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An uncoupled limit model for a high-contrast problem in a thin multi-structure.

Abstract

We investigate a degenerating elliptic problem in a multi-structure Ω_ε of \mathbb{R}^3 , in the framework of the thermal stationary conduction with highly contrasting diffusivity. Precisely, Ω_ε consists of a fixed base Ω^- surmounted by a thin cylinder Ω_ε^+ with height 1 and cross-section with a small diameter of order ε . Moreover, Ω_ε^+ contains a cylindrical core, always with height 1 and cross-section with diameter of order ε , with conductivity of order 1, surrounded by a ring with conductivity of order ε^2 . Also Ω^- has conductivity of order ε^2 . By assuming that the temperature is zero on the top and on the bottom of the boundary of Ω_ε , while the flux is zero on the remaining part of the boundary, under a suitable choice of the source term we prove that the limit problem, as ε vanishes, boils down to two uncoupled problems: one in Ω^- and one in Ω_1^+ . Moreover, the problem in Ω_1^+ is nonlocal.

Keywords: Degenerating problem, multi-structure, anisotropic, reduction of dimension, nonlocal problem.

AMS Classification: 35B25; 35B27; 35B40

1 Introduction

We investigate a degenerating elliptic problem in a multi-structure Ω_ε of \mathbb{R}^3 , in the framework of the thermal stationary conduction with highly contrasting diffusivity.

Precisely, for a small parameter ε in $]0, 1]$, we consider a multi-structure Ω_ε in \mathbb{R}^3 consisting of a fixed base Ω^- surmounted by a thin cylinder Ω_ε^+ with height 1 and cross-section with diameter of order ε (see Fig. 1).

We assume that Ω_ε is filled with two different materials. Specifically, Ω_ε^+ contains a cylindrical core F_ε , always with height 1 and cross-section with diameter of order ε ,

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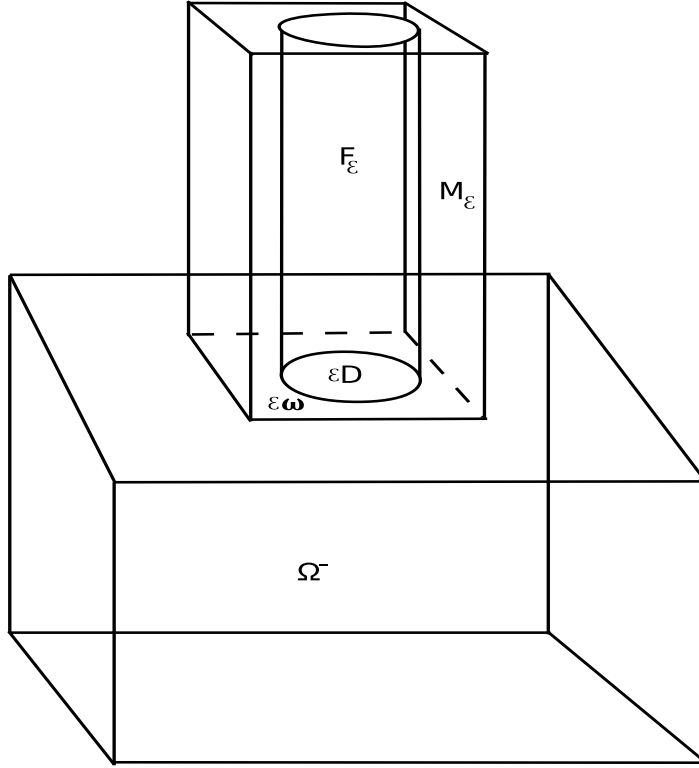


Figure 1: The domain Ω_ε

filled with a material with conductivity of order 1, surrounded by an insulation material in $M_\varepsilon = \text{int}(\Omega_\varepsilon^+ \setminus \bar{F}_\varepsilon)$ with conductivity of order ε^2 . Also Ω^- is filled with a material with conductivity of order ε^2 . Namely, the diffusivity matrix is given by

$$A_\varepsilon(X', x_3) = (\chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon})(X', x_3) A\left(\frac{X'}{\varepsilon}, x_3\right) + \varepsilon^2 \chi_{\Omega^-}(X', x_3) A(X', x_3), \quad \text{a.e. in } \Omega_\varepsilon,$$

where A is a uniformly elliptic matrix with coefficients in $L^\infty(\Omega^+)$, X' denotes a generic element (X_1, X_2) of \mathbb{R}^2 , and $\Omega^+ = \Omega_1$ (i.e., Ω_ε^+ with $\varepsilon = 1$).

We assume that the source term is of order 1 in Ω_ε^+ and of order ε^2 in Ω^- . Namely, the source term is given by

$$f_\varepsilon(X', x_3) = f\left(\frac{X'}{\varepsilon}, x_3\right) \chi_{\Omega_\varepsilon^+}(X', x_3) + \varepsilon^2 f \chi_{\Omega^-}(X', x_3), \quad \text{a.e. in } \Omega_\varepsilon,$$

with $f \in L^2(\Omega^+ \cup \Omega^-)$.

By assuming that the temperature is zero on the top and on the bottom Γ_ε^D of the boundary of Ω_ε , while the flux is zero on the remaining part Γ_ε^N of the boundary, we are

interested in the following problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \bar{u}_\varepsilon) = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla \bar{u}_\varepsilon \cdot \nu = 0 & \text{on } \Gamma_N^\varepsilon, \\ \bar{u}_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^D, \end{cases}$$

where ν denotes the exterior unit normal to $\partial\Omega_\varepsilon$.

A numerical analysis of this problem could be not available since the required discretization step could become too small for the machine precision. So an asymptotic analysis of the problem, as ε vanishes, is needed.

First note that a similar problem was studied in [?], in the case where Ω^- is filled with a material with conductivity of order 1. In this case the authors proved that the diffusion through the base Ω^- is not seen at the limit, and only the diffusion in thin cylinder Ω_ε^+ is taken into account. Moreover, they proved that the diffusion in Ω_ε^+ occurs mainly along the vertical axis of its conductive part and it is described especially by a function v depending only on the vertical variable x_3 . More precisely, they proved that the limit temperature involves also other two functions which are related to the diffusion in the insulation part. Hence the limit problem appears to be nonlocal. About nonlocal limit problems arising from contrasting diffusivity in thin cylinders see also [?] and [?].

The novelty of our paper is the presence of material with conductivity of order ε^2 also in Ω^- . In this case we prove that also a diffusion through the base Ω^- appears at the limit, but it is completely independent of the limit diffusion along thin cylinder Ω_ε^+ . The limit behavior in the thin cylinder is almost the same as in [?]. Then, two uncoupled problems appear at the limit, but some boundary conditions in the two limit problem retain memory of the initial junction. We refer to Section ?? for the formulation of the problem and the statement of the main results (see Theorem ??, Theorem ??, and Remark ??).

By following [?], the problem is rescaled from Ω_ε^+ to Ω^+ . Precisely, setting

$$u_\varepsilon^+(x', x_3) = \bar{u}_\varepsilon(\varepsilon x', x_3) \quad \text{a.e. in } \Omega^+, \quad u_\varepsilon^-(x', x_3) = \bar{u}_\varepsilon(x', x_3) \quad \text{a.e. in } \Omega^-,$$

the following interface condition

$$u_\varepsilon^+(x', 0) = u_\varepsilon^-(\varepsilon x', 0) \tag{1}$$

appears on the surface separating Ω^+ and Ω^- .

Section ?? is devoted to obtaining some sharp *a priori* norm-estimates for $(u_\varepsilon^+, u_\varepsilon^-)$. These estimate are sharp in the sense that they depend on the zone where the material has a low conductivity, or a high conductivity. Then, a first convergence result is obtained.

From a mathematical point of view, one of the main difficulties of this paper is to build suitable test functions that, roughly speaking, at the same time take into account the interface condition in (??), but they also allow to split the problem in two uncoupled

problems at the limit. These test functions are build in Section ?? and they are based on a density result proved in Section ?. Section ? is devoted to the proof of the main results (i.e., Theorem ? and Theorem ?).

We refer to [?] for problems in thin multi-structures. We refer to [?], [?], and [?] for other problems in the same multi-structure of this paper.

Numerous studies have been carried out on the homogenization of high-contrast problems. We just recall the founding work of T. Arbogast, J. Douglas, and U. Hornung [?] (see also [?], [?], [?], and [?]). Here we just focalize the reader's attention on the homogenization of high-contrast problems in fibered structures: [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?] and [?].

For spectral stiff problems in domains surrounded by thin layers, we refer to [?], [?], [?] and references therein.

2 Setting of the problem and main results

In what follows, X' (resp. x') denotes a generic element (X_1, X_2) (resp. (x_1, x_2)) of \mathbb{R}^2 , (X', x_3) (resp. (x', x_3)) denotes a generic element $X = (X_1, X_2, X_3)$ (resp. (x_1, x_2, x_3)) of \mathbb{R}^3 , $\nabla'v$ denotes $\left(\frac{\partial v}{\partial X_1}, \frac{\partial v}{\partial X_2}\right)$ or $\left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\right)$, as appropriate, v' denotes $\frac{\partial v}{\partial x_3}$ if v depends only on x_3 , and χ_A denotes the characteristic function of a set A . Moreover, ε denotes a parameter taking values in a vanishing sequence in $]0, 1[$.

For each ε , we consider the multi-structure in \mathbb{R}^3 (see Fig. 1)

$$\Omega_\varepsilon = \Omega^- \cup (\varepsilon\omega \times \{0\}) \cup \Omega_\varepsilon^+,$$

consisting of the fixed base

$$\Omega^- =]-1, 1[^2 \times]-1, 0[,$$

surmounted by the thin cylinder

$$\Omega_\varepsilon^+ = \varepsilon\omega \times I,$$

where

$$\omega = \left] -\frac{1}{2}, \frac{1}{2} \right[^2, \quad I =]0, 1[.$$

The set $\varepsilon\omega \times \{0\}$ is the surface separating Ω^- from Ω_ε^+ .

We assume that Ω_ε is filled with two different materials. Precisely, Ω_ε^+ contains a cylindrical core

$$F_\varepsilon = \varepsilon D \times I$$

filled with a material with conductivity of order 1, where D is the circle in \mathbb{R}^2 with centre the origin and radius $r \in]0, \frac{1}{2}[$, surrounded by an insulation material in

$$M_\varepsilon = \text{int}(\Omega_\varepsilon^+ \setminus \bar{F}_\varepsilon)$$

with conductivity of order ε^2 . Also Ω^- is filled with a material with conductivity of order ε^2 .

For what follows, we set

$$\Omega^+ = \omega \times I \text{ and } \Omega = \Omega^- \cup (\omega \times \{0\}) \cup \Omega^+.$$

The aim of this paper is to study the asymptotic behaviour, as ε tends to zero, of the weak solution to the following problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \bar{u}_\varepsilon) = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla \bar{u}_\varepsilon \cdot \nu = 0 & \text{on } \Gamma_\varepsilon^D, \\ \bar{u}_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^N, \end{cases} \quad (2)$$

where ν denotes the exterior unit normal to $\partial\Omega_\varepsilon$, Γ_ε^D denotes the top and the bottom of Ω_ε , while Γ_ε^N denotes the remaining part of the boundary of Ω_ε , i.e.

$$\Gamma_\varepsilon^D = \Gamma^- \cup \Gamma_\varepsilon^+, \quad \Gamma^- =]-1, 1[^2 \times \{-1\}, \quad \Gamma_\varepsilon^+ = \left] -\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right[\times \{1\}, \quad \text{and } \Gamma_\varepsilon^N = \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon^D;$$

A_ε is the diffusivity matrix defined by

$$A_\varepsilon(X', x_3) = (\chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon})(X', x_3) A\left(\frac{X'}{\varepsilon}, x_3\right) + \varepsilon^2 \chi_{\Omega^-}(X', x_3) A(X', x_3), \quad \text{a.e. in } \Omega_\varepsilon, \quad (3)$$

with A a 3×3 matrix-valued function on Ω such that

$$\begin{cases} A \in (L^\infty(\Omega))^{3 \times 3}, \\ \exists \alpha \in]0, +\infty[: A(x) \lambda \lambda \geq \alpha |\lambda|^2, \quad \text{a.e. } x \in \Omega, \quad \forall \lambda \in \mathbb{R}^3; \end{cases} \quad (4)$$

and f_ε is the source term defined by

$$f_\varepsilon(X', x_3) = f\left(\frac{X'}{\varepsilon}, x_3\right) \chi_{\Omega_\varepsilon^+}(X', x_3) + \varepsilon^2 f(X', x_3) \chi_{\Omega^-}(X', x_3), \quad \text{a.e. in } \Omega_\varepsilon,$$

with

$$f \in L^2(\Omega). \quad (5)$$

The weak formulation of problem (??) is given by

$$\left\{ \begin{array}{l} \bar{u}_\varepsilon \in H^1(\Omega_\varepsilon), \quad \bar{u}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon^D, \\ \int_{F_\varepsilon} A\left(\frac{X'}{\varepsilon}, x_3\right) \nabla \bar{u}_\varepsilon \nabla v dX' dx_3 + \varepsilon^2 \int_{M_\varepsilon} A\left(\frac{X'}{\varepsilon}, x_3\right) \nabla \bar{u}_\varepsilon \nabla v dX' dx_3 \\ + \varepsilon^2 \int_{\Omega^-} A(X', x_3) \nabla \bar{u}_\varepsilon \nabla v dX' dx_3 \\ = \int_{\Omega_\varepsilon^+} f\left(\frac{X'}{\varepsilon}, x_3\right) v dX' dx_3 + \int_{\Omega^-} f(X', x_3) v dX' dx_3, \\ \forall v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \Gamma_\varepsilon^D. \end{array} \right. \quad (6)$$

The Lax-Milgram Theorem ensures the existence and the uniqueness of the solution to problem (??).

As it is usual (see [?], [?]), problem (??) will be reformulated on the fixed domain Ω through the following map

$$T_\varepsilon : (x', x_3) \in \Omega \rightarrow (\varepsilon x', x_3) \chi_{\Omega^+ \cup (\omega \times \{0\})} + (x', x_3) \chi_{\Omega^-} \in \Omega_\varepsilon.$$

Precisely, setting

$$u_\varepsilon(x', x_3) = \begin{cases} u_\varepsilon^+(x', x_3) = \bar{u}_\varepsilon(\varepsilon x', x_3) & \text{a.e. in } \Omega^+ \cup (\omega \times \{0\}), \\ u_\varepsilon^-(x', x_3) = \bar{u}_\varepsilon(x', x_3) & \text{a.e. in } \Omega^-, \end{cases} \quad (7)$$

the following interface condition

$$u_\varepsilon^+(x', 0) = u_\varepsilon^-(\varepsilon x', 0) \quad \text{a.e. in } \omega,$$

holds true. Consequently, introducing the space

$$H_{D,\varepsilon}^1(\Omega) = \left\{ \phi_\varepsilon = (\phi_\varepsilon^+, \phi_\varepsilon^-) : \phi_\varepsilon^+ \in H^1(\Omega^+), \quad \phi_\varepsilon^- \in H^1(\Omega^-), \right. \\ \left. \phi_\varepsilon^+ = 0 \text{ on } \Gamma^+, \quad \phi_\varepsilon^- = 0 \text{ on } \Gamma^-, \quad \phi_\varepsilon^+(x', 0) = \phi_\varepsilon^-(\varepsilon x', 0) \text{ a.e. } x' \in \omega \right\},$$

where

$$\Gamma^+ = \left] -\frac{1}{2}, \frac{1}{2} \right[\times \{1\},$$

problem (??) is rescaled in the following one

$$\left\{ \begin{array}{l} u_\varepsilon \in H_{D,\varepsilon}^1(\Omega), \\ \int_F A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' \phi_\varepsilon^+ \\ \frac{\partial \phi_\varepsilon^+}{\partial x_3} \end{pmatrix} dx + \int_M A \begin{pmatrix} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \nabla' \phi_\varepsilon^+ \\ \frac{\partial \phi_\varepsilon^+}{\partial x_3} \end{pmatrix} dx \\ + \int_{\Omega^-} A \nabla u_\varepsilon^- \cdot \nabla \phi_\varepsilon^- dx = \int_{\Omega^+} f \phi_\varepsilon^+ dx + \int_{\Omega^-} f \phi_\varepsilon^- dx, \quad \forall \phi_\varepsilon \in H_{D,\varepsilon}^1(\Omega), \end{array} \right. \quad (8)$$

where

$$F = D \times I \text{ and } M = \Omega^+ \setminus \overline{F}.$$

To stating the main result of this paper, the following spaces are needed.

$$\mathcal{U} = \{\phi \in L^2(I; H^1(\omega)) : \phi|_F = 0\}, \quad (9)$$

$$H_d^1(I) = \{\phi \in H^1(I) : \phi(1) = 0\}, \quad (10)$$

$$\mathcal{W} = \left\{ \phi \in L^2(I; H^1(D)) : \int_D \phi(x', x_3) dx' = 0 \text{ a.e. } x_3 \in I \right\}, \quad (11)$$

$$H_d^1(\Omega^-) = \{\phi \in H^1(\Omega^-) : \phi(x', -1) = 0 \text{ a.e. } x' \in (-1, 1)^2\}, \quad (12)$$

In what follows, the subscript t will refer to test functions.

The main result of this paper is the following one.

Theorem 2.1. *For every ε , let u_ε be the unique weak solution to problem (??), under assumptions (??) and (??). Then, there exist a quadruple (u^+, v, w, u^-) in $\mathcal{U} \times H_d^1(I) \times \mathcal{W} \times H_d^1(\Omega^-)$ such that*

$$u_\varepsilon^+ \rightarrow u^+ + v \text{ strongly in } L^2(M), \quad (13)$$

$$\nabla' u_\varepsilon^+ \rightarrow \nabla' u^+ \text{ strongly in } (L^2(M))^2, \quad (14)$$

$$\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \rightarrow 0 \text{ strongly in } L^2(M), \quad (15)$$

$$u_\varepsilon^+ \rightarrow v \text{ strongly in } H^1(F), \quad (16)$$

$$\frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \rightarrow \nabla' w \text{ strongly in } (L^2(F))^2, \quad (17)$$

$$u_\varepsilon^- \rightarrow u^- \text{ strongly in } H^1(\Omega^-), \quad (18)$$

as ε tends to zero, where (u^+, v, w) is the unique solution to the following problem

$$\begin{cases} (u^+, v, w) \in \mathcal{U} \times H_d^1(I) \times \mathcal{W}, \\ \int_F A \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} \begin{pmatrix} \nabla' w_t \\ v'_t \end{pmatrix} dx + \int_M A \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u_t^+ \\ 0 \end{pmatrix} dx \\ = \int_{\Omega^+} f(u_t^+(x) + v_t(x_3)) dx, \quad \forall (u_t^+, v_t, w_t) \in \mathcal{U} \times H_d^1(I) \times \mathcal{W}, \end{cases} \quad (19)$$

while u^- is the unique solution to the following problem

$$\begin{cases} u^- \in H_d^1(\Omega^-), \\ \int_{\Omega^-} A \nabla u^- \nabla u_t^- dx = \int_{\Omega^-} f u_t^- dx, \quad \forall u_t^- \in H_d^1(\Omega^-). \end{cases} \quad (20)$$

In order to highlight the nonlocal character of the diffusion in the upper part of the domain, now another representation of problem (??) will be given. Moreover, Theorem ?? will be translated in terms of the asymptotic behavior of the solution \bar{u}_ε of problem (??).

Theorem 2.2. *For every ε , let \bar{u}_ε be the unique solution to (??), under assumptions (??) and (??). Then*

$$\begin{aligned} & \frac{1}{\varepsilon^2 |\omega|} \int_{\varepsilon\omega} \bar{u}_\varepsilon(X', \cdot) dX' \rightarrow \\ & v(\cdot) + \frac{1}{|\omega|} \left(\frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f(x', \cdot) dx' \int_{\omega \setminus D} \hat{u}(x', \cdot) dx' + \int_{\omega \setminus D} \tilde{u}(x', \cdot) dx' \right) \end{aligned} \quad (21)$$

strongly in $L^2(I)$, as $\varepsilon \rightarrow 0$,

where \tilde{u} is the unique solution to the following problem

$$\begin{cases} \tilde{u} \in \mathcal{U}, \\ \int_M A \begin{pmatrix} \nabla' \tilde{u} \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' \tilde{u}_t \\ 0 \end{pmatrix} dx = \int_M \left(f - \frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f dx' \right) \tilde{u}_t dx, \quad \forall \tilde{u}_t \in \mathcal{U}, \end{cases} \quad (22)$$

for a.e. x_3 in I , $\hat{u}(\cdot, x_3)$ is the unique solution to the following problem

$$\begin{cases} \hat{u}(\cdot, x_3) \in H^1(\omega), \quad \hat{u}(x', x_3) = 0 \text{ a.e. } x' \in D, \\ \int_{\omega \setminus D} A \begin{pmatrix} \nabla' \hat{u} \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' \hat{u}_t \\ 0 \end{pmatrix} dx' = \int_{\omega \setminus D} \hat{u}_t dx', \\ \forall \hat{u}_t \in H^1(\omega), \quad \hat{u}_t(x', x_3) = 0 \text{ a.e. } x' \in D, \end{cases} \quad (23)$$

and v is the unique solution to the following problem

$$\begin{cases} v \in H_d^1(I), \\ \int_I a_0(x_3)v'(x_3)v_t'(x_3)dx_3 = \int_I \left(\int_\omega f(x', x_3) dx' \right) v_t(x_3)dx_3, \quad \forall v_t \in H_d^1(I). \end{cases} \quad (24)$$

In (??) the coefficient a_0 is defined by

$$a_0(x_3) = \int_D \left(\sum_{j=1}^2 a_{3j}(x', x_3) \frac{\partial \hat{w}}{\partial x_j}(x', x_3) + a_{33}(x', x_3) \right) dx', \quad \text{a.e. } x_3 \in I,$$

where, for a.e. x_3 in I , $\hat{w}(\cdot, x_3)$ is the unique solution to the elementary equation

$$\begin{cases} \hat{w}(\cdot, x_3) \in H^1(D), \quad \int_D \hat{w}(x', x_3)dx' = 0, \\ \int_D A \begin{pmatrix} \nabla' \hat{w} \\ 1 \end{pmatrix} \begin{pmatrix} \nabla' \hat{w}_t \\ 0 \end{pmatrix} dx' = 0, \quad \forall \hat{w}_t \in H^1(D). \end{cases} \quad (25)$$

Moreover, \hat{w} belongs to $\mathcal{W} \cap L^\infty(I; H^1(D))$ and \hat{u} belongs to $\mathcal{U} \cap L^\infty(I; H^1(\omega))$.

Remark 2.1. From (??) it appears clearly that the diffusion in the fixed base Ω^- of the multi-domain is completely independent of the diffusion along the thin cylinder Ω_ε^+ .

The right-hand side of (??) represents the limit cross-section average of the temperature in the upper part of the multi-domain. The term v appearing in (??) means that the diffusion in the thin cylinder occurs mainly along the vertical axis of its conductive part. On the other side, v is not sufficient to determine the average of the limit temperature since such average requires to compute \hat{u} and \tilde{u} which are related to the diffusion in the insulation part M of Ω^+ ; hence the limit problem is not local. Moreover, the function \hat{w} which depends only on the diffusivity matrix is due to the anisotropy of the conductive part of the thin cylinder, and it is taken into account in the effective diffusivity coefficient $a_0(x_3)$. When the conductive part is made up of an isotropic material for which the entries of the matrix A are such that $A_{13} = A_{23} = 0$, then $w = 0$ (for instance, [?], [?] and [?]).

Note also that the interface condition disappears at the limit giving rise to Neumann boundary conditions on v and u^- on the interface $x' = 0$.

Finally, we point out that in the terminology of correctors, the strong convergences stated in the Theorem 1.1 provide a corrector for u_ε which means that u_ε^+ behaves as $u^+ + v + \varepsilon w$ while u_ε^- behaves as u^- .

3 A priori norm-estimates and convergence results

This section is devoted to proving some *a priori* norm-estimates for the solutions to problem (??). First, we recall a well-known result, the proof of which is just stated for the sake of clarity.

Lemma 3.1. *Let A be a bounded connected open set of \mathbb{R}^n with Lipschitz boundary and B be a measurable subset of A with positive Lebesgue-measure. Let p in $[1, +\infty]$. Then, there exists C in $]0, +\infty[$ such that*

$$\left\| u - \frac{1}{|B|} \int_B u \, dx \right\|_{L^p(A)} \leq C \|\nabla u\|_{L^p(A)}, \quad \forall u \in W^{1,p}(A). \quad (26)$$

Proof. Set

$$u_A = \frac{1}{|A|} \int_A u \, dx, \quad u_B = \frac{1}{|B|} \int_B u \, dx.$$

Assume p in $[1, +\infty[$. The Poncaré-Wirtingen inequality provides the existence of C_1 in $]0, +\infty[$ such that

$$\begin{aligned} \|u - u_B\|_{L^p(A)} &\leq \|u - u_A\|_{L^p(A)} + \|u_A - u_B\|_{L^p(A)} \leq C_1 \|\nabla u\|_{L^p(A)} + |u_A - u_B| |A|^{\frac{1}{p}}, \\ \forall u \in W^{1,p}(A). \end{aligned} \quad (27)$$

In the other side, the Hölder inequality and again the Poncaré-Wirtingen inequality give

$$\begin{aligned} |u_A - u_B| &\leq \frac{1}{|B|} \int_B |u - u_A| \, dx \leq \frac{1}{|B|} \int_A |u - u_A| \, dx \leq \frac{|A|^{1-\frac{1}{p}}}{|B|} \|u - u_A\|_{L^p(A)} \\ &\leq \frac{|A|^{1-\frac{1}{p}}}{|B|} C_1 \|\nabla u\|_{L^p(A)}, \quad \forall u \in W^{1,p}(A). \end{aligned} \quad (28)$$

Eventually, (??) follows from (??) and (??).

The proof in the case $p = +\infty$ is similar. \square

Proposition 3.1. *For every ε , let u_ε be the solution to problem (??). Then there exists a positive constant C , independent of ε , such that*

$$\|u_\varepsilon^+\|_{L^2(\Omega^+)} \leq C, \quad (29)$$

$$\left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(F)} \leq C, \quad (30)$$

$$\frac{1}{\varepsilon} \|\nabla' u_\varepsilon^+\|_{(L^2(F))^2} \leq C, \quad (31)$$

$$\varepsilon \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(M)} \leq C, \quad (32)$$

$$\|\nabla' u_\varepsilon^+\|_{(L^2(M))^2} \leq C, \quad (33)$$

$$\|u_\varepsilon^-\|_{L^2(\Omega^-)} \leq C, \quad (34)$$

$$\|\nabla' u_\varepsilon^-\|_{(L^2(\Omega^-))^2} \leq C, \quad (35)$$

for every ε .

Proof. Obviously,

$$\begin{aligned} \|u_\varepsilon^+\|_{L^2(\Omega^+)}^2 &\leq 2 \int_{\Omega^+} \left| u_\varepsilon^+(x', x_3) - \frac{1}{|D|} \int_D u_\varepsilon^+(y', x_3) dy' \right|^2 dx' dx_3 \\ &\quad + 2 \int_{\Omega^+} \left| \frac{1}{|D|} \int_D u_\varepsilon^+(y', x_3) dy' \right|^2 dx' dx_3, \quad \forall \varepsilon. \end{aligned} \quad (36)$$

As far as the first integral in (36) is concerned, Lemma 2.2 ensures the existence of C in $]0, +\infty[$ such that

$$\int_\omega \left| u_\varepsilon^+(x', x_3) - \frac{1}{|D|} \int_D u_\varepsilon^+(y', x_3) dy' \right|^2 dx' \leq C \int_\omega |\nabla' u_\varepsilon^+(x', x_3)|^2 dx', \quad \forall \varepsilon, \text{ a.e. } x_3 \in I,$$

which implies

$$\int_{\Omega^+} \left| u_\varepsilon^+(x', x_3) - \frac{1}{|D|} \int_D u_\varepsilon^+(y', x_3) dy' \right|^2 dx' dx_3 \leq C \int_{\Omega^+} |\nabla' u_\varepsilon^+(x', x_3)|^2 dx' dx_3. \quad \forall \varepsilon, \quad (37)$$

by an integration on x_3 in I .

As far as the second integral in (36) is concerned, the Cauchy-Schwarz inequality and the Poincaré inequality provide that

$$\int_{\Omega^+} \left| \frac{1}{|D|} \int_D u_\varepsilon^+(y', x_3) dy' \right|^2 dx' dx_3 \leq |\omega| \int_F |u_\varepsilon^+|^2 dx \leq |\omega| \int_F \left| \frac{\partial u_\varepsilon^+}{\partial x_3} \right|^2 dx, \quad \forall \varepsilon. \quad (38)$$

Combining (37), (38), and (39) gives the existence of C in $]0, +\infty[$ such that

$$\|u_\varepsilon^+\|_{L^2(\Omega^+)}^2 \leq C \left(\|\nabla' u_\varepsilon^+\|_{(L^2(\Omega^+))^3}^2 + \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(F)}^2 \right), \quad \forall \varepsilon. \quad (39)$$

Choosing u_ε as test function in (2.1), thanks to (39) and the Cauchy-Schwarz inequality, get to

$$\frac{\alpha}{\varepsilon^2} \|\nabla' u_\varepsilon^+\|_{(L^2(F))^2}^2 + \alpha \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(F)}^2 + \alpha \|\nabla' u_\varepsilon^+\|_{(L^2(M))^2}^2 + \alpha \varepsilon^2 \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(M)}^2 \quad (40)$$

$$+ \alpha \|\nabla u_\varepsilon^-\|_{(L^2(\Omega^-))^3}^2 \leq \|f\|_{L^2(\Omega)} (\|u_\varepsilon^+\|_{L^2(\Omega^+)} + \|u_\varepsilon^-\|_{L^2(\Omega^-)}), \quad \forall \varepsilon.$$

Combining (??), (??) and Poincaré inequality on Ω^- , leads to existence of C in $]0, +\infty[$ such that

$$\begin{aligned}
& \left(\frac{1}{\varepsilon} \|\nabla' u_\varepsilon^+\|_{(L^2(F))^2} + \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(F)} + \|\nabla' u_\varepsilon^+\|_{(L^2(M))^2} + \varepsilon \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(M)} \right. \\
& \left. + \|\nabla u_\varepsilon^-\|_{(L^2(\Omega^-))^3} \right)^2 \\
& \leq C \left(\frac{1}{\varepsilon} \|\nabla' u_\varepsilon^+\|_{(L^2(F))^2} + \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(F)} + \|\nabla' u_\varepsilon^+\|_{(L^2(M))^2} + \varepsilon \left\| \frac{\partial u_\varepsilon^+}{\partial x_3} \right\|_{L^2(M)} \right. \\
& \left. + \|\nabla u_\varepsilon^-\|_{(L^2(\Omega^-))^3} \right), \quad \forall \varepsilon,
\end{aligned} \tag{41}$$

Eventually, estimates (??), (??), (??), (??), and (??) follow from (??). Estimate (??) follows from (??), (??), (??), and (??). Estimates (??) follows from (??) and the Poincaré inequality in Ω^- . \square

Proposition ?? allows us to obtain the following convergence result.

Proposition 3.2. *For every ε , let u_ε be the solution to problem (??). Let \mathcal{U} , $H_d^1(I)$, \mathcal{W} , and $H_d^1(\Omega^-)$ be the spaces defined in (??), (??), (??), and (??). Then, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, a quadruple (u^+, v, w, u^-) in $\mathcal{U} \times H_d^1(I) \times \mathcal{W} \times H_d^1(\Omega^-)$, depending possibly on the selected subsequence, such that*

$$u_\varepsilon^+ \rightharpoonup u^+ + v \text{ weakly in } L^2(M), \tag{42}$$

$$\nabla' u_\varepsilon^+ \rightharpoonup \nabla' u^+ \text{ weakly in } (L^2(M))^2, \tag{43}$$

$$\varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \rightharpoonup 0 \text{ weakly in } L^2(M), \tag{44}$$

$$u_\varepsilon^+ \rightharpoonup v \text{ weakly in } H^1(F), \tag{45}$$

$$\frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \rightharpoonup \nabla' w \text{ weakly in } (L^2(F))^2, \tag{46}$$

$$u_\varepsilon^- \rightharpoonup u^- \text{ weakly in } H^1(\Omega^-), \tag{47}$$

as $\varepsilon \rightarrow 0$.

Proof. Estimate (??), (??) and (??) provide the existence of a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and

$$u_0 \in L^2(I; H^1(\omega)), \tag{48}$$

depending possibly on the selected subsequence, such that

$$u_\varepsilon^+ \rightharpoonup u_0 \text{ weakly in } L^2(\Omega^+), \text{ as } \varepsilon \rightarrow 0, \quad (49)$$

$$\nabla' u_\varepsilon^+ \rightharpoonup \nabla' u_0 \text{ weakly in } (L^2(\Omega^+))^2, \text{ as } \varepsilon \rightarrow 0, \quad (50)$$

and

$$\nabla' u_0 = 0, \text{ a.e. in } F. \quad (51)$$

Equation (51) implies that in F u_0 depends only on x_3 , i.e. there exists a function

$$v \in L^2(I) \quad (52)$$

such that

$$u_0(x) = v(x_3), \text{ a.e. in } F. \quad (53)$$

Consequently, setting

$$u^+ : x = (x', x_3) \in \Omega^+ \rightarrow u^+(x) = u_0(x) - v(x_3), \quad (54)$$

thanks to (49), (50) and (51) one has that

$$u^+ \in \mathcal{U} \quad (55)$$

and convergences (49) and (50) can be rewritten as

$$u_\varepsilon^+ \rightharpoonup u^+ + v \text{ weakly in } L^2(\Omega^+), \text{ as } \varepsilon \rightarrow 0, \quad (56)$$

$$\nabla' u_\varepsilon^+ \rightharpoonup \nabla' u^+ \text{ weakly in } (L^2(\Omega^+))^2, \text{ as } \varepsilon \rightarrow 0. \quad (57)$$

Now, let us prove that

$$v \in H^1(I). \quad (58)$$

Indeed, (51) provides the existence of a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and of a function

$$u_1 \in L^2(\Omega^+), \quad (59)$$

depending possibly on the selected subsequence, such that

$$\frac{\partial u_\varepsilon^+}{\partial x_3} \chi_F \rightharpoonup u_1 \chi_F \text{ weakly in } L^2(\Omega^+), \text{ as } \varepsilon \rightarrow 0. \quad (60)$$

Moreover, since u belongs to \mathcal{U} , (51) gives that

$$u_\varepsilon^+ \chi_F \rightharpoonup v \chi_F \text{ weakly in } L^2(\Omega^+), \text{ as } \varepsilon \rightarrow 0, \quad (61)$$

which implies

$$\frac{\partial u_\varepsilon^+}{\partial x_3} \chi_F = \frac{\partial}{\partial x_3} (u_\varepsilon^+ \chi_F) \rightarrow \frac{\partial}{\partial x_3} (v \chi_F) = v' \chi_F \text{ in } \mathcal{D}'(\Omega^+), \text{ as } \varepsilon \rightarrow 0, \quad (62)$$

since χ_F is independent of x_3 .

From (??) and (??) it follows that

$$u_1(x) = v'(x_3), \text{ a.e. in } F, \quad (63)$$

which combined with (??) and (??) proves (??).

Now, let us prove that

$$v(1) = 0. \quad (64)$$

Indeed, thanks to (??), (??), and (??)

$$\begin{cases} \int_{\Omega^+} u_\varepsilon^+ \chi_F \varphi \, dx \rightarrow \int_{\Omega^+} v \chi_F \varphi \, dx, \\ \int_{\Omega^+} \frac{\partial u_\varepsilon^+}{\partial x_3} \chi_F \varphi \, dx \rightarrow \int_{\Omega^+} v' \chi_F \varphi \, dx, \\ \text{as } \varepsilon \rightarrow 0, \quad \forall \varphi \in L^2(I), \end{cases} \quad (65)$$

that is

$$\begin{cases} \int_I \left(\int_D u_\varepsilon^+ \, dx' \right) \varphi \, dx_3 \rightarrow |D| \int_I v \varphi \, dx_3, \\ \int_I \left(\int_D u_\varepsilon^+ \, dx' \right)' \varphi \, dx_3 = \int_I \left(\int_D \frac{\partial u_\varepsilon^+}{\partial x_3} \, dx' \right) \varphi \, dx_3 \rightarrow |D| \int_I v' \varphi \, dx_3, \\ \text{as } \varepsilon \rightarrow 0, \quad \forall \varphi \in L^2(I). \end{cases} \quad (66)$$

This means that

$$\int_D u_\varepsilon^+ \, dx' \rightharpoonup |D|v \text{ weakly in } H^1(I), \quad (67)$$

which implies (??), since

$$\int_D u_\varepsilon^+(x', 1) \, dx' = 0, \quad \forall \varepsilon. \quad (68)$$

At this stage, we have proved (??), (??), (??), (??), and (??). Indeed, (??) and (??) follow from (??), (??), (??), (??) and (??). Convergence (??) follows from (??) and (??). Convergence (??) follows from (??), (??), and (??). Eventually, (??) is an immediate consequence of estimates (??) and (??).

Let us prove (??).

For every ε , let us set

$$w_\varepsilon : x = (x', x_3) \in F \rightarrow \frac{u_\varepsilon(x)}{\varepsilon} - \frac{1}{|D|} \int_D \frac{u_\varepsilon(x', x_3)}{\varepsilon} \, dx'. \quad (69)$$

Thanks to (??),

$$\|\nabla' w_\varepsilon\|_{(L^2(F))^2} \leq C, \quad \forall \varepsilon. \quad (70)$$

On the other side, thanks to the Poincaré-Wirtinger inequality there exists c in $]0, +\infty[$ such that

$$\int_D |w_\varepsilon(x', x_3)|^2 dx' \leq c \int_D |\nabla' w_\varepsilon(x', x_3)|^2 dx', \quad \text{a.e. } x_3 \in I, \quad \forall \varepsilon, \quad (71)$$

since

$$\int_D w_\varepsilon(x', x_3) dx' = 0, \quad \text{a.e. } x_3 \in I, \quad \forall \varepsilon. \quad (72)$$

Integrating (??) on x_3 over I and using (??) give

$$\|w_\varepsilon\|_{L^2(F)} \leq \sqrt{c}C, \quad \forall \varepsilon. \quad (73)$$

Hence (??), (??), and (??) provide the existence of w in $L^2(I; \mathcal{W})$ such that

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } L^2(I; \mathcal{W}), \quad \text{as } \varepsilon \rightarrow 0, \quad (74)$$

which implies (??). □

4 Density results

This section is devoted to prove a density result, which proof is obtained by adapting the proof of Proposition 3.1. in [?].

Let us set

$$\begin{aligned} \tilde{V} = & \left\{ (v^+, v^-) \in C^1([0, 1]) \times W^{1,\infty}(\Omega^-) : \right. \\ & \exists I_{v^-} \subset \mathbb{R}^3 \text{ neighborhood of } (0', 0) \text{ depending on } v^- : v^- = v^+(0), \text{ in } I_{v^-} \cap \Omega^-, \\ & \left. v^+(1) = 0, \quad v^-(x', -1) = 0, \text{ a.e. } x' \in]-1, 1[^2 \right\}. \end{aligned} \quad (75)$$

Proposition 4.1. *Let \tilde{V} be defined in (??) respectively. Then \tilde{V} is dense in $H_d^1([0, 1]) \times H_d^1(\Omega^-)$ in H^1 -norm.*

Proof. Let us set

$$\begin{aligned} \hat{V} = & \left\{ (v^+, v^-) \in C^1([0, 1]) \times C^1(\overline{\Omega^-}) : \right. \\ & \left. v^+(1) = 0, \quad v^-(x', -1) = 0, \text{ a.e. } x' \in]-1, 1[^2 \right\}. \end{aligned} \quad (76)$$

To obtain the thesis it is enough to prove that \tilde{V} is dense in \widehat{V} in H^1 - norm.

Let $(v^+, v^-) \in \widehat{V}$. Let us prove that there exists a sequence $\{(v_n^+, v_n^-)\}_{n \in \mathbb{N}}$ in \tilde{V} such that

$$v_n^+ \rightarrow v^+ \text{ in } H^1(]0, 1[), \quad (77)$$

$$v_n^- \rightarrow v^- \text{ in } H^1(\Omega^-). \quad (78)$$

To this aim, for every n in \mathbb{N} , let us set

$$B\left(\frac{1}{n}\right) = \left\{ x = (x', x_3) : |x| < \frac{1}{n}, \quad x_3 < 0 \right\},$$

$$B\left(\frac{2}{n}\right) = \left\{ x = (x', x_3) : |x| < \frac{2}{n}, \quad x_3 < 0 \right\},$$

and

$$\varphi_n : x \in \overline{B\left(\frac{2}{n}\right)} \setminus \overline{B\left(\frac{1}{n}\right)} \rightarrow \varphi_n(x) = n \operatorname{dist}\left(x, \partial B\left(\frac{2}{n}\right)\right) = 2 - n|x|.$$

Obviously,

$$\left\{ \begin{array}{l} \varphi_n \in C^1(\overline{B\left(\frac{2}{n}\right)} \setminus \overline{B\left(\frac{1}{n}\right)}), \quad \forall n, \\ 0 \leq \varphi_n \leq 1, \text{ in } \overline{B\left(\frac{2}{n}\right)} \setminus \overline{B\left(\frac{1}{n}\right)}, \quad \forall n, \\ |\nabla \varphi_n| = n, \text{ in } B\left(\frac{2}{n}\right) \setminus \overline{B\left(\frac{1}{n}\right)}, \quad \forall n, \\ \varphi_n = 1, \text{ on } \partial B\left(\frac{1}{n}\right), \quad \forall n, \\ \varphi_n = 0, \text{ on } \partial B\left(\frac{2}{n}\right), \quad \forall n. \end{array} \right. \quad (79)$$

Now, for every n in \mathbb{N} , let us set

$$v_n^+ = v^+, \text{ in } \Omega^+, \quad (80)$$

$$v_n^- = \begin{cases} v^+(0), & \text{in } \overline{B\left(\frac{1}{n}\right)}, \\ \varphi_n v^+(0) + (1 - \varphi_n) v^-, & \text{in } \overline{B\left(\frac{2}{n}\right)} \setminus \overline{B\left(\frac{1}{n}\right)}, \\ v^-, & \text{in } \Omega^- \setminus \overline{B\left(\frac{1}{n}\right)}, \end{cases} \quad (81)$$

Obviously, the sequence $\{(v_n^+, v_n^-)\}_{n \in \mathbb{N}}$ belongs to \tilde{V} and convergence (??) is satisfied. To complete the proof it remains to prove that also converge (??) is satisfied. Indeed, (??) provides that

$$\begin{aligned} \int_{\Omega^-} |v_n^- - v^-|^2 dx &= \int_{B\left(\frac{1}{n}\right)} |v^+(0) - v^-|^2 dx + \int_{B\left(\frac{2}{n}\right) \setminus B\left(\frac{1}{n}\right)} |\varphi|^2 |v^+(0) - v^-|^2 dx \\ &\leq \|v^+(0) - v^-\|_{L^\infty(\Omega^-)}^2 \frac{16\pi}{3n^3} \rightarrow 0, \text{ as } n \rightarrow +\infty, \end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega^-} |\nabla v_n^- - \nabla v^-|^2 dx \\
&= \int_{B(\frac{1}{n})} |\nabla v^-|^2 dx + \int_{B(\frac{2}{n}) \setminus B(\frac{1}{n})} |-\varphi_n \nabla v^- + (v^+(0) - v^-) \nabla \varphi_n|^2 dx \\
&\leq 2 \|\nabla v^-\|_{(L^\infty(\Omega^-))^3}^2 \frac{16\pi}{3n^3} + 2 \|v^+(0) - v^-\|_{L^\infty(\Omega^-)}^2 \int_{B(\frac{2}{n}) \setminus B(\frac{1}{n})} |\nabla \varphi_n|^2 dx \\
&\leq 2 \|\nabla v^-\|_{(L^\infty(\Omega^-))^3}^2 \frac{16\pi}{3n^3} + 2 \|v^+(0) - v^-\|_{L^\infty(\Omega^-)}^2 \frac{16\pi}{3n} \longrightarrow 0, \text{ as } n \rightarrow +\infty,
\end{aligned}$$

which prove (??). □

5 Proof of Theorem ?? and Theorem ??

5.1 Proof of Theorem ??

Proposition ?? ensures the existence of a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and of a quadruple (u^+, v, w, u^-) in $\mathcal{U} \times H_d^1(I) \times \mathcal{W} \times H_d^1(\Omega^-)$, depending possibly on the selected subsequence, satisfying (??)÷(??).

Let \tilde{V} be defined in (??). Moreover, let us set

$$\mathcal{U}_{\text{reg}} = \{\phi \in C^1(\overline{\Omega^+}) : \phi = 0 \text{ in } F, \quad \phi(\cdot, 1) = 0 \text{ in } \omega\} \quad (82)$$

and

$$C_d^1(\overline{\Omega^+}) = \{\phi \in C^1(\overline{\Omega^+}) : \phi(\cdot, 1) = 0 \text{ in } \omega\}. \quad (83)$$

Now, in (??) choosing

$$\phi_\varepsilon(x', x_3) = \begin{cases} \varepsilon w_t(x) + u_t^+(x) + v_t(x_3), & \text{in } \omega \times (\eta_\varepsilon, 1), \\ \left[\varepsilon w_t(x', \eta_\varepsilon) + u_t^+(x', \eta_\varepsilon) + v_t(\eta_\varepsilon) \right] \frac{x_3}{\eta_\varepsilon} + u_t^-(\varepsilon x', 0) \frac{\eta_\varepsilon - x_3}{\eta_\varepsilon}, & \text{in } \omega \times (0, \eta_\varepsilon), \\ u_t^-(x), & \text{in } \Omega^-, \end{cases} \quad (84)$$

as test function, with

$$u_t^+ \in \mathcal{U}_{\text{reg}}, \quad (v_t, u_t^-) \in \tilde{V}, \quad w_t \in C_d^1(\overline{\Omega^+}), \quad (85)$$

and $\{\eta_\varepsilon\}_\varepsilon$ a decreasing sequence of positive numbers included in $]0, 1[$, such that

$$\lim_{\varepsilon \rightarrow 0} \eta_\varepsilon = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\sqrt{\eta_\varepsilon}} = 0, \quad (86)$$

gives

$$I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + I_{4,\varepsilon} + I_{5,\varepsilon} = I_{6,\varepsilon} + I_{7,\varepsilon} + I_8 \quad \forall \varepsilon, \quad (87)$$

with

$$\begin{aligned} I_{1,\varepsilon} &= \int_{\eta_\varepsilon}^1 \int_{\omega} \chi_{FA} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \nabla' w_t \\ \varepsilon \frac{\partial w_t}{\partial x_3} + v_t' \end{pmatrix} dx' dx_3, \\ I_{2,\varepsilon} &= \int_0^{\eta_\varepsilon} \int_{\omega} \chi_{FA} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \nabla' w_t(x', \eta_\varepsilon) \frac{x_3}{\eta_\varepsilon} + (\nabla' u_t^-)(\varepsilon x', 0) \frac{\eta_\varepsilon - x_3}{\eta_\varepsilon} \\ \frac{1}{\eta_\varepsilon} [\varepsilon w_t(x', \eta_\varepsilon) + v_t(\eta_\varepsilon) - u_t^-(\varepsilon x', 0)] \end{pmatrix} dx' dx_3, \\ I_{3,\varepsilon} &= \int_{\eta_\varepsilon}^1 \int_{\omega} \chi_{MA} \begin{pmatrix} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \varepsilon \nabla' w_t + \nabla' u_t^+ \\ \varepsilon^2 \frac{\partial w_t}{\partial x_3} + \varepsilon \frac{\partial u_t^+}{\partial x_3} + \varepsilon v_t' \end{pmatrix} dx' dx_3, \\ I_{4,\varepsilon} &= \int_0^{\eta_\varepsilon} \int_{\omega} \chi_{MA} \begin{pmatrix} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} [\varepsilon \nabla' w_t(x', \eta_\varepsilon) + \nabla' u_t^+(x', \eta_\varepsilon)] \frac{x_3}{\eta_\varepsilon} + \varepsilon \nabla' u_t^-(\varepsilon x', 0) \frac{\eta_\varepsilon - x_3}{\eta_\varepsilon} \\ \frac{\varepsilon}{\eta_\varepsilon} [\varepsilon w_t(x', \eta_\varepsilon) + u_t^+(x', \eta_\varepsilon) + v_t(\eta_\varepsilon) - u_t^-(\varepsilon x', 0)] \end{pmatrix} dx' dx_3, \\ I_{5,\varepsilon} &= \int_{\Omega^-} A \nabla u_\varepsilon^- \nabla u_t^- dx, \\ I_{6,\varepsilon} &= \int_{\eta_\varepsilon}^1 \int_{\omega} f [\varepsilon w_t + u_t^+ + v_t] dx' dx_3, \\ I_{7,\varepsilon} &= \int_0^{\eta_\varepsilon} \int_{\omega} f [\varepsilon w_t(x', \eta_\varepsilon) + u_t^+(x', \eta_\varepsilon) + v_t(\eta_\varepsilon)] \frac{x_3}{\eta_\varepsilon} + u_t^-(\varepsilon x', 0) \frac{\eta_\varepsilon - x_3}{\eta_\varepsilon} dx' dx_3, \\ I_8 &= \int_{\Omega^-} f u_t^- dx. \end{aligned}$$

Now, let us pass to the limit, as ε tends to zero, in each term of (??).

Convergences (??) and (??) provide that

$$\lim_{\varepsilon \rightarrow 0} I_{1,\varepsilon} = \int_F A \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} \begin{pmatrix} \nabla' w_t \\ v_t' \end{pmatrix} dx. \quad (88)$$

Since (v_t, u_t^-) belong to \tilde{V} , one has that

$$\frac{1}{\eta_\varepsilon} |v_t(\eta_\varepsilon) - u_t^-(\varepsilon x', 0)| = \frac{1}{\eta_\varepsilon} |v_t(\eta_\varepsilon) - v_t(0)| \leq \|v_t'\|_{L^\infty(0,1)}, \quad \forall x' \in \omega,$$

$$\nabla' u_t^-(\varepsilon x', 0) = 0, \quad \forall x' \in \omega,$$

for ε small enough. Consequently, thanks also to (??), (??), and (??), one obtains

$$\lim_{\varepsilon \rightarrow 0} I_{2,\varepsilon} = 0. \quad (89)$$

Convergences (??) and (??) give

$$\lim_{\varepsilon \rightarrow 0} I_{3,\varepsilon} = \int_M A \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u_t^+ \\ 0 \end{pmatrix} dx. \quad (90)$$

Using (??), (??), and (??) provides that

$$\lim_{\varepsilon \rightarrow 0} I_{4,\varepsilon} = 0. \quad (91)$$

Convergence (??) provides that

$$\lim_{\varepsilon \rightarrow 0} I_{5,\varepsilon} = \int_{\Omega^-} A \nabla u^- \nabla u_t^- dx. \quad (92)$$

Moreover, it is obviously that

$$\lim_{\varepsilon \rightarrow 0} I_{6,\varepsilon} = \int_{\Omega^+} f(u_t^+ + v_t) dx, \quad (93)$$

$$\lim_{\varepsilon \rightarrow 0} I_{7,\varepsilon} = 0. \quad (94)$$

Eventually, passing to the limit, as ε tends to 0, in (??), and using (??) \div (??), get

$$\left\{ \begin{array}{l} (u^+, v, w, u^-) \in \mathcal{U} \times H_d^1(I) \times \mathcal{W} \times H_d^1(\Omega^-), \\ \int_F A \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} \begin{pmatrix} \nabla' w_t \\ v_t' \end{pmatrix} dx + \int_M A \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u_t^+ \\ 0 \end{pmatrix} dx + \int_{\Omega^-} A \nabla u^- \nabla u_t^- dx \\ = \int_{\Omega^+} f(u_t^+ + v_t) dx + \int_{\Omega^-} f u_t^- dx, \\ \forall u_t^+ \in \mathcal{U}_{\text{reg}}, \quad \forall (v_t, u_t^-) \in \tilde{V}, \quad \forall w_t \in C_d^1(\overline{\Omega^+}). \end{array} \right.$$

Consequently, Proposition ?? and classical density results ensure that (u^+, v, w) solves problem (??) and u^- solves problem ?. Since these problems have a unique solution, convergences (??) \div (??) hold true for the whole sequence.

It remains to prove that weak convergences (??) \div (??) are in fact strong convergences, that is convergences (??) \div (??) hold true.

To this aim, first let us prove the convergence of energies. Indeed, passing to the limit in (??), as ε tends to zero, with test function $\phi_\varepsilon = u_\varepsilon$, using convergences (??), (??),

and (??), and choosing $(u_t^+, v_t, w_t) = (u^+, v, w)$ as test function in (??) and $u_t^- = u^-$ as test function in (??) give the convergence of energies

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left(\int_M A \begin{pmatrix} \nabla' u_\varepsilon^+ \\ \varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \nabla' u_\varepsilon^+ \\ \varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} dx + \int_F A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ \\ \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} dx + \int_{\Omega^-} A \nabla u_\varepsilon^- \nabla u_\varepsilon^- dx \right) \\ &= \int_F A \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} dx + \int_M A \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} dx + \int_{\Omega^-} A \nabla u^- \nabla u^- dx. \end{aligned}$$

Now, for every ε set

$$\begin{aligned} J_\varepsilon &= \int_M A \begin{pmatrix} \nabla' u_\varepsilon^+ - \nabla' u^+ \\ \varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} \begin{pmatrix} \nabla' u_\varepsilon^+ - \nabla' u^+ \\ \varepsilon \frac{\partial u_\varepsilon^+}{\partial x_3} \end{pmatrix} dx \\ &+ \int_F A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ - \nabla' w \\ \frac{\partial u_\varepsilon^+}{\partial x_3} - v' \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon^+ - \nabla' w \\ \frac{\partial u_\varepsilon^+}{\partial x_3} - v' \end{pmatrix} dx \\ &+ \int_{\Omega^-} A (\nabla u_\varepsilon^- - \nabla u^-) (\nabla u_\varepsilon^- - \nabla u^-) dx. \end{aligned}$$

The convergence of the energies and convergences (??)÷(??) provide that

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon = 0,$$

which, thanks to (??), implies all strong convergences stated in Theorem ??, except the first one (??). The proof of (??) will follow from (??), (??), (??), and the following Lemma ?? applied with $z = u_\varepsilon^+ - (u^+ + v)$. Although Lemma ?? is proved in [?], we reproduce its proof for the convenience of the reader.

Lemma 5.1. *There exists K in $]0, +\infty[$ such that*

$$\begin{cases} \forall z \in L^2(I; H^1(\omega)) \cap L^2(D; H_d^1(I)), \\ \|z\|_{L^2(\Omega^+)} \leq K (\|\nabla' z\|_{(L^2(\Omega^+))^2} + \left\| \frac{\partial z}{\partial x_3} \right\|_{L^2(F)}). \end{cases} \quad (95)$$

Proof. We argue by contradiction. Assuming (??) false implies the existence of a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $L^2(0, L; H^1(\omega)) \cap L^2(D; H_d^1(I))$ such that

$$\begin{cases} \|z_n\|_{L^2(\Omega^+)} = 1 \quad \forall n, \\ \left(\|\nabla' z_n\|_{L^2(\Omega^+)} + \left\| \frac{\partial z_n}{\partial x_3} \right\|_{L^2(F)} \right) \rightarrow 0, \text{ as } n \rightarrow +\infty. \end{cases} \quad (96)$$

Lemma ?? ensures existence of a constant C in $]0, +\infty[$ such that

$$\int_{\omega} \left| z_n(x', x_3) - \frac{1}{|D|} \int_D z_n(y', x_3) dy' \right|^2 dx' \leq C^2 \int_{\omega} |\nabla' z_n(x', x_3)|^2 dx', \text{ a.e. } x_3, \forall n \in \mathbb{N},$$

from which, integrating over x_3 in I , it follows that

$$\left(\int_{\Omega^+} \left| z_n(x', x_3) - \frac{1}{|D|} \int_D z_n(y', x_3) dy' \right|^2 dx' dx_3 \right)^{\frac{1}{2}} \leq C \|\nabla' z_n\|_{L^2(\Omega^+)}, \quad \forall n \in \mathbb{N}. \quad (97)$$

On the other side, first applying Hölder inequality and then the Poincaré inequality ensure the existence of C in $]0, +\infty[$ such that

$$\begin{aligned} & \left(\int_{\Omega^+} \left| \frac{1}{|D|} \int_D z_n(y', x_3) dy' \right|^2 dx' dx_3 \right)^{\frac{1}{2}} \leq |\omega|^{\frac{1}{2}} |D|^{-\frac{1}{2}} \|z_n\|_{L^2(F)} \\ & \leq |\omega|^{\frac{1}{2}} |D|^{-\frac{1}{2}} C \left\| \frac{\partial z_n}{\partial x_3} \right\|_{L^2(F)}, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (98)$$

Combining (??), (??), and (??) leads to the contradiction:

$$1 = \|z_n\|_{L^2(\Omega^+)} \leq C \|\nabla' z_n\|_{L^2(\Omega^+)} + |\omega|^{\frac{1}{2}} |D|^{-\frac{1}{2}} C \left\| \frac{\partial z_n}{\partial x_3} \right\|_{L^2(F)} \longrightarrow 0, \text{ as } n \rightarrow +\infty.$$

□

5.2 Proof of Theorem ??

Taking $(u_t^+, v_t) = (0, 0)$ in (??) provides that w is the unique solution to following problem

$$\begin{cases} w \in \mathcal{W}, \\ \int_I \int_D A \begin{pmatrix} \nabla' w \\ v' \end{pmatrix} \begin{pmatrix} \nabla' w_t \\ 0 \end{pmatrix} dx' dx_3 = 0, \quad \forall w_t \in \mathcal{W}. \end{cases} \quad (99)$$

On the other hand, multiplying equation in (??) by $\phi(x_3)v'(x_3)$, with ϕ in $L^2(I)$, integrating over I , and using a density argument provide that $\hat{w}v'$ solves the equation in (??). Moreover, one can prove that \hat{w} belongs to $L^\infty(I, H^1(D))$ (for instance, compare the proof of (2.28) in [?]). Consequently, $\hat{w}v'$ belongs to $L^2(I; H^1(D))$. Then, the uniqueness of the solution to problem (??) ensure that

$$w(x', x_3) = \hat{w}(x', x_3)v'(x_3), \text{ a.e. } (x', x_3) \in F.$$

Replacing w by $\hat{w}v'$ and choosing $(u_t^+, w_t) = (0, 0)$ in the equation in (??) provide that v is the unique solution to problem (??).

Choosing $(u_t, 0, 0)$, with u_t in \mathcal{U} , as test function in (??) implies that u^+ is the unique solution to the following problem

$$\begin{cases} u^+ \in \mathcal{U}, \\ \int_M A \begin{pmatrix} \nabla' u^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u_t \\ 0 \end{pmatrix} dx = \int_M f u_t dx, \quad \forall u_t \in \mathcal{U}. \end{cases} \quad (100)$$

Splitting f in the following way

$$f(x', x_3) = \frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f(x', x_3) dx' + f(x', x_3) - \frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f(x', x_3) dx', \quad \text{a.e. } (x', x_3) \in M,$$

and taking advantage of the linearity ensure that

$$u^+(x) = u_0^+(x) + \tilde{u}(x), \quad \text{a.e. } x \in M, \quad (101)$$

where u_0^+ is the unique solution to

$$\begin{cases} u_0^+ \in \mathcal{U}, \\ \int_M A \begin{pmatrix} \nabla' u_0^+ \\ 0 \end{pmatrix} \begin{pmatrix} \nabla' u_t \\ 0 \end{pmatrix} dx = \int_M \left(\frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f(x', x_3) dx' \right) u_t dx, \quad \forall u_t \in \mathcal{U}, \end{cases} \quad (102)$$

and \tilde{u} is the unique solution to (??). Arguing exactly as we have done for w and bearing in mind that $M = (\omega \setminus D) \times I$ infer that

$$u_0^+(x', x_3) = \hat{u}(x', x_3) \frac{1}{|\omega \setminus D|} \int_{\omega \setminus D} f(x', x_3) dx', \quad \text{a.e. } (x', x_3) \in M \quad (103)$$

where \hat{u} is the unique solution to (??).

Now, for each ε let \bar{u}_ε be the unique solution to (??). Then, (??), (??), and a change of variable ensure that

$$\frac{1}{\varepsilon^2} \int_{\varepsilon\omega} \bar{u}_\varepsilon(X', \cdot) dX' \rightarrow \int_{\omega \setminus D} u^+(x', \cdot) dx' + |\omega|v(\cdot) \quad \text{strongly in } L^2(I), \quad \text{as } \varepsilon \rightarrow 0. \quad (104)$$

Eventually, (??) follows from (??), (??), and (??).

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