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RESEARCH ARTICLE

# The one-dimensional model for an elliptic equation in a perforated thin anisotropic heterogeneous three-dimensional structure

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In this paper, we investigate the one-dimensional model for a thin three-dimensional structure  $\hat{\Omega}_\varepsilon$  in the framework of the thermal conduction. The structure is characterized by two small positive parameters  $\varepsilon$  and  $r_\varepsilon$ . The first parameter  $\varepsilon$  corresponds to the thickness of the structure while the second one characterizes the thickness of its core  $T_\varepsilon$  which plays the role of a “hole.” The structure is assumed to be heterogeneous and anisotropic, and we deal with three cases related to the limit  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(r_\varepsilon)| = k$ ,  $k \in \{0, 1, +\infty\}$ . We exhibit the “strange” term appearing in the one-dimensional model in the critical case  $k = 1$ , and we highlight the effect of the anisotropy on the form of the corrector for  $u_\varepsilon$ .

## KEYWORDS

anisotropy, corrector, hole, strange term, thin structure

## MSC CLASSIFICATION

35B25, 35B27, 35B40, 76M50, 74K10

## 1 | INTRODUCTION

As in the homogenization models intended to give a global description (at the macroscopic scale) of a phenomenon, looking for a one-dimensional or two-dimensional model for a problem posed in a three-dimensional structure is a classical problem well-known by engineers and physicists. This approach is motivated in particular by the need to simplify numerical calculations. In this context, we propose to study the one-dimensional model associated with the following equation:

$$\begin{cases} -\operatorname{div} A(x) \nabla \bar{u}_\varepsilon = f & \text{in } \hat{\Omega}_\varepsilon, \\ \bar{u}_\varepsilon = 0 & \text{on } \partial \hat{\Omega}_\varepsilon^D, \\ A(x) \nabla \bar{u}_\varepsilon \cdot n = 0 & \text{on } \partial \hat{\Omega}_\varepsilon^N, \end{cases} \quad (1.1)$$

where  $A$  denotes a  $3 \times 3$  matrix submitted to classical assumptions (see below) and  $\hat{\Omega}_\varepsilon$  denotes a thin three-dimensional heterogeneous structure with a boundary  $\partial \hat{\Omega}_\varepsilon = \partial \hat{\Omega}_\varepsilon^D \cup \partial \hat{\Omega}_\varepsilon^N$ . More precisely, given two positive sequences  $\varepsilon$  and  $r_\varepsilon$  both tending to zero in such a way that  $r_\varepsilon \ll \varepsilon$ , the domain  $\hat{\Omega}_\varepsilon$  is described as follows.

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$$\hat{\Omega}_\varepsilon = \varepsilon Y \times (0, L) \setminus \hat{T}_\varepsilon, \quad Y = \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \right)^2, \quad L > 0, \quad \hat{T}_\varepsilon = r_\varepsilon \bar{D} \times (0, L), \quad (1.2)$$

where  $\bar{D}(0, d)$  is the closed disk of radius  $0 < d < \frac{1}{2}$ .

Hence,  $\hat{\Omega}_\varepsilon$  denotes the parallelepiped  $\varepsilon Y \times (0, L)$  from which we remove the small cylinder  $r_\varepsilon \bar{D} \times (0, L)$  (the hole). The generic point  $x \in \mathbb{R}^3$  is denoted as  $x = (x', x_3)$  with  $x' = (x_1, x_2) \in \mathbb{R}^2$ ; we assume homogeneous Dirichlet boundary condition on  $\partial \hat{T}_\varepsilon$  and also on  $\partial \hat{\Omega}_\varepsilon^D$  ( $D$  stands for Dirichlet condition) which denotes the union of the lower ( $x_3 = 0$ ) and the upper ( $x_3 = L$ ) faces of the parallelepiped; hence,  $\partial \hat{\Omega}_\varepsilon^D := \{x = (x', x_3) \in \hat{\Omega}_\varepsilon, \text{ such that } x_3 = 0 \text{ or } x_3 = L\}$ ; the rest of the boundary (the outer lateral one) is devoted to Neumann condition and denoted by  $\partial \hat{\Omega}_\varepsilon^N = \partial \hat{\Omega}_\varepsilon \setminus (\partial \hat{\Omega}_\varepsilon^D \cup \partial \hat{T}_\varepsilon)$ .

Introducing the classical change of variables and unknowns

$$x' = \varepsilon y, \quad u_\varepsilon(y, x_3) := \bar{u}_\varepsilon(\varepsilon y, x_3) \quad \forall y \in Y, \quad (1.3)$$

and setting

$$\Omega := Y \times (0, L), \quad T_\varepsilon = \frac{1}{\varepsilon} \hat{T}_\varepsilon = \frac{r_\varepsilon}{\varepsilon} \bar{D} \times (0, L), \quad \Omega_\varepsilon = \Omega \setminus T_\varepsilon, \quad (1.4)$$

$$H_D^1(\Omega_\varepsilon) := \{u \in H^1(\Omega), u(y, 0) = u(y, L) = 0, u = 0 \text{ on } \partial T_\varepsilon\}, \quad (1.5)$$

we get easily from (1.1) the following variational equation satisfied by  $u_\varepsilon$ :

$$\begin{cases} u_\varepsilon \in H_D^1(\Omega_\varepsilon), \\ \int_{\Omega_\varepsilon} A(\varepsilon y, x_3) \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' \phi \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} dy dx_3 = \int_{\Omega_\varepsilon} f(\varepsilon y, x_3) \phi(y, x_3) dy dx_3, \\ \forall \phi \in H_D^1(\Omega_\varepsilon), \end{cases} \quad (1.6)$$

where  $\nabla'$  denotes the gradient with respect to the two first variables  $y = (y_1, y_2)$ .

*Remark 1.1.* For the sake of simplicity, we have assumed that the hole has a cylindrical form with a straight section  $\frac{r_\varepsilon}{\varepsilon} \bar{D}$  although the study may be performed assuming only the hole defined as a cylinder with a section  $d_\varepsilon T$ ,  $T$  being a closed set of  $\mathbb{R}^2$  such that there exists  $\lambda \in (0, 1/2)$  satisfying  $D_\lambda \subset T \subset Y$  where  $D_\lambda$  denotes the disk with radius  $\lambda$  centered at the origin.

In the sequel, functions of  $H_D^1(\Omega_\varepsilon)$  are implicitly extended by zero inside the hole so that it may be considered as elements of

$$H_D^1(\Omega) := \{u \in H^1(\Omega), u(y, 0) = u(y, L) = 0\}. \quad (1.7)$$

Under classical hypotheses on the matrix  $A$  and the source term  $f$  (see below), existence and uniqueness of the solution  $u_\varepsilon$  of (1.9) for fixed  $\varepsilon$  are an immediate consequence of the Lax–Milgram Theorem.

For the sake of brevity, we introduce the notation

$$\nabla^\varepsilon \phi := \begin{pmatrix} \frac{1}{\varepsilon} \nabla' \phi \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix}, \quad \forall \phi \in H^1(\Omega), \quad (1.8)$$

in such a way that (1.6) may be simply written as

$$u_\varepsilon \in H_D^1(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} A(\varepsilon y, x_3) \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon \phi dy dx_3 = \int_{\Omega_\varepsilon} f(\varepsilon y, x_3) \phi dy dx_3, \quad \forall \phi \in H_D^1(\Omega_\varepsilon). \quad (1.9)$$

*Remark 1.2.* In the sequel and for the sake of brevity, the study will concern only the sequence  $u_\varepsilon$  defined on the fixed domain  $\Omega$  from which one can deduce the behavior of the average over  $\varepsilon Y$  of different quantities related to the sequence  $\bar{u}_\varepsilon$  defined on the variable domain  $\Omega_\varepsilon$  as it was done for instance in Gaudiello and Sili.<sup>1</sup>

The isotropic setting for a thin structure having a hole was addressed in Murat and Sili,<sup>2</sup> and the analysis was based on the use of test function introduced in the study of periodic homogenization problems in perforated domains; see Cioranescu and Murat.<sup>3</sup> The asymptotic analysis shows that for the critical size of the hole  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(r_\varepsilon)| = 1$ , a zero order term appears in the one-dimensional limit equation obtained from (1.9) by letting  $\varepsilon \rightarrow 0$ . In the literature, this term is sometimes called “strange term.”

In the case of a simple reduction of dimension without a hole, see Murat and Sili,<sup>4,5</sup> it is known that the anisotropy of the material generally leads to the introduction of additional terms in the limit diffusion coefficients. More precisely, the limit of the sequence of the rescaled transverse temperature gradients  $\frac{1}{\varepsilon} \nabla' u_\varepsilon$  requires more attention since their limit which is proved to be still a gradient  $\nabla' w$  (the gradient with respect to the two first horizontal variable) is in fact the quantity that takes into account the anisotropy of the material at the limit. For orthotropic media (including isotropic ones), the entries of the diffusion matrix are such that  $A_{13} = A_{23} = 0$ . In that case, the limit  $\nabla' w$  reduces to zero. Similar situation arises in the framework of linear elasticity; see Sili.<sup>5</sup> In the terminology of correctors (see Bensoussan et al. and Tartar<sup>6,7</sup>), the anisotropy introduces the additional term  $\varepsilon w$  in the corrector of  $u_\varepsilon$  since in some sense,  $u_\varepsilon$  behaves like  $u_\varepsilon \sim u(x_3) + \varepsilon w$  as proved in Murat and Sili<sup>4</sup> and Sili.<sup>8</sup>

In this work, we aim to investigate the effect of the anisotropy in the limit diffusion when the structure contains a hole.

Unfortunately, the test function used in Murat and Sili<sup>2</sup> which is an adaptation to the reduction of dimension problem of the test function already used in Cioranescu and Murat<sup>3</sup> for the homogenization in domains with holes is not suitable for the anisotropic case, and thus, it cannot be used in the asymptotic analysis.

Following an idea introduced in Casado-Diaz<sup>9,10</sup> in the study of the homogenization of monotone operators in domains with holes and based on a judicious adaptation of the two-scale convergence method of Arbogast, Douglas and Hornung (see their work<sup>11</sup>) which was developed later byNguetseng<sup>12</sup> and Allaire,<sup>13</sup> we identify the limit one-dimensional model, and we prove the strong convergence of the sequence  $u_\varepsilon$  for the norm of  $H^1(\Omega)$  associated to the operator  $\nabla^\varepsilon$ ; namely, setting  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(r_\varepsilon)| = k$ , we prove that  $\|\nabla^\varepsilon (u_\varepsilon - (u(x_3) + \varepsilon w)w_\varepsilon(x))\|_{(L^2(\Omega))^3} \rightarrow 0$  where  $w_\varepsilon = 0$  if  $k = 0$ ,  $w_\varepsilon = 1$  if  $k = +\infty$ , and  $w_\varepsilon$  is defined in (1.17) in the critical case  $k = 1$ . Note that for  $k = 1$ , the sequence  $\varepsilon / \sqrt{|\ln(d_\varepsilon)|}$  ( $d_\varepsilon$  defined in 1.11) occurring in the definition of  $w_\varepsilon$  is equivalent to  $\varepsilon^2$  for small  $\varepsilon$ . The convergence result implies in particular that  $\|u_\varepsilon - (u(x_3) + \varepsilon w)w_\varepsilon\|_{H^1(\Omega)} \rightarrow 0$  and that the sequence of transversal temperature gradients  $\frac{1}{\varepsilon} \nabla' u_\varepsilon$  behaves as  $w_\varepsilon \nabla' w + (u + \varepsilon w) \frac{1}{\varepsilon} \nabla' w_\varepsilon$ . We establish this corrector result under a weak assumption on the regularity of the matrix  $A$ . The limit problem is given by (1.21) where  $\mu$  is defined by (1.16) in the case  $k = 1$  and  $\mu = 0$  in the case  $k = +\infty$ . For  $k = 0$ , the sequence  $u_\varepsilon$  converges strongly to zero in  $H^1(\Omega)$ .

In the case  $k = +\infty$ , our result means that the hole does not affect the form of the limit problem nor that of the corrector which is identical to the one found in Murat and Sili<sup>4,5</sup> and Sili.<sup>8</sup>

We end this introduction pointing out once again that a method a priori intended for the study of homogenization problems is successfully applied to the study of a dimension reduction problem. The main reason is related to the fact that we consider here a thin structure the configuration of which may be identified as a representative cell of a periodic homogenization of a composite fibered medium as pointed out in Paroni and Sili<sup>14</sup> and Murat and Sili.<sup>2</sup> Note that the corresponding homogenization problems were also addressed in the last two references, and the homogenized problem is shown to be a copy of the one-dimensional limit problem obtained in the reduction of dimension occurring locally in each cell. The comparison between the homogenized problem and the local reduction of dimension is however not possible in the case of homogenization with oscillating boundaries due to the effect induced by the oscillations of the boundaries; see previous studies.<sup>1,15,16</sup>

Before stating our main result, we make more precise our assumptions. We assume the following:

$$\begin{cases} f \in L^2(\Omega) \text{ and } f \text{ is continuous with respect to the variable } y, \\ A \in (L^\infty(\Omega))^{3 \times 3} \text{ and } A \text{ is continuous with respect to the variable } y, \\ \text{there exists } c > 0, \text{ such that } A\xi\xi \geq c|\xi|^2 \forall \xi \in \mathbb{R}^3. \end{cases} \quad (1.10)$$

As in Casado-Diaz,<sup>10</sup> we now introduce the following change of variables and unknowns already used in periodic homogenization which allows to deal with a sequence of functions  $\hat{u}_\varepsilon$  which are constant with respect to the macroscopic variable in each cell.

Setting

$$\begin{cases} d_\varepsilon = \frac{r_\varepsilon}{\varepsilon} d, \\ k = \lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(r_\varepsilon)|, \\ \delta_\varepsilon(y) = 2 - \frac{\ln(|y|)}{\ln(d_\varepsilon)} \text{ a.e. } y \in Y, \\ R_\varepsilon = 2 + \frac{\ln(2)}{\ln(d_\varepsilon)}, \\ \theta(y) = \arctan\left(\frac{y_2}{y_1}\right) \text{ a.e. } y \in Y, \\ z = (\cos(\theta), \sin(\theta)) \forall \theta \in (0, 2\pi), \end{cases} \quad (1.11)$$

the function  $\hat{u}_\varepsilon$  is defined by

$$\hat{u}_\varepsilon(r, \theta, x_3) := u_\varepsilon(d_\varepsilon^{2-r} z, x_3) \text{ a.e. } (x_3, r, \theta) \in (0, L) \times (0, R_\varepsilon) \times (0, 2\pi). \quad (1.12)$$

Hence, the new function  $\hat{u}_\varepsilon$  depends only on the variables  $r, \theta$  and  $x_3$ ; it depends implicitly on  $y$  through the change of variables  $y = d_\varepsilon^{2-r} z$ , ( $z = (\cos(\theta), \sin(\theta))$ ),  $r = \delta_\varepsilon(y)$ , where  $\delta_\varepsilon$  is a radial function according to (1.11).

Note that this change of variables allows us to derive a strong compactness result on the sequence  $\hat{u}_\varepsilon$ . Indeed, for instance in the case  $k = 1$  for which  $|\ln(r_\varepsilon)|$  is equivalent to  $1/\varepsilon^2$  and according to (2.4) below, we get a priori estimates on the sequence  $\frac{1}{\varepsilon^2} \frac{\partial \hat{u}_\varepsilon}{\partial \theta}$  in  $L^2((0, R) \times (0, 2\pi) \times (0, L))$  for all  $0 < R < 2$ ; a priori estimate of this kind on the sequence  $u_\varepsilon$  is out of reach.

We will use the following notations:

For almost all  $x_3 \in (0, L)$ , let us define the  $3 \times 3$  matrix  $A^0(x_3)$  and the  $2 \times 2$  matrices  $A'(y, x_3)$  and  $A'^0(x_3)$  by

$$\begin{cases} A_{ij}^0(x_3) = A_{ij}(0, x_3), \quad A'_{\alpha\beta}(y, x_3) = A_{\alpha\beta}(y, x_3), \\ \text{and } A'^0_{\alpha\beta}(x_3) = A_{\alpha\beta}(0, x_3) \text{ a.e. } (y, x_3) \in Y \times (0, L), \end{cases} \quad (1.13)$$

where the Greek indices run over  $\{1, 2\}$  and the Latin indices run over  $\{1, 2, 3\}$ .

We set

$$\xi := \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}, \quad (1.14)$$

where  $\theta$  ranges over  $(0, 2\pi)$ . The tangential gradient of a function  $u(r, \theta)$  at the point  $rz$ ,  $r > 0$  is therefore defined by the vector  $\partial u / \partial \theta(r, \theta) \xi$ .

The space  $H_m^1(0, 2\pi)$  is defined as the subspace of  $H^1(0, 2\pi)$  of functions with zero average over  $(0, 2\pi)$ . In order to build the corrector, that is, an approximation for the sequence  $u_\varepsilon$  in some strong topology and also the strange term arising in the one-dimensional model, we need to introduce the unique solution  $(\hat{u}_0, \hat{u}_1)$  of the variational problem

$$\begin{cases} (\hat{u}_0, \hat{u}_1) \in L^\infty(0, L; H^1(1, 2)) \times L^\infty(0, L; L^2(1, 2; H_m^1(0, 2\pi))), \\ \hat{u}_0(1, x_3) = 0, \quad \hat{u}_0(2, x_3) = 1, \\ \text{a.e. } x_3 \in (0, L), \quad \int_0^{2\pi} \int_1^2 A'^0(x_3) \left( \frac{\partial \hat{u}_0}{\partial r} z + \frac{\partial \hat{u}_1}{\partial \theta} \xi \right) \left( \frac{\partial \bar{u}_0}{\partial r} z + \frac{\partial \bar{u}_1}{\partial \theta} \xi \right) dr d\theta = 0, \\ \forall (\bar{u}_0, \bar{u}_1) \in H_0^1(1, 2) \times L^2((1, 2; H^1(0, 2\pi))). \end{cases} \quad (1.15)$$

The strange term arising at the limit is then a function  $\mu(x_3)$  of the vertical variable defined by

$$\mu(x_3) := \int_0^{2\pi} \int_1^2 A'^0(x_3) \left( \frac{\partial \hat{u}_0}{\partial r} z + \frac{\partial \hat{u}_1}{\partial \theta} \xi \right) z dr d\theta, \text{ a.e. } x_3 \in (0, L), \quad (1.16)$$

while the corrector will be obtained with the help of the sequence

$$w_\varepsilon(y, x_3) = \hat{u}_0(\delta_\varepsilon(y), x_3) + \frac{\varepsilon}{\sqrt{|\ln(d_\varepsilon)|}} \hat{u}_1(\delta_\varepsilon(y), \theta(y), x_3) \text{ a.e. } (y, x_3) \in \Omega. \quad (1.17)$$

*Remark 1.3.* Note that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln(d_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln(r_\varepsilon)$  is always true and that  $\frac{\varepsilon}{\sqrt{|\ln(d_\varepsilon)|}} \simeq \varepsilon^2$  in the case  $k = 1$ , so that for  $k = 1$ ,  $w_\varepsilon$  may be equivalently written as  $w_\varepsilon(y, x_3) = \hat{u}_0(\delta_\varepsilon(y), x_3) + \varepsilon^2 \hat{u}_1(\delta_\varepsilon(y), \theta(y), x_3)$ .

We also need to define the elementary equation which allows to give a simple expression of the limit diffusivity coefficient when dealing with anisotropic materials.

$$\begin{cases} \hat{w} \in L^\infty(0, L; H_m^1(Y)), \\ \int_Y A^0(x_3) \begin{pmatrix} \nabla' \hat{w} \\ 1 \end{pmatrix} \begin{pmatrix} \nabla' \hat{w} \\ 0 \end{pmatrix} dy = 0 \quad \forall \hat{w} \in H^1(Y), \text{ a.e. } x_3 \in (0, L). \end{cases} \quad (1.18)$$

*Remark 1.4.* For  $x_3$  given in  $(0, L)$ , the existence and uniqueness of  $\hat{w}$  is a consequence of the Lax–Milgram Theorem applied in the space  $H_m^1(Y)$  equipped with the norm  $\|\nabla' u\|_{(L^2(Y))^2}$ . Using the hypotheses (1.10) on the matrix  $A$ , one can prove that in fact  $\hat{w} \in L^\infty(0, L; H_m^1(Y))$ ; see also Sili.<sup>8</sup> We also can check that  $w = 0$  if the matrix  $A$  fulfills  $A_{13} = A_{23} = 0$  as it is the case for isotropic materials; see Sili.<sup>8</sup>

The limit diffusivity coefficient is then defined by

$$a_0(x_3) := \int_Y A^0(x_3) \begin{pmatrix} \nabla' \hat{w} \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dy. \quad (1.19)$$

Setting

$$\tilde{f}(x_3) := f(0, x_3) \quad x_3 \in (0, L), \quad (1.20)$$

our main result may be stated as follows:

**Theorem 1.5.** *Assume (1.10) and set  $k := \lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(r_\varepsilon)|$ . Then,*

*If  $k = 1$ , the sequence  $u_\varepsilon$  of (1.9) converges weakly in  $H_D^1(\Omega)$  to the unique solution  $u$  of the one-dimensional problem*

$$u \in H_0^1(0, L), \quad -\frac{d}{dx_3} a_0(x_3) \frac{du}{dx_3} + \mu(x_3)u = \tilde{f} \text{ in } (0, L), \quad (1.21)$$

*where  $\mu \in L^\infty(0, L)$ ,  $\mu > 0$  a.e. in  $(0, L)$ , is defined by (1.16).*

*If  $k = +\infty$ , the sequence  $u_\varepsilon$  converges weakly in  $H_D^1(\Omega)$  to the unique solution of*

$$u \in H_0^1(0, L), \quad -\frac{d}{dx_3} a_0(x_3) \frac{du}{dx_3} = \tilde{f} \text{ in } (0, L). \quad (1.22)$$

*If  $k = 0$ , the sequence  $u_\varepsilon$  converges strongly to zero in  $H^1(\Omega)$ .*

*The following corrector result holds: assuming  $(\partial_{x_3} \hat{u}_0, \partial_{x_3} \hat{u}_1) \in L^2((0, 2) \times (0, L)) \times L^2((0, 2) \times (0, 2\pi) \times (0, L))$ ,  $\hat{w}$  defined by (1.18) such that  $\hat{w} \in H^1(\Omega)$ , one has*

$$\|\nabla^\varepsilon \left( u_\varepsilon - \left( u + \varepsilon \hat{w} \frac{du}{dx_3} \right) w_\varepsilon \right)\|_{(L^2(\Omega))^3} \rightarrow 0. \quad (1.23)$$

*Moreover, if the solution  $u$  of (1.21) or (1.22) is such that  $u \in H_0^2(0, L)$ , then the following convergence holds true:*

$$\|u_\varepsilon - \left( u + \varepsilon \hat{w} \frac{du}{dx_3} \right) w_\varepsilon\|_{H^1(\Omega)} \rightarrow 0. \quad (1.24)$$

*Remark 1.6.* Theorem 1.5 states that the strong convergence in  $H^1(\Omega)$  of the sequence  $u_\varepsilon$  occurs without additional assumptions only in the trivial case  $k = 0$ . For the two other cases  $k = 1$  and  $k = +\infty$ , the convergence is in general a weak convergence; however, if the pairs  $(\hat{u}_0, \hat{u}_1)$  and  $u, \hat{w}$  are sufficiently regular, then one has the corrector result (1.23) from which one can deduce (1.24) with the help of the Poincaré inequality and then the strong convergence of  $u_\varepsilon$  in  $H^1(\Omega)$  since the sequence  $w_\varepsilon$  strongly converges to 1 in  $H^1(\Omega)$ . By construction,  $\hat{u}_0$  and  $\hat{u}_1$  are  $H^1$  with respect to  $r$  and  $\theta$ , and thus, the regularity hypothesis on the derivative with respect to  $x_3$  of  $\hat{u}_0$  and  $\hat{u}_1$  makes the sequence  $w_\varepsilon$  defined

in (1.17) converge strongly to 1 in  $H^1(\Omega)$ . Note that such regularity hypothesis is not out of reach since it is ensured as soon as the entries  $A_{\alpha\beta}$  are not depending on  $x_3$  leading to  $\hat{u}_0$  and  $\hat{u}_1$  constant with respect to  $x_3$  as shown by (1.15). In a similar manner, the regularity assumptions made on  $\hat{w}$  are reached at least as soon as the matrix does not depend on  $x_3$ .

The hypothesis  $du/dx_3 \in H_0^1(0, L)$  allows to deduce (1.24) from the convergence of the gradients (1.23) with the help of the Poincaré inequality; that hypothesis is reached in case of a regular matrix  $A$ .

*Remark 1.7.* The corrector result states that  $u_\varepsilon$  behaves as  $(u + \varepsilon w)w_\varepsilon$  where  $w = \hat{w}(du/dx_3)$ . Equivalently, one can say that  $u_\varepsilon$  behaves as  $uw_\varepsilon + \varepsilon w$ . Indeed, one has  $(u + \varepsilon w)w_\varepsilon - (uw_\varepsilon + \varepsilon w) = (w_\varepsilon - 1)\varepsilon w$  while under the same hypotheses on  $w$  stated in Theorem 1.5, one can check that  $(w_\varepsilon - 1)\varepsilon w$  strongly converges to zero in  $H^1(\Omega)$  for the norm associated to  $\nabla^\varepsilon$ ; that is,  $\nabla^\varepsilon((w_\varepsilon - 1)\varepsilon w)$  strongly converges to zero in  $(L^2(\Omega))^3$ . In the case of the Laplacian, it is known that the corrector takes the form  $uw_\varepsilon$ ; see Cioranescu and Murat<sup>3</sup> and Murat and Sili.<sup>2</sup> Hence, the role of the anisotropy appears here in the corrector through the term  $\varepsilon w$  or equivalently  $\varepsilon w w_\varepsilon$ ,  $w_\varepsilon$  being a sequence which tends to 1 in the  $H^1(\Omega)$  norm.

Finally, in order to complete the study, we link the approach followed here with the one using capacities, see Dal Maso and Garroni.<sup>17,18</sup> For that aim, we set

$$A'^\varepsilon \in \mathbb{M}^{2 \times 2}(\mathbb{R}), \quad A'_{\alpha\beta}{}^\varepsilon(r, \theta, x_3) = A_{\alpha\beta}(d_\varepsilon^{2-r}z, x_3), \quad (1.25)$$

and then we introduce the following variational problem:

$$\begin{cases} \hat{c}_\varepsilon \in L^\infty(0, L; H^1((0, 2) \times (0, 2\pi))), \\ \hat{c}_\varepsilon = 0 \text{ a.e. in } (0, 1) \times (0, 2\pi) \times (0, L), \quad \hat{c}_\varepsilon = 1 \text{ a.e. in } (R_\varepsilon, 2) \times (0, 2\pi) \times (0, L), \\ \text{a.e. } x_3 \in (0, L), \quad \int_0^{2\pi} \int_1^2 A'^\varepsilon \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) dr d\theta = 0, \\ \forall \bar{c} \in H^1((1, 2) \times (0, 2\pi)), \quad \bar{c}(1, \theta) = \bar{c}(R_\varepsilon, \theta) = 0, \text{ a.e. in } (0, 2\pi). \end{cases} \quad (1.26)$$

Regarding the sequence  $\hat{c}_\varepsilon$ , we prove the following theorem.

**Theorem 1.8.** *For each  $\varepsilon$ , the variational problem (1.26) admits a unique solution  $\hat{c}_\varepsilon$ . For almost all  $x_3 \in (0, L)$ , the sequence  $(\hat{c}_\varepsilon(\cdot, x_3), |\ln(d_\varepsilon)|\hat{c}_\varepsilon(\cdot, x_3))_\varepsilon$  converges weakly in  $H^1((0, 2) \times (0, 2\pi)) \times L^2(0, 2; H^1(0, 2\pi))$  to the unique solution  $(\hat{u}_0, \hat{u}_1)$  of (1.15).*

Let  $c_\varepsilon$  be the sequence defined by  $c_\varepsilon(y, x_3) = \hat{c}_\varepsilon(\delta_\varepsilon(y), \theta(y), x_3)$ . Denoting by  $D_{d_\varepsilon}$  the disk of radius  $d_\varepsilon$ , the sequence  $c_\varepsilon$  satisfies the following:

$$\begin{cases} \text{a.e. } x_3 \in (0, L), \\ c_\varepsilon = 0 \text{ in } D_{d_\varepsilon}, \quad -\text{div}_y A' \nabla' c_\varepsilon = 0 \text{ in } \mathcal{D}'(Y \setminus D_{d_\varepsilon}), \\ c_\varepsilon = 1 \text{ in } \partial Y, \\ \mu(x_3) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{Y \setminus B_{d_\varepsilon}} A'(y, x_3) \nabla' c_\varepsilon \nabla' c_\varepsilon dy. \end{cases} \quad (1.27)$$

*Remark 1.9.* Theorem 1.8 states that the extra term  $\mu$  given by the formula (1.16) is also given as a limit of capacities according to the general result of Dal Maso and Garroni.<sup>17,18</sup>

In the following section, we will prove some a priori estimates which will be used in the last section to prove Theorem 1.5. As announced above, we now establish some a priori estimates in particular those leading to a strong compactness result on the sequence  $\hat{u}_\varepsilon$ .

## 2 | A PRIORI ESTIMATES

**Proposition 2.1.** *The sequence  $u_\varepsilon$  extended by zero to the whole  $\Omega$  satisfies the a priori estimate*

$$\left\| \frac{1}{\varepsilon} \nabla' u_\varepsilon \right\|_{L^2(\Omega)} + \left\| \frac{\partial u_\varepsilon}{\partial x_3} \right\|_{L^2(\Omega)} \leq C, \quad (2.1)$$

and there exist a subsequence of  $\varepsilon$  (still denoted by  $\varepsilon$ ) and two functions  $u \in H_0^1(0, L)$ ,  $w \in L^2(0, L; H_m^1(Y))$  such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } H^1(\Omega), \quad (2.2)$$

$$\frac{1}{\varepsilon} \nabla' u_\varepsilon \rightharpoonup \nabla' w \text{ weakly in } L^2(\Omega). \quad (2.3)$$

The sequence  $\hat{u}_\varepsilon$  satisfies the following a priori estimates:

$$\begin{cases} \int_0^L \int_0^{R_\varepsilon} \int_0^{2\pi} \left( \frac{1}{\varepsilon^2 |\ln(d_\varepsilon)|} \left| \frac{\partial \hat{u}_\varepsilon}{\partial r} \right|^2 + \frac{|\ln(d_\varepsilon)|}{\varepsilon^2} \left| \frac{\partial \hat{u}_\varepsilon}{\partial \theta} \right|^2 \right) d\theta dr dx_3 \leq C, \\ \int_0^L \int_0^{R_\varepsilon} \int_0^{2\pi} \left| \frac{\partial \hat{u}_\varepsilon}{\partial x_3} \right|^2 d_\varepsilon^{2(2-r)} |\ln(d_\varepsilon)| d\theta dr dx_3 \leq C, \end{cases} \quad (2.4)$$

and there exist a subsequence of  $\varepsilon$  and two functions  $(u_0, u_1)$  such that

$$\begin{cases} u_0 \in L^2(0, L; H^1(0, 2)), \\ u_0(r, x_3) = 0 \text{ a.e. } x_3 \in (0, L), \forall r \in [0, 1], \\ \forall R, 0 < R < 2, \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \hat{u}_\varepsilon \rightharpoonup u_0 \text{ weakly in } L^2(0, L; H^1((0, R) \times (0, 2\pi))), \\ u_1 \in L^2((0, 2) \times (0, L); H_m^1(0, 2\pi)), \\ \forall R, 0 < R < 2, \frac{\sqrt{|\ln(d_\varepsilon)|}}{\varepsilon} \frac{\partial \hat{u}_\varepsilon}{\partial \theta} \rightharpoonup \frac{\partial u_1}{\partial \theta} \text{ weakly in } L^2((0, R) \times (0, 2\pi) \times (0, L)); \\ \text{In the case } k = 1, u_0(2, x_3) = u(x_3) \text{ a.e. } x_3 \in (0, L) \text{ and } u = 0 \text{ in } (0, L) \text{ if } k = 0, \\ \forall R, 0 < R < 2, d_\varepsilon^{(2-r)} \sqrt{|\ln(d_\varepsilon)|} \frac{\partial \hat{u}_\varepsilon}{\partial x_3} \rightharpoonup 0 \text{ weakly in } L^2((0, L) \times (0, R) \times (0, 2\pi)). \end{cases} \quad (2.5)$$

In the case  $k = 0$ , the sequence  $u_\varepsilon$  converges strongly to zero in  $H^1(\Omega)$ .

*Proof.* Choosing  $u_\varepsilon$  as test function in (1.9), we get thanks to the coerciveness of the matrix  $A$ ,

$$\left\| \frac{1}{\varepsilon} \nabla' u_\varepsilon \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial u_\varepsilon}{\partial x_3} \right\|_{L^2(\Omega)}^2 \leq C \int_\Omega f(\varepsilon y, x_3) u_\varepsilon dy dx_3. \quad (2.6)$$

Since  $u_\varepsilon$  vanishes on the part  $\Gamma_D$  of the boundary of  $\Omega$  defined by

$$\Gamma_D := \{u \in H^1(\Omega), u(y, 0) = u(y, L) = 0, \text{ a.e. } y \in Y\}, \quad (2.7)$$

the use of the Cauchy–Schwarz inequality in the last integral combined with the Poincaré inequality allows to get easily the estimate (2.1). Therefore,  $u_\varepsilon$  is bounded in  $H_D^1(\Omega)$ , and there exist a subsequence of  $\varepsilon$  and a function  $u \in H_D^1(\Omega)$  such that (2.2) holds true. The domain  $\Omega$  being connected, the fact that  $u = u(x_3) \in H_0^1(0, L)$  is a consequence of the estimate (2.1) which implies that  $\nabla' u_\varepsilon$  strongly converges to zero in  $L^2(\Omega)$  and thus,  $\nabla' u = 0$  in  $\Omega$ .

From (2.1) and the Poincaré–Wirtinger inequality, we deduce that the sequence  $w_\varepsilon := \frac{1}{\varepsilon} u_\varepsilon - \int_Y \frac{1}{\varepsilon} u_\varepsilon dy$  is bounded in  $L^2(0, L; H_m^1(Y))$ , and thus, there exist  $w \in L^2(0, L; H_m^1(Y))$  and a subsequence of  $\varepsilon$  satisfying (2.3).

Estimate (2.4) is a consequence of the above a priori estimate on  $u_\varepsilon$  and the change of variables (1.12) which transforms  $dy dx_3$  into  $dy dx_3 = -d_\varepsilon^{2(2-r)} \ln(d_\varepsilon) dr d\theta dx_3$ . Indeed, elementary calculations lead to the following equalities valid for  $(y, x_3) \in \Omega$  and  $(r, \theta) \in (0, R_\varepsilon) \times (0, 2\pi)$ ,

$$\frac{\partial u_\varepsilon}{\partial x_3}(y, x_3) = \frac{\partial \hat{u}_\varepsilon}{\partial x_3}(r, \theta, x_3), \quad (2.8)$$

$$\frac{\partial u_\varepsilon}{\partial y_1}(y, x_3) = \frac{1}{d_\varepsilon^{2-r}} \left( -\frac{1}{\ln(d_\varepsilon)} \frac{\partial \hat{u}_\varepsilon}{\partial r}(r, \theta, x_3) \cos(\theta) - \frac{\partial \hat{u}_\varepsilon}{\partial \theta}(r, \theta, x_3) \sin(\theta) \right), \quad (2.9)$$

and similarly,

$$\frac{\partial u_\varepsilon}{\partial y_2}(y, x_3) = \frac{1}{d_\varepsilon^{2-r}} \left( -\frac{1}{\ln(d_\varepsilon)} \frac{\partial \hat{u}_\varepsilon}{\partial r}(r, \theta, x_3) \sin(\theta) + \frac{\partial \hat{u}_\varepsilon}{\partial \theta}(r, \theta, x_3) \cos(\theta) \right). \quad (2.10)$$



In other words, for  $\varepsilon$  small enough, one can write

$$\nabla' u_\varepsilon(y, x_3) = \frac{1}{d_\varepsilon^{2-r}} \left( \frac{1}{|\ln(d_\varepsilon)|} \frac{\partial \hat{u}_\varepsilon}{\partial r}(r, \theta, x_3) \zeta + \frac{\partial \hat{u}_\varepsilon}{\partial \theta}(r, \theta, x_3) \xi \right), \quad \forall (y, x_3) \in \Omega, \quad (2.11)$$

where  $\zeta$  and  $\xi$  are defined in (1.11) and (1.14), respectively.

Estimate (2.4) is thus nothing but estimate (2.1) written in terms of  $\hat{u}_\varepsilon$ .

On the other hand, one can check easily that  $\hat{u}_\varepsilon = 0$  if  $0 < r \leq 1$  since  $u_\varepsilon = 0$  for  $|y| \leq d_\varepsilon$  so that estimate (2.4) shows that the sequence  $\frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \hat{u}_\varepsilon$  is bounded in  $L^2(0, L; H^1((0, 2) \times (0, 2\pi)))$ .

Remark also that  $\frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \frac{\partial \hat{u}_\varepsilon}{\partial \theta} = \frac{1}{|\ln(d_\varepsilon)|} \frac{\sqrt{|\ln(d_\varepsilon)|}}{\varepsilon} \frac{\partial \hat{u}_\varepsilon}{\partial \theta}$  converges strongly to zero in  $L^2((0, L) \times (0, R) \times (0, 2\pi))$  for all  $R \in (0, 2)$  by virtue of the estimate (2.4). Hence, there exists a function  $u_0(r, \theta, x_3) \in L^2(0, L; H^1((0, 2) \times (0, 2\pi)))$  such that for a subsequence of  $\varepsilon$ , the first convergence in (2.5) holds true. Moreover,  $u_0(r, \theta, x_3) = u_0(r, x_3) = 0$  for  $(r, \theta, x_3) \in (0, 1) \times (0, 2\pi) \times (0, L)$ . Regarding  $u_0$ , it only remains to prove the equality  $u_0(2, x_3) = u(x_3)$ . This will be done below. The existence of  $u_1$  and a subsequence such that the last convergence of (2.5) holds is deduced from the part of the estimate (2.4) related to the derivative with respect to  $\theta$ .

Noting that the sequence  $d_\varepsilon^{2(2-r)} \sqrt{|\ln(d_\varepsilon)|}$  converges uniformly to zero (with respect to  $r$ ) in each interval  $[0, R]$  with  $0 < R < 2$ , we derive easily the last convergence of (2.5) from the first one.

Let us now prove the property  $u_0(2, x_3) = u(x_3)$  a.e. in  $(0, L)$ . The proof is given in Casado-Diaz<sup>10</sup> for  $k = 1$  in the framework of the periodic homogenization; we reproduce it here including the case  $k = 0$  for the convenience of the reader.

First, by the Rellich–Kondrachov's Theorem, one can assume that  $u_\varepsilon$  converges strongly to  $u$  in  $L^2(\Omega)$  so that using once again the change of variables (1.12), we infer

$$\int_\Omega |u_\varepsilon - u|^2 dx = |\ln(d_\varepsilon)| \int_0^L \int_0^{R_\varepsilon} \int_0^{2\pi} |\hat{u}_\varepsilon - u|^2 d_\varepsilon^{2(2-r)} dr d\theta dx_3 \rightarrow 0. \quad (2.12)$$

Let us fix a constant  $\eta$  such that  $0 < \eta < 1/2$ . Defining  $R'_\varepsilon$  by  $R'_\varepsilon := 2 - \frac{\ln(\eta)}{\ln(d_\varepsilon)} < R_\varepsilon$ , one can check that  $d_\varepsilon^{2(2-r)} \geq \eta^2$  for all  $r \geq R'_\varepsilon$ . In addition, one has  $\ln(d_\varepsilon) = \ln(2\eta)/R_\varepsilon - R'_\varepsilon$  in such a way that (2.12) implies

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{R_\varepsilon - R'_\varepsilon} \int_0^L \int_{R'_\varepsilon}^{R_\varepsilon} \int_0^{2\pi} |\hat{u}_\varepsilon - u|^2 dr d\theta dx_3 = 0. \quad (2.13)$$

Assume now that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln(d_\varepsilon) = 1$ . Then the function  $u_0$  and the related convergence of (2.5) still hold for the sequence  $\hat{u}_\varepsilon$  since obviously  $\hat{u}_\varepsilon = \varepsilon \sqrt{|\ln(d_\varepsilon)|} \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \hat{u}_\varepsilon$ . To continue the proof, we need the following lemma the proof of which is based on elementary arguments.

**Lemma 2.2.** *There exists a positive constant  $C$  such that for every  $r_1, r_2, r_3$  satisfying  $0 \leq r_1 \leq r_2 < r_3$  and for every  $v \in H^1(r_1, r_3)$ , one has*

$$|v(r_1)|^2 \leq C \left( \frac{1}{r_3 - r_2} \int_{r_2}^{r_3} |v(r)|^2 dr + (r_3 - r_1) \int_{r_1}^{r_3} \left| \frac{d}{dr} v(r) \right|^2 dr \right). \quad (2.14)$$

Let  $\delta$  be such that  $0 < \delta < 2$  and  $\varepsilon$  be sufficiently small so that  $R'_\varepsilon \geq 2 - \delta$ . We apply (2.14) with  $v := \hat{u}_\varepsilon(\cdot, \theta, x_3) - u(x_3)$  for almost all  $(\theta, x_3) \in (0, 2\pi) \times (0, L)$  and with  $r_1 = 2 - \delta$ ,  $r_2 = R'_\varepsilon$ ,  $r_3 = R_\varepsilon$  and then we integrate the corresponding inequality with respect to  $(\theta, x_3) \in (0, 2\pi) \times (0, L)$  to get

$$\begin{cases} \int_0^L \int_0^{2\pi} |\hat{u}_\varepsilon(2 - \delta, \theta, x_3) - u(x_3)|^2 d\theta dx_3 \\ \leq C \left( \frac{1}{R_\varepsilon - R'_\varepsilon} \int_0^L \int_0^{2\pi} \int_{R'_\varepsilon}^{R_\varepsilon} |\hat{u}_\varepsilon(r, \theta, x_3) - u(x_3)|^2 dr d\theta dx_3 + \right. \\ \left. + (R_\varepsilon + \delta - 2) \int_0^L \int_0^{2\pi} \int_0^{R_\varepsilon} \left| \frac{\partial}{\partial r} \hat{u}_\varepsilon(r, \theta, x_3) \right|^2 dr \right). \end{cases} \quad (2.15)$$

From the weak lower semi-continuity of the norm and thanks to (2.13), we derive the following inequality by passing to the limit in (2.15):

$$2\pi \int_0^L |u_0(2 - \delta, x_3) - u(x_3)|^2 dx_3 \leq C\delta, \quad (2.16)$$

which leads to the equality

$$\int_0^L |u_0(2, x_3) - u(x_3)|^2 dx_3 = 0 \quad (2.17)$$

and then  $u_0(2, x_3) = u(x_3)$  in  $(0, L)$ .

In the case  $k := \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln(d_\varepsilon) = 0$ , we remark that the first convergence in (2.5) holds true for  $\hat{u}_\varepsilon = \varepsilon \sqrt{|\ln(d_\varepsilon)|} \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \hat{u}_\varepsilon$  with the corresponding limit  $u_0 = 0$ . The same arguments as those used for  $k = 1$  lead to (2.17) with  $u_0 = 0$ , and this allows to conclude that  $u = 0$  in the case  $k = 0$ . It is then easy to deduce the strong convergence of  $u_\varepsilon$  to zero in  $H^1(\Omega)$  from the coerciveness of the matrix  $A$  by passing to the limit in the following equality:

$$\int_\Omega A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon u_\varepsilon dy dx_3 = \int_\Omega f(\varepsilon y, x_3) u_\varepsilon dy dx_3. \quad (2.18)$$

□

### 3 | PROOF OF THE MAIN RESULTS

We start by introducing the appropriate test function which will be used in order to pass to the limit in (1.9) when  $k$  takes the values 1 or  $+\infty$ . Note that the case  $k = 0$  is not concerned by what follows since the sequence  $u_\varepsilon$  of solutions of (1.9) converges strongly to zero in  $H^1(\Omega)$  as proved in Proposition 2.1.

#### 3.1 | Test function

Following Casado-Diaz,<sup>10</sup> we choose two functions  $v_0$  and  $v_1$ ,  $v_0 \in C^\infty([0, +\infty[)$  and  $v_1 \in C^\infty([0, +\infty[ \times (0, 2\pi))$  such that there exists  $\delta \in \mathbb{R}$ ,  $0 < \delta < 1$  in such a way  $v_0 = 0$  in  $[0, 1]$ ,  $v_0 = 1$  in  $[2 - \delta, +\infty[$ ,  $v_1 = 1$  in  $[0, 1] \cup [2 - \delta, +\infty[ \times (0, 2\pi)$ . We now set

$$v_\varepsilon(y) = v_0(\delta_\varepsilon(y)) + \frac{\varepsilon}{\sqrt{|\ln(d_\varepsilon)|}} v_1(\delta_\varepsilon(y), \theta(y)) \text{ a.e. } y \in Y, \quad (3.1)$$

where  $\delta_\varepsilon(y)$  and  $\theta(y)$  are defined by (1.11).

Regarding this function, we prove the following result.

**Proposition 3.1.** *The sequence  $v_\varepsilon$  defined by (3.1) belongs to  $H^1(Y)$ , vanishes in  $T_\varepsilon$ , and satisfies the inequality*

$$\int_Y \frac{1}{\varepsilon^2} |\nabla' v_\varepsilon|^2 dy \leq C \int_1^{2-\delta} \int_0^{2\pi} \left( \frac{1}{\varepsilon^2 |\ln(d_\varepsilon)|} \left| \frac{d}{dr} v_0 \right|^2 + \frac{1}{|\ln(d_\varepsilon)|^2} \left| \frac{\partial}{\partial r} v_1 \right|^2 + \left| \frac{\partial}{\partial \theta} v_1 \right|^2 \right) dr d\theta. \quad (3.2)$$

For  $k = 1$  or  $k = +\infty$ , there exists a subsequence of  $\varepsilon$  such that  $v_\varepsilon$  converges strongly to 1 in  $H^1(Y)$ .

*Proof.* Taking into account (1.11) and (1.14), we get with the help of the change of variables  $y = d_\varepsilon^{2-r} z$ ,

$$\frac{\partial}{\partial y_\alpha} \delta_\varepsilon(y) = \frac{z_\alpha}{d_\varepsilon^{2-r} |\ln(d_\varepsilon)|}; \quad \frac{\partial}{\partial y_\alpha} \theta(y) = \frac{\xi_\alpha}{d_\varepsilon^{2-r}}, \text{ a.e. } y \in Y, \forall \alpha = 1, 2. \quad (3.3)$$

Due to (3.1) and (3.3), the change of variables in the integral on the left of (3.2) allows to deduce easily the inequality (3.2). On the other hand, by construction of  $v_0$  and  $v_1$  and since  $|y| \leq d_\varepsilon$  implies  $0 \leq \delta_\varepsilon(y) \leq 1$ , we get  $v_\varepsilon = 0$  in  $T_\varepsilon$ . Also one can check that the sequence  $v_\varepsilon$  is bounded in  $L^2(Y)$  which combined with (3.2) implies the boundedness of  $v_\varepsilon$  in  $H^1(Y)$  for  $k = 1$  or  $k = +\infty$ . Hence, one can assume up to extracting a subsequence that  $v_\varepsilon$  converges weakly to some  $v \in H^1(Y)$ . We obtain that  $v = 1$  by checking that  $\int_Y |v_\varepsilon - 1|^2 dy \rightarrow 0$ . Since  $|y| \geq d_\varepsilon^\delta$  implies  $\delta_\varepsilon(y) \geq 2 - \delta$  and since  $v_0 = 1$  in  $[2 - \delta, +\infty[$ , we obtain the latter convergence by writing the integral as a sum  $\int_{|y| \leq d_\varepsilon^\delta} |v_\varepsilon - 1|^2 dy + \int_{|y| \geq d_\varepsilon^\delta} |v_\varepsilon - 1|^2 dy$ . The convergence of  $v_\varepsilon$  in  $H^1(Y)$  is a strong convergence since  $\nabla' v_\varepsilon$  converges strongly to zero in  $L^2(Y)$  according to the estimate (3.2) in which the right hand side is bounded for  $k = 1$  or  $k = +\infty$  so that  $\nabla' v_\varepsilon$  is bounded in  $(L^2(\Omega))^2$  by  $\varepsilon C$  where  $C$  is a constant. □

### 3.2 | Passing to the limit in (1.9)

In this subsection, we use the convention on repeated indices. The indices  $\alpha, \beta$  take the values 1 or 2. We take a test function in (1.9) in the form  $(\bar{u}(x_3) + \varepsilon \bar{w})v_\varepsilon(y)$  where  $\bar{u} \in H_0^1(0, L)$  and  $\bar{w} \in H_D^1(\Omega)$ . Clearly, such test function is admissible in view of the properties of  $v_\varepsilon$ . For the sake of brevity, we denote the entries of the matrix by  $A_{\alpha\beta}$  instead of  $A_{\alpha\beta}(\varepsilon y, x_3)$  while the derivative with respect to  $x_i$  ( $i=1,2,3$ ) will be denoted by  $\partial_i$  instead of  $\partial/\partial x_i$ .

We write explicitly the left hand side of (1.9), and we get

$$\left\{ \begin{aligned} & \int_{\Omega} A(\varepsilon y, x_3) \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' v_\varepsilon (\bar{u}(x_3) + \varepsilon \bar{w}) + v_\varepsilon \nabla' \bar{w} \\ \left( \frac{d\bar{u}}{dx_3} + \varepsilon \partial_3 \bar{w} \right) v_\varepsilon \end{pmatrix} dy dx_3 \\ & = \int_{\Omega} \left( A_{\alpha\beta} \frac{1}{\varepsilon} \partial_\beta u_\varepsilon + A_{\alpha 3} \partial_3 u_\varepsilon \right) \left( \frac{1}{\varepsilon} \partial_\alpha v_\varepsilon (\bar{u} + \varepsilon \bar{w}) + v_\varepsilon \partial_\alpha \bar{w} \right) \\ & \quad + \left( A_{3\alpha} \frac{1}{\varepsilon} \partial_\alpha u_\varepsilon + A_{33} \partial_3 u_\varepsilon \right) \left( \frac{d\bar{u}}{dx_3} + \varepsilon \partial_3 \bar{w} \right) v_\varepsilon dy dx_3. \end{aligned} \right. \quad (3.4)$$

Note that due to the fact that  $v_\varepsilon = 0$  in  $T_\varepsilon$ , the integral over  $\Omega_\varepsilon$  in (1.9) reduces to an integral over  $\Omega$ . We pass to the limit  $\varepsilon \rightarrow 0$  in each term of the right hand side of (3.4).

Note also that (3.2) implies that  $\nabla' v_\varepsilon$  converges strongly to zero in  $L^2(Y)$  so that one can assume that for a subsequence of  $\varepsilon$ ,  $v_\varepsilon$  converges strongly to 1 in  $H^1(Y)$ . Using the latter together with the convergences (2.2) and (2.3), we get the following limits:

$$\left\{ \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha\beta} \frac{1}{\varepsilon} \partial_\beta u_\varepsilon \partial_\alpha v_\varepsilon \bar{w} dy dx_3 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha 3} \partial_3 u_\varepsilon \partial_\alpha v_\varepsilon \bar{w} dy dx_3 = 0, \\ & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{3\alpha} \frac{1}{\varepsilon} \partial_\alpha u_\varepsilon \varepsilon \partial_3 \bar{w} v_\varepsilon dy dx_3 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{33} \partial_3 u_\varepsilon \varepsilon \partial_3 \bar{w} v_\varepsilon dy dx_3 = 0, \end{aligned} \right. \quad (3.5)$$

$$\left\{ \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha\beta} \frac{1}{\varepsilon} \partial_\beta u_\varepsilon v_\varepsilon \partial_\alpha \bar{w} dy dx_3 = \int_{\Omega} A_{\alpha\beta}^0 \partial_\beta w \partial_\alpha \bar{w} dy dx_3, \\ & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha 3} \partial_3 u_\varepsilon v_\varepsilon \partial_\alpha \bar{w} dy dx_3 = \int_{\Omega} A_{\alpha 3}^0 \frac{du}{dx_3} \partial_\alpha \bar{w} dy dx_3, \end{aligned} \right. \quad (3.6)$$

$$\left\{ \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{3\alpha} \frac{1}{\varepsilon} \partial_\alpha u_\varepsilon v_\varepsilon \frac{d\bar{u}}{dx_3} dy dx_3 = \int_{\Omega} A_{3\alpha}^0 \partial_\alpha w \frac{d\bar{u}}{dx_3} dy dx_3, \\ & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{33} \partial_3 u_\varepsilon \frac{d\bar{u}}{dx_3} v_\varepsilon dy dx_3 = \int_{\Omega} A_{33}^0 \frac{du}{dx_3} \frac{d\bar{u}}{dx_3} dy dx_3. \end{aligned} \right. \quad (3.7)$$

Hence, it remains to compute the limits of two integrals; for the first one, using the above change of variables, the definition of  $v_\varepsilon$ , the continuity of the matrix  $A$  with respect to  $y$ , the second a priori estimate of (2.4), we get easily

$$\left\{ \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha 3} \partial_3 u_\varepsilon \frac{1}{\varepsilon} \partial_\alpha v_\varepsilon \bar{u} dy dx_3 \\ & = \lim_{\varepsilon \rightarrow 0} \int_0^L \int_0^{R_\varepsilon} \int_0^{2\pi} A_{3\alpha} (d_\varepsilon^{2-r} z, x_3) \partial_3 \hat{u}_\varepsilon d_\varepsilon^{2-r} \sqrt{|\ln(d_\varepsilon)|} \left[ \left( \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \frac{dv_0}{dr} \right. \right. \\ & \quad \left. \left. + \frac{1}{|\ln(d_\varepsilon)|} \frac{dv_1}{dr} \right) z_\alpha + \frac{dv_1}{d\theta} \xi_\alpha \right] \bar{u} d\theta dr dx_3 = 0. \end{aligned} \right. \quad (3.8)$$

We now deal with the last limit arising in the right hand side of (3.4), namely,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha\beta} \frac{1}{\varepsilon} \partial_\beta u_\varepsilon \frac{1}{\varepsilon} \partial_\alpha v_\varepsilon \bar{u} dy dx_3. \quad (3.9)$$

In view of (2.11), (3.1), and (3.3), the following equality holds true:

$$\left\{ \begin{aligned} & \frac{1}{\varepsilon} \nabla' u_\varepsilon \frac{1}{\varepsilon} \nabla' v_\varepsilon d_\varepsilon^{2(2-r)} |\ln(d_\varepsilon)| \\ & = \left( \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \partial_r \hat{u}_\varepsilon z + \frac{\sqrt{|\ln(d_\varepsilon)|}}{\varepsilon} \partial_\theta \hat{u}_\varepsilon \xi \right) \left( \left( \frac{1}{\varepsilon \sqrt{|\ln(d_\varepsilon)|}} \frac{dv_0}{dr} + \frac{1}{|\ln(d_\varepsilon)|} \partial_r v_1 \right) z + \partial_\theta v_1 \xi \right). \end{aligned} \right. \quad (3.10)$$

Hence, using again the change of variables  $y = d_\varepsilon^{2-r}z$  in (3.9), we can compute the limit (3.9) to get with the help of (3.14) and (2.5)

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A_{\alpha\beta} \frac{1}{\varepsilon} \partial_\beta u_\varepsilon \frac{1}{\varepsilon} \partial_\alpha v_\varepsilon \bar{u} \, dy dx_3 \\ = \int_0^L \int_0^{2\pi} \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \left( \eta \frac{dv_0}{dr} z + \partial_\theta v_1 \xi \right) d\theta dr \bar{u}(x_3) dx_3, \end{cases} \quad (3.11)$$

where we have set  $\eta = 1$  if  $k = 1$  and  $\eta = 0$  if  $k = +\infty$ . We continue the proof assuming that  $k = 1$  and thus  $\eta = 1$  since choosing  $v_1 = 0$  in the definition (3.11), we see that the limit (3.11) reduces to zero and therefore the limit problem in the case the limit problem in the case  $k = +\infty$  which corresponds to  $\eta = 0$  is the following equation:

$$\begin{cases} (u, w) \in H_0^1(0, L) \times L^2(0, L; H_m^1(Y)), \\ \int_{\Omega} A^0(x_3) \begin{pmatrix} \nabla' w \\ \frac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \nabla' \bar{w} \\ \frac{d\bar{u}}{dx_3} \end{pmatrix} dy dx_3 = \int_{\Omega} \tilde{f}(x_3) \bar{u}(x_3) dy dx_3, \\ \forall (\bar{u}, \bar{w}) \in H_0^1(0, L) \times L^2(0, L; H_m^1(Y)). \end{cases} \quad (3.12)$$

At this stage of the proof, we have proved thanks to (3.4), (3.5), (3.6), (3.7), (3.8), and (3.11) that the passing to the limit in (1.9) in the case  $\eta = 1$  leads to the following equation:

$$\begin{cases} \int_{\Omega} A^0(x_3) \begin{pmatrix} \nabla' w \\ \frac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \nabla' \bar{w} \\ \frac{d\bar{u}}{dx_3} \end{pmatrix} dy dx_3 \\ + \int_0^L \int_0^{2\pi} \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \left( \frac{dv_0}{dr} z + \partial_\theta v_1 \xi \right) d\theta dr \bar{u}(x_3) dx_3 \\ = \int_{\Omega} \tilde{f}(x_3) \bar{u}(x_3) dy dx_3 \end{cases}. \quad (3.13)$$

Due to the fact that  $v_0$  and  $v_1$  vanish in  $[0, 1]$  according to (3.1), one can see that the second integral of the right hand side in (3.13) reduces to an integral over  $(1, 2)$  with respect to  $r$ ; moreover, we can extend equation (3.13) by density to all  $(v_0, v_1) \in H^1(1, 2) \times L^2(1, 2; H^1(0, 2\pi))$ ,  $v_0(1) = 0$ ,  $v_0(2) = 1$ ,  $(\bar{u}, \bar{w}) \in H_0^1(0, L) \times L^2(0, L; H_m^1(Y))$ . Hence, setting

$$h(x_3) = \int_1^2 \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \left( \frac{dv_0}{dr} z + \partial_\theta v_1 \xi \right) d\theta dr, \quad (3.14)$$

Equation (3.13) takes the following form:

$$\int_{\Omega} A^0(x_3) \begin{pmatrix} \nabla' w \\ \frac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \nabla' \bar{w} \\ \frac{d\bar{u}}{dx_3} \end{pmatrix} dy dx_3 + \int_0^L h(x_3) \bar{u}(x_3) dx_3 = \int_{\Omega} \tilde{f}(x_3) \bar{u}(x_3) dy dx_3. \quad (3.15)$$

On the other hand, for all  $\bar{v}_0 \in H_0^1(1, 2)$ , the function  $v_0 + \bar{v}_0$  satisfies the same hypotheses as those satisfied by  $v_0$ ; therefore, one can deduce from (3.15) that for all  $\bar{v}_0 \in H_0^1(1, 2)$ , the following system holds true:

$$\begin{cases} \text{a.e. } x_3 \in (0, L), \\ \int_1^2 \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \frac{dv_0}{dr} z d\theta dr \\ = \int_1^2 \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \left( \frac{d}{dr} (v_0 + \bar{v}_0) z + \partial_\theta v_1 \xi \right) d\theta dr \end{cases} \quad (3.16)$$

so that

$$\begin{cases} \text{a.e. } x_3 \in (0, L), \\ \int_1^2 \int_0^{2\pi} A'^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) \left( \frac{d}{dr} \bar{v}_0 z + \partial_\theta v_1 \xi \right) d\theta dr = 0, \\ \forall (\bar{v}_0, v_1) \in H_0^1(1, 2) \times L^2(1, 2; H^1(0, 2\pi)). \end{cases} \quad (3.17)$$

Hence, the pair  $(u_0(x_3), u_1(\cdot, \cdot, x_3)) \in L^2(0, L; H^1(1, 2)) \times L^2(0, L; L^2(1, 2; H^1(1, 2)))$  is a solution of (3.17), and moreover, it satisfies  $u_0(x_3, 2) = u(x_3)$ , a.e. in  $(0, L)$ . The uniqueness of the solution  $(u_0^s, u_1^s)$  of (3.17) satisfying  $u_0^s(2) = s$  for a fixed  $s \in \mathbb{R}$  allows to conclude that the pair  $(u_0(x_3), u_1(\cdot, \cdot, x_3))$  is given by  $(u_0(x_3), u_1(\cdot, \cdot, x_3)) = (\hat{u}_0, \hat{u}_1)u(x_3)$  where  $(\hat{u}_0, \hat{u}_1)$  is the unique solution of (1.15).

Note that (3.15) holds true for  $h(x_3)$  defined by (3.14) with arbitrary  $(v_0, v_1) \in H^1(1, 2) \times L^2(1, 2; H^1(0, 2\pi))$  such that  $v_0(1) = 0$ ,  $v_0(2) = 1$ . In particular, (3.15) is still valid when choosing  $v_1 = 0$  and  $v_0(r) = r - 1$ . Therefore, turning back to (3.15) and bearing in mind (1.16), we derive the following equation in the critical case  $k = 1$ :

$$\begin{cases} (u, w) \in H_0^1(0, L) \times L^2(0, L; H_m^1(Y)), \\ \int_{\Omega} A^0(x_3) \begin{pmatrix} \nabla' w \\ \frac{du}{dx_3} \end{pmatrix} \begin{pmatrix} \nabla' \bar{w} \\ \frac{d\bar{u}}{dx_3} \end{pmatrix} dy dx_3 + \int_0^L \mu(x_3) u(x_3) \bar{u}(x_3) dx_3 = \int_{\Omega} \tilde{f}(x_3) \bar{u}(x_3) dy dx_3, \\ \forall (\bar{u}, \bar{w}) \in H_0^1(0, L) \times L^2(0, L; H_m^1(Y)). \end{cases} \quad (3.18)$$

Finally, taking  $\bar{u} = 0$  in (3.18) and arguing as for  $(u_0, u_1)$  above, we get thanks to a uniqueness argument (see Sili<sup>8</sup>) that  $w$  is given by

$$w(y, x_3) = \hat{w}(y, x_3) \frac{du}{dx_3}(x_3) \quad (3.19)$$

where  $\hat{w}$  is the unique solution of (1.18).

Using (3.19) in (3.18) together with the definition (1.19) and after choosing  $\bar{w} = 0$ , we derive equation (1.21).

Let us notice that replacing  $(u_0, u_1)$  by  $(\hat{u}_0, \hat{u}_1)u(x_3)$  and choosing  $(v_0, v_1) = (\hat{u}_0, \hat{u}_1)$  in (3.14), we derive thanks to (3.15) and (3.18) the following equation:

$$\begin{cases} \int_0^L \int_1^2 \int_0^{2\pi} A'^0(x_3) (\partial_r \hat{u}_0 z + \partial_{\theta} \hat{u}_1 \xi) (\partial_r \hat{u}_0 z + \partial_{\theta} \hat{u}_1 \xi) d\theta dr u(x_3) \bar{u}(x_3) dx_3 \\ = \int_0^L \mu(x_3) u(x_3) \bar{u}(x_3) dx_3, \quad \forall \bar{u} \in H_0^1(0, L). \end{cases} \quad (3.20)$$

Taking  $\bar{u} = u$  in (3.20), we conclude with the help of the coerciveness of the matrix  $A'_0$  that  $\mu(x_3) > 0$  almost everywhere in  $(0, L)$ .

Finally, the limit Equation (1.22) corresponding to the case  $k = +\infty$  is obtained in a similar way by choosing  $\bar{w} = 0$  in (3.12) and then replacing  $w$  thanks to (3.19).

It remains to prove the corrector result which is the purpose of the following subsection.

### 3.3 | Proof of the corrector result

We consider the critical case  $k = 1$ ; the case  $k = +\infty$  is simpler, and the calculations are similar. We introduce the sequence

$$I_{\varepsilon} := \int_{\Omega} A \nabla^{\varepsilon} (u_{\varepsilon} - (u + \varepsilon w) w_{\varepsilon}) \nabla^{\varepsilon} (u_{\varepsilon} - (u + \varepsilon w) w_{\varepsilon}) dy dx_3, \quad (3.21)$$

and we will prove that  $\lim_{\varepsilon \rightarrow 0} I_{\varepsilon} = 0$ ; the coerciveness of the matrix  $A$  will then allow to obtain the corrector result stated in (1.5) since  $w$  is given by (3.19).

Note that in view of the regularity hypothesis on the solution  $(\hat{u}_0, \hat{u}_1)$  of (1.15), one can check easily that  $w_{\varepsilon}$  strongly converges to 1 in  $H^1(\Omega)$ . The latter convergence property of  $w_{\varepsilon}$  will be used several times in the calculation of the limit of  $I_{\varepsilon}$ . We will also use (2.11) and the following expression derived from the definition of  $w_{\varepsilon}$  given in (1.17):

$$\nabla' w_{\varepsilon}(y, x_3) = \frac{1}{d_{\varepsilon}^{2-r} |\ln(d_{\varepsilon})|} \partial_r \hat{u}_0 z + \frac{\varepsilon}{d_{\varepsilon}^{2-r} \sqrt{|\ln(d_{\varepsilon})|}} \left( \frac{1}{|\ln(d_{\varepsilon})|} \partial_r \hat{u}_1 z + \partial_{\theta} \hat{u}_1 \xi \right), \quad (3.22)$$

where  $z$  and  $\xi$  are defined in (1.11) and (1.14), respectively.

In order to simplify the exposition, we perform the calculations assuming the matrix  $A$  to be symmetric which allows us to gather some terms, but the calculation without this assumption is quite similar. We split the last integral into several parts, and we get

$$\begin{cases} I_\varepsilon = \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon u_\varepsilon dy dx_3 - 2 \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (u w_\varepsilon) dy dx_3 \\ \quad - 2\varepsilon \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (w w_\varepsilon) dy dx_3 + \int_{\Omega} A \nabla^\varepsilon ((u + \varepsilon w) w_\varepsilon) ((u + \varepsilon w) w_\varepsilon) dy dx_3 \\ = \int_{\Omega} f u_\varepsilon dy dx_3 - 2 \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (u w_\varepsilon) dy dx_3 \\ \quad - 2\varepsilon \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (w w_\varepsilon) dy dx_3 + \int_{\Omega} A \nabla^\varepsilon ((u + \varepsilon w) w_\varepsilon) \nabla^\varepsilon ((u + \varepsilon w) w_\varepsilon) dy dx_3 \end{cases} \quad (3.23)$$

We discuss in detail the limit of the second integral in the right hand side of the last equality, the limits of the other integrals are studied in a similar way, and we will only indicate these limits.

Recalling the definition (1.8) of the operator  $\nabla^\varepsilon$ , we get

$$\begin{cases} \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (u w_\varepsilon) dy dx_3 = \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \begin{pmatrix} \frac{1}{\varepsilon} \nabla' w_\varepsilon u \\ \frac{du}{dx_3} w_\varepsilon + u \frac{\partial w_\varepsilon}{\partial x_3} \end{pmatrix} dy dx_3 \\ = \int_{\Omega} A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' w_\varepsilon u \\ 0 \end{pmatrix} + \int_{\Omega} A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} w_\varepsilon + u \frac{\partial w_\varepsilon}{\partial x_3} \end{pmatrix} \\ + \int_{\Omega} A \begin{pmatrix} 0 \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' w_\varepsilon u \\ 0 \end{pmatrix} + \int_{\Omega} A \begin{pmatrix} 0 \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} w_\varepsilon + u \frac{\partial w_\varepsilon}{\partial x_3} \end{pmatrix}. \end{cases} \quad (3.24)$$

Thanks to the change of variable  $y = d_\varepsilon^{2-r} z$ , (2.11), (3.22), the property  $(u_0, u_1) = (\hat{u}_0, \hat{u}_1)u$ , (3.20) with  $\bar{u} = u$ , and Proposition 2.1, we get

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' w_\varepsilon u \\ 0 \end{pmatrix} dy dx_3 = \\ = \lim_{\varepsilon \rightarrow 0} \int_0^L \int_0^{2\pi} \int_0^{R_\varepsilon} A(d_\varepsilon^{2-r} z, x_3) \begin{pmatrix} \frac{1}{\varepsilon^{2|\ln(d_\varepsilon)|}} \partial_r \hat{u}_\varepsilon z + \frac{1}{\varepsilon^2} \partial_\theta \hat{u}_\varepsilon \xi \\ 0 \end{pmatrix} \\ \times \begin{pmatrix} \partial_r \hat{u}_0 z + \frac{\varepsilon}{\sqrt{|\ln(d_\varepsilon)|}} (\partial_r \hat{u}_1 z + |\ln(d_\varepsilon)| \partial_\theta \hat{u}_1 \xi) \\ 0 \end{pmatrix} dr d\theta u(x_3) dx_3 = \\ = \int_0^L \int_0^{2\pi} \int_1^2 A(0, x_3) \begin{pmatrix} \partial_r u_0 z + \partial_\theta u_1 \xi \\ 0 \end{pmatrix} \begin{pmatrix} \partial_r \hat{u}_0 z + \partial_\theta \hat{u}_1 \xi \\ 0 \end{pmatrix} dr d\theta u(x_3) dx_3 \\ = \int_0^L \int_0^{2\pi} \int_1^2 A^0(x_3) (\partial_r u_0 z + \partial_\theta u_1 \xi) (\partial_r \hat{u}_0 z + \partial_\theta \hat{u}_1 \xi) dr d\theta u(x_3) dx_3 = \\ = \int_0^L \mu(x_3) u^2(x_3) dx_3. \end{cases} \quad (3.25)$$

For the other integrals in the right hand side of the last equality in (3.24), we get easily by the use of Proposition 2.1 and the strong convergence of  $w_\varepsilon$  to 1 in  $H^1(\Omega)$ ,

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \begin{pmatrix} \frac{1}{\varepsilon} \nabla' u_\varepsilon \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} w_\varepsilon + u \frac{\partial w_\varepsilon}{\partial x_3} \end{pmatrix} dy dx_3 = \int_{\Omega} A \begin{pmatrix} \nabla' w \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} \end{pmatrix} dy dx_3, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \begin{pmatrix} 0 \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} w_\varepsilon + u \frac{\partial w_\varepsilon}{\partial x_3} \end{pmatrix} dy dx_3 = \int_{\Omega} A \begin{pmatrix} 0 \\ \frac{du}{dx_3} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{du}{dx_3} \end{pmatrix} dy dx_3, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \begin{pmatrix} 0 \\ \frac{\partial u_\varepsilon}{\partial x_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon} \nabla' w_\varepsilon u \\ 0 \end{pmatrix} dy dx_3 = 0. \end{cases} \quad (3.26)$$

For the third integral arising in the last right hand side of (3.23), we use once again the fact that  $\nabla' w_\varepsilon$  converges strongly to zero in  $L^2(\Omega)$ . Hence, we have

$$\left\{ \begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \nabla^\varepsilon (w w_\varepsilon) dy dx_3 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \nabla^\varepsilon u_\varepsilon \left( \begin{array}{c} w \nabla' w_\varepsilon + w_\varepsilon \nabla' w \\ \varepsilon w \frac{\partial w_\varepsilon}{\partial x_3} + \varepsilon w_\varepsilon \frac{\partial w}{\partial x_3} \end{array} \right) dy dx_3 \\ &= \int_{\Omega} A^0(x_3) \left( \begin{array}{c} \nabla' w \\ \frac{du}{dx_3} \end{array} \right) \left( \begin{array}{c} \nabla' w \\ 0 \end{array} \right) dy dx_3. \end{aligned} \right. \quad (3.27)$$

Finally, we compute the limit of the last integral of (3.23), and we get through similar calculations of that used in (3.25)

$$\left\{ \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A (\nabla^\varepsilon (u w_\varepsilon) + \varepsilon \nabla^\varepsilon (w w_\varepsilon)) (\nabla^\varepsilon (u w_\varepsilon) + \varepsilon \nabla^\varepsilon (w w_\varepsilon)) dy dx_3 \\ = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A \nabla^\varepsilon (u w_\varepsilon) \nabla^\varepsilon (u w_\varepsilon) + 2\varepsilon \int_{\Omega} A \nabla^\varepsilon (u w_\varepsilon) \nabla^\varepsilon (w w_\varepsilon) \\ + \varepsilon^2 \int_{\Omega} A \nabla^\varepsilon (w w_\varepsilon) \nabla^\varepsilon (w w_\varepsilon) = \int_0^L \mu u^2 dx_3 + \int_{\Omega} A^0(x_3) \left( \begin{array}{c} 0 \\ \frac{du}{dx_3} \end{array} \right) \left( \begin{array}{c} 0 \\ \frac{du}{dx_3} \end{array} \right) \\ + 2 \int_{\Omega} A^0(x_3) \left( \begin{array}{c} 0 \\ \frac{du}{dx_3} \end{array} \right) \left( \begin{array}{c} \nabla' w \\ 0 \end{array} \right) dy dx_3 + \int_{\Omega} A^0(x_3) \left( \begin{array}{c} \nabla' w \\ 0 \end{array} \right) \left( \begin{array}{c} \nabla' w \\ 0 \end{array} \right) dy dx_3. \end{aligned} \right. \quad (3.28)$$

Summarizing the above limits and using the following property derived from (3.18) by choosing  $\bar{u} = 0$  and  $\bar{w} = w$ ,

$$\int_{\Omega} A^0(x_3) \left( \begin{array}{c} \nabla' w \\ \frac{du}{dx_3} \end{array} \right) \left( \begin{array}{c} \nabla' w \\ 0 \end{array} \right) dy dx_3 = 0, \quad (3.29)$$

we get  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$ . Under the additional assumption  $u \in H_0^2(0, L)$ , the function  $u_\varepsilon - (u + \varepsilon w)w_\varepsilon$  vanishes over  $\Gamma_D$  defined by (2.7) in such a way that the Poincaré inequality allows to deduce the convergence (1.24) from (1.23). The proof of Theorem 1.5 is now complete.

### 3.4 | Proof of Theorem 1.8

The existence and uniqueness of  $\hat{c}_\varepsilon$  may be obtained applying the Lax–Milgram Theorem in the subspace  $H$  of functions  $u$  of  $H^1((0, 2) \times (0, 2\pi))$  such that  $u = 0$  a.e. in  $(0, 1) \times (0, 2\pi) \cup (R_\varepsilon, 2) \times (0, 2\pi)$  equipped with the norm  $\|u\|_H^2 = \int_0^{2\pi} \int_0^2 |\partial_r u z + |\ln(d_\varepsilon)| \partial_\theta u \xi|^2 dr d\theta$ . Indeed, in a classical way, one can prove first the existence of  $\hat{d}_\varepsilon$  unique solution of

$$\left\{ \begin{aligned} \hat{d}_\varepsilon &\in L^\infty(0, L; H^1((0, 2) \times (0, 2\pi))), \\ \hat{d}_\varepsilon &= 0 \text{ a.e. in } (0, 1) \times (0, 2\pi) \times (0, L), \quad \hat{d}_\varepsilon = 0 \text{ a.e. in } (R_\varepsilon, 2) \times (0, 2\pi) \times (0, L), \\ \text{a.e. } x_3 \in (0, L), \int_0^{2\pi} \int_1^2 A^{r\varepsilon} \left( \frac{\partial \hat{d}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{d}_\varepsilon}{\partial \theta} \xi \right) \left( \frac{\partial \bar{c}}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \bar{c}}{\partial \theta} \xi \right) dr d\theta &= \\ - \int_0^{2\pi} \int_1^2 A^{r\varepsilon} \left( \frac{1}{R_\varepsilon - 1} z \right) \left( \frac{\partial \bar{c}}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \bar{c}}{\partial \theta} \xi \right) dr d\theta, \\ \forall \bar{c} \in H^1((1, 2) \times (0, 2\pi)), \hat{c}(1, \theta) = \hat{c}(2, \theta) = 0, \text{ a.e. in } (0, 2\pi). \end{aligned} \right. \quad (3.30)$$

One can then define  $\hat{c}_\varepsilon$  by  $\hat{c}_\varepsilon = \hat{d}_\varepsilon + r/R_\varepsilon - 1 - 1/R_\varepsilon - 1$  in  $(1, R_\varepsilon)$ ,  $\hat{c}_\varepsilon = 0$  in  $(0, 1)$ ,  $\hat{c}_\varepsilon = 1$  in  $(R_\varepsilon, 2)$ , which is therefore the unique solution of (1.26).

Taking  $\hat{c}_\varepsilon$  as test function in (1.26), we get that  $(\hat{c}_\varepsilon(\cdot, x_3))_\varepsilon$  is bounded in  $H^1((0, 2) \times (0, 2\pi))$  while  $(|\ln(d_\varepsilon)| \hat{c}_\varepsilon(\cdot, x_3))_\varepsilon$  is bounded in  $L^2(0, 2; H^1((0, 2\pi)))$ . One can then pass to the limit in (1.26) by choosing test function in the form  $\bar{u}_0 + \frac{1}{|\ln(d_\varepsilon)|} \bar{u}_1$  with regular  $\bar{u}_0$  and  $\bar{u}_1$ , and we conclude thanks to the uniqueness of the solution  $(\hat{u}_0, \hat{u}_1)$  of (1.15) and a density argument that  $\hat{c}_\varepsilon(\cdot, x_3)$  converges weakly to  $\hat{u}_0(\cdot, x_3)$  in  $H^1((0, 2) \times (0, 2\pi))$  and that  $|\ln(d_\varepsilon)| \hat{c}_\varepsilon$  converges weakly to  $\hat{u}_1(\cdot, x_3)$  in  $L^2(0, 2; H^1((0, 2\pi)))$  for almost all  $x_3 \in (0, L)$ . Hence,

$$\left\{ \begin{aligned} \partial_r \hat{c}_\varepsilon &\rightharpoonup \partial_r \hat{u}_0 \text{ weakly in } L^2((0, 2) \times (0, 2\pi) \times (0, L)), \\ \partial_\theta \hat{c}_\varepsilon &\rightharpoonup \partial_\theta \hat{u}_1 \text{ weakly in } L^2((0, 2) \times (0, 2\pi) \times (0, L)). \end{aligned} \right. \quad (3.31)$$



To get the last equality of (1.27), we take  $\bar{c} = \hat{c}_\varepsilon - (r/(R_\varepsilon - 1) - 1/(R_\varepsilon - 1))$  extended by 0 in  $((0, 1) \cup (R_\varepsilon, 2)) \times (0, 2\pi)$  as test function in (1.26); we infer

$$\begin{cases} \int_0^{2\pi} \int_1^2 A'^\varepsilon \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) dr d\theta = \\ \int_0^{2\pi} \int_1^2 A'^\varepsilon \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) \left( \frac{1}{R_\varepsilon - 1} z \right) dr d\theta. \end{cases} \quad (3.32)$$

On the other hand, a simple calculation using the change of variables  $y = d_\varepsilon^{2-r} z$  and the equality  $c_\varepsilon(y, x_3) = \hat{c}_\varepsilon(\delta_\varepsilon(y), \theta(y), x_3)$  shows that

$$\begin{cases} \int_{Y \setminus B_{d_\varepsilon}} A'(y, x_3) \nabla' c_\varepsilon \nabla' c_\varepsilon dy = \\ \frac{1}{|\ln(d_\varepsilon)|} \int_0^{2\pi} \int_1^2 A'^\varepsilon \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) \left( \frac{\partial \hat{c}_\varepsilon}{\partial r} z + |\ln(d_\varepsilon)| \frac{\partial \hat{c}_\varepsilon}{\partial \theta} \xi \right) dr d\theta. \end{cases} \quad (3.33)$$

Hence, dividing by  $\varepsilon^2$  in (3.33) and using (3.32) together with the convergence (3.31) and the fact that  $R_\varepsilon \rightarrow 2$ , we obtain the last equality of (1.27) in the case  $k := \lim_{\varepsilon \rightarrow 0} \varepsilon^2 |\ln(d_\varepsilon)| = 1$ . Note that the result still holds true in the case  $k = +\infty$  since the passage to the limit in that case gives  $\mu = 0$ .

One can check easily that  $c_\varepsilon(y, x_3) = 1$  for  $|y| = 1/2$  as a consequence of  $\hat{c}_\varepsilon(R_\varepsilon, \theta, x_3) = 1$  and similarly  $c_\varepsilon(y, x_3) = 0$  for  $y \in D_{d_\varepsilon}$  due to the equality  $\hat{c}_\varepsilon = 0$  for  $r \in (0, 1)$ .

Finally, the equation  $-\operatorname{div}_y A' \nabla' c_\varepsilon = 0$  in  $D'(Y \setminus \bar{D}_{d_\varepsilon})$  is a consequence of the variational equation (1.26) and the above change of variables. The proof of Theorem 1.8 is complete.

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## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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