Limit models for thin heterogeneous structures with high contrast

A. Gaudiello, Ali Sili

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A. Gaudiello† A. Sili‡

Abstract

We investigate two linear diffusion problems, with strongly contrasting diffusivity, in a thin heterogeneous cylinder with a small cross-section of radius h_n, n in $\mathbb{N}$. In this cylinder we distinguish an inner cylindrical core $\hat{C}_n$ with cross-section of radius $r_n << h_n$ and its complementary annulus $\hat{I}_n$ and we treat two complementary cases. In the first case we consider a low conductivity of order δ_n^2 in the core $\hat{C}_n$ and a conductivity of order 1 in the annulus $\hat{I}_n$; the opposite situation in the second case. We study the asymptotic behavior of these problems as $h_n \to 0$, $r_n \to 0$, $\frac{r_n}{h_n} \to 0$, and $\delta_n \to 0$. In the first case we prove that the inner core has not any influence on the limit behavior. In the second case, we pinpoint three different limit regimes depending on the ratio $\mu = \lim_n \frac{\delta_n}{h_n}$, according to $\mu = 0$, $0 < \mu < +\infty$, or $\mu = +\infty$. We obtain L^2-strong convergence for the solution and its gradient. We examine the limit problems and compare them with other models.

Keywords: Thin structure; contrasting diffusivity; degenerating equation; correctors.

2020 AMS subject classifications: 35J25, 35J70

1 Introduction

In this paper we investigate two linear diffusion problems, with strongly contrasting diffusivity, in a thin heterogeneous cylinder.

In what follows, x' (resp. y') denotes a generic element (x_1, x_2) (resp. (y_1, y_2)) of $\mathbb{R}^2$, moreover (x', x_3) (resp. (y', x_3)) denotes a generic element $x = (x_1, x_2, x_3)$ (resp. (y_1, y_2, x_3)) of $\mathbb{R}^3$.

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†Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Napoli, Italia. e-mail: gaudiell@unina.it He is also member of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the "Istituto Nazionale di Alta Matematica (INdAM)".

‡Institut de Mathématiques de Marseille (I2M), UMR 7373, Aix-Marseille Université, CNRS, Centrale Marseille, CMI, 39 rue F. Joliot-Curie, 13453 Marseille cedex 13, France. e-mail: ali.sili@univ-amu.fr

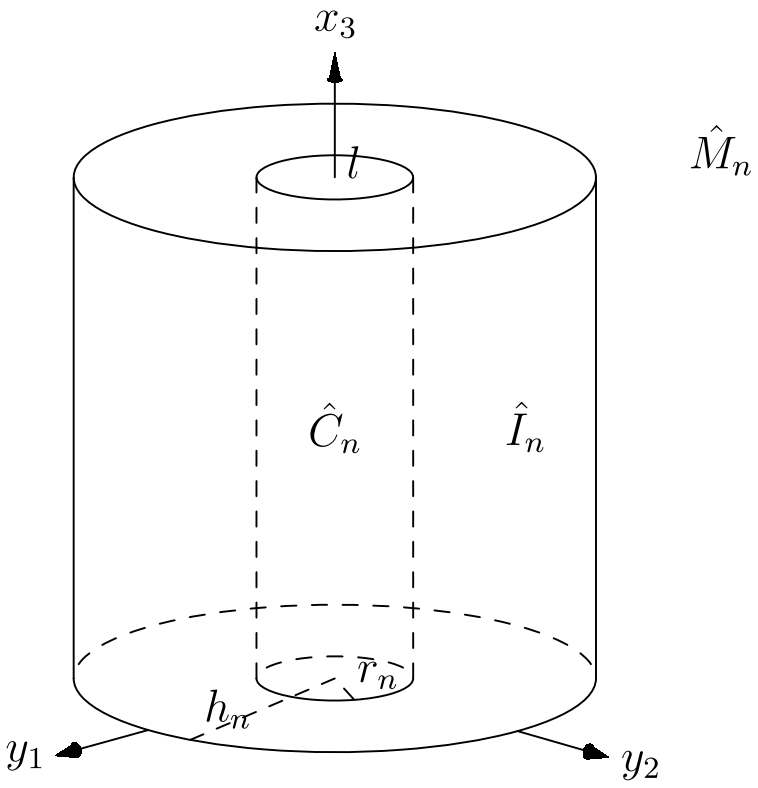

Figure 1: Thin cylinder $\hat{M}_n$

For every n in $\mathbb{N}$ we consider a thin cylinder $\hat{M}_n$ of $\mathbb{R}^3$ with height l and with a small cross-section h_nM'. In $\hat{M}_n$ we distinguish two thin cylinders with the same height l: an inner core $\hat{C}_n$ with small cross-section r_nC' and its complementary annulus $\hat{I}_n = \left(h_nM' \setminus r_n\overline{C'}\right) \times]0, l[$ (see Figure 1), where M' and C' are two open connected sets of $\mathbb{R}^2$ with Lipschitz boundaries such that $0 \in C' \subset\subset M'$, and h_n and r_n are two small parameters taking values in two vanishing sequences of positive numbers such that

$$r_n << h_n.$$

In $\hat{M}_n$ we consider the following boundary value problem:

$$\begin{cases} -\text{div}\left(g_n(y')A\left(x_3\right)D\widehat{u}_n(y',x_3)\right) + b\left(x_3\right)\widehat{u}_n(y',x_3) = \widehat{f}_n(y',x_3), \text{ in } \hat{M}_n, \\ \\ \left(g_n(y')A\left(x_3\right)D\widehat{u}_n(y',x_3)\right)\cdot\nu = 0, \text{ on } \partial\hat{M}_n, \end{cases} \tag{1.1}$$

where A is a coercive matrix with coefficients in $L^\infty(]0,l[)$, b is a coefficient in $L^\infty(]0,l[)$ which is bounded from below away from zero, $\widehat{f}_n$ is a function in $L^2(\hat{M}_n)$, and ν denotes the exterior unit normal to $\partial\hat{M}_n$. For the sake of simplicity in this introduction we assume that the matrix A and the coefficient b depend only on the vertical variable x_3, but in the paper we treat a more general case with A and b depending on (y', x_3). As far as function g_n is concerned, we shall consider two cases:

$$I) \quad g_n = \delta_n^2\chi_{\hat{C}_n} + \chi_{\hat{I}_n}, \qquad II) \quad g_n = \chi_{\hat{C}_n} + \delta_n^2\chi_{\hat{I}_n},$$

where also δ_n is a small parameter taking values in a vanishing sequence of positive numbers, and $\chi_{\hat{C}_n}$ and $\chi_{\hat{I}_n}$ denote the characteristic function of $\hat{C}_n$ and $\hat{I}_n$, respectively. In the first case, $\hat{M}_n$ has a very small insulation inner cylindrical core $\hat{C}_n$ with low conductivity of order δ_n^2 enveloped by the annulus $\hat{I}_n$ with conductivity of order 1. In the second case, we have the opposite situation, i.e., $\hat{M}_n$ has a very small inner cylindrical core $\hat{C}_n$ with conductivity of order 1 enveloped by the insulation annulus $\hat{I}_n$ with low conductivity of order δ_n^2.

In this paper we study the asymptotic behavior of problem (1.1) as $h_n \to 0$, $r_n \to 0$, $\dfrac{r_n}{h_n} \to 0$, $\delta_n \to 0$, and under suitable assumptions on $\widehat{f}_n$. Precisely, we assume that the rescaling of $\widehat{f}_n$ on $M'\times]0,l[$, i.e., $f_n(x',x_3) = \widehat{f}_n(h_n x', x_3)$, is L^2-strongly converging to a function $f(x',x_3)$.

We point out that the peculiarity of this paper is the assumption $\dfrac{r_n}{h_n} \to 0$ (i.e., $r_n << h_n$) combined with the lost of coercivity of the problems when h_n and r_n vanish, due to the vanishing coefficient δ_n^2 in front of the divergence term.

• In the first case (i.e., when $g_n = \delta_n^2 \chi_{\hat{C}_n} + \chi_{\hat{I}_n}$), using the Cauchy-Schwarz inequality, Theorem 3.2 provides that

$$⨍_{h_n M'} \widehat{u}_n(x',\cdot)dx' \longrightarrow u \text{ strongly in } L^2(]0,l[), \tag{1.2}$$

and $u \in H^1(]0,l[)$ is the unique weak solution to

$$\begin{cases} -\dfrac{d}{dx_3}\left(a_0 \dfrac{du}{dx_3}\right) + |M'|bu = f_0, \text{ in }]0,l[, \\ \\ \left(a_0 \dfrac{du}{dx_3}\right)(0) = 0 = \left(a_0 \dfrac{du}{dx_3}\right)(l), \end{cases}$$

where for a.e. x_3 in $]0,l[$

$$a_0(x_3) = \int_{M'} A(x_3) \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} dx', \qquad f_0(x_3) = \int_{M'} f(x',x_3)dx',$$

and $\widetilde{z}(\cdot,x_3) \in H^1(M')$ is the unique weak solution to

$$\begin{cases} \displaystyle\int_{M'} \widetilde{z}(x',x_3)dx' = 0, \\ \\ \text{div}_{x'}\left(A'(x_3)D_{x'}\widetilde{z}\right) = 0, \text{ in } M', \\ \\ \left(A'(x_3)D_{x'}\widetilde{z} + \begin{pmatrix} A_{13}(x_3) \\ A_{23}(x_3) \end{pmatrix}\right) \cdot \nu' = 0, \text{ on } \partial M', \end{cases}$$

with $A' = (A_{\alpha\beta})_{\alpha,\beta=1,2}$ and ν' denoting the unit exterior normal to $\partial M'$.

Note that $a_0(x_3) = |M'|A_{33}(x_3)$ if $A_{13} = A_{23} = 0$ (since in this case $\widetilde{z} = 0$). In particular, $a_0 = |M'|$ if $A = I$.

• In the second case (i.e., when $g_n = \chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}$), the limit behavior of problem (1.1) depends on the

$$\lim_n \frac{\delta_n}{h_n} = \mu \in [0, +\infty].$$

Precisely, using the Cauchy-Schwarz inequality, Theorem 4.1 provides that

- if $\mu \in]0, +\infty[$, then

$$⨍_{h_n M'} \widehat{u}_n(x', \cdot) dx' \longrightarrow ⨍_{M'} u(x', \cdot) dx' \text{ strongly in } L^2(]0, l[),$$

where, for a.e. x_3 in $]0, l[$, $u(\cdot, x_3) \in H^1(M')$ is the unique weak solution to

$$\begin{cases} -\mu^2 \mathrm{div}_{x'} \left(A'(x_3) D_{x'} u(\cdot, x_3) \right) + b(x_3) u(\cdot, x_3) = f(\cdot, x_3), \text{ in } M', \\ \\ \left(A'(x_3) D_{x'} u(\cdot, x_3) \right) \cdot \nu' = 0, \text{ on } \partial M'. \end{cases} \tag{1.3}$$

- If $\mu = 0$ or $\mu = +\infty$, then

$$⨍_{h_n M'} \widehat{u}_n(x', \cdot) dx' - \frac{1}{b} ⨍_{h_n M'} \hat{f}_n(x', \cdot) dx' \longrightarrow 0 \text{ strongly in } L^2(]0, l[).$$

or equivalently,

$$⨍_{h_n M'} \widehat{u}_n(x', \cdot) dx' \longrightarrow \frac{1}{b} ⨍_{M'} f(x', \cdot) dx' \text{ strongly in } L^2(]0, l[).$$

More generally, in Theorem 3.2 and in Theorem 4.1 we consider the case where the matrix A and the coefficient b depend also on y' and we prove L^2-strong convergence results for $\widehat{u}_n$ and for its gradient. It is not difficult to extend these results to monotone operators.

As far as the first case is concerned (i.e., $g_n = \delta_n^2 \chi_{\hat{C}_n} + \chi_{\hat{I}_n}$), let us point out that convergence (1.2) is the same result which one obtains studying the limit behavior of the following problem (for instance, compare [10] ,[15], [16], and [19])

$$\begin{cases} -\mathrm{div} \left(A\left(x_3\right) D\widehat{u}_n(y', x_3) \right) + b\left(x_3\right) \widehat{u}_n(y', x_3) = \widehat{f}_n(y', x_3), \text{ in } \hat{M}_n, \\ \\ \left(A\left(x_3\right) D\widehat{u}_n(y', x_3) \right) \cdot \nu = 0, \text{ on } \partial \hat{M}_n, \end{cases}$$

i.e., roughly speaking, in a first approximation (in $L^2(]0, l[)$-norm) of the average of the solution on the cross-section of the thin cylinder, the presence of a very small insulation cylindrical core $\hat{C}_n$ with a small conductivity does not give any contribution with respect to considering a conductivity of order 1 everywhere in the original problem. Moreover, in Theorem 3.2 we shall show that $\widehat{u}_n(y', x_3)$ and its gradient behave as $u(x_3) + h_n \frac{du}{dx_3}(x_3) \widetilde{z}\left(\frac{y'}{h_n}, x_3 \right)$ in $\widehat{I}_n$ which is a corrector result.

As far as the second case is concerned (i.e., $g_n = \chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}$), h_n is a critical size for δ_n. Precisely, if $\delta_n \simeq h_n$ (i.e., $\mu \in]0, +\infty[$), then the $L^2(]0, l[)$-limit of the average of $\widehat{u}_n$ on the cross-section of the thin cylinder is $⨍_{M'} u(x', \cdot) dx'$, with u the solution to problem

(1.3). Moreover in Theorem 4.1 we shall show that $\widehat{u}_n(y', x_3)$ and its gradient with respect to y' behave as $u\left(\frac{y'}{h_n}, x_3\right)$ in $\widehat{M}_n$. If $\delta_n << h_n$ or $\delta_n >> h_n$ (i.e., $\mu = 0$ or $\mu = +\infty$), the $L^2(]0,l[)$-limi is given by $\frac{1}{b}\fint_{M'} f(x', \cdot)dx'$ (formally, this result can be obtained choosing $\mu = 0$ or $\mu = +\infty$ in problem (1.3)). Indeed, this last result depends on the fact that in this introduction we are assuming b independent of y'. In Theorem (4.1) we shall prove that, more generally (i.e., A and b depend also on y'), the case $\mu = 0$ is different from the case $\mu = +\infty$.

For the sake of completeness and for better understand our result in the second case, we recall what happens when $r_n \simeq h_n \simeq \delta_n$ in the case $g_n = \chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}$. Actually, if for instance one assumes $r_n = rh_n$, $r > 0$, then the limit problem is a nonlocal one in the sense that the limit diffusion is described by a nonlocal system of two equations in which a two-dimensional problem like (1.3), posed in that case on $M' \setminus rC'$, is associated with a one-dimensional problem involving the diffusion along the vertical axis x_3 of the core. This result was proved in [20] for the Laplacian equation with the homogeneous Dirichlet boundary condition on the extremities of the cylinder and the homogeneous Neumann boundary condition elsewhere (also compare [22] for monotone operators), and in [21] for the linearized elasticity system in a thin beam clamped at the extremities. It is easily seen that the nonlocal effect still appears when the homogeneous Neumann boundary condition is assumed everywhere (just a zero order term in the equation is needed in order to obtain *a priori* estimates). On the other hand, the results are similar in the opposite situation, i.e., $g_n = \delta_n^2 \chi_{\hat{C}_n} + \chi_{\hat{I}_n}$ and $r_n \simeq h_n \simeq \delta_n$.

Our present result states that if the radius r_n of the core is negligible compared to h_n and if $g_n = \chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}$ with $\delta_n \simeq h_n$, then the nonlocal effect disappears since the limit diffusion inside the core is not perceived and only the diffusion outside the core (which is described by (1.3)) gives a contribution in the limit process.

Eventually, we point that when $r_n \simeq h_n$, but δ_n has a different behavior, it is an easy task to prove that the limit problem is local. We also refer to [6] for nonlocal limits in problems with high contrast in a thin cylinder.

In the homogenization setting, there is a vast literature on problems with highly contrasting coefficients since the founding work of T. Arbogast, J. Douglas, and U. Hornung [1] (see also [4] and [5]). Here, we just recall those papers devoted to the homogenization of fibered structures with high contrast: [2], [3], [7], [9], [11], [17], [18], [20], [21], [22]. Let us point out that as far as the periodic homogenization of fibered media is concerned, the homogenized problem has the same form as the limit problem obtained by the corresponding reduction of dimension occurring in the reference cell, as remarked in [20].

Our paper is structured as follows: in Section 2 we introduce some notations and we fix the assumptions on A and b depending on (y', x_3). In Section 3 we study the asymptotic behavior of the problem with a low conductivity in the core of the structure. After rescaling the solution on $M = M' \times]0, l[$ by $u_n(x) = \widehat{u}(h_n x', x_3)$, we prove that $\delta_n \left(\frac{1}{h_n} D_{x'} u_n, D_{x_3} u_n\right) \chi_{C_n}$ is bounded in $L^2(M)$ and, up to a subsequence, u_n converges to u weakly in $L^2(M)$ and $\left(\frac{1}{h_n} D_{x'} u_n, D_{x_3} u_n\right) \chi_{I_n}$ converges to $\xi = (\xi', \xi_3)$ weakly in $\left(L^2(M)\right)^3$, where C_n and I_n denote the scaling of $\hat{C}_n$ and $\hat{I}_n$, respectively. It should be noted that C_n and I_n still depend on n

since $\hat{C}_n$ and $\hat{I}_n$ are dependent also on r_n (and not only on h_n). It is not difficult to show that u is independent of x' and $\xi_3 = D_{x_3}u$, but it is quite complicated to write ξ' in terms of u. To do that we need to adapt an uniform extension operator introduced in [12]. Eventually, using the convergence of the energies allows us to obtain the strong convergence in $L^2(M)$ for u_n and for its rescaled gradient (in particular we prove that $\delta_n \left(\frac{1}{h_n} D_{x'} u_n, D_{x_3} u_n\right) \chi_{C_n}$ converges to zero strongly in $L^2(M)$). In Section 4 we study the asymptotic behavior of the problem with a low conductivity in the annulus of the structure. At first we rewrite the problem so that it can be splited in a part on $\widehat{M}_n$ and a part on $\widehat{C}_n$. Then we introduce two different rescalings $u_n(x) = \widehat{u}(h_n x', x_3)$ in $M = M' \times]0, l[$ and $\overline{u}_n(x) = \widehat{u}_n(r_n x', x_3)$ in $C = C' \times]0, l[$, and we write the problem satisfied by the couple $(u_n, \overline{u}_n)$. *A priori estimates* suggest us that the limit problem depends on the ratio $\mu = \lim_n \frac{\delta_n}{h_n}$ and three different regimes appear according to $\mu = 0$, $0 < \mu < +\infty$, or $\mu = +\infty$. The limit problems are obtained passing to the limit in the equation with suitable test functions which, roughly speaking, take into account the relation between u_n and $\overline{u}_n$, and using suitable density results. Eventually, thanks to the convergence of the energies we obtain the strong convergences.

2 Definitions and notation

Let $\{h_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ be two sequences in $]0, 1]$ such that

$$\lim_n r_n = 0, \quad \lim_n h_n = 0, \quad \lim_n \frac{r_n}{h_n} = 0. \tag{2.1}$$

Let

$$0 \in C' \subset\subset M' \subseteq \left]-\frac{1}{2}, \frac{1}{2}\right[^2$$

be two open connected sets with Lipschitz boundaries and let $l \in]0, +\infty[$.

For every $n \in \mathbb{N}$ set

$$\hat{M}_n = h_n M' \times]0, l[, \quad \hat{C}_n = r_n C' \times]0, l[, \quad \hat{I}_n = \left(h_n M' \setminus r_n \overline{C'}\right) \times]0, l[.$$

Set

$$M = M' \times]0, l[, \tag{2.2}$$

and let A and b be such that

$$\begin{cases} A \in (L^\infty(M))^{3\times 3}, \\ \\ \exists \lambda \in]0, +\infty[: \ A(x)\,\xi\,\xi \geq \lambda\,|\xi|^2, \text{ a.e. } x \in M, \quad \forall\, \xi \in \mathbb{R}^3. \end{cases} \tag{2.3}$$

$$\exists \gamma \in]0, +\infty[: \ b \in L^\infty(M), \quad b(x) \geq \gamma, \text{ a.e. } x \in M. \tag{2.4}$$

Let $\{\delta_n\}_{n\in\mathbb{N}} \subset]0, 1]$ be such that

$$\lim_n \delta_n = 0, \tag{2.5}$$

and for every $n \in \mathbb{N}$ let

$$\widehat{f}_n \in L^2(\hat{M}_n). \tag{2.6}$$

In what follows, α, β belong to $\{1, 2\}$ and repeated indices imply the summation.

3 Problem I: a low conductivity in the core of the structure

For every $n \in \mathbb{N}$ consider the following problem

$$\begin{cases} -\mathrm{div}\left(\left(\delta_n^2 \chi_{\hat{C}_n} + \chi_{\hat{I}_n}\right) A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n\right) + b\left(\frac{y'}{h_n}, x_3\right)\widehat{u}_n = \widehat{f}_n, \text{ in } \hat{M}_n, \\ \left(\delta_n^2 \chi_{\hat{C}_n} + \chi_{\hat{I}_n}\right)\left(A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n\right)\cdot \nu = 0, \text{ on } \partial\hat{M}_n, \end{cases} \tag{3.1}$$

where ν denotes the exterior unit normal to $\partial\hat{M}_n$.

The weak formulation of (3.1) is

$$\begin{cases} \widehat{u}_n \in H^1\left(\hat{M}_n\right), \\ \delta_n^2 \displaystyle\int_{\hat{C}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 + \int_{\hat{I}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 \\ + \displaystyle\int_{\hat{M}_n} b\left(\frac{y'}{h_n}, x_3\right)\widehat{u}_n\widehat{\varphi} dy' dx_3 = \int_{\hat{M}_n} \widehat{f}_n \widehat{\varphi} dy' dx_3, \quad \forall \widehat{\varphi} \in H^1\left(\hat{M}_n\right). \end{cases} \tag{3.2}$$

which admits a unique solution by virtue of the Lax-Milgram Theorem.

The goal of this section is to study the asymptotic behavior of this problem, as n diverges. To this aim set

$$f_n : (x', x_3) \in M \to \widehat{f}_n(h_n x', x_3), \tag{3.3}$$

and assume that

$$f_n \to f \text{ strongly in } L^2(M). \tag{3.4}$$

Let $H^1_m(M') = \left\{ v \in H^1(M') : \displaystyle\int_{M'} v dx' = 0 \right\}$. The Lax-Milgram Theorem ensures that for a.e. x_3 in $]0, l[$ the following problem

$$\begin{cases} \widetilde{z}(\cdot, x_3) \in H^1_m(M'), \\ \displaystyle\int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 0 \end{pmatrix} \begin{pmatrix} D_{x'}\psi \\ 0 \end{pmatrix} dx' = \\ - \displaystyle\int_{M'} A \begin{pmatrix} 0' \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\psi \\ 0 \end{pmatrix} dx', \quad \forall \psi \in H^1(M'), \end{cases} \tag{3.5}$$

admits a unique solution. Moreover, the measurability of A ensures (compare [16], pages 749-50) that the function

$$x_3 \in]0, l[\to \widetilde{z}(\cdot, x_3) \in H^1_m(M')$$

is measurable. Furthermore, choosing $\psi = \widetilde{z}(\cdot, x_3)$ as test function in (3.5) provides

$$\sup_{x_3 \in]0,l[} \|D_{x'}\widetilde{z}(\cdot, x_3)\|_{(L^2(M'))^2} < +\infty,$$

i.e.

$$\widetilde{z} \in L^\infty(0,l,H^1_m(M')). \tag{3.6}$$

Set

$$a_0 : x_3 \in]0,l[\to \int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} dx'. \tag{3.7}$$

$$b_0 : x_3 \in]0,l[\to \int_{M'} b dx', \tag{3.8}$$

and

$$f_0 : x_3 \in]0,l[\to \int_{M'} f dx'. \tag{3.9}$$

Then (2.3), (2.4), (3.4), and (3.6) provide that

$$a_0 \in L^\infty(]0,l[), \quad b_0 \in L^\infty(]0,l[), \quad f_0 \in L^2(]0,l[),$$

and

$$a_0(x_3) \geq \lambda |M'|, \quad b_0(x_3) \geq \gamma |M'|, \quad \text{a.e. } x_3 \in]0,l[.$$

Consequently, the Lax-Milgram Theorem ensures that the following problem

$$\begin{cases} u \in H^1(]0,l[), \\ \\ \displaystyle\int_0^l a_0 \frac{du}{dx_3}\frac{dv}{dx_3} dx_3 + \int_0^l b_0 uv dx_3 = \int_0^l f_0 v dx_3, \quad \forall v \in H^1(]0,l[), \end{cases} \tag{3.10}$$

admits a unique solution which is the weak solution to the following one

$$\begin{cases} \displaystyle -\frac{d}{dx_3}\left(a_0 \frac{du}{dx_3}\right) + b_0 u = f_0, \text{ in }]0,l[, \\ \\ \displaystyle \left(a_0 \frac{du}{dx_3}\right)(0) = 0 = \left(a_0 \frac{du}{dx_3}\right)(l). \end{cases} \tag{3.11}$$

Remark 3.1. *If A is independent of x_3, then also $\widetilde{z}$ is independent of x_3. Consequently, a_0 is a positive constant.*

If the matrix A is such that $A_{13} = A_{23} = 0$, then $\widetilde{z} = 0$. Consequently, $a_0 = \int_{M'} A_{33} dx'$. In particular, $a_0 = |M'|$ if $A = I$.

Theorem 3.2. *Assume* (2.1), (2.3), (2.4), (2.5), (2.6), (3.3), (3.4), *For every $n \in \mathbb{N}$ let $\widehat{u}_n$ be the unique weak solution to problem* (3.1). *Let $\widetilde{z} \in L^\infty(0,l,H^1_m(M'))$ be the unique solution to* (3.5). *Let a_0, b_0, and f_0 be defined by* (3.7), (3.8), *and* (3.9), *respectively. Let u be the unique weak solution to* (3.11). *Then*

$$\begin{cases} \displaystyle \lim_n \frac{1}{h_n} \|\widehat{u}_n - u\|_{L^2(\widehat{M}_n)} = 0, \\ \\ \displaystyle \lim_n \frac{1}{h_n} \|\delta_n D\widehat{u}_n\|_{\left(L^2(\widehat{C}_n)\right)^3} = 0, \\ \\ \displaystyle \lim_n \frac{1}{h_n} \left(\left\| D_{y'}\widehat{u}_n - \frac{du}{dx_3}(D_{x'}\widetilde{z})\left(\frac{y'}{h_n}, x_3\right) \right\|_{\left(L^2(\widehat{I}_n)\right)^2} + \left\| D_{x_3}\widehat{u}_n - \frac{du}{dx_3} \right\|_{L^2(\widehat{I}_n)} \right) = 0. \end{cases} \tag{3.12}$$

Remark 3.3. *Let us point out that the last convergence provides a corrector since it means that* $\widehat{u}_n$ *behaves as* $u(x_3) + h_n \dfrac{du}{dx_3} \widetilde{z}\left(\dfrac{y'}{h_n}, x_3\right)$ *in* $\widehat{I}_n$.

Proof. The proof will be divided into nine steps.

Step 1. The first step is devoted to reformulating Problem (3.2).

Problem (3.2) can be reformulated through the map

$$(x', x_3) \in M \to (h_n x', x_3) \in \hat{M}_n.$$

Precisely, for every $n \in \mathbb{N}$ set

$$C_n = \frac{r_n}{h_n} C' \times]0, l[, \quad I_n = \left(M' \setminus \frac{r_n}{h_n} \overline{C'}\right) \times]0, l[,$$

and

$$D_{h_n} : \varphi \in H^1(M) \to \begin{pmatrix} \dfrac{1}{h_n} D_{x_1} \varphi \\ \dfrac{1}{h_n} D_{x_2} \varphi \\ D_{x_3} \varphi \end{pmatrix} \in (L^2(M))^3. \tag{3.13}$$

Then u_n defined by

$$u_n : (x', x_3) \in M \to \widehat{u}_n(h_n x', x_3) \tag{3.14}$$

is the unique solution to

$$\begin{cases} u_n \in H^1(M), \\ \delta_n^2 \displaystyle\int_{C_n} A D_{h_n} u_n D_{h_n} \varphi dx + \int_{I_n} A D_{h_n} u_n D_{h_n} \varphi dx + \int_M b u_n \varphi dx \\ = \displaystyle\int_M f_n \varphi dx, \quad \forall \varphi \in H^1(M). \end{cases} \tag{3.15}$$

Step 2. This step is devoted to proving the existence of a subsequence of $\mathbb{N}$*, still denoted by* $\mathbb{N}$*,* $u \in L^2(M)$*, and* $\xi = (\xi', \xi_3) \in \left(L^2(M)\right)^2 \times L^2(M)$ *(which can depend on the subsequence) such that*

$$u_n \rightharpoonup u \text{ weakly in } L^2(M) \tag{3.16}$$

and

$$\chi_{I_n} D_{h_n} u_n \rightharpoonup \xi = (\xi', \xi_3) \text{ weakly in } \left(L^2(M)\right)^3. \tag{3.17}$$

Choosing $\varphi = u_n$ as test function in (3.15) and using (2.3), (2.4), (3.4), and the Young inequality provide the existence of a positive constant c such that

$$\|u_n\|_{L^2(M)} \leq c, \quad \forall n \in \mathbb{N}, \tag{3.18}$$

$$\|\chi_{C_n} \delta_n D_{h_n} u_n\|_{(L^2(M))^3} \leq c, \quad \forall n \in \mathbb{N}, \tag{3.19}$$

$$\|\chi_{I_n} D_{h_n} u_n\|_{(L^2(M))^3} \leq c, \quad \forall n \in \mathbb{N}. \tag{3.20}$$

Eventually, (3.16) and (3.17)follow from (3.18) and (3.20).

Step 3. This step is devoted to proving that

$$\xi_3 = D_{x_3} u, \ \textit{a.e. in } M. \tag{3.21}$$

and

$$u \in H^1(]0, l[). \tag{3.22}$$

Combining (3.16) with the fact that (thanks to the last limit in (2.1))

$$\chi_{I_n} \to 1 \text{ a.e. in } M \tag{3.23}$$

gives

$$\chi_{I_n} u_n \rightharpoonup u \text{ weakly in } L^2(M). \tag{3.24}$$

Then, (3.21) follows from (3.17), (3.24), and from the fact that χ_{I_n} is independent of x_3.

Now, let us prove that

$$u \text{ is independent of } x' \text{ a.e. in } M. \tag{3.25}$$

To this aim, let us prove that

$$u \text{ is independent of } x' \text{ a.e. in } A_k = \left\{ x = (x', x_3) \in M : |x'| > \frac{1}{k} \right\}, \quad \forall k \in \mathbb{N}, \tag{3.26}$$

which implies(3.25) since $\{A_k\}_{k\in\mathbb{N}}$ is an increasing sequence of open sets such that $\bigcup_{k\in\mathbb{N}} A_k = M$.

Let us fix $k \in \mathbb{N}$. Since for n large enough A_k is included in I_n, (3.20) provides that

$$\|D_{x'} u_n\|_{(L^2(A_k))^2} \leq \|D_{x'} u_n\|_{(L^2(I_n))^2} \leq c h_n, \text{ for } n \text{ large enough},$$

which implies

$$\|D_{x'} u\|_{(L^2(A_k))^2} = 0,$$

thanks to (3.16).

Eventually, (3.22) follows from (3.16), (3.17), (3.21), and (3.25).

Step 4. This step is devoted to proving the existence of z in $L^2(0, l, H^1_m(M'))$ such that

$$\xi' = D_{x'} z, \ \textit{a.e. in } M. \tag{3.27}$$

For every $n \in \mathbb{N}$ let $Q_n \in \mathcal{L}\left(H^1\left(h_n M' \setminus r_n \overline{C'}\right), H^1(h_n M')\right)$ be such that

$$Q_n \varphi = \varphi, \text{ a.e. in } h_n M' \setminus r_n \overline{C'}, \quad \forall \varphi \in H^1\left(h_n M' \setminus r_n \overline{C'}\right), \tag{3.28}$$

and

$$\begin{aligned}&\exists c \in]0,+\infty[\quad : \int_{h_nM'} |D_{y'}(Q_n\varphi)|^2 dy' \\ &\leq c \int_{h_nM'\setminus r_n\overline{C'}} |D_{y'}\varphi|^2 dy', \quad \forall \varphi \in H^1\left(h_nM' \setminus r_n\overline{C'}\right), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{3.29}$$

The existence of such a family is proved in the proof of Lemma A1 in [12] when $M' = \left]-\frac{1}{2},\frac{1}{2}\right[^2$.

Now, for every $n \in \mathbb{N}$ and for a.e. x_3 in $]0,l[$ set

$$z_n(x',x_3) = \left(Q_n\left(\widehat{u}_n(\cdot,x_3)_{|_{h_nM'\setminus r_n\overline{C'}}}\right)\right)(h_nx'), \text{ a.e. } x' \in M', \tag{3.30}$$

where $\widehat{u}_n$ is the unique solution to (3.2).

At first remark that

$$z_n \in L^2(0,l,H^1(M')), \quad \forall n \in \mathbb{N}. \tag{3.31}$$

Indeed, $\widehat{u}_n \in H^1\left(\hat{M}_n\right) \subset H^1(\hat{I}_n)$. Consequently $\widehat{u}_n \in L^2\left(0,l,H^1\left(h_nM' \setminus r_n\overline{C'}\right)\right)$. Then, since $Q_n \in \mathcal{L}\left(H^1\left(h_nM' \setminus r_n\overline{C'}\right), H^1(h_nM')\right)$, one deduces that (see Proposition 1.2.2 in [13]) $Q_n\left(\widehat{u}_n(\cdot,x_3)_{|_{h_nM'\setminus r_n\overline{C'}}}\right) \in L^2(0,l,H^1(h_nM'))$, which implies (3.31).

Now, let us prove that there exist a subsequence of $\mathbb{N}$, still denoted by n, and z in $L^2(0,l,H^1_m(M'))$ (which can depend on the subsequence) such that

$$\chi_{I_n}\frac{1}{h_n}D_{x'}z_n \rightharpoonup D_{x'}z \text{ weakly in } \left(L^2(M)\right)^2. \tag{3.32}$$

Combining (3.30) with (3.29) and (3.14) provides that for every $n \in \mathbb{N}$ and for a.e. x_3 in $]0,l[$

$$\begin{aligned}&\int_{M'} |D_{x'}z_n(x',x_3)|^2 dx' = \int_{M'} \left|D_{x'}\left(\left(Q_n\left(\widehat{u}_n(\cdot,x_3)_{|_{h_nM'\setminus r_n\overline{C'}}}\right)\right)(h_nx')\right)\right|^2 dx' \\ &= \int_{h_nM'} \left|D_{y'}\left(Q_n\left(\widehat{u}_n(\cdot,x_3)_{|_{h_nM'\setminus r_n\overline{C'}}}\right)\right)(y')\right|^2 dy' \\ &\leq c\int_{h_nM'\setminus r_n\overline{C'}} |D_{y'}\widehat{u}_n(y',x_3)|^2 dy' = c\int_{M'\setminus\frac{r_n}{h_n}\overline{C'}} |D_{x'}\left(\widehat{u}_n(h_nx',x_3)\right)|^2 dx' \\ &= c\int_{M'\setminus\frac{r_n}{h_n}\overline{C'}} |D_{x'}\left(u_n(x',x_3)\right)|^2 dx',\end{aligned}$$

which implies, by an integration on $x_3 \in]0,l[$, that

$$\int_M |D_{x'}z_n|^2 dx \leq c\int_{I_n} |D_{x'}u_n|^2 dx, \quad \forall n \in \mathbb{N}, \tag{3.33}$$

where c is the positive constant independent of $n \in \mathbb{N}$ given in (3.29). Then, combining (3.33) with (3.20) provides the existence of a positive constant c such that

$$\left\| \frac{1}{h_n} D_{x'} z_n \right\|_{(L^2(M))^2} \leq c, \quad \forall n \in \mathbb{N},$$

which implies the existence of a subsequence of $\mathbb{N}$, still denoted by n, and z in $L^2(0,l,H^1_m(M'))$ (which can depend on the subsequence) such that

$$\frac{1}{h_n} D_{x'} z_n \rightharpoonup D_{x'} z \text{ weakly in } \left(L^2(M)\right)^2. \tag{3.34}$$

Limit (3.32) follows from (3.23) and (3.34).

Now, remark that (3.30), (3.28), and (3.14) provide that for every $n \in \mathbb{N}$ and for a.e. x_3 in $]0,l[$

$$\begin{aligned} z_n(x',x_3) &= \left(Q_n \left(\widehat{u}_n(\cdot,x_3)_{|_{h_n M' \setminus r_n \overline{C'}}} \right) \right)(h_n x') \\ &= \widehat{u}_n(h_n x', x_3) = u_n(x',x_3), \text{ a.e. } x' \in M' \setminus \tfrac{r_n}{h_n}\overline{C'}. \end{aligned} \tag{3.35}$$

Eventually, (3.27) follows from (3.17), (3.32) and from the fact that

$$\chi_{I_n} \frac{1}{h_n} D_{x'} z_n = \chi_{I_n} \frac{1}{h_n} D_{x'} u_n, \text{ a.e. in } M, \quad \forall n \in \mathbb{N},$$

due to (3.35).

Step 5. *This step is devoted to proving that*

$$\begin{aligned} &\int_M A \begin{pmatrix} D_{x'} z \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} D_{x'} w \\ \dfrac{dv}{dx_3} \end{pmatrix} dx + \int_M buv dx = \int_M fv dx, \\ &\forall v \in H^1(]0,l[), \quad \forall w \in H^1(M). \end{aligned} \tag{3.36}$$

To this aim, let us rewrite the equation in (3.15) in the following way

$$\begin{aligned} &\delta_n^2 \int_{C_n} A D_{h_n} u_n D_{h_n} \varphi dx + \int_{I_n} \Bigg[\left(A_{\alpha\beta} \frac{1}{h_n} D_{x_\beta} u_n + A_{\alpha 3} D_{x_3} u_n \right) \frac{1}{h_n} D_{x_\alpha} \varphi \\ &+ \left(A_{3\alpha} \frac{1}{h_n} D_{x_\alpha} u_n + A_{33} D_{x_3} u_n \right) D_{x_3} \varphi \Bigg] dx + \int_M b u_n \varphi dx = \int_M f_n \varphi dx, \quad \forall \varphi \in H^1(M). \end{aligned} \tag{3.37}$$

Choosing $\varphi = v + h_n w$, with $v, w \in H^1(M)$ and v independent of x', as test function in

(3.37) gives

$$\delta_n^2 \int_{C_n} AD_{h_n} u_n \begin{pmatrix} D_{x'} w \\ \dfrac{dv}{dx_3} + h_n D_{x_3} w \end{pmatrix} dx + \int_{I_n} \left[\left(A_{\alpha\beta} \frac{1}{h_n} D_{x_\beta} u_n + A_{\alpha 3} D_{x_3} u_n \right) D_{x_\alpha} w \right.$$

$$\left. + \left(A_{3\alpha} \frac{1}{h_n} D_{x_\alpha} u_n + A_{33} D_{x_3} u_n \right) \left(\frac{dv}{dx_3} + h_n D_{x_3} w \right) \right] dx + \int_M b u_n (v + h_n w) dx$$

$$= \int_M f_n (v + h_n w) dx, \quad \forall v \in H^1(]0,l[), \quad \forall w \in H^1(M). \tag{3.38}$$

Passing to the limit, as n diverges, in (3.38) and using (2.5), (3.4), (3.19), (3.16), (3.17), (3.21), and (3.27) provide

$$\int_M \left[\left(A_{\alpha\beta} D_{x_\beta} z + A_{\alpha 3} \frac{du}{dx_3} \right) D_{x_\alpha} w + \left(A_{3\alpha} D_{x_\alpha} z + A_{33} \frac{du}{dx_3} \right) \frac{dv}{dx_3} \right] dx + \int_M buv dx$$

$$= \int_M fv dx, \quad \forall v \in H^1(]0,l[), \quad \forall w \in H^1(M),$$

i.e. (3.36).

Step 6. *This step is devoted to identifying z in terms of u, i.e. to proving that*

$$z = \frac{du}{dx_3} \widetilde{z}, \text{ in } L^2(0,l,H^1_m(M')), \tag{3.39}$$

where, for a.e. x_3 in $]0,l[$, $\widetilde{z}(\cdot,x_3)$ is the unique solution to (3.5)

Choosing $v = 0$ and $w = \psi\phi$, with $\psi \in H^1(M')$ and $\phi \in C_0^\infty(]0,l[)$, as test functions in (3.36) gives

$$\int_0^l \left[\int_{M'} A \begin{pmatrix} D_{x'} z \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} D_{x'} \psi \\ 0 \end{pmatrix} dx' \right] \phi dx_3 = 0, \quad \forall \psi \in H^1(M'), \quad \forall \phi \in C_0^\infty(]0,l[).$$

Consequently, since $H^1(M')$ is separable, for a.e. x_3 in $]0,l[$ one has

$$\int_{M'} A \begin{pmatrix} D_{x'} z \\ 0 \end{pmatrix} \begin{pmatrix} D_{x'} \psi \\ 0 \end{pmatrix} dx' = - \int_{M'} A \begin{pmatrix} 0' \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} D_{x'} \psi \\ 0 \end{pmatrix} dx', \quad \forall \psi \in H^1(M'). \tag{3.40}$$

Remark that (3.6) ensures that

$$\frac{du}{dx_3} \widetilde{z} \in L^2(0,l,H^1_m(M')).$$

Moreover, multiplying both sides of (3.5) by $\dfrac{du}{dx_3}$ implies that for a.e. x_3 in $]0,l[$ $\dfrac{du}{dx_3}\widetilde{z}$ solves (3.40). Since for a.e. x_3 in $]0,l[$ this equation admits a unique solution in $H^1_m(M')$, one has that for a.e. x_3 in $]0,l[$

$$z = \frac{du}{dx_3}\widetilde{z}, \text{ a.e. in } M',$$

i.e. (3.39).

Step 7. This step is devoted to identifying u, i.e. to proving that u is the unique solution to (3.10).

Choosing $w = 0$ as test function in (3.36) and using (3.39) give

$$\int_M A \begin{pmatrix} \dfrac{du}{dx_3} D_{x'}\widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} 0' \\ \dfrac{dv}{dx_3} \end{pmatrix} dx + \int_M buv dx = \int_M fv dx, \quad \forall v \in H^1(]0,l[),$$

i.e.

$$\int_0^l \left(\int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} 0' \\ 1 \end{pmatrix} dx' \right) \frac{du}{dx_3}\frac{dv}{dx_3} dx_3 + \int_0^l \left(\int_{M'} b dx' \right) uv dx_3$$

$$= \int_0^l \left(\int_{M'} f dx' \right) v dx_3, \quad \forall v \in H^1(]0,l[),$$

which can be rewritten as

$$\int_0^l \overline{a}\frac{du}{dx_3}\frac{dv}{dx_3} dx_3 + \int_0^l b_0 uv dx_3 = \int_0^l f_0 v dx_3, \quad \forall v \in H^1(]0,l[), \tag{3.41}$$

where

$$\overline{a} = \int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} 0' \\ 1 \end{pmatrix} dx', \text{ a.e. in }]0,l[, \tag{3.42}$$

and b_0 is defined in (3.8).

Remark that

$$\overline{a} = a_0, \text{ a.e. in }]0,l[, \tag{3.43}$$

with a_0 defined in (3.7). Indeed

$$a_0 = \int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} dx' = \int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \left(\begin{pmatrix} 0' \\ 1 \end{pmatrix} + \begin{pmatrix} D_{x'}\widetilde{z} \\ 0 \end{pmatrix} \right) dx'$$

$$= \overline{a} + \int_{M'} A \begin{pmatrix} D_{x'}\widetilde{z} \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\widetilde{z} \\ 0 \end{pmatrix} dx'. \tag{3.44}$$

On the other side, choosing $\psi = \widetilde{z}(\cdot, x_3)$ as test function in (3.5) provides that the last integral in (3.44) is zero. So (3.43) holds true. Then, u is the unique solution to (3.10). Consequently, all previous convergences hold true for the whole sequence.

Step 8. This step is devoted to proving that

$$\lim_n \left[\int_{C_n} |\delta_n D_{h_n} u_n|^2 \, dx + \int_{I_n} \left| D_{h_n} u_n - \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \right|^2 dx + \int_M |u_n - u|^2 dx \right] = 0. \quad (3.45)$$

Passing to the limit in (3.15) with $\varphi = u_n$ as test function, using (3.4) and (3.16), and choosing $v = u$ as test function in (3.10) (which is an admissible test function thanks to (3.22)) provide the convergence of the energies

$$\begin{aligned} &\lim_n \left(\delta_n^2 \int_{C_n} A D_{h_n} u_n D_{h_n} u_n dx + \int_{I_n} A D_{h_n} u_n D_{h_n} u_n dx + \int_M b |u_n|^2 dx \right) \\ &= \int_0^l a_0 \left| \frac{du}{dx_3} \right|^2 dx_3 + \int_0^l b_0 |u|^2 dx_3. \end{aligned} \quad (3.46)$$

Now, (2.3), (2.4) (3.16), (3.17), (3.21), (3.23), (3.25), (3.27), (3.39), (3.8), (3.7), and (3.46) provide

$$\begin{aligned} &\limsup_n \left[\lambda \int_{C_n} |\delta_n D_{h_n} u_n|^2 \, dx + \lambda \int_{I_n} \left| D_{h_n} u_n - \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \right|^2 dx + \gamma \int_M |u_n - u|^2 dx \right] \\ &\leq \limsup_n \left[\delta_n^2 \int_{C_n} A D_{h_n} u_n D_{h_n} u_n dx \right. \\ &+ \int_M A \left(\chi_{I_n} D_{h_n} u_n - \chi_{I_n} \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \right) \left(\chi_{I_n} D_{h_n} u_n - \chi_{I_n} \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \right) dx \\ &\left. + \int_M b \, |u_n - u|^2 \, dx \right] \\ &= \limsup_n \left[\delta_n^2 \int_{C_n} A D_{h_n} u_n D_{h_n} u_n dx + \int_{I_n} A D_{h_n} u_n D_{h_n} u_n dx + \int_M b |u_n|^2 dx \right. \\ &- \int_M A \chi_{I_n} D_{h_n} u_n \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} dx - \int_M A \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \chi_{I_n} D_{h_n} u_n dx \\ &\left. + \int_M \chi_{I_n} A \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} dx - 2 \int_M b u_n u dx + \int_M b \, |u|^2 \, dx \right] = 0, \end{aligned}$$

i.e. (3.45) holds true.

Remark that the third last integral in the last formula tends to

$$\int_M A \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} dx$$

thanks to the Lebesgue dominated convergence theorem, since (3.23) holds true and (2.3) and (3.39) ensure that

$$A \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \dfrac{du}{dx_3} D_{x'} \widetilde{z} \\ \dfrac{du}{dx_3} \end{pmatrix} \in L^1(M).$$

Step 9. Eventually, (3.12) *follows from* (3.45) *by a changing of variable.* □

Remark 3.4. *If assumption* (3.4) *is replaced with*

$$f_n \rightharpoonup f \text{ weakly in } L^2(M).$$

One can prove all the previous result until the convergence of the energies (3.46) *excluded. To prove the convergence of the energies, strong convergence* (3.4) *is needed. Then, to obtain convergences in* (3.12)*, assumption* (3.4) *is needed.*

4 Problem II: a low conductivity in the annulus of the structure

For every $n \in \mathbb{N}$ consider the following problem

$$\begin{cases} -\mathrm{div}\left(\left(\chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}\right) A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n\right) + b\left(\frac{y'}{h_n}, x_3\right) \widehat{u}_n = \widehat{f}_n \text{ in } \hat{M}_n, \\ \left(\chi_{\hat{C}_n} + \delta_n^2 \chi_{\hat{I}_n}\right) \left(A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n\right) \cdot \nu = 0 \text{ on } \partial \hat{M}_n, \end{cases} \tag{4.1}$$

where $\hat{M}_n$, $\hat{C}_n$, $\hat{I}_n$, δ_n, A, b, $\widehat{f}_n$, and ν are defined in the previous section. The weak formulation of (4.1) is

$$\begin{cases} \widehat{u}_n \in H^1\left(\hat{M}_n\right), \\ \displaystyle\int_{\hat{C}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 + \delta_n^2 \int_{\hat{I}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 \\ \displaystyle + \int_{\hat{M}_n} b\left(\frac{y'}{h_n}, x_3\right) \widehat{u}_n \widehat{\varphi} dy' dx_3 = \int_{\hat{M}_n} \widehat{f}_n \widehat{\varphi} dy' dx_3, \quad \forall \widehat{\varphi} \in H^1\left(\hat{M}_n\right). \end{cases} \tag{4.2}$$

which admits a unique solution by virtue of the Lax-Milgram Theorem.

The goal of this section is to study the asymptotic behavior of this problem as n diverges. To this aim for a.e. x_3 in $]0,l[$, consider the following problem

$$\begin{cases} -\mu^2 \mathrm{div}_{x'} \left(A'(\cdot, x_3) D_{x'} u(\cdot, x_3) \right) + b(\cdot, x_3) u(\cdot, x_3) = f(\cdot, x_3), \text{ in } M', \\ \\ \left(A_{\alpha\beta}(\cdot, x_3) D_{x'} u(\cdot, x_3) \right) \cdot \nu' = 0, \text{ on } \partial M', \end{cases} \tag{4.3}$$

where $A' = \left(A_{\alpha\beta} \right)_{\alpha,\beta=1,2}$ and ν' denotes the unit exterior normal to $\partial M'$. For a.e. x_3 in $]0,l[$ the weak formulation to (4.3) is

$$\begin{cases} u(\cdot, x_3) \in H^1(M'), \\ \\ \mu^2 \displaystyle\int_{M'} A_{\alpha\beta}(\cdot, x_3) D_{x_\beta} u(\cdot, x_3) D_{x_\alpha} w dx' + \int_{M'} b(\cdot, x_3) u(\cdot, x_3) w dx' \\ \\ = \displaystyle\int_{M'} f(\cdot, x_3) w dx', \quad \forall w \in H^1(M'). \end{cases} \tag{4.4}$$

which admits a unique solution by virtue of the Lax-Milgram Theorem.

Theorem 4.1. *Assume* (2.1), (2.3), (2.4), (2.5), (2.6), (3.3), (3.4), *For every* $n \in \mathbb{N}$ *let* $\widehat{u}_n$ *be the unique weak solution to problem* (4.1).

i) If

$$\lim_n \frac{\delta_n}{h_n} = \mu \in]0, +\infty[, \tag{4.5}$$

then

$$\begin{cases} \displaystyle\lim_n \frac{1}{h_n} \left\| \widehat{u}_n - u\left(\frac{y'}{h_n}, x_3 \right) \right\|_{L^2(\widehat{M}_n)} = 0, \\ \\ \displaystyle\lim_n \left\| D\widehat{u}_n - \left(D_{x'} \left(u\left(\frac{y'}{h_n}, x_3 \right) \right), 0 \right) \right\|_{L^2(\widehat{M}_n)} = 0, \end{cases} \tag{4.6}$$

where for a.e. x_3 *in* $]0,l[$, $u(\cdot, x_3)$ *is the unique weak solution to* (4.3).

ii) If

$$\lim_n \frac{\delta_n}{h_n} = 0, \tag{4.7}$$

then

$$\lim_n \frac{1}{h_n} \left\| \widehat{u}_n - \frac{f\left(\frac{y'}{h_n}, x_3 \right)}{b\left(\frac{y'}{h_n}, x_3 \right)} \right\|_{L^2(\widehat{M}_n)} = 0 = \lim_n \frac{1}{h_n} \left\| \widehat{u}_n - \frac{\widehat{f}_n}{b\left(\frac{y'}{h_n}, x_3 \right)} \right\|_{L^2(\widehat{M}_n)}. \tag{4.8}$$

iii) If

$$\lim_n \frac{\delta_n}{h_n} = +\infty, \tag{4.9}$$

then

$$\lim_n \frac{1}{h_n} \left\| \widehat{u}_n - \frac{\int_{M'} f(x',\cdot)dx'}{\int_{M'} b(x',\cdot)dx'} \right\|_{L^2(\widehat{M}_n)} = 0 = \lim_n \frac{1}{h_n} \left\| \widehat{u}_n - \frac{\int_{h_n M'} \widehat{f}_n(y',\cdot)dy'}{\int_{h_n M'} b(\frac{y'}{h_n},\cdot)dy'} \right\|_{L^2(\widehat{M}_n)}. \quad (4.10)$$

Remark 4.2. *Let us point out that* (4.6) *provides a corrector since it means that* $\widehat{u}_n$ *and its gradient with respect to* y' *behave as* $u\left(\frac{y'}{h_n}, x_3\right)$ *in* $\widehat{M}_n$*, when* $.\mu \in]0,+\infty[$.

Proof. *Step 1. This step is devoted to reformulating Problem* (4.2).

Problem (4.2) can be rewritten in the following way

$$\begin{cases} \widehat{u}_n \in H^1\left(\hat{M}_n\right), \\ \left(1-\delta_n^2\right) \int_{\hat{C}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 + \delta_n^2 \int_{\hat{M}_n} A\left(\frac{y'}{h_n}, x_3\right) D\widehat{u}_n D\widehat{\varphi} dy' dx_3 \\ + \int_{\hat{M}_n} b\left(\frac{y'}{h_n}, x_3\right) \widehat{u}_n \widehat{\varphi} dy' dx_3 = \int_{\hat{M}_n} \widehat{f}_n \widehat{\varphi} dy' dx_3, \quad \forall \widehat{\varphi} \in H^1\left(\hat{M}_n\right). \end{cases} \quad (4.11)$$

Let M be defined in (2.2) and for every $n \in \mathbb{N}$ let D_{h_n} be defined in (3.13),

$$f_n : (x', x_3) \in M \to \widehat{f}_n(h_n x', x_3), \quad (4.12)$$

and

$$u_n : (x', x_3) \in M \to \widehat{u}_n(h_n x', x_3). \quad (4.13)$$

Moreover, let

$$C = C' \times]0, l[,$$

and for every $n \in \mathbb{N}$ set

$$D_{r_n} : \varphi \in H^1(C) \to \begin{pmatrix} \frac{1}{r_n} D_{x_1}\varphi \\ \frac{1}{r_n} D_{x_2}\varphi \\ D_{x_3}\varphi \end{pmatrix} \in (L^2(C))^3,$$

and

$$\overline{u}_n : (x', x_3) \in C \to \widehat{u}_n(r_n x', x_3). \quad (4.14)$$

Note that the pair $(u_n, \overline{u}_n)$ satisfies the equality

$$\overline{u}_n(x', x_3) = \widehat{u}_n(r_n x', x_3) = \widehat{u}_n\left(h_n \frac{r_n}{h_n} x', x_3\right) = u_n\left(\frac{r_n}{h_n} x', x_3\right),$$

so that if we set for every $n \in \mathbb{N}$

$$V_n = \left\{ (\varphi, \overline{\varphi}) \in H^1(M) \times H^1(C) \ : \ \overline{\varphi}(x', x_3) = \varphi\left(\frac{r_n}{h_n} x', x_3\right), \text{ a.e. in } C \right\}, \quad (4.15)$$

then $(u_n, \overline{u}_n)$ is the unique solution to

$$\begin{cases} (u_n, \overline{u}_n) \in V_n, \\ (1-\delta_n^2)\left(\dfrac{r_n}{h_n}\right)^2 \displaystyle\int_C A\left(\frac{r_n}{h_n}x', x_3\right) D_{r_n}\overline{u}_n D_{r_n}\overline{\varphi} dx + \delta_n^2 \int_M A D_{h_n} u_n D_{h_n}\varphi dx \\ + \displaystyle\int_M b u_n \varphi dx = \int_M f_n \varphi dx, \quad \forall (\varphi, \overline{\varphi}) \in V_n. \end{cases} \tag{4.16}$$

which can be also rewritten in the following way

$$\begin{cases} (u_n, \overline{u}_n) \in V_n, \\ (1-\delta_n^2)\left(\dfrac{r_n}{h_n}\right)^2 \displaystyle\int_C \Bigg[\left(A_{\alpha\beta}\left(\frac{r_n}{h_n}x', x_3\right)\frac{1}{r_n} D_{x_\beta}\overline{u}_n + A_{\alpha 3}\left(\frac{r_n}{h_n}x', x_3\right) D_{x_3}\overline{u}_n \right)\frac{1}{r_n} D_{x_\alpha}\overline{\varphi} \\ + \left(A_{3\alpha}\left(\dfrac{r_n}{h_n}x', x_3\right)\dfrac{1}{r_n} D_{x_\alpha}\overline{u}_n + A_{33}\left(\dfrac{r_n}{h_n}x', x_3\right) D_{x_3}\overline{u}_n \right) D_{x_3}\overline{\varphi} \Bigg] dx \\ +\delta_n^2 \displaystyle\int_M \Bigg[\left(A_{\alpha\beta}\frac{1}{h_n} D_{x_\beta} u_n + A_{\alpha 3} D_{x_3} u_n \right)\frac{1}{h_n} D_{x_\alpha}\varphi \\ + \left(A_{3\alpha}\dfrac{1}{h_n} D_{x_\alpha} u_n + A_{33} D_{x_3} u_n \right) D_{x_3}\varphi \Bigg] dx + \displaystyle\int_M b u_n \varphi dx \\ = \displaystyle\int_M f_n \varphi dx, \quad \forall (\varphi, \overline{\varphi}) \in V_n. \end{cases} \tag{4.17}$$

Step 2. This step is devoted to obtaining a priori estimates for problem (4.16).
Choosing $(\varphi, \overline{\varphi}) = (u_n, \overline{u}_n)$ as test function in (4.16) and using (2.3), (2.4), (2.5), (3.4), and the Young inequality provide the existence of a positive constant c such that

$$\|u_n\|_{L^2(M)} \leq c, \quad \forall n \in \mathbb{N}, \tag{4.18}$$

$$\left\| \frac{1}{h_n} D_{x'}\overline{u}_n \right\|_{(L^2(C))^2} \leq c, \quad \forall n \in \mathbb{N}, \tag{4.19}$$

$$\left\| \frac{r_n}{h_n} D_{x_3}\overline{u}_n \right\|_{L^2(C)} \leq c, \quad \forall n \in \mathbb{N}, \tag{4.20}$$

$$\left\| \frac{\delta_n}{h_n} D_{x'} u_n \right\|_{(L^2(M))^2} \leq c, \quad \forall n \in \mathbb{N}, \tag{4.21}$$

$$\|\delta_n D_{x_3} u_n\|_{L^2(M)} \leq c, \quad \forall n \in \mathbb{N}. \tag{4.22}$$

Step 3. This step is devoted to choosing suitable test functions in (4.17).

Let $w \in C^\infty(\overline{M'})$ be such that w is constant in a neighborhood of $0'$ and let $v \in C_0^\infty(]0,l[)$. For every $n \in \mathbb{N}$ set

$$\overline{\varphi}_n : (x', x_3) \in C \to w\left(\frac{r_n}{h_n} x'\right) v(x_3).$$

Then,

$$(wv, \overline{\varphi}_n) \in V_n, \quad \forall n \in \mathbb{N},$$

and

$$w\left(\frac{r_n}{h_n} x'\right) \text{ is constant a.e. in } C', \text{ for } n \text{ large enough,}$$

thanks to (2.1). Consequently, choosing $(\varphi, \overline{\varphi}) = (wv, \overline{\varphi}_n)$ as test function in (4.17) gives

$$\begin{aligned}
&(1-\delta_n^2)\frac{r_n}{h_n}\int_C \left(A_{3\alpha}\left(\frac{r_n}{h_n}x', x_3\right)\frac{1}{h_n} D_{x_\alpha}\overline{u}_n + A_{33}\left(\frac{r_n}{h_n}x', x_3\right)\frac{r_n}{h_n} D_{x_3}\overline{u}_n\right) D_{x_3} v(x_3) w\left(\frac{r_n}{h_n}x'\right) dx \\
&+ \int_M \Bigg[\left(A_{\alpha\beta}\frac{\delta_n}{h_n} D_{x_\beta} u_n + \delta_n A_{\alpha 3} D_{x_3} u_n\right)\frac{\delta_n}{h_n} D_{x_\alpha} w(x') v(x_3) \\
&+ \left(A_{3\alpha}\frac{\delta_n}{h_n} D_{x_\alpha} u_n + A_{33}\frac{\delta_n}{h_n} h_n D_{x_3} u_n\right)\delta_n D_{x_3} v(x_3) w(x')\Bigg] dx + \int_M b u_n w v dx \\
&= \int_M f_n w v dx, \text{ for } n \text{ large enough.}
\end{aligned} \tag{4.23}$$

Step 4. This step is devoted to prove case i).

Estimates (4.18), (4.21) and (4.22) combined with (4.5) provide that

$$h_n D_{x_3} u_n \rightharpoonup 0 \text{ weakly in } L^2(M), \tag{4.24}$$

and that there exist a subsequence of $\mathbb{N}$, still denoted by $\mathbb{N}$, and u in $L^2(0,l,H^1_m(M'))$ (which can depend on the subsequence) such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(0,l,H^1(M')). \tag{4.25}$$

Passing to the limit in (4.23), as $n \to +\infty$, and using (2.1), (2.3), (2.4), (2.5), (3.4), (4.5), (4.19), (4.20), (4.24), and (4.25) provide

$$\mu^2 \int_M A_{\alpha\beta} D_{x_\beta} u D_{x_\alpha} w v dx + \int_M b u w v dx = \int_M f w v dx,$$

$$\forall w \in C^\infty(\overline{M'}) : w \text{ is constant in a neighborhood of } 0', \quad \forall v \in C_0^\infty(]0,l[).$$

Consequently, by a density argument (compare the proof of Proposition 3.1, in the case $p \leq N-1$, in [14]) one has

$$\mu^2 \int_M A_{\alpha\beta} D_{x_\beta} u D_{x_\alpha} w v dx + \int_M buwvdx = \int_M fwvdx,$$

$$\forall w \in H^1(M'), \quad \forall v \in C_0^\infty(]0,l[),$$

which implies that for a.e. x_3 in $]0,l[$ $u(\cdot, x_3)$ solves (4.4), since $H^1(M')$ is separable. Note that (4.25) holds true for the whole sequence since Problem (4.4) admits a unique solution.

Passing to the limit in (4.16) with $\varphi = (u_n, \overline{u}_n)$ as test function, using (3.4) and (4.25), and choosing $w = u(\cdot., x_3)$ as test function in (4.4) provide the convergence of the energies

$$\lim_n \left[(1-\delta_n^2) \left(\frac{r_n}{h_n} \right)^2 \int_C A\left(\frac{r_n}{h_n} x', x_3 \right) D_{r_n} \overline{u}_n D_{r_n} \overline{u}_n dx \right.$$

$$\left. + \delta_n^2 \int_M A D_{h_n} u_n D_{h_n} u_n dx + \int_M b|u_n|^2 dx \right] = \mu^2 \int_M A_{\alpha\beta} D_{x_\beta} u D_{x_\alpha} u dx + \int_M bu^2 dx \qquad (4.26)$$

$$= \mu^2 \int_M A \begin{pmatrix} D_{x'} u \\ 0 \end{pmatrix} \begin{pmatrix} D_{x'} u \\ 0 \end{pmatrix} dx + \int_M bu^2 dx.$$

Now, (2.3), (2.4), (2.5), (4.5), (4.24), (4.25), and (4.26) provide

$$\limsup_n \left[\lambda \int_M \left| \delta_n D_{h_n} u_n - \mu \begin{pmatrix} D_{x'} u \\ 0 \end{pmatrix} \right|^2 dx + \gamma \int_M |u_n - u|^2 dx \right]$$

$$\leq \limsup_n \left[(1-\delta_n^2) \left(\frac{r_n}{h_n} \right)^2 \int_C A\left(\frac{r_n}{h_n} x', x_3 \right) D_{r_n} \overline{u}_n D_{r_n} \overline{u}_n dx \right.$$

$$+ \int_M A \left(\delta_n D_{h_n} u_n - \mu \begin{pmatrix} D_{x'} u \\ 0 \end{pmatrix} \right) \left(\delta_n D_{h_n} u_n - \mu \begin{pmatrix} D_{x'} u \\ 0 \end{pmatrix} \right) dx$$

$$\left. + \int_M b \left| u_n - u \right|^2 dx \right]$$

$$= \limsup_n \left[(1-\delta_n^2) \left(\frac{r_n}{h_n}\right)^2 \int_C A\left(\frac{r_n}{h_n}x', x_3\right) D_{r_n}\overline{u}_n D_{r_n}\overline{u}_n dx \right.$$

$$+ \int_M \delta_n^2 A D_{h_n} u_n D_{h_n} u_n dx + \int_M b|u_n|^2 dx$$

$$-\mu\delta_n \int_M A D_{h_n} u_n \begin{pmatrix} D_{x'}u \\ 0 \end{pmatrix} dx - \mu\delta_n \int_M A \begin{pmatrix} D_{x'}u \\ 0 \end{pmatrix} D_{h_n} u_n dx$$

$$\left. +\mu^2 \int_M A \begin{pmatrix} D_{x'}u \\ 0 \end{pmatrix} \begin{pmatrix} D_{x'}u \\ 0 \end{pmatrix} dx - 2\int_M b u_n u dx + \int_M b\,|u|^2\,dx \right] = 0,$$

which implies

$$\lim_n \left[\int_M \left| \begin{pmatrix} D_{x'}u_n, \\ h_n D_{x_3} u_n \end{pmatrix} - \begin{pmatrix} D_{x'}u \\ 0 \end{pmatrix} \right|^2 dx + \gamma \int_M |u_n - u|^2 dx \right] = 0, \tag{4.27}$$

thanks to (4.5).

Eventually, (4.6) follows from (4.27) by a changing of variable.

Step 5. This step is devoted to prove case ii).

Estimate (4.18) provides the existence of a subsequence of $\mathbb{N}$, still denoted by n, and u in $L^2(M)$ (which can depend on the subsequence) such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(M). \tag{4.28}$$

Passing to the limit in (4.23), as $n \to +\infty$, and using (2.1), (2.3), (2.4), (2.5), (3.4), (4.7), (4.19), (4.20), (4.21), (4.22), and (4.28) provide

$$\int_M buwv dx = \int_M fwv dx,$$

$$\forall w \in C^\infty(\overline{M'}) : w \text{ is constant in a neighborhood of } 0', \quad \forall v \in C_0^\infty(]0, l[).$$

which implies that

$$u = \frac{f}{b}, \text{ a.e. in } M, \tag{4.29}$$

by a density argument. Note that (4.29) ensures that (4.28) holds true for the whole sequence.

Passing to the limit in (4.16) with $\varphi = (u_n, \overline{u}_n)$ as test function, using (2.3), (2.4), (3.4),

(4.28), a l.s.c argument, and (4.29) provide

$$\int_M \frac{f^2}{b}dx = \int_M b|u|^2dx$$

$$\leq \liminf_n \int_M b|u_n|^2dx \leq \lim_n \left[(1-\delta_n^2)\left(\frac{r_n}{h_n}\right)^2 \int_C A\left(\frac{r_n}{h_n}x', x_3\right) D_{r_n}\overline{u}_n D_{r_n}\overline{u}_n dx\right.$$

$$\left.+\delta_n^2 \int_M AD_{h_n}u_n D_{h_n}u_n dx + \int_M b|u_n|^2dx\right] = \lim_n \int_M f_n u_n dx$$

$$= \int_M fudx = \int_M \frac{f^2}{b}dx = \int_M b|u|^2dx,$$

i.e.

$$\lim_n \int_M |\sqrt{b}u_n|^2dx = \int_M |\sqrt{b}u|^2dx.$$

Consequently, thanks to (4.28) and (2.4),

$$\sqrt{b}u_n \to \sqrt{b}u \text{ strongly in } L^2(M),$$

which implies

$$u_n \to u = \frac{f}{b} \text{ strongly in } L^2(M), \tag{4.30}$$

thanks to (4.29) and (2.4), too.

Now, combining (4.30) with (3.4) ensures that

$$u_n - \frac{f_n}{b} \to 0 \text{ strongly in } L^2(M). \tag{4.31}$$

Eventually, (4.8) follows from (4.30) and (4.31) by a changing of variable.

Step 6. This step is devoted to prove case iii).

Estimates (4.18) and (4.21) provide the existence of a subsequence of $\mathbb{N}$, still denoted by n, and u in $L^2(M)$ independent of x' (which can depend on the subsequence) such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(M). \tag{4.32}$$

Passing to the limit in (4.23), as $n \to +\infty$, with $w = 1$ a.e. in M' and using (2.1), (2.3), (2.4), (2.5), (3.4), (4.9), (4.19), (4.20), (4.21), (4.22), and (4.32) provide

$$\int_M buvdx = \int_M fvdx, \quad \forall v \in C_0^\infty(]0,l[).$$

which implies that

$$u(x_3) = \frac{\int_{M'} f(x', x_3)dx'}{\int_{M'} b(x', x_3)dx'}, \text{ for a.e. } x_3 \in]0,l[. \tag{4.33}$$

Note that (4.33) ensures that (4.32) holds true for the whole sequence.

Passing to the limit in (4.16) with $\varphi = (u_n, \overline{u}_n)$ as test function, using (2.3), (2.4), (3.4), (4.32), a l.s.c argument, and (4.33) provide

$$\int_0^l \left| \frac{\int_{M'} f dx'}{\sqrt{\int_{M'} b dx'}} \right|^2 dx_3 = \int_M b|u|^2 dx$$

$$\leq \liminf \int_M b|u_n|^2 dx \leq \lim_n \left[(1-\delta_n^2) \left(\frac{r_n}{h_n}\right)^2 \int_C A\left(\frac{r_n}{h_n}x', x_3\right) D_{r_n}\overline{u}_n D_{r_n}\overline{u}_n dx \right.$$

$$\left. +\delta_n^2 \int_M A D_{h_n} u_n D_{h_n} u_n dx + \int_M b|u_n|^2 dx \right] = \lim_n \int_M f_n u_n dx$$

$$= \int_M f u dx = \int_0^l \left| \frac{\int_{M'} f dx'}{\sqrt{\int_{M'} b dx'}} \right|^2 dx_3 = \int_M |\sqrt{b}u|^2 dx,$$

i.e.

$$\lim_n \int_M |\sqrt{b}u_n|^2 dx = \int_M |\sqrt{b}u|^2 dx.$$

Consequently, thanks to (4.32) and (2.4),

$$\sqrt{b}u_n \to \sqrt{b}u \text{ strongly in } L^2(M),$$

which implies

$$u_n \to u = \frac{\int_{M'} f(x', \cdot) dx'}{\int_{M'} b(x', \cdot) dx'} \text{ strongly in } L^2(M), \tag{4.34}$$

thanks to (4.33)and (2.4), too.

On the other side, (3.4) provides

$$\int_{M'} f_n(x', \cdot) dx' \to \int_{M'} f(x', \cdot) dx' \text{ strongly in } L^2(M).$$

Consequently, again thanks to (2.4),

$$\frac{\int_{M'} f_n(x', \cdot) dx'}{\int_{M'} b(x', \cdot) dx'} \to \frac{\int_{M'} f(x', \cdot) dx'}{\int_{M'} b(x', \cdot) dx'} \text{ strongly in } L^2(M). \tag{4.35}$$

Then, combining (4.34) with (4.35) gives

$$u_n - \frac{\int_{M'} f_n(x', \cdot) dx'}{\int_{M'} b(x', \cdot) dx'} \to 0 \text{ strongly in } L^2(M), \tag{4.36}$$

Eventually, (4.10) follows from (4.34) and (4.36) by a changing of variable.

□

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