containing another thinner parallelepiped of size $r_\varepsilon \ll \varepsilon$ playing a role of a «hole ». As in the homogenization, the one-dimensional limit problem involves a "strange" term. In addition both limit problems are of the same structure. In the second example, the geometry is similar but now we assume a high contrast between the order 1 of the conductivity in the small parallelepiped of size $r_\varepsilon := r\varepsilon$, ($r > 0$ fixed) and the conductivity in the big parallelepiped assumed to be of order ε^2. We prove that the limit problem is a nonlocal problem and that it has the same structure as the corresponding periodic homogenized problem.**

1. **Introduction.** During the last years, the study of the homogenization of composite heterogeneous media has given rise to an extensive literature and a significant part of that works was designed to the homogenization of media characterized by high heterogeneities, (see [2], [3]), [4], [5], [6], [8], [14]). The pioneer work for problems of this kind was done in [2] in the study of the double porosity model of single phase flow. Thereafter, the main idea of [2] was taken up in [1] and [12] to give rise to the two-scale convergence method. Fibered media is an example of composite heterogeneous media with high contrasting properties since usually the material constituting the fibers is very different from the material around it. For instance, in elasticity one can consider rigid fibers immersed in a soft matrix while in the framework of heat conduction one can consider fibers with high conductivity surrounded

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by a material with a poor conductivity. Among the first works devoted to the homogenization of such composite media one can quote [7] where the homogenization process was performed using asymptotic expansions.

From the mathematical point of view, the contrast between the properties of the two materials leads to a degenerate problem in the sense that in general it leads to a lack of compactness. Indeed, the operators under consideration are in general not uniformly bounded (see [5]) or not uniformly coercive with respect to the small parameter (see [16]).

In general, the configuration domain of such media may be described by a domain Ω of $\mathbb{R}^3$ which is a periodic replication with a period of size ε of a set of Y_i^ε, $i \in I_\varepsilon$.

More precisely, Y_i^ε is assumed to be the union of a fiber F_i^ε with its complement M_i^ε (M stands for matrix) in Y_i^ε.

It is well known that the homogenization process in such degenerate problems gives rise to homogenized problems with a different form from the equation at the microscopic level since memory effects, strange terms or nonlocal effects may appear at the limit (see [3], [5], [6], [8], [9], [11], [16]).

The aim of the present work is to show that in the case of fibered media such effects at the limit are not due to the homogenization process itself but to the local structure of the composite media; more precisely, we show that the form of the homogenized problem is already determined by the study of the $3d-1d$ reduction of dimension which occurs locally. To illustrate that, we give here two examples in the framework of the thermal conduction. In the first one we show that one can obtain an extra term (or a strange term, see [11], [3]) in the study of the reduction of dimension $3d-1d$ (see [13]). To prove that result, we consider in section 2 below a thin domain $\Omega^\varepsilon := \varepsilon Y \times (0,L)$ as a copy of the local cell arising in a periodic fibered medium for which the configuration domain Ω is the union of $\dfrac{1}{\varepsilon^2}$ such cells, in other words, $\Omega := \bigcup_{i\in I_\varepsilon} \big((\varepsilon Y + \varepsilon i)\times(0,L)\big)$. In fact, in this example the "fibers" play a role of vertical parallel holes and we prove that a strange term already appears at the limit in the $3d-1d$ problem under exactly the same assumptions on the size of the hole as the ones assumed in the corresponding homogenization problem. Furthermore, it is shown that the structure of the homogenized problem is very close to the one of the limit problem obtained in the reduction of dimension $3d-1d$. The homogenization problem will be considered in section 3.

In section 4, we consider another example for which the reduction of dimension $3d-1d$ leads to a limit problem involving a nonlocal effect; the comparison with the corresponding homogenization problem is made in section 5 and once again the similarity between the two limit problems is pointed out.

One can explain the similarity between the homogenization problem and the reduction of dimension problem by the fact that the geometry of the fibered medium is such that the homogenization process implies a local reduction of dimension so that for such media, the homogenization may be viewed as a repeat of local reductions of dimension. More precisely, when we assume that the domain Ω is made from a single cell of size ε, $\Omega = \Omega^\varepsilon := Y^\varepsilon := \varepsilon Y \times (0,L)$ where for example Y is the square defined by $Y := \left]-\frac{1}{2},\frac{1}{2}\right[^2$ and $L>0$, one can denote the variable x in Ω^ε by $x=(x',x_3)=(\varepsilon y, x_3)$, $y\in Y$, $x_3\in(0,L)$, in such a way that $x'=\varepsilon y$. This classical dilation transforms the reduction of dimension problem posed in Ω^ε into a singular perturbation problem posed on the fixed domain Ω (see [11]) and the limit

problem (as ε goes to zero) is written in terms of (y, x_3). When we study the periodic homogenization with $\Omega := \bigcup_{i\in I_\varepsilon} \big((\varepsilon Y + \varepsilon i) \times (0, L)\big)$, the variable x in Ω writes as $x = (x', x_3)$ with $x' = \varepsilon y + \varepsilon i$, $y \in Y$, $i \in I_\varepsilon$, $x_3 \in (0, L)$. It is well known (see [1], [10]) that the homogenized problem may be formulated in terms of the variables (y, x). Of course, it is not reasonable to expect a homogenized problem involving only x_3 as a macroscopic variable since we have many cells distributed in the plane (x_1, x_2) so that several reductions of dimension occur. For that reason, the role of the part $x' = (x_1, x_2)$ of the macroscopic variable in the homogenized problem is restricted to that of a parameter so that the main variables are still (y, x_3) as in the $1d$- model obtained after the reduction of dimension $3d - 1d$. Remarkably, dealing with one small cylinder or $\dfrac{1}{\varepsilon^2}$ small cylinders does not affect the form of the limit problem.

For the sake of simplicity and brevity we consider in this work the case of the Laplacian but the results remain valid for other more general operators as we will show in coming works.

2. **Strange term in the reduction of dimension.** We consider a thin structure described as follows. Let ε, r_ε be two decreasing sequences of positive numbers such that $\lim_{\varepsilon\to 0} \dfrac{r_\varepsilon}{\varepsilon} = 0$ and let $\hat{\Omega}_\varepsilon$, $\hat{F}_\varepsilon$, $\hat{M}_\varepsilon$ be respectively the parallelepiped, the fiber and the matrix defined by

$$\begin{cases} \hat{\Omega}_\varepsilon = \varepsilon Y \times (0, L), \;\; Y = \big(]-\frac{1}{2}, \frac{1}{2}[\big)^2, \;\; L > 0, \hat{F}_\varepsilon = r_\varepsilon \overline{D} \times (0, L) \subset \hat{\Omega}_\varepsilon, \\ \\ \text{where } \overline{D}(0, r) \text{ is the closed disk of radius } 0 < r < \frac{1}{2}. \end{cases} \tag{1}$$

For the sake of brevity, in the sequel the interval $(0, L)$ is sometimes denoted by I. We will assume Dirichlet condition on the part

$$\partial\hat{\Omega}_\varepsilon^D = \{(x', x_3) \in \mathbb{R}^3 : \, x_3 = 0 \text{ or } x_3 = L \text{ or } x' \in r_\varepsilon \partial\overline{D}\} \tag{2}$$

of the boundary of $\hat{\Omega}_\varepsilon$ and Newmann condition on the rest

$$\partial\hat{\Omega}_\varepsilon^N = \partial\hat{\Omega}_\varepsilon \setminus \; \partial\hat{\Omega}_\varepsilon^D \tag{3}$$

of the boundary. Remark that $\partial\hat{\Omega}_\varepsilon^D$ is made of the upper and the lower faces of the parallelepiped together with the boundary of the fiber $\hat{F}_\varepsilon$ which plays here the role of a hole. We consider the following equation:

$$\begin{cases} -\Delta \bar{u}_\varepsilon = f \quad \text{in } \; \hat{\Omega}_\varepsilon, \\ \\ \bar{u}_\varepsilon = 0 \quad \text{on } \partial\hat{\Omega}_\varepsilon^D, \\ \\ \dfrac{\partial \bar{u}_\varepsilon}{\partial n} = 0 \quad \text{on } \partial\hat{\Omega}_\varepsilon^N. \end{cases} \tag{4}$$

Introduce the change of variables $x' = \varepsilon y$, $u_\varepsilon(y, x_3) := \bar{u}_\varepsilon(\varepsilon y, x_3)$ and set

$$\Omega := Y \times (0, L), \;\; F_\varepsilon = \frac{1}{\varepsilon}\hat{F}_\varepsilon, \;\; \Omega_\varepsilon = \Omega \setminus F_\varepsilon, \tag{5}$$

$$H_D^1(\Omega_\varepsilon) := \{u \in H^1(\Omega), \;\; u(y, 0) = u(y, L) = 0, \;\; u = 0 \text{ on } \partial F_\varepsilon\} \tag{6}$$

. Finally, denote by ∇' the gradient with respect to the two first variables $y = (y_1, y_2)$. Similarly, Δ' denotes the Laplacian with respect to the same variables.

With these notations, one deduces easily that (4) is transformed into an equation satisfied by u_ε with the following variational formulation

$$\begin{cases} u_\varepsilon \in H^1_D(\Omega_\varepsilon), \ \displaystyle\int_{\Omega_\varepsilon} \Big(\frac{1}{\varepsilon^2}\nabla' u_\varepsilon \nabla'\phi + \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \phi}{\partial x_3}\Big)\, dydx_3 = \int_{\Omega_\varepsilon} f\phi\, dydx_3, \\ \forall\, \phi \in H^1_D(\Omega_\varepsilon). \end{cases} \tag{7}$$

Assuming the source term $f \in L^2(\Omega)$, problem (7) is well-posed; in addition, u_ε may be extended by zero inside the hole F_ε so that the new sequence still denoted by u_ε belongs to $H^1_D(\Omega) := \{u \in H^1(\Omega),\ u(y,0) = u(y,L) = 0\}$. Furthermore, taking $\phi = u_\varepsilon$ in (7), we obtain easily that there exist $u \in H^1_0(0,L)$ (subspace of functions in $H^1_D(\Omega)$ depending only on the variable x_3) and a subsequence (still denoted by ε) such that:

$$u_\varepsilon \rightharpoonup u \ \text{ weakly in } \ H^1_D(\Omega). \tag{8}$$

The limit function u depends on the size of the hole and it is characterized by the following theorem. Indeed, defining $\tilde{f}(x_3) := \displaystyle\int_Y f(y,x_3)\, dy$, we have the following:

Theorem 2.1.

$$\begin{cases} \textit{If there exists } k > 0 \textit{ such that } r_\varepsilon \sim e^{-\frac{k}{\varepsilon^2}}, \textit{the function } u \\ \textit{is the unique solution of the one-dimensional problem} \\ u \in H^1_0(0,L), \ -\dfrac{d^2u}{dx_3^2} + \dfrac{2\pi r}{k}u = \tilde{f} \ \ in\,(0,L), \\ if\, r_\varepsilon \ll e^{-\frac{k}{\varepsilon^2}}, u \textit{ is the unique solution of} \\ u \in H^1_0(0,L), -\dfrac{d^2u}{dx_3^2} = \tilde{f} \ \ in\,(0,L), \\ if\, r_\varepsilon \gg e^{-\frac{k}{\varepsilon^2}}, then\, u = 0. \end{cases} \tag{9}$$

Sketch of the Proof. Introduce the following function (see [11])

$$w_\varepsilon(y) = \begin{cases} 0 \ \text{ in } \frac{r_\varepsilon}{\varepsilon}D, \\ \frac{\ln(|y|)-\ln(\frac{r_\varepsilon}{\varepsilon})}{\ln(r)-\ln(\frac{r_\varepsilon}{\varepsilon})} \ \text{ in } D \setminus \frac{r_\varepsilon}{\varepsilon}D, \\ 1 \ \text{ in } Y \setminus D. \end{cases} \tag{10}$$

Then: $\dfrac{\partial w_\varepsilon}{\partial n} = \dfrac{1}{\ln(\frac{\varepsilon}{r_\varepsilon})}$ on ∂D, $-\Delta' w_\varepsilon = 0$ in $D\setminus\dfrac{r_\varepsilon}{\varepsilon}D$, $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(Y)$. Use as a test function in (7): $\phi(y,x_3) = w_\varepsilon(y)\bar{u}(x_3)$, $\bar{u} \in H^1_0(I)$.Then

$$\int_0^L \int_{D\setminus\frac{r_\varepsilon}{\varepsilon}D} \frac{1}{\varepsilon^2}\nabla' u_\varepsilon \nabla' w_\varepsilon \bar{u} dydx_3 + \int_\Omega \frac{\partial u_\varepsilon}{\partial x_3} w_\varepsilon \frac{d\bar{u}}{dx_3}\, dydx_3 = \int_\Omega f w_\varepsilon \bar{u}\, dydx_3. \tag{11}$$

Integrating by parts with respect to y the first integral on the left, and bearing in mind the property $-\Delta' w_\varepsilon = 0$ in $D \setminus \frac{r_\varepsilon}{\varepsilon} D$, we see that it reduces to the boundary integral: $\displaystyle\int_0^L \int_{\partial D} \frac{1}{\varepsilon^2} \frac{\partial w_\varepsilon}{\partial n} u_\varepsilon d\sigma \bar{u} dx_3 = \int_I \int_{\partial D} \frac{1}{\varepsilon^2} \frac{1}{\ln(\frac{\varepsilon}{r_\varepsilon})} u_\varepsilon d\sigma \bar{u} dx_3$.

Hence, we have the following conclusions in terms of the thickness of the hole:

if $r_\varepsilon = e^{\frac{-k}{\varepsilon^2}}$, the limit of the last integral is equal to $\displaystyle\frac{1}{k}|\partial D| \int_0^L u\bar{u}\ dx_3 = \frac{2\pi r}{k} \int_0^L u\bar{u}\ dx_3$, so that the limit equation is the first equation of (9).

If $r_\varepsilon \ll e^{-\frac{k}{\varepsilon^2}}$, the limit of that integral is equal to zero in such a way that the extra term disappears and the limit equation is now the second equation arising in (9).

Finally, let us prove that $u = 0$ under the assumption $r_\varepsilon \gg e^{-\frac{k}{\varepsilon^2}}$. It is well known that for given $\varepsilon > 0$, the eigenvalues of the operator $\displaystyle -\frac{1}{\varepsilon^2}\Delta' - \frac{\partial^2}{\partial x_3^2}$ with eigenvectors in $H^1_D(\Omega_\varepsilon)$ is an increasing positive sequence of real numbers and that for $f \in L^2(\Omega)$, the solution of (7) is given by $\displaystyle u_\varepsilon = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^\varepsilon}(f, v_i^\varepsilon) v_i^\varepsilon$ where $(\lambda_i^\varepsilon, v_i^\varepsilon)_{i\geq 1}$ is the sequence of the eigenelements under consideration and where $(,)$ denotes the scalar product of $L^2(\Omega)$. As a consequence, we deduce the estimate

$$\| u_\varepsilon \|_{L^2(\Omega)} \leq \frac{1}{\lambda_1^\varepsilon} \| f \|_{L^2(\Omega)} . \tag{12}$$

On the other hand, the first eigenvalue is given by $\displaystyle\lambda_1^\varepsilon = \min_{v \in H^1_D(\Omega_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\varepsilon} \left(\frac{1}{\varepsilon^2}|\nabla' v|^2 + |\frac{\partial v}{\partial x_3}|^2\right)\ dx}{\displaystyle\int_{\Omega_\varepsilon} |v|^2\ dx.}$.

As an immediate consequence, we derive the following inequality

$$\lambda_1^\varepsilon \geq \min_{v \in H^1_D(\Omega_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_0^L \int_{Y\setminus F_\varepsilon} \frac{1}{\varepsilon^2}|\nabla' v|^2\ dy dx_3}{\displaystyle\int_0^L \int_{Y\setminus F_\varepsilon} |v|^2\ dy dx_3}. \tag{13}$$

On the other hand, since for $x_3 \in (0, L)$ any function $v \in H^1_D(\Omega_\varepsilon)$ is such that $v(., x_3) \in H^1(Y)$ and $v(., x_3) = 0$ on $\partial(\frac{r_\varepsilon}{\varepsilon} D)$, we can use the following estimate proved in [14], pages 44-45:

$$\min_{v \in H^1(Y)\setminus\{0\},} \frac{\displaystyle\int_{Y\setminus F_\varepsilon} \frac{1}{\varepsilon^2}|\nabla' v|^2\ dy}{\displaystyle\int_{Y\setminus F_\varepsilon} |v|^2\ dy} \geq \frac{1}{\varepsilon^2} \frac{1}{\displaystyle\int_{\frac{r_\varepsilon}{\varepsilon}}^{\frac{1}{2}} t\ dt \int_{\frac{r_\varepsilon}{\varepsilon}}^{\frac{1}{2}} \frac{1}{t}\ dt}. \tag{14}$$

An easy computation shows that the quantity in the right hand side of (32) goes to $+\infty$ if $r_\varepsilon \gg e^{-\frac{k}{\varepsilon^2}}$ and then the result follows from (13) and (12). □

3. **Strange term in the homogenization.** In order to make easy the comparison with the results of the last section, we keep here analogous notations; in particular, $Y \times (0, L)$ and $\bar{D}(0, r)$ denote respectively the parallelepiped and the disk defined in the previous section. We assume now that Ω is the configuration domain of a set of parallelepipeds $Y_\varepsilon^i := (\varepsilon Y + \varepsilon i) \times (0, L)$ distributed with a period εY in the horizontal x'-directions; each parallelepiped Y_ε^i contains another small parallelepiped playing the role of a hole $F_\varepsilon^i := (r_\varepsilon \bar{D} + \varepsilon i) \times (0, L)$. The set of all the holes contained in Ω will be denoted by F_ε. We then define

$$\Omega_\varepsilon = \Omega \setminus F_\varepsilon, \quad F_\varepsilon = \bigcup_{i \in I_\varepsilon} F_\varepsilon^i, \tag{15}$$

where I_ε denotes as usual $I_\varepsilon := \{i = (i_1, i_2) \in \mathbb{Z}^2,\ Y_\varepsilon^i \subset \Omega\}$. Note that the analogous configuration domain in the reduction of the dimension problem is $(\varepsilon Y \setminus r_\varepsilon \bar{D}) \times (0, L)$ denoted there by $\hat{\Omega}_\varepsilon$ (to be distinguished from Ω_ε denoting there the domain obtained after scaling). In other words, the real domain (before scaling) of the $3d - 1d$ problem is nothing but the representative cell of the present homogenization problem.

The equation we want to homogenize is the following

$$u_\varepsilon \in H_D^1(\Omega_\varepsilon), \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \phi \, dx = \int_{\Omega_\varepsilon} f \phi \, dx, \, \forall \, \phi \in H_D^1(\Omega_\varepsilon), \tag{16}$$

where $H_D^1(\Omega_\varepsilon) := \{u \in H^1(\Omega),\ u(x', 0) = u(x', L) = 0,\ u = 0 \text{ on } \partial F_\varepsilon\}$.
To give the limit problem, we need to define

$$\Gamma_D := \{x = (x', x_3) \in \Omega,\ x_3 = 0 \text{ or } x_3 = L\},\ \Gamma_N := \partial\Omega \setminus \Gamma_D, \tag{17}$$

together with the space

$$H_D^1(\Omega) := \{u \in H^1(\Omega),\ u = 0 \text{ on } \Gamma_D\}. \tag{18}$$

We consider a sequence δ_ε of positive numbers such that

$$r_\varepsilon \ll \delta_\varepsilon \ll \varepsilon, \quad \lim_{\varepsilon \to 0} \varepsilon^2 \ln(\delta_\varepsilon) = 0. \tag{19}$$

For example if $r_\varepsilon = e^{-\frac{k}{\varepsilon^2}},\ k > 0$, one can take $\delta_\varepsilon = e^{-\frac{k}{\varepsilon}}$. Denote by C_{δ_ε} the circle of radius $\frac{\delta_\varepsilon}{\varepsilon}$ centered at the origin and set $C_{\delta_\varepsilon}^i = \varepsilon C_{\delta_\varepsilon} + \varepsilon i$. Finally, we introduce the sequence

$$\tilde{u}_\varepsilon = \sum_{i \in I_\varepsilon} \frac{1}{2\pi\delta_\varepsilon} \int_{C_{\delta_\varepsilon}^i} u_\varepsilon \, d\sigma \, \chi_{Y_\varepsilon^i}(x'). \tag{20}$$

The following estimate, proved in [3], will be helpful in the sequel.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\int_\Omega |u_\varepsilon - \tilde{u}_\varepsilon|^2 \, dx \le C\varepsilon^2 \big(1 + |\ln(\frac{\delta_\varepsilon \sqrt{2}}{\varepsilon})|\big) \int_\Omega |\nabla u_\varepsilon|^2 \, dx. \tag{21}$$

We now state the main result of this section through the following theorem.

Theorem 3.2. *There exists $u \in H_D^1(\Omega)$ such that the sequence of solutions of (16) may be extended by zero to $H_D^1(\Omega)$ on a sequence still denoted by u_ε such that*

$$u_\varepsilon \rightharpoonup u \ \text{ weakly in } \ H_D^1(\Omega). \tag{22}$$

Furthermore, u is characterized by

$$\begin{cases} \textit{If there exists } k>0 \textit{ such that } r_\varepsilon \sim e^{-\frac{k}{\varepsilon^2}}, \\ \textit{then } u \textit{ is the unique solution} \\ \textit{of the homogenized problem} \\ u \in H^1_D(\Omega),\ \ -\Delta u + \dfrac{2\pi}{k} u = f \ \ \textit{in } \Omega, \\ \qquad\qquad \dfrac{\partial u}{\partial n} = 0 \ \ \textit{on } \Gamma_N. \\ If\, r_\varepsilon \ll e^{-\frac{k}{\varepsilon^2}}, \\ \textit{then } u \textit{ is the unique solution of} \\ u \in H^1_D(\Omega),\ -\Delta u = f \ \ \textit{in } \Omega, \\ \qquad\qquad \dfrac{\partial u}{\partial n} = 0 \ \ \textit{on } \Gamma_N. \\ If\, r_\varepsilon \gg e^{-\frac{k}{\varepsilon^2}}, \textit{then } u = 0. \end{cases} \tag{23}$$

,

Proof. First of all, we remark that the sequence u_ε of solutions of (16) extended by zero to the holes is bounded in $H^1_D(\Omega)$; this can be seen using $\phi = u_\varepsilon$ as test function in (16) and then applying the Cauchy-Schwarz inequality in the right hand side. Therefore, we can assume that there exists $u \in H^1_D(\Omega)$ such that (up to a subsequence) $u_\varepsilon \rightharpoonup u$ weakly in $H^1_D(\Omega)$. We want to find the equation satisfied by u.

Let $\phi \in \mathcal{C}^\infty(\Omega) \cap H^1_D(\Omega)$. Define the following sequence of piece-wise constant functions

$$\phi_\varepsilon = \sum_{i \in I_\varepsilon} \frac{1}{\pi r_\varepsilon^2} \int_{D^i_{r_\varepsilon}} \phi(x', x_3)\, dx' \chi_{Y^i_\varepsilon}(x'), \tag{24}$$

where $D^i_{r_\varepsilon}$ denotes the disk of center $\varepsilon(i_1, i_2)$ and of radius r_ε. Using the regularity of ϕ, one can check easily by the use of the mean value Theorem that there exists a constant $C > 0$ such that $|\phi - \phi_\varepsilon| \le C\varepsilon$ in Ω.

Define the function $d_\varepsilon(x') := \text{dist}\left(x'; \{\varepsilon i,\ i = (i_1, i_2) \in \mathbb{Z}^2\}\right)$ and denote by $D^i_{\delta_\varepsilon}$ the disk with center εi and with radius δ_ε, then we will use the following function defined for each $i \in \mathbb{Z}^2$ such that $Y^i_\varepsilon \subset \Omega$ (i.e., $i \in I_\varepsilon$) by:

$$w_\varepsilon(x') = \begin{cases} 0 \ \text{ in } D^i_{r_\varepsilon}, \\ \frac{\ln(d_\varepsilon(x')) - \ln(r_\varepsilon)}{\ln(\delta_\varepsilon) - \ln(r_\varepsilon)} \ \text{ in } D^i_{\delta_\varepsilon} \setminus D^i_{r_\varepsilon}, \\ 1 \ \text{ in } Y^i_\varepsilon \setminus D^i_{\delta_\varepsilon}. \end{cases} \tag{25}$$

In view of the definition of w_ε, one can check easily by the use of cylindrical coordinates in the tube $U_\varepsilon^i := \{(x', x_3) \in Y_\varepsilon^i,\ r_\varepsilon < d_\varepsilon(x') < \delta_\varepsilon,\ x_3 \in (0, L)\}$ the following estimate:

$$\int_\Omega |\nabla' w_\varepsilon(x')|^2 \, dx \leq \frac{C}{\varepsilon^2 \ln(\frac{\delta_\varepsilon}{r_\varepsilon})}. \tag{26}$$

Hence, the second equality arising in (19) allows us to conclude that w_ε is bounded in $H^1(\Omega)$ under the hypothesis $r_\varepsilon \approx e^{-\frac{k}{\varepsilon^2}}$ or the hypothesis $r_\varepsilon \ll e^{-\frac{k}{\varepsilon^2}}$. Note also that the latest hypothesis which corresponds to "small holes" implies that the sequence w_ε which is bounded in $H^1(\Omega)$ strongly converges to 1 in $H^1(\Omega)$. If we assume only the hypothesis $r_\varepsilon \approx e^{-\frac{k}{\varepsilon^2}}$, then w_ε only converges to 1 in the weak topology of $H^1(\Omega)$, (see [9]).

We now choose a test function in the form $\phi(x) w_\varepsilon(x')$ in equation (16) with $\phi \in H_D^1(\Omega)$ and we get

$$\begin{cases} \displaystyle\int_\Omega \nabla u_\varepsilon \nabla \big(w_\varepsilon \phi\big) \, dx = \int_\Omega \nabla u_\varepsilon \nabla w_\varepsilon (\phi - \phi_\varepsilon) \, dx + \int_\Omega w_\varepsilon \nabla u_\varepsilon \nabla \phi \, dx + \\ \displaystyle\qquad\qquad + \int_\Omega \phi_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon \, dx. \end{cases} \tag{27}$$

Due to the inequality $\sup_{x \in \Omega} |\phi - \phi_\varepsilon| \leq C\varepsilon$ pointed out above and due to the boundedness in $H^1(\Omega)$ of the sequences u_ε and w_ε, the first integral in the right hand side of (27) clearly tends to zero, while the second one converges to $\displaystyle\int_\Omega \nabla u \nabla \phi \, dx$ since one can assume the strong convergence to 1 in $L^2(\Omega)$ of the sequence w_ε by the Rellich Theorem. Hence, it only remains to compute the limit of the third integral. In view of the definition of w_ε, that integral reduces to the integral on the tube $U_\varepsilon^i := \{(x', x_3) \in Y_\varepsilon^i,\ r_\varepsilon < d_\varepsilon(x') < \delta_\varepsilon,\ x_3 \in (0, L)\}$. Using cylindrical coordinates $(r, \theta, x_3) \in (r_\varepsilon, \delta_\varepsilon) \times (0, 2\pi) \times (0, L)$ and the definition (25) of w_ε, it is not difficult to check that one can write in terms of (r, θ, x_3)

$$\nabla' u_\varepsilon \nabla' w_\varepsilon = \frac{1}{r \ln(\frac{\delta_\varepsilon}{r_\varepsilon})} \frac{\partial u_\varepsilon}{\partial r}(r, \theta, x_3) \text{ in } U_\varepsilon^i, \tag{28}$$

so that bearing in mind the formula (24) and the fact that $u_\varepsilon(r_\varepsilon,\theta,x_3)=0$, we get:

$$\begin{cases} \displaystyle\int_\Omega \phi_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon \ dx = \sum_{i\in I_\varepsilon}\int_{U_\varepsilon^i} \phi_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon \ dx \\ \displaystyle= \sum_{i\in I_\varepsilon}\int_0^L\int_0^{2\pi}\int_{r_\varepsilon}^{\delta_\varepsilon} \frac{1}{r\ln(\frac{\delta_\varepsilon}{r_\varepsilon})}\frac{\partial u_\varepsilon}{\partial r}(r,\theta,x_3)\phi_\varepsilon(x_3)\ r\ dr\ d\theta dx_3 \\ \displaystyle= \frac{1}{\ln(\frac{\delta_\varepsilon}{r_\varepsilon})}\sum_{i\in I_\varepsilon}\int_0^L\int_0^{2\pi} u_\varepsilon(\delta_\varepsilon,\theta,x_3)\phi_\varepsilon(x_3)\ d\theta dx_3 \\ \displaystyle= \frac{2\pi}{\ln(\frac{\delta_\varepsilon}{r_\varepsilon})}\sum_{i\in I_\varepsilon}\int_0^L\int_0^{2\pi} \frac{1}{2\pi}u_\varepsilon(\delta_\varepsilon,\theta,x_3)\phi_\varepsilon(x_3)\ d\theta dx_3 \\ \displaystyle= \frac{2\pi}{\varepsilon^2\ln(\frac{\delta_\varepsilon}{r_\varepsilon})}\sum_{i\in I_\varepsilon}\int_0^L\int_{Y_\varepsilon^i} \tilde{u}_\varepsilon(x',x_3)\phi_\varepsilon(x)\ dx'dx_3 \\ \displaystyle= \frac{2\pi}{\varepsilon^2\ln(\frac{\delta_\varepsilon}{r_\varepsilon})}\int_\Omega \tilde{u}_\varepsilon(x',x_3)\phi_\varepsilon(x)\ dx'dx_3, \end{cases} \tag{29}$$

where $\tilde{u}_\varepsilon$ is defined by (20). On the other hand, from Lemma 3.1 and the property $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ we deduce that $\tilde{u}_\varepsilon \to u$ strongly in $L^2(\Omega)$ since u_ε converges strongly to u in $L^2(\Omega)$. Finally using the property $\phi_\varepsilon \to \phi$ strongly in $L^2(\Omega)$, we obtain the first two items of the theorem by passing to the limit in the last term of (29).

We now prove the last part of the theorem using the same idea as in the $3d-1d$ problem showing that the first eigenvalue λ_ε^1 of the operator under consideration tends to infinity when $\varepsilon\to 0$. We establish a lower bound of λ_ε^1 as follows

$$\lambda_1^\varepsilon = \min_{v\in H_D^1(\Omega_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\varepsilon}\left(|\nabla' v|^2 + |\frac{\partial v}{\partial x_3}|^2\right)\ dx}{\displaystyle\int_{\Omega_\varepsilon}|v|^2\ dx} \geq \min_{v\in H_D^1(\Omega_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\varepsilon}|\nabla' v|^2\ dx}{\displaystyle\int_{\Omega_\varepsilon}|v|^2\ dx}. \tag{30}$$

In each cell $Y_\varepsilon^i \subset \Omega$, one can write for all $v\in H_D^1(\Omega_\varepsilon)$ by the use of cylindrical coordinates as above

$$\int_{Y_\varepsilon^i}|\nabla' v|^2 dx \geq \int_0^L\int_0^{2\pi}\int_{r_\varepsilon}^{\varepsilon}|\frac{\partial v}{\partial r}|^2 r\ dr\ d\theta\ dx_3. \tag{31}$$

As a consequence, we infer

$$\frac{\displaystyle\int_{Y_\varepsilon^i}|\nabla' v|^2 dx}{\displaystyle\int_{Y_\varepsilon^i}|v|^2 dx} \geq \frac{\displaystyle\int_0^L\int_0^{2\pi}\int_{r_\varepsilon}^{\varepsilon}|\frac{\partial v}{\partial r}|^2 r\ dr\ d\theta\ dx_3}{\displaystyle\int_0^L\int_0^{2\pi}\int_{r_\varepsilon}^{\varepsilon}|v|^2 r\ dr\ d\theta\ dx_3}. \tag{32}$$

Using the following inequality proved in [14],

$$\int_{r_\varepsilon}^{\varepsilon}|\frac{\partial v}{\partial r}|^2 r\ dr \geq \frac{1}{\gamma(\varepsilon)}\int_{r_\varepsilon}^{\varepsilon}|v(r,\theta,x_3)|^2 r\ dr, \ \text{ a.e. } (\theta,x_3)\in(0,2\pi)\times(0,L), \tag{33}$$

where $\gamma(\varepsilon) := \displaystyle\int_{r_\varepsilon}^{\varepsilon} r\ dr \int_{r_\varepsilon}^{\varepsilon} \frac{1}{r}\ dr$, we get by integrating over $(0, 2\pi) \times (0, L)$ and summing up over $i \in I_\varepsilon$,

$$\min_{v \in H^1_D(\Omega_\varepsilon)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla' v|^2\ dx}{\displaystyle\int_{\Omega_\varepsilon} |v|^2\ dx} \geq \frac{1}{\gamma(\varepsilon)}. \tag{34}$$

One can check that the constant $\gamma(\varepsilon)$ goes to zero under the hypothesis $r_\varepsilon \gg e^{-\frac{k}{\varepsilon^2}}$ and then the conclusion of the last part of the theorem is a consequence of the inequality (12) which still holds true in the homogenization problem.

□

We now give the second example regarding nonlocal effects in the limit problem. We begin with the reduction of dimension.

4. **Nonlocal effects in the reduction of dimension.** In order to describe the heterogeneities of the medium, we need here to introduce some other notations and change slightly those of section 2. In this section $D(0, r)$ still denotes the disk defined in the previous sections, Ω_ε is defined by $\Omega_\varepsilon = \varepsilon Y \times (0, L) = \varepsilon Y \times I$, Y being defined by (1) and $I := (0, L)$. In addition, we introduce the sets $M_\varepsilon = \Omega_\varepsilon \setminus F_\varepsilon$ where now $F_\varepsilon := \varepsilon \overline{D} \times (0, L)$ in such a way that $\Omega_\varepsilon = M_\varepsilon \cup F_\varepsilon$. Similarly, we put $F := D \times I, \quad M := \Omega \setminus F$, in such a way that $\Omega = Y \times (0, L) = M \cup F$. We still denote by $H^1_D(\Omega)$ the space $H^1_D(\Omega) := \{u \in H^1(\Omega),\ u(y, 0) = u(y, L) = 0\}$.

We then consider the problem

$$\begin{cases} u_\varepsilon \in H^1_D(\Omega), \\ \displaystyle\int_\Omega \big(\chi_F + \varepsilon^2 \chi_M\big)\big(\frac{1}{\varepsilon^2}\nabla' u_\varepsilon \nabla' \phi + \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \phi}{\partial x_3}\big)\ dydx_3 = \int_\Omega f\phi\ dydx_3, \\ \forall\ \phi \in H^1_D(\Omega). \end{cases} \tag{35}$$

Clearly, problem (35) is the variational formulation in the fixed domain Ω of the equation satisfied by $u_\varepsilon(y, x_3) := \bar{u}_\varepsilon(\varepsilon y, x_3)$ and obtained from the equation posed in the variable domain Ω_ε and satisfied by $\bar{u}_\varepsilon$. Loosely speaking, equation (35) takes into account the contrast between the diffusion in the fiber which is assumed to be of order 1 (before scaling) and the diffusion in the matrix (outside the fiber) assumed to be of order ε^2.

As regards of the asymptotic behaviour of the sequence u_ε, the main result may be stated as follows.

Theorem 4.1. *The sequence u_ε of solutions of (35) is such that:*

$$\begin{cases} u_\varepsilon \to u_0(x) + v(x_3) \quad \textit{strongly in } L^2(I; H^1(Y)), \\ \frac{1}{\varepsilon}\nabla' u_\varepsilon \chi_F \to 0 \quad \textit{strongly in } \big(L^2(\Omega)\big)^2,\ \frac{\partial u_\varepsilon}{\partial x_3}\chi_F \to \frac{dv}{dx_3}\chi_F \quad \textit{in } L^2(\Omega), \\ \nabla' u_\varepsilon \chi_M \to \nabla' u_0 \chi_M \quad \textit{strongly in } \big(L^2(\Omega)\big)^2,\ \varepsilon\frac{\partial u_\varepsilon}{\partial x_3}\chi_M \to 0 \quad \textit{in } L^2(\Omega), \end{cases} \tag{36}$$

where the pair (u_0, v) *is the unique solution of*

$$\begin{cases} (u_0, v) \in \{\phi \in L^2(I; H^1(Y)),\ \phi(., x_3) = 0 \ in\, D\} \times H_0^1(I), \\ \displaystyle\int_\Omega \Big(\frac{dv}{dx_3}\frac{d\bar{v}}{dx_3}\chi_F + \nabla' u_0 \nabla' \bar{u}\chi_M\Big)\ dydx_3 = \int_\Omega f(\bar{u} + \bar{v})\ dydx_3, \\ \forall\, (\bar{u}, \bar{v}) \in \{\phi \in L^2(I; H^1(Y)),\ \phi = 0 \ in\, D\} \times H_0^1(I). \end{cases} \tag{37}$$

Furthermore, the convergence of the energies $E_\varepsilon := \displaystyle\int_\Omega \big(\chi_F + \varepsilon^2\chi_M\big)\big(\frac{1}{\varepsilon^2}\nabla' u_\varepsilon \nabla' u_\varepsilon + \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_\varepsilon}{\partial x_3}\big)\ dydx_3$ *towards the limit energy* $E_0 := \displaystyle\int_\Omega \Big(\frac{dv}{dx_3}\frac{dv}{dx_3}\chi_F + \nabla' u_0 \nabla' u_0 \chi_M\Big)\ dydx_3$ *holds true.*

Proof. Taking $\phi = u_\varepsilon$ in equation (35), we get

$$E_\varepsilon = \int_\Omega f u_\varepsilon\ dydx_3 \le C \parallel u_\varepsilon \parallel_{L^2(\Omega)}. \tag{38}$$

By the assumption $u_\varepsilon(y, L) = u_\varepsilon(y, 0) = 0$ one can apply the one dimensional Poincaré inequality in $(0, L)$ to the function $u_\varepsilon(y, .)$ for a given $y \in Y$ to obtain

$$\int_0^L |u_\varepsilon(y, x_3)|^2\ dx_3 \le C \int_0^L |\frac{\partial u_\varepsilon}{\partial x_3}(y, x_3)|^2\ dx_3, \text{ a.e. in } Y. \tag{39}$$

Integrating (39) with respect to $y \in D$ and bearing in mind that $F = D \times (0, L)$, we get

$$\parallel u_\varepsilon \parallel^2_{L^2(F)} \le C \parallel \frac{\partial u_\varepsilon}{\partial x_3}\chi_F \parallel^2_{L^2(\Omega)} \le CE_\varepsilon. \tag{40}$$

On the other hand, for a given $x_3 \in (0, L)$ the Poincaré-Wirtinger inequality gives the estimate

$$\parallel u_\varepsilon(., x_3) - \frac{1}{|D|}\int_D u_\varepsilon(., x_3)\ dy \parallel_{L^2(Y)} \le C \parallel \nabla' u_\varepsilon(., x_3) \parallel_{L^2(Y)}. \tag{41}$$

Integrating (41) with respect to $x_3 \in (0, L)$, we derive with the help of (40) and the fact that $\parallel \nabla' u_\varepsilon \parallel^2_{L^2(\Omega)} \le E_\varepsilon$ for sufficiently small ε, the estimate

$$\parallel u_\varepsilon \parallel^2_{L^2(\Omega} \le CE_\varepsilon. \tag{42}$$

Turning back to equation (35) and applying the Young inequality in the right hand side, we deduce that $E_\varepsilon \le C$ which in turn implies by virtue of (42) that

$$\parallel u_\varepsilon \parallel_{L^2(\Omega} \le C. \tag{43}$$

From estimate (43) and the boundedness of the sequence E_ε, we deduce, up a subsequence still denoted by ε, the existence of $u_0 \in L^2(I; H^1(Y))$ such that

$$u_\varepsilon \rightharpoonup u_{00}(y, x_3) \quad \text{weakly in } L^2(I; H^1(Y)), \tag{44}$$

which together with the estimates (which are consequences of $E_\varepsilon \le C$)

$$\begin{cases} \parallel \frac{1}{\varepsilon}\nabla' u_\varepsilon \chi_F \parallel_{(L^2(\Omega))^2} \le C, \\ \parallel u_\varepsilon \parallel_{H^1(F)} \le C, \end{cases} \tag{45}$$

easily implies the existence of $v \in H_0^1(I)$ such that

$$u_{00}(y, x_3) = v(x_3) \quad \text{in } F. \tag{46}$$

Therefore we obtain all the convergences stated in (36) except the convergence $\frac{1}{\varepsilon}\nabla' u_\varepsilon \chi_F \rightharpoonup 0$ replacing the strong convergence by the weak convergence, if we define u_0 by

$$u_0(y, x_3) = u_{00}(y, x_3) - v(x_3) \quad \text{in}\, \Omega. \tag{47}$$

To prove that the weak convergences are actually strong convergences, we first identify the limit problem and then we will prove the convergence of the sequence E_ε to E_0. The strong convergence of u_ε in $L^2(\Omega)$ will be derived from the convergence of the sequence E_ε with the help of a kind of Poincaré-Wirtinger inequality as it will be seen below.

The first estimate in (45) amounts to say that the sequence defined by $w_\varepsilon := \frac{1}{\varepsilon}(u_\varepsilon - \frac{1}{|D|}\int_D u_\varepsilon\, dy)$ is bounded in $L^2(I; H^1_m(D))$ where $H^1_m(D)$ is the subspace of functions in $H^1(D)$ with zero average. Hence, one can assume that for a subsequence at least, it converges weakly in $L^2(I; H^1_m(D))$ to some w. Taking a test function ϕ in (35) in the form $\phi = \bar{u} + \bar{v} + \varepsilon\bar{w}$ with $\bar{u} \in \mathcal{D}(\Omega)$ such that $\bar{u} = 0$ in F and $\bar{v} \in H^1_0(I)$, $\bar{w} \in \mathcal{D}(\Omega)$, we can pass to the limit to find the equation

$$\int_\Omega \big((\nabla' w \nabla' \bar{w} + \frac{dv}{dx_3}\frac{d\bar{v}}{dx_3})\chi_F + \nabla' u_0 \nabla' \bar{u}\chi_M\big)\, dydx_3 = \int_\Omega f(\bar{u} + \bar{v})\, dydx_3. \tag{48}$$

By a density argument we can choose $\bar{u} = \bar{v} = 0$ and $\bar{w} = w$ in (48) so that we get $w = 0$ and equation (37) is obtained. Note that by the same, the convergence $\frac{1}{\varepsilon}\nabla' u_\varepsilon \chi_F \rightharpoonup 0$ is proved.

Introduce now the sequence

$$X_\varepsilon := \int_\Omega \big((\frac{1}{\varepsilon^2}|\nabla' u_\varepsilon|^2 + (\frac{\partial u_\varepsilon}{\partial x_3} - \frac{dv}{dx_3})^2)\chi_F + (|\nabla' u_\varepsilon - \nabla' u_0|^2 + \varepsilon^2|\frac{\partial u_\varepsilon}{\partial x_3}|^2)\chi_M\big). \tag{49}$$

Choosing $\phi = u_\varepsilon$ in (35) and $(\bar{u}, \bar{v}) = (u_0, v)$ in (37) and thanks to the previous weak convergences, we show that the limit of X_ε is zero so that the stated strong convergences take place. To prove the strong convergence in $L^2(\Omega)$ of the sequence u_ε, we will use the following Poincaré-Wirtinger type inequality: there exists a positive constant C such that

$$\begin{cases} \| u \|_{L^2(\Omega)} \leq C\big(\| \nabla' u \|_{L^2(\Omega)} + \| \dfrac{\partial u}{\partial x_3} \|_{L^2(F)}\big), \\ \\ \forall\, u \in L^2(I; H^1(Y)) \cap L^2(D; H^1_0(I)). \end{cases} \tag{50}$$

The proof may be done arguing by contradiction. Assume that there exists a sequence u_n in that space such that $\| u_n \|_{L^2(\Omega)} = 1$ for all n while the sequence $\| \nabla' u_n \|_{L^2(\Omega)} + \| \frac{\partial u_n}{\partial x_3} \|_{L^2(F)}$ goes to zero. Then the bidimensional Poincaré-Wirtinger inequality applied to the function $u_n(., x_3)$ for $x_3 \in I$ together with an integration over I of such inequality allows one to get the estimate

$$\| u_n - \frac{1}{|D|}\int_D u_n\, dy \|_{L^2(\Omega)} \leq C \| \nabla' u_n \|_{L^2(\Omega)},\ \forall n. \tag{51}$$

On the other hand, the one dimensional Poincaré Inequality applied to the function $u_n(y, .)$ for $y \in D$ and then an integration with respect to $y \in D$ of that inequality lead to the estimate

$$\| \frac{1}{|D|}\int_D u_n\, dy \|_{L^2(\Omega)} \leq C \| \frac{\partial u_n}{\partial x_3} \|_{L^2(\Omega)},\ \forall n. \tag{52}$$

From (51) and (52), we deduce that $\| u_n \|_{L^2(\Omega)}$ goes to zero and this is a contradiction with the assumption.

Applying (50) to the sequence $u_\varepsilon - (u_0 + v)$, (recall that $u_0 = 0$ in F) we get

$$\left\{ \begin{array}{l} \| u_\varepsilon - (u_0 + v) \|_{L^2(\Omega)} \leq C \Big(\| \nabla' u_\varepsilon - \nabla' u_0 \|_{L^2(M)} + \\ \\ + \| \nabla' u_\varepsilon \|_{L^2(F)} + \| \frac{\partial u_\varepsilon}{\partial x_3} - \frac{dv}{dx_3} \|_{L^2(F)} \Big). \end{array} \right. \tag{53}$$

Applying the previous strong convergences, the right hand side of (53) tends to zero so that the proof of the theorem is now complete. □

One can highlight the nonlocal character of the previous equation. Indeed, define $u(x_3) := \displaystyle\int_Y \big(u_0(x) + v(x_3)\big)\, dy = \int_{Y \setminus D} u_0(y, x_3)\, dy + v(x_3)$. Let $\hat{u}$ be the unique solution of

$$\left\{ \begin{array}{l} -\Delta' \hat{u} = 1 \quad \text{in } Y \setminus D, \\ \\ \hat{u} = 0 \quad \text{on } \partial D, \\ \\ \dfrac{\partial \hat{u}}{\partial n} = 0 \quad \text{on } \partial Y. \end{array} \right. \tag{54}$$

Define $m := \displaystyle\int_{Y \setminus D} \hat{u}\, dy > 0$. Then, we have the following result.

Theorem 4.2. *Assume that $f(y, x_3) = f(x_3)$ does not depend on the variable y. Then u_0 given in (36) may be written as $u_0(x) = f(x_3)\hat{u}(y)$ and the sequence $\bar{u}_\varepsilon$ defined in the variable domain $\Omega_\varepsilon = \varepsilon Y \times I$ (recall that $u_\varepsilon(y, x_3) = \bar{u}_\varepsilon(\varepsilon y, x_3)$ for $(y, x_3) \in \Omega$) is such that*

$$\left\{ \begin{array}{l} \dfrac{1}{|\varepsilon Y|} \displaystyle\int_{\varepsilon Y} \bar{u}_\varepsilon(x', x_3)\, dx' \to u \quad \textit{strongly in } \ L^2(I), \\ \\ \dfrac{1}{|\varepsilon M|} \displaystyle\int_{\varepsilon M} \varepsilon \nabla' \bar{u}_\varepsilon(x', x_3)\, dx' \to \dfrac{1}{|M|} \displaystyle\int_M \nabla' u_0(y, x_3)\, dy \\ \\ \textit{strongly in } \ L^2(I), \\ \\ \dfrac{1}{|\varepsilon D|} \displaystyle\int_{\varepsilon D} \dfrac{\partial \bar{u}_\varepsilon}{\partial x_3}(x)\, dx' \to \dfrac{dv}{dx_3} \ \ \textit{strongly in } \ L^2(I). \end{array} \right. \tag{55}$$

The pair (u, v) is the unique solution of the nonlocal one-dimensional problem

$$\left\{ \begin{array}{l} (u, v) \in L^2(I) \times H_0^1(I), \ u(x_3) - v(x_3) = m f(x_3), \\ \\ -|D| \dfrac{d^2 v}{dx_3^2} = f(x_3) \quad \textit{in } I. \end{array} \right. \tag{56}$$

Proof. Making the change of variable $x' = \varepsilon y$, the strong convergences (55) become immediate consequences of the convergences (36).

The second equation of (56) is obtained from the equation (37) by choosing $\bar{u} = 0$ and taking into account the fact that $F = D \times I$.

On the other hand, one can check that the function $f(x_3)\hat{u}(y)$ where $\hat{u}$ is the solution of (54), solves the same equation as u_0, that is the equation obtained from (37) by choosing $\bar{v} = 0$. By virtue of the uniqueness of u_0, we conclude that $u_0(y, x_3) = f(x_3)\hat{u}(y)$ and then the first equation of (56) is nothing but the equality $u(x_3) := \displaystyle\int_{Y\setminus D} u_0(y, x_3)\ dy + v(x_3)$ which defines u. □

5. **Nonlocal effects in the homogenization.** To describe the geometry of the medium, we need further notations.

Let $D(0, r)$ be the disk defined in the previous sections and let ω be the square $\omega :=]-1, 1[^2$. Assume that $\Omega := \omega \times (0, L) = \omega \times I$ is now the configuration domain of a set F_ε of cylindrical parallel fibers periodically distributed with a period $\varepsilon Y = \varepsilon\big(]-\frac{1}{2}, \frac{1}{2}[\big)^2$ in the x'-horizontal directions which are surrounded by a poor conductor occupying the matrix M_ε in such a way that

$$\Omega = F_\varepsilon \cup M_\varepsilon, \quad F_\varepsilon = \bigcup_{i\in I_\varepsilon} F^i_\varepsilon, \quad F^i_\varepsilon = (\varepsilon\bar{D}(0, r) + \varepsilon i) \times I \tag{57}$$

$$M_\varepsilon = \Omega \setminus F_\varepsilon. \tag{58}$$

Hence, the medium is now a periodic replication of the one arising in the previous section. The equation we want to homogenize is the following

$$u_\varepsilon \in H^1_D(\Omega),\ \int_\Omega \big(\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon}\big)\nabla u_\varepsilon \nabla\phi\ dx = \int_\Omega f(x)\phi(x)\ dx,\ \forall\ \phi \in H^1_D(\Omega), \tag{59}$$

where $H^1_D(\Omega)$ is still the space defined in the beginning of the previous section.

For the sake of brevity, we consider only the case of a source term not depending on the microscopic variable but one can handle also that general case as pointed out in the Remark 5.1 below.

Before stating the main result, we recall the definition of two-scale convergence (see [12], [1]) a well adapted tool for periodic homogenization. A sequence $t_\varepsilon \in L^2(\Omega)$ two scale converges to a function $t \in L^2(\Omega \times Y)$ if $\displaystyle\int_\Omega t_\varepsilon(x)\phi(x, \frac{x'}{\varepsilon})\ dx \to \int_\Omega\int_Y t(x, y)\phi(x, y)\ dxdy,\ \forall\,\phi \in L^2(\Omega; C_\#(Y))$ where $C_\#(Y)$ denotes the space of functions which are continuous and Y-periodic. It is known that every bounded sequence in $L^2(\Omega)$ admits a two-scale converging subsequence.

In the sequel, the notation $\rightharpoonup\rightharpoonup$ will stand for the two-scale convergence. The main result may be stated as follows.

Theorem 5.1. *The sequence u_ε of solutions of (59) is such that :*

$$u_\varepsilon \rightharpoonup\rightharpoonup u_0(x, y) + v(x), \tag{60}$$

$$\frac{1}{\varepsilon}\nabla' u_\varepsilon \chi_{F_\varepsilon} \rightharpoonup 0 \ \textit{weakly in}\, L^2(\Omega), \tag{61}$$

$$\frac{\partial u_\varepsilon}{\partial x_3}\chi_{F_\varepsilon} \rightharpoonup |D|\frac{\partial v}{\partial x_3} \quad \textit{weakly in}\ L^2(\Omega), \tag{62}$$

$$\varepsilon\nabla' u_\varepsilon \chi_{M_\varepsilon} \rightharpoonup\rightharpoonup \nabla'_y u_0 \chi_{Y\setminus D},\ \ \varepsilon\frac{\partial u_\varepsilon}{\partial x_3}\chi_{M_\varepsilon} \rightharpoonup\rightharpoonup 0, \tag{63}$$

where the pair $(u_0, v) \in \{\phi \in L^2(\Omega; H^1_\#(Y)),\ \phi(x,.) = 0 \ in\, D\} \times L^2(\omega; H^1_0(I))$ *is the unique solution of*

$$\begin{cases} \displaystyle\int_{\Omega\times Y} \Big(\frac{\partial v}{\partial x_3}\frac{\partial \bar{v}}{\partial x_3}\chi_D(y) + \nabla' u_0 \nabla' \bar{u}\chi_{\omega\setminus D}\Big)\ dxdy = \int_{\Omega\times Y} f(x)(\bar{u}+\bar{v})\ dxdy, \\ \forall\, (\bar{u},\bar{v}) \in \{\phi \in L^2(\Omega; H^1_\#(Y)),\ \phi(x,.) = 0 \ in\, D\} \times L^2(\omega; H^1_0(I)). \end{cases} \tag{64}$$

Furthermore, by the same approach already used in the $3d-1d$ problem, one can eliminate the microscopic variable y to derive a formulation of the limit problem involving only the macroscopic variable x. Indeed, let $\hat{u}$ be the unique solution of

$$\begin{cases} -\Delta'_y \hat{u} = 1 \quad \text{in } Y\setminus D, \\ \hat{u} = 0 \quad \text{on } \partial D, \\ \hat{u} \text{ is } Y-\text{periodic}. \end{cases} \tag{65}$$

Define $m := \displaystyle\int_{Y\setminus D} \hat{u}\ dy > 0$.

Theorem 5.2. *The function* u_0 *given in (60) may be written as* $u_0(x,y) = f(x)\hat{u}(y)$ *and the sequence* u_ε *is such that*

$$u_\varepsilon(x) \rightharpoonup u := \int_{Y\setminus D} u_0(x,y)\ dy + v(x) \quad in\ \ L^2(\Omega), \tag{66}$$

the pair (u,v) *is the unique solution of the nonlocal homogenized problem*

$$\begin{cases} (u,v) \in L^2(\Omega)\times L^2(\omega; H^1_0(I)),\ u(x) - v(x) = mf(x), \\ -|D|\dfrac{\partial^2 v}{\partial x_3^2} = f(x) \quad in\ \Omega. \end{cases} \tag{67}$$

In the homogenization setting we cannot reasonably expect strong convergences similar to those obtained in the reduction of dimension due to the oscillations induced by the microscopic variable. However, we have the following result.

Theorem 5.3. *Let* u_ε *be the sequence of solutions of (59) and let* (u_0, v) *be the pair defined by (64) , then the sequence*

$$X_\varepsilon = \int_\Omega \Big((|\frac{\partial v}{\partial x_3} - \frac{\partial u_\varepsilon}{\partial x_3}|^2 + |\frac{1}{\varepsilon}\nabla' u_\varepsilon|^2)\chi_D(\frac{x'}{\varepsilon}) + (|\nabla'_y u_0(x,\frac{x'}{\varepsilon}) - \nabla' u_\varepsilon|^2 + |\varepsilon\frac{\partial u_\varepsilon}{\partial x_3}|^2)\chi_{Y\setminus D}(\frac{x'}{\varepsilon}) \tag{68}$$

converges to zero.

Remark that this result is a corrector result meaning that u_ε may be approximated by $v(x) + u_0(x, \frac{x'}{\varepsilon})$.

Due to the close similarities between the reduction of dimension problem and the homogenization problem and for the sake of brevity, we will give only the outline of the proof of Theorem 5.1.

Proof. As above we take $\phi = u_\varepsilon$ in the equation (59) and then we prove in the same way that the L^2-norm of u_ε is dominated up to a positive constant by the energy which is now defined by $E_\varepsilon := \displaystyle\int_\Omega \big(\chi_{F_\varepsilon} + \varepsilon^2\chi_{M_\varepsilon}\big)\big(\nabla' u_\varepsilon \nabla' u_\varepsilon + \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial u_\varepsilon}{\partial x_3}\big)\ dx'dx_3$.

Once again, we use the Poincaré-Wirtinger inequality in the reference cell Y to obtain the inequality

$$\int_Y |u - \frac{1}{|D|}\int_D u\ dy|^2\ dy \leq C\int_Y |\nabla u|^2\ dy,\ \ \forall u \in H^1(Y). \tag{69}$$

For given ε, $i \in I_\varepsilon$ and $x_3 \in (0, L)$, we choose $u(y) := u_\varepsilon(\varepsilon y + \varepsilon i, x_3)$ in (69) and then we make the change of variables $x' = \varepsilon y + \varepsilon i$ in such a way we derive the inequality

$$\int_{Y_\varepsilon^i} |u_\varepsilon - \frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} u_\varepsilon\ dx'|^2\ dx' \leq C\int_{Y_\varepsilon^i} \varepsilon^2|\nabla' u_\varepsilon|^2\ dx'. \tag{70}$$

On the other hand, similarly to the proof in the reduction of dimension problem and due to the Dirichlet boundary condition $u_\varepsilon(x', 0) = u_\varepsilon(x', L) = 0$, one can apply the one dimensional Poincaré inequality to the sequence $u_\varepsilon(x', .)$ for almost all $x' \in \omega$ to get after an integration with respect to x' and after summing up over $i \in I_\varepsilon$,

$$\int_\Omega \sum_{i\in I_\varepsilon} |\frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} u_\varepsilon\ dx'|^2\ dx \leq C\sum_{i\in I_\varepsilon}\int_0^L\int_{D_\varepsilon^i} |\frac{\partial u_\varepsilon}{\partial x_3}|^2\ dx = C\int_\Omega |\frac{\partial u_\varepsilon}{\partial x_3}|^2\chi_{F_\varepsilon}\ dx. \tag{71}$$

Summing up over $i \in I_\varepsilon$ and integrating over $(0, L)$ the inequality (70), we get with the helph of (71)

$$\int_\Omega |u_\varepsilon|^2\ dx \leq C\int_\Omega \big(|\frac{\partial u_\varepsilon}{\partial x_3}|^2\chi_{F_\varepsilon} + \varepsilon^2|\nabla' u_\varepsilon|^2\big)\ dx \leq CE_\varepsilon. \tag{72}$$

Using (72) in equation (59) in which we take $\phi = u_\varepsilon$, we derive easily the estimate $E_\varepsilon \leq C$. In particular the last inequality implies that the sequence $\dfrac{\partial u_\varepsilon}{\partial x_3}\chi_{F_\varepsilon}$ is bounded in $L^2(\Omega)$ so that $u_\varepsilon\chi_{F_\varepsilon}$ is bounded in $L^2(\omega; H_0^1(0, L))$. Therefore one can extract a subsequence still denoted ε and find functions $u_{00} \in L^2(\Omega; H_\#^1(Y))$, $K \in (L^2(\Omega \times Y))^2$, $v \in L^2(\omega; H_0^1(0, L))$ such that the following convergences hold true:

$$u_\varepsilon \rightharpoonup\rightharpoonup u_{00},\quad \varepsilon\nabla' u_\varepsilon \rightharpoonup\rightharpoonup \nabla'_y u_{00},\ \frac{1}{\varepsilon}\nabla' u_\varepsilon\chi_{F_\varepsilon} \rightharpoonup\rightharpoonup K\chi_D(y), \tag{73}$$

Since $\nabla' u_\varepsilon\chi_{F_\varepsilon}$ strongly converges to zero in $L^2(\Omega)$, we deduce from (73) that $u_{00}(x, y) = v(x)$ in $\Omega \times D$ with some $v \in L^2(\Omega)$.

In addition, there exists $\bar{v} \in L^2(\omega; H_0^1(0, L))$ such that

$$u_\varepsilon\chi_{F_\varepsilon} \rightharpoonup \bar{v}\ \text{ weakly in }\ L^2(\omega; H_0^1(0, L))),\quad \frac{\partial u_\varepsilon}{\partial x_3}\chi_{F_\varepsilon} \rightharpoonup \frac{\partial \bar{v}}{\partial x_3}\quad \text{weakly in }\ L^2(\Omega). \tag{74}$$

From the first convergence of (73) we deduce $\bar{v}(x) := \displaystyle\int_Y u_{00}(x, y)\chi_D(y)\ dy = |D|v(x)$. Hence, $v \in L^2(\omega; H_0^1(0, L))$.

On the other hand, the Poincaré-Wirtinger inequality applied in $H^1(D)$ and the same change of variables as in the proof of (70) leads to

$$\int_{D_\varepsilon^i} |u_\varepsilon - \frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} u_\varepsilon\ dx'|^2\ dx' \leq C\int_{D_\varepsilon^i} \varepsilon^2|\nabla' u_\varepsilon|^2\ dx'. \tag{75}$$

Using the boundedness in $L^2(\Omega)$ of the sequence $\dfrac{1}{\varepsilon}\nabla' u_\varepsilon\chi_{F_\varepsilon}$, we deduce from (75) that the sequence $w_\varepsilon := \displaystyle\sum_{i\in I_\varepsilon}\frac{1}{\varepsilon}(u_\varepsilon - \frac{1}{|D_\varepsilon^i|}\int_{D_\varepsilon^i} u_\varepsilon\ dx')\chi_{D_\varepsilon^i}$ is bounded in $L^2(0, L; H_m^1(D))$

and one can assume possibly by extracting a subsequence that it converges weakly in that space to some w. We then prove that the last limit K arising in (73) is equal to $\nabla'_y w$.

The function defined by $u_0(x,y) := u_{00}(x,y) - v(x)$ in $\Omega \times Y$ satisfies the convergence (60). Taking in (59) a test function in the form $\phi(x) = \bar{u}_0(x, \frac{x'}{\varepsilon}) + \bar{v}(x) + \bar{w}(x, \frac{x'}{\varepsilon})$ with regular $\bar{u}_0$, $\bar{w}$, $\bar{v}$ and passing to the limit, we obtain the limit equation(64) by a density argument. Choosing $\bar{u}_0 = \bar{v} = 0$ and $\bar{w} = w$ in that equation, we conclude that $w = 0$ completing the proof of Theorem 5.1.

□

Remark 1. In order to emphasize the nonlocal effect at the limit, we have assumed in Theorem 4.2 and in Theorem 5.2 that the function f does not depend on the variable y. One can handle the general case of a source term $f(y,x)$ depending also on the variable y; in this case, setting $\hat{f}(x) := \int_Y f(y,x)\, dy$, one can check easily that the solution $u_0(x,y)$ of (60) takes the form $u_0(x,y) = \hat{f}(x)\hat{u}(y) + \hat{u}_0(x,y)$ where $\hat{u}$ is the solution of (65) and where $\hat{u}_0(x,y)$ is the solution of (60) but with the right hand side given by $f(y,x) - \int_Y f(y,x)\, dy$. This is due to the simple remark that any right hand side $f(y,x)$ of (60) may be written as $f(y,x) = f(y,x) - \int_Y f(y,x)\, dy + \int_Y f(y,x)\, dy$ so that the solution u_0 is the superposition of the two solutions. Of course the same remark holds true in the problem of reduction of dimension. In both cases, the nonlocal effect is due to the term $\hat{f}(x)\hat{u}(y)$.

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