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Abstract The object of the present paper is the study of the joint lifetime of d components subject to a common stressful external environment. Out of the stressing environment, the components are independent and the lifetime of each component is characterized by its failure (hazard) rate function. The impact of the external environment is modelled through an increase in the individual failure rates of the components. The failure rate increments due to the environment increase over time and they are dependent among components. The evolution of the joint failure rate increments is modelled by a non negative multivariate additive process, which include Lévy processes and non-homogeneous compound Poisson processes, hence encompassing several models from the previous literature. A full form expression is provided for the multivariate survival function with respect to the intensity measure of a general additive process, using the construction of an additive process from a Poisson random measure (or Poisson point process). The results are next specialized to Lévy processes and other additive processes (time-scaled Lévy processes, extended Lévy processes and shock models), thus providing simple and easily computable expressions. All results are provided under the assumption that the additive process has bounded variations, but it is possible to relax this assumption by means of approximation procedures, as is shown for the last model of this paper.

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1 Introduction

Using the vocabulary of reliability theory, the object of the present paper is the study of the joint lifetime of d components subject to a common stressful external environment. Out of the stressing environment, the components are assumed to be independent and the lifetime of each component is characterized by its failure (hazard) rate function.

Most classically, the environment can act through shocks on the components. In that case, each shock may lead to possibly simultaneous failures, also called common cause failures in the reliability literature. Classical models for possibly simultaneous failures are the binomial failure rate (BFR) model [3,28] and the Marshall–Olkin multivariate exponential distribution family [20]. As noticed by [25, p. 223], "even if common cause failures are caused by a common cause, they do not need to occur at the same time. A rather long time between failures does not necessarily mean that there is no dependency between the failure events". This drawback of the two previous classical models for common cause failures has lead to the development of cumulative shock models, where a shock simultaneously increases some intrinsic characteristics (hazard rate, deterioration level, age, etc.) of the components, leading to some possibly delayed failures among components although due to a common cause (see [19] for an overview on shock models and [21] for the study of a particular cumulative shock model).

An approach for a realistic shock model with mixed effect (possible instantaneous failure or sudden increase of a characteristic) has been studied in [6] in the univariate case, where a shock can be fatal to the system with a probability depending on the shock's arrival time, and where a non fatal shock increases the system failure rate of a random increment. This model has been extended to the bivariate case in [22] , where more references on shock models for reliability can also be found. In both [6] and [22], the shocks arrive according to a non-homogeneous Poisson process $(N(t))_{t\geq 0}$, and the successive increments in the failure rate due to the shocks are assumed to be independent and identically distributed (i.i.d.). The process describing the increment of the failure rates due to the shocks hence appear as a non-homogeneous compound Poisson process (univariate [6] or bivariate [22]). Given the bivariate non-homogeneous compound Poisson process in [22], the two components are assumed to be conditionally independent, and the conditional failure rate of each component is the sum of its intrinsic failure rate and of the corresponding margin in the compound Poisson process. We then say that the bivariate non-homogeneous compound Poisson process stands for the *conditional hazard rate increment process*, which appears as a specific type of covariate process, in the vocabulary of statistics literature.

In practice, the impact of the external environment on the components do not always arrive through isolated shocks: in [27] for instance, the authors consider an extended gamma process as conditional hazard rate increment process for the modeling of a "cumulative effect of the environment" (in a bivariate setting) and in [29], the authors envision a process constructed as a function of a standard gamma process (in a univariate setting). Extended gamma processes have also been used as prior in a Bayesian context [11], leading to similar computations as in [27].

Noticing that non-homogeneous compound Poisson processes and (standard / extended) gamma processes are specific additive processes [26], we here suggest to consider multivariate additive processes as conditional hazard rate increment processes, thus leading to a unified view for different models from the previous literature. Note that this model mostly leads to lifetimes with increasing hazard rates, that is to lifetimes with the Increasing Failure Rate (IFR) property in the vocabulary of reliability theory. This means that with such a model, the components are not supposed to recover from the impact of the environment on their health statement. This IFR property is quite common an assumption in reliability theory since the pioneering work by Barlow & Proschan (see, e.g., [4]). Indeed, it means that the components are aging over time and that they cannot regenerate themselves (that is out of maintenance actions). This is typically the case of industrial systems, where, for instance, the wear and tear of a wind turbine blade induced by a high wind on some time period cannot be recovered without any repair and induces a failure rate that remains higher from the high wind period. The model hence is well adapted to model multivariate lifetimes in the context of reliability theory, where component lifetimes usually exhibit aging (IFR) properties and where an external environment commonly has a lasting adverse effect on the failure rate.

The point of the paper is the study of the multivariate lifetime induced by the suggested model. The paper is organized as follows: we begin with some technical reminders in Section 2. We next present the model for a general multivariate additive process in Section 3 and provide the expression for the multivariate survival function with respect to the intensity measure of the additive process. The results are specialized in Section 4 to the case of (homogeneous) univariate and multivariate Lévy processes, and in Section 5 to other additives processes (time-scaled Lévy processes, extended Lévy processes and shock models). The paper ends with concluding remarks and perspectives in Section 6.

2 Technical reminders

Let $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ be a multivariate non negative additive process, which is assumed to be right-continuous with left-side limits, with no loss of generality. We recall that, following [26, Def. 1.6], the process $\mathbf{X}$ is such that:

- $\mathbf{X}(0) = 0$ a.s.,
- $\mathbf{X}$ has independent increments,
- $\mathbf{X}$ is stochastically continuous.

Based on its non negativity, the process $\mathbf{X}$ is known to be (componentwise) non decreasing. In the specific case where the increments of $\mathbf{X}$ are homogeneous, that is if the distribution of $\mathbf{X}(t+s) - \mathbf{X}(t)$ depends only on s for all $s,t > 0$, then $\mathbf{X}$ is a Lévy process.

In all the following, the process $\mathbf{X}$ is assumed to have bounded variations, not to be almost surely equal to zero ($\mathbb{P}(\mathbf{X} \neq \mathbf{0}) > 0$) and to have no drift. Hence $\mathbf{X}$ is a pure jump process. We now introduce some notations and make a small review on the

jump structure of such multivariate non negative additive processes (without drift). We refer to [26] for more details.

First, let us recall that for each $t>0$, the distribution of the random vector $\mathbf{X}(t)$ is infinitely divisible and we set $\rho_t(d\mathbf{u})$ to be its Lévy measure, where $\rho_t(d\mathbf{u})$ is a (non null) Radon measure concentrated on $\mathbb{R}_+^d\setminus\{\mathbf{0}\}$ such that

$$\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho_t(d\mathbf{u})<+\infty, \tag{1}$$

based on the bounded variations of $\mathbf{X}$.

The Laplace transform of $\mathbf{X}(t)$ can be expressed as:

$$\mathscr{L}_{\mathbf{X}(t)}(\mathbf{y})=\mathbb{E}\left(e^{-\langle\mathbf{y},\mathbf{X}(t)\rangle}\right)=\exp\left[-\int_{\mathbb{R}_+^d}\left(1-e^{-\langle\mathbf{y},\mathbf{x}\rangle}\right)\rho_t(d\mathbf{x})\right]$$

for all $\mathbf{y}\in\mathbb{R}_+^d$.

In the specific case where $\mathbf{X}$ is a Lévy process, it is enough to specify the Lévy measure ρ_1 of $\mathbf{X}(1)$, which is known to control the jump structure of all the process (as $\rho_t=t\ \rho_1$ for all $t>0$). In that case, $\rho=\rho_1$ is called the Lévy measure of the process $\mathbf{X}$.

Coming back to the general case of an additive process and following the notations of [26, Theorem 19.3], we now define the measure $\tilde{\rho}$ on $F=(0,\infty)\times\left(\mathbb{R}_+^d\setminus\{\mathbf{0}\}\right)$ by

$$\tilde{\rho}\left((0,t]\times B\right)=\rho_t(B)\ \text{ for all } B\in\mathscr{B}\left(\mathbb{R}_+^d\setminus\{\mathbf{0}\}\right), \tag{2}$$

which we call intensity measure of $\mathbf{X}$**.** (In the case of a Lévy process, then $\tilde{\rho}(ds,d\mathbf{x})=ds\ \rho(d\mathbf{x})$).

We also set

$$J(C)=\#\left\{s>0:(s,\Delta\mathbf{X}(s))\in C\right\}\ \text{ for all } C\in\mathscr{B}(F),$$

where $\Delta\mathbf{X}(s):=\mathbf{X}(s)-\mathbf{X}(s^-)$ for all $s>0$. Setting $\{T_n,n\in\mathbb{N}\}$ to be the jump times of $\mathbf{X}$ (which are know to be almost surely countably many, see [26, Theorem 21.3]), we can also write

$$J(ds,d\mathbf{x})=\sum_{n\in\mathbb{N}}\delta_{(T_n,\Delta\mathbf{X}(T_n))}(ds,d\mathbf{x})$$

with probability 1.

With such notations, we know from [26, Theorem 19.3] that:

- J is a Poisson random measure on F with intensity measure $\tilde{\rho}$;
- $\mathbf{X}(t)=\int_{(0,t]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{x}\,J(ds,d\mathbf{x})$ for all $t>0$ almost surely (or to be more specific, there exists a sample space on which this is true).

Hence, any non negative additive process can be written as an integral with respect to a Poisson random measure. This will be the key point to compute the survival function of the multivariate lifetime in the following.

Finally, we end this technical section by a summary of our assumptions:

Assumption 1 *The process* $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ *is a multivariate non negative additive process such that:*

- $\mathbf{X}$ *is right-continuous with left-side limits;*
- $\mathbf{X}$ *has bounded variations (which means that* (1) *is true);*
- $\mathbf{X}$ *is not almost surely equal to zero (*$\mathbb{P}(\mathbf{X} \neq \mathbf{0}) > 0$*);*
- $\mathbf{X}$ *has no drift.*

3 The multivariate lifetime model

3.1 The dependence model

Let us consider d components. Out of the stressing environment, these components are independent and for $i = 1, \cdots, d$, the intrinsic hazard rate function of the $i-$th component is $h_i(t), t \geq 0$ and its cumulative hazard rate function is $H_i(t) = \int_0^t h_i(s)\, ds$, $t \geq 0$.

The components are in a common stressing environment which increases their hazard rates.

We set $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ to be a multivariate non negative additive process fulfilling Assumption 1.

For each $i = 1, \cdots, d$, the conditional hazard rate of the $i-$th item given $\mathbf{X}$ is

$$h_{X_i}(t) = h_i(t) + X_i(t) \tag{3}$$

for all $t > 0$.

The random variable $X_i(t)$ hence corresponds to the increment in the hazard rate of component i due to the stressing environment. In the following, the process $\mathbf{X}$ is called *conditional hazard rate increment process*.

The corresponding conditional cumulative hazard rate is provided by

$$H_{X_i}(s_i) = H_i(s_i) + \int_0^{s_i} X_i(t)\ dt \tag{4}$$

for all $s_i > 0$.

Given $(\mathbf{X}(t))_{t\geq 0}$, the components are assumed to be independent. All the dependence between the components hence comes from the fact that the processes $(X_i(t))_{t\geq 0}$, $i = 1, \cdots, d$ are dependent.

We now introduce the multivariate lifetime

$$\tau = (\tau_1, \cdots, \tau_d)$$

and the corresponding multivariate survival function:

$$\bar{F}_\tau(\mathbf{s}) = \mathbb{P}\left[\bigcap_{i=1}^{d} \{\tau_i > s_i\}\right], \text{ for all } \mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d.$$

The object of the paper is the study of the multivariate lifetime τ.

3.2 The multivariate survival function

Our aim here is to see how to compute $\bar{F}_\tau(\mathbf{s})$ in a general setting. For a better understanding, we begin with a preliminary analysis.

Lemma 1 *Let $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ be a multivariate non negative additive process fulfilling Assumption 1. We have*

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s})} G(\mathbf{s})$$

with

$$H(\mathbf{s}) = \sum_{i=1}^{d} H_i(s_i), \tag{5}$$

$$G(\mathbf{s}) = \mathbb{E}\left[e^{-\sum_{i=1}^{d}\int_0^{s_i} X_i(t)dt}\right], \tag{6}$$

for all $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d$.

Proof Conditionning by $\mathbf{X}$ and based on the conditional independence assumption, we have:

$$\begin{aligned}
\bar{F}_\tau(\mathbf{s}) &= \mathbb{P}\left(\bigcap_{i=1}^{d}\{\tau_i > s_i\}\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^{d}\{\tau_i > s_i\}\,|\mathbf{X}\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{d}\mathbb{P}(\tau_i > s_i|\mathbf{X})\right].
\end{aligned}$$

Now remembering (4), we get:

$$\begin{aligned}
\bar{F}_\tau(\mathbf{s}) &= \mathbb{E}\left[\prod_{i=1}^{d} e^{-H_i(s_i)} e^{-\int_0^{s_i} X_i(t)\,dt}\right] \\
&= e^{-\sum_{i=1}^{d} H_i(s_i)}\,\mathbb{E}\left[e^{-\sum_{i=1}^{d}\int_0^{s_i} X_i(t)\,dt}\right],
\end{aligned}$$

which provides the result. □

The main point hence is to see how to compute $G(\mathbf{s})$, which is done in the next theorem.

Theorem 2 *Let $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ be a multivariate non negative additive process fulfilling Assumption 1 with intensity measure $\tilde{\rho}(dt, d\mathbf{x})$. Then:*

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s})-K(\mathbf{s})} \tag{7}$$

for all $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d$ where $H(s)$ is defined by (5) and

$$K(\mathbf{s}) = \int_F \left(1 - e^{-\sum_{i=1}^{d}(s_i - t)^+ x_i}\right) \tilde{\rho}(dt, d\mathbf{x}) \tag{8}$$

with $\mathbf{x} = (x_1, \cdots, x_d)$ and $F = (0,\infty) \times \left(\mathbb{R}_+^d \setminus \{\mathbf{0}\}\right)$ in the integral.

Proof Let $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d$. The point is to compute $G(\mathbf{s})$, as defined by (6).

Let us first note that

$$\begin{aligned}\int_0^{s_i} X_i(t)\ dt &= \int_0^{s_i} \left(\int_{(0,t]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}} x_i\, J(dr, d\mathbf{x}) \right) dt \\ &= \int_F \left(\mathbf{1}_{\{r\le s_i\}} \int_r^{s_i} dt \right)\ x_i\, J(dr, d\mathbf{x}) \\ &= \int_F (s_i - r)^+\ x_i\, J(dr, d\mathbf{x}) \end{aligned} \tag{9}$$

for all $i = 1, \cdots, d$ and all $s_i \ge 0$.

For each $\mathbf{s} \in \mathbb{R}_+^d$, let us set

$$\varphi_{\mathbf{s}}(r, \mathbf{x}) = \sum_{i=1}^d (s_i - r)^+\ x_i \tag{10}$$

with $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}_+^d$.

Based on (9), we have:

$$\sum_{i=1}^d \int_0^{s_i} X_i(t)\, dt = \int_F \varphi_{\mathbf{s}}(r, \mathbf{x})\ J(dr, d\mathbf{x})$$

and using the definition of G, we get

$$G(\mathbf{s}) = \mathbb{E}\left[e^{-\int_F \varphi_{\mathbf{s}}(r,\mathbf{x})\ J(dr,d\mathbf{x})} \right].$$

Now our aim is to use Theorem 2.7 from [18] to compute $G(\mathbf{s})$. For that, we have to check that

$$\int_F \min(\varphi_{\mathbf{s}}(r, \mathbf{x}), 1)\ \tilde{\rho}(dr, d\mathbf{x}) < \infty.$$

Let us first note that based on the definition of $\varphi_{\mathbf{s}}$ given in (10), we have $\varphi_{\mathbf{s}}(r, \mathbf{x}) = 0$ as soon as $r > S = \|\mathbf{s}\|_\infty$. Also

$$0 \le \varphi_{\mathbf{s}}(r, \mathbf{x}) \le d\ S\ \|\mathbf{x}\|_\infty$$

for all $r \le S$ and all $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}_+^d$. This provides

$$\begin{aligned}\int_F \min(\varphi_{\mathbf{s}}(r, \mathbf{x}), 1)\ \tilde{\rho}(dr, d\mathbf{x}) &= \int_{(0,S]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \min(\varphi_{\mathbf{s}}(r, \mathbf{x}), 1)\ \tilde{\rho}(dr, d\mathbf{x}) \\ &\le \int_{(0,S]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \min(d\ S\ \|\mathbf{x}\|_\infty, 1)\ \tilde{\rho}(dr, d\mathbf{x}) \\ &= \int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \min(d\ S\ \|\mathbf{x}\|_\infty, 1)\ \rho_S(d\mathbf{x}) \end{aligned}$$

based on the definition of $\tilde{\rho}$ given in (2).

Now, it is easy to check that

$$\min(d\ S\ \|\mathbf{x}\|_\infty, 1) \le \max(d\ S, 1)\ \ \min(\|\mathbf{x}\|_\infty, 1)$$

which entails that

$$\int_F \min\left(\varphi_{\mathbf{s}}(r,\mathbf{x}),1\right)\ \tilde{\rho}\left(dr,d\mathbf{x}\right) \leq \max\left(d\ S,1\right)\ \int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \min\left(\|\mathbf{x}\|_\infty,1\right)\ \rho_S\left(d\mathbf{x}\right) < +\infty$$

based on the technical condition (1).

This allows us to derive that

$$\begin{aligned} G(\mathbf{s}) &= \exp\left[-\int_F \left(1-e^{-\varphi_{\mathbf{s}}(r,\mathbf{x})}\right)\ \tilde{\rho}\left(dr,d\mathbf{x}\right)\right] \\ &= \exp\left[-\int_F \left(1-e^{-\sum_{i=1}^d (s_i-r)^+\ x_i}\right)\ \tilde{\rho}\left(dr,d\mathbf{x}\right)\right] \end{aligned}$$

from [18, Theorem 2.7] (where the result is provided for the Fourier transform but the arguments still hold for the Laplace transform), which achieves this proof. □

Remark 1 In the univariate case, the previous theorem writes down

$$\bar{F}_\tau(s) = \exp\left(-\int_0^s h(r)\,dr - \int_{(0,s]\times\mathbb{R}_+^*} \left(1-e^{-(s-r)x}\right)\tilde{\rho}\left(dr,dx\right)\right)$$

which mostly is [17, Theorem 3.1], except from the fact that the terms W_i's of the quoted theorem are not present here and the fact that contrary to the quoted paper, we do not assume $\tilde{\rho}\left(dr,dx\right)$ to be of the shape $\tilde{\rho}\left(dr,dx\right) = dr\ \tilde{\rho}_r\left(dx\right)$. Also, the result is given here in a multivariate setting whereas it is given in a univariate setting in [17, Theorem 3.1].

The remaining of the paper is devoted to the study of the multivariate lifetime, considering different kinds of multivariate additive processes.

4 Lévy processes as conditional hazard rate increment processes

Let $\mathbf{Y} = \left(\mathbf{Y}(t)\right)_{t\geq 0}$ be a multivariate non negative Lévy process, which is assumed to be right-continuous with left-side limits, to have bounded variations and no drift, and to be not almost surely equal to zero. Then $\mathbf{Y}$ is a specific additive process fulfilling Assumption 1.

Let ρ be its Lévy measure. The Lévy measure of $\mathbf{Y}_t$ then is $\rho_t = t\,\rho$ and $\tilde{\rho}\left(ds,d\mathbf{u}\right) = ds\ \rho\left(d\mathbf{u}\right)$, which implies that Theorem 2 writes down

$$\bar{F}_\tau(\mathbf{v}) = e^{-H(\mathbf{v})-K(\mathbf{v})} \tag{11}$$

with

$$K(\mathbf{v}) = \int_F \left(1-e^{-\sum_{i=1}^d (v_i-s)^+ x_i}\right) ds\ \rho\left(d\mathbf{x}\right) \tag{12}$$

for all $\mathbf{v} = (v_1,\cdots,v_d) \in \mathbb{R}_+^d$, where we recall that $F = (0,\infty)\times\left(\mathbb{R}_+^d\setminus\{\mathbf{0}\}\right)$.

For a better understanding, we first explore the univariate case.

4.1 Case of a univariate Lévy process

We first give some expressions for the survival and hazard rate fonctions with respect to the Laplace transform of $Y(1)$ for a univariate Lévy process $(Y(t))_{t\geq 0}$ with Lévy measure ρ, where we recall that

$$\mathscr{L}_{Y(1)}(u) = \exp\left(-\int_{\mathbb{R}_+^*}\left(1-e^{-ux}\right)\,\rho(dx)\right) \tag{13}$$

for all $u \geq 0$ (see e.g. [18]).

Proposition 1 *Let $Y = (Y(t))_{t\geq 0}$ be a univariate non negative Lévy process fulfilling Assumption 1, with Lévy measure ρ. Then:*

The hazard rate and survival functions of the lifetime τ are provided by

$$\begin{aligned} h_\tau(v) &= h(v) - \ln\left(\mathscr{L}_{Y(1)}(v)\right), \\ \bar{F}_\tau(v) &= e^{-H(v)-K(v)} \end{aligned}$$

for all $v \geq 0$, respectively, where

$$K(v) = -\int_0^v \ln\left(\mathscr{L}_{Y(1)}(u)\right) du,\ \forall v \geq 0. \tag{14}$$

Proof Based on (12), we have

$$\begin{aligned} K(v) &= \int_{\mathbb{R}_+\times\mathbb{R}_+^*}\left(1-e^{-(v-s)^+x}\right)\,ds\,\rho(dx) \\ &= \int_0^v\left(\int_{\mathbb{R}_+^*}\left(1-e^{-(v-s)x}\right)\,\rho(dx)\right)ds \\ &= \int_0^v\left(\int_{\mathbb{R}_+^*}\left(1-e^{-ux}\right)\,\rho(dx)\right)du \\ &= -\int_0^v \ln\left(\mathscr{L}_{Y(1)}(u)\right)du \end{aligned}$$

with $u = v-s$ in the third line and using (13) for the last line. This provides the result for $\bar{F}_\tau(v)$, from where we easily derive the result for $h_\tau(v)$, using that

$$h_\tau(v) = \frac{f_\tau(v)}{\bar{F}_\tau(v)} = -\frac{\bar{F}_\tau'(v)}{\bar{F}_\tau(v)}.$$

□

We now provide other expressions for the survival and hazard rate functions of the lifetime τ with respect to both ρ and corresponding tail integral function, that can be of specific interest in the multivariate case later on.

Proposition 2 *Let $Y=(Y(t))_{t\geq 0}$ be a univariate non negative Lévy process fulfilling Assumption 1, with Lévy measure ρ. The survival function of the lifetime τ is provided by*

$$\bar{F}_\tau(v)=e^{-H(v)-K(v)}$$

with

$$K(v)=\int_{\mathbb{R}_+^*}\frac{1}{x}\left(e^{-vx}+vx-1\right)\,\rho(dx)=\int_0^\infty U(y)\,\gamma_v(y)\,dy,\ \forall v\geq 0,$$

where

$$U(y)=\int_{(y,\infty)}\rho(dx),\ \textit{for all } y>0 \tag{15}$$

is the tail integral function of ρ and

$$\gamma_v(y)=\frac{1}{y^2}\left(1-e^{-vy}-vye^{-vy}\right),\ \forall y,v>0. \tag{16}$$

The corresponding hazard rate function is given by

$$h_\tau(v)=h(v)+\int_{\mathbb{R}_+^*}\left(1-e^{-vx}\right)\,\rho(dx)=h(v)+v\int_0^\infty U(y)e^{-vy}\,dy \tag{17}$$

for all $v>0$.

Proof The first expression of $h_\tau(v)$ is a direct consequence of Proposition 1 and (13), from where we can also write

$$\begin{aligned}K(v)&=\int_{\mathbb{R}_+^*}\left(\int_0^v\left(1-e^{-ux}\right)du\right)\,\rho(dx) \qquad (18)\\&=\int_{\mathbb{R}_+^*}\frac{1}{x}\left(e^{-vx}+vx-1\right)\,\rho(dx),\end{aligned}$$

which provides the first expression for $K(v)$.

Using that

$$1-e^{-ux}=\int_0^x ue^{-yu}dy, \tag{19}$$

we can also write (18) as

$$K(v)=\int_{(0,v)\times(0,\infty)}\left(\int_0^x ue^{-yu}dy\right)du\,\rho(dx).$$

Based on Fubini's theorem for the first and third lines, we get:

$$\begin{aligned}K(v)&=\int_{(0,v)\times(0,\infty)}\left(\int_{(y,\infty)}\rho(dx)\right)ue^{-uy}du\,dy\\&=\int_{(0,v)\times(0,\infty)}U(y)ue^{-uy}du\,dy\\&=\int_{\mathbb{R}_+^*}U(y)\left(\int_0^v ue^{-uy}du\right)\,dy\\&=\int_{\mathbb{R}_+^*}U(y)\frac{1}{y^2}\left(1-e^{-vy}-vye^{-vy}\right)\,dy \qquad (20)\end{aligned}$$

which is the second expression for $K(v)$.

Using again (19), we also have

$$\begin{aligned}\int_{\mathbb{R}_+^*}\left(1-e^{-vx}\right)\,\rho\left(dx\right) &= \int_{\mathbb{R}_+^*}\left(\int_0^x v\,e^{-vy}dy\right)\,\rho\left(dx\right)\\ &= v\int_{\mathbb{R}_+^*}\left(\int_{(y,\infty)}\rho\left(dx\right)\right)\,e^{-vy}dy\\ &= v\int_{\mathbb{R}_+^*}U\left(y\right)\,e^{-vy}dy,\end{aligned}$$

which provides the second expression for $h_\tau(v)$ (which can also be obtained through derivation of $\bar{F}_\tau(v)$). □

The two previous propositions provide us with several expressions for $\bar{F}_\tau(v)$ and $h_\tau(v)$ (and hence also for the p.d.f. $f_\tau(v)=h_\tau(v)\bar{F}_\tau(v)$), which allows to chose the most appropriate one, according to the context.

Remark 2 The first expression of the hazard rate function in Proposition 2 was already given in [17, Corollary 3.3], as well as (18) in [17, Corollary 3.2] (considering $a=0$ in the quoted paper and omitting the terms $h(t)$ and $H(t)$).

Remark 3 Assuming $m_1=\mathbb{E}\left(Y\left(1\right)\right)=\int_{\mathbb{R}_+^*}x\,\rho\left(dx\right)$ and $m_2=var\left(Y\left(1\right)\right)=\int_{\mathbb{R}_+^*}x^2\,\rho\left(dx\right)$ to be finite (see [9, Proposition 3.13] for their expression with respect to ρ), it is easy to check that

$$h_\tau(v)\underset{v\to 0^+}{=}h(v)+\int_{\mathbb{R}_+^*}\left(vx+x^2\,O\left(v^2\right)\right)\,\rho\left(dx\right)\underset{v\to 0^+}{=}h(v)+vm_1+m_2O\left(v^2\right).$$

Hence, the increment in the hazard rate due to the stressing environment mostly is a linear function of v when v is small.

The result of the previous proposition is now applied to the case of a homogeneous gamma process, which is a Lévy process such that $Y(t)\sim\Gamma(at,b)$ with $a,b>0$, where

$$f_{\Gamma(at,b)}\left(y\right)=\frac{b^{at}}{\Gamma\left(at\right)}y^{at-1}e^{-by}\mathbf{1}_{\mathbb{R}_+}\left(y\right)$$

for all $t>0$, with $\mathbb{E}\left(Y\left(t\right)\right)=at/b$, $var\left(Y\left(t\right)\right)=at/b^2$, Lévy measure $\rho_{\Gamma(a,b)}\left(dx\right)=ae^{-bx}/x\,dx$ and Laplace transform

$$\mathscr{L}_{Y(1)}\left(v\right)=\left(\frac{b}{b+v}\right)^a,\ \forall v\geq 0 \tag{21}$$

(see e.g. [1, page 6]).

Corollary 1 (Univariate homogeneous gamma process) *Let $\left(Y\left(t\right)\right)_{t\geq 0}$ be a univariate homogeneous gamma process with $Y(t)\sim\Gamma(at,b)$. Then*

$$\begin{aligned}\bar{F}_\tau(v)&=e^{-H(v)+at}\left(\frac{b}{b+v}\right)^{a(b+v)},\\ h_\tau(v)&=h(v)+a\ln\left(\frac{b+v}{b}\right).\end{aligned}$$

Proof The expression for $h_\tau(v)$ is directly obtained from Proposition 1 and (21).

The expression of $K(v)$ in the same proposition provides

$$K(v) = \int_0^v a \ln\left(\frac{b+u}{b}\right) du = -av - \ln\left(\left(\frac{b}{b+v}\right)^{a(b+v)}\right)$$

and the result for $\bar{F}_\tau(v)$. □

We now look at the case of an inverse gaussian process $(Y_t)_{t\geq 0}$, which is a Lévy process such that $Y(t) \sim IG(at,b)$ with $a,b>0$, where

$$f_{IG(at,b)}(x) = \frac{at}{(2\pi x^3)^{\frac{1}{2}}} \exp\left(-\frac{b^2}{2}x + abt - \frac{(at)^2}{2}\frac{1}{x}\right)$$

for all $t>0$, with $\mathbb{E}(Y(t)) = at/b$, $var(Y(t)) = at/b^3$, Lévy measure

$$\rho_{IG(a,b)}(dx) = \frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{b^2}{2}x\right) \mathbf{1}_{\mathbb{R}_+^*}(x)\ dx$$

and Laplace transform

$$\mathscr{L}_{Y(1)}(v) = \exp\left(-a\left(\sqrt{b^2+2v}-b\right)\right),\ \forall v \geq 0 \tag{22}$$

(see e.g. [15, pages 595 and 658]).

Corollary 2 (Inverse gaussian process) *Let $(Y(t))_{t\geq 0}$ be an inverse gaussian process with $Y(t) \sim IG(at,b)$. Then*

$$\bar{F}_\tau(v) = \exp\left(-H(v) - a\left(\frac{1}{3}\left(\left(b^2+2v\right)^{\frac{3}{2}} - b^3\right) - bv\right)\right)$$
$$h_\tau(v) = h(v) + a\left(\sqrt{b^2+2v}-b\right).$$

Proof The expression for $h_\tau(t)$ is directly obtained from Proposition 1 and (22).

The expression of $K(v)$ in the same proposition gives

$$\begin{aligned} K(v) &= \int_0^v a\left(\sqrt{b^2+2u}-b\right) du \\ &= a\int_b^{\sqrt{b^2+2v}} (x-b)\ x\, dx \\ &= a\left(\frac{1}{3}\left(\left(b^2+2v\right)^{\frac{3}{2}} - b^3\right) - bv\right) \end{aligned}$$

with $x = \sqrt{b^2+2u}$ in the second line, which provides the result for $\bar{F}_\tau(v)$. □

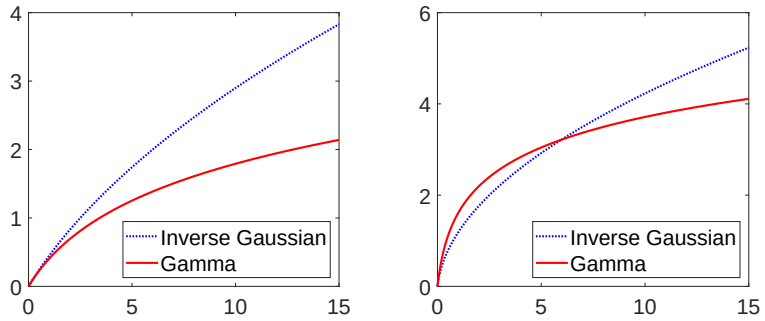

Fig. 1 The increment in the unconditional failure rate for a gamma and an inverse gaussian increment process, with $a = 1$ and $b = 2$ (left), $b = 0.25$ (right).

Remark 4 Any (homogenous) Lévy process has a linear trend in the sense that $\mathbb{E}(Y(t)) = Ct$ for some $C > 0$. However, it can be seen from the previous examples that the increment in the unconditional hazard rate due to the stressing environment can have much less impact than a linear function, as it asymptotically behaves like a logarithm function in the case of a gamma process and as a square root function in the case of an inverse gaussian process.

The increment in the unconditional hazard rate is plotted in Figure 1 for the case of a gamma and an inverse gaussian increment process which share the same mean function $\mathbb{E}(Y(t)) = at/b$ with $a = 1$ and $b = 2$ (left), $b = 0.25$ (right). As can be seen, the increment in the unconditional hazard rate can be larger when t is small for a gamma process than for a inverse gaussian process but after a while, it becomes larger for the inverse gaussian process (in coherence with the asymptotic behaviour).

4.2 Case of a multivariate Lévy process

We now come to the multi-dimensional case, where we recall that $\tilde{\rho}(dt, d\mathbf{x}) = dt\, \rho(d\mathbf{y})$, and where the Laplace transform of $\mathbf{Y}(1)$ is provided by

$$\mathscr{L}_{\mathbf{Y}(1)}(\mathbf{v}) = \exp\left(-\int_{\mathbb{R}_+^d \setminus \{\mathbf{0}\}} \left(1 - e^{-\sum_{i=1}^d v_i x_i}\right) \rho(d\mathbf{y})\right) \tag{23}$$

for all $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}_+^d$. This allows to provide new expressions for $\bar{F}_\tau(\mathbf{v})$ and $h_\tau(\mathbf{v})$ in this specific setting.

Proposition 3 *Let $\mathbf{Y} = (\mathbf{Y}(t))_{t \geq 0}$ be a multivariate non negative Lévy process fulfilling Assumption 1. We have*

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s}) - K(\mathbf{s})} \tag{24}$$

for all $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d$ where $H(s)$ is defined by (5) and

$$K(\mathbf{s}) = -\int_0^\infty \ln\left(\mathscr{L}_{\mathbf{Y}(1)}\left((s_1 - t)^+, \cdots, (s_d - t)^+\right)\right) dt$$

for all $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}_+^d$.

Proof The result is a direct consequence from (12) and (23). □

A common way to model a multivariate non negative Lévy process is to construct it through superposition (see [5]). We next provide the result for the bivariate case.

Corollary 3 *Let $(Z_i(t))_{t\geq 0}$, $i=1,2,3$ be three independent univariate non negative Lévy processes fulfilling Assumption 1. We set*

$$\begin{cases} Y_1(t) = Z_1(t) + Z_3(t) \\ Y_2(t) = Z_2(t) + Z_3(t) \end{cases} \tag{25}$$

for all $t \geq 0$. Then $\mathbf{Y} = (\mathbf{Y}(t))_{t\geq 0}$ be a bivariate non negative Lévy process and the survival function is provided by (24) with

$$\begin{aligned} K(\mathbf{s}) = &-\int_0^{s_1} \ln\left(\mathscr{L}_{Z_1(1)}(t)\right) dt - \int_0^{s_2} \ln\left(\mathscr{L}_{Z_2(1)}(t)\right) dt \\ &- \int_0^{\max(s_1,s_2)} \ln \mathscr{L}_{Z_3(1)}\left((s_1-t)^+ + (s_2-t)^+\right) \lambda(t)\, dt. \end{aligned}$$

Proof We have

$$\begin{aligned} \mathscr{L}_{\mathbf{Y}(1)}(s_1,s_2) &= \mathbb{E}\left(e^{-s_1 Y_1(1) - s_2 Y_2(1)}\right) \\ &= \mathbb{E}\left(e^{-s_1 Z_1(1) - s_2 Z_2(1) - (s_1+s_2) Z_3(1)}\right) \\ &= \mathscr{L}_{Z_1(1)}(s_1)\, \mathscr{L}_{Z_2(1)}(s_2)\, \mathscr{L}_{Z_3(1)}(s_1+s_2) \end{aligned}$$

due to the independence assumption.

Hence

$$\begin{aligned} K(\mathbf{s}) = &-\int_0^{\infty} \ln\left(\mathscr{L}_{Z_1(1)}\left((s_1-t)^+\right)\right) dt - \int_0^{\infty} \ln\left(\mathscr{L}_{Z_2(1)}\left((s_2-t)^+\right)\right) dt \\ &- \int_0^{\infty} \ln\left(\mathscr{L}_{Z_3(1)}\left((s_1-t)^+ + (s_2-t)^+\right)\right) dt, \end{aligned}$$

which easily provides the result. □

The previous corollary is next specialized to the case of gamma processes.

Corollary 4 *Let $(Z_i(t))_{t\geq 0}$, $i=1,2,3$ be three independent univariate gamma processes, such that $Z_i(t) \sim \Gamma(a_i t, b_i)$ with $a_i, b_i > 0$, $i=1,2,3$, and let $(Y_i(t))_{t\geq 0}$, $i=1,2$ be constructed through (25). Then*

$$\begin{aligned} \bar{F}_{\tau}(\mathbf{s}) = &\, e^{-H(\mathbf{s}) + a_1 s_1 + a_2 s_2 + a_3 \max(s_1,s_2)} \left(\frac{b_1}{b_1+s_1}\right)^{a_1(b_1+s_1)} \left(\frac{b_2}{b_2+s_2}\right)^{a_2(b_2+s_2)} \\ &\times \frac{b_3^{a_3(b_3+\max(s_1,s_2))}}{(b_3+|s_2-s_1|)^{a_3(b_3+|s_2-s_1|)/2}\,(b_3+s_1+s_2)^{a_3(b_3+s_1+s_2)/2}} \end{aligned} \tag{26}$$

for all $\mathbf{s} = (s_1,s_2) \in \mathbb{R}_+^2$.

Proof Based on the result for univariate gamma processes given in Corollary 1, the only point is to compute

$$I(\mathbf{s}) = \int_0^{\max(s_1,s_2)} \ln \mathscr{L}_{Z_3(1)} \left((s_1 - t)^+ + (s_2 - t)^+ \right) \lambda(t)\, dt.$$

Assume that $s_1 < s_2$ and let us write

$$I(\mathbf{s}) = I_1(\mathbf{s}) + I_2(\mathbf{s})$$

with

$$\begin{aligned} I_1(\mathbf{s}) &= \int_0^{s_1} \ln \mathscr{L}_{Z_3(1)}(s_1 + s_2 - 2t)\, dt \\ &= a_3 \int_0^{s_1} \ln \left(\frac{b_3}{b_3 + s_1 + s_2 - 2t} \right) dt \\ &= a_3 \left(s_1 + \ln \left(b_3^{s_1} \frac{(b_3 - s_1 + s_2)^{(b_3 - s_1 + s_2)/2}}{(b_3 + s_1 + s_2)^{(b_3 + s_1 + s_2)/2}} \right) \right) \end{aligned}$$

and

$$\begin{aligned} I_2(\mathbf{s}) &= \int_{s_1}^{s_2} \ln \mathscr{L}_{Z_3(1)}(s_2 - t)\, dt \\ &= a_3 \int_{s_1}^{s_2} \ln \left(\frac{b_3}{b_3 + s_2 - t} \right) dt \\ &= a_3 \left((s_2 - s_1) + \ln \left(\left(\frac{b_3}{b_3 + s_2 - s_1} \right)^{(b_3 + s_2 - s_1)} \right) \right). \end{aligned}$$

This provides

$$I(\mathbf{s}) = a_3 s_2 + \ln \left(\frac{b_3^{a_3(b_3+s_2)}}{(b_3 - s_1 + s_2)^{a_3(b_3+s_2-s_1)/2} (b_3 + s_1 + s_2)^{a_3(b_3+s_1+s_2)/2}} \right)$$

Similar results can be obtained in the case $s_2 \geq s_1$, and the two cases are next synthesized in (26). □

Another common way to model the dependence between the marginal processes in a Lévy process is to use a Lévy copula. We limit the study to the bidimensional case for sake of simpicity, but the results could clearly be extended. Following [9, 16], we recall that a (two-dimensional non-negative) Lévy-copula is any 2-increasing function $\mathbf{C} : [0,\infty]^2 \to [0,\infty]$ such that $\mathbf{C}(x,\infty) = \mathbf{C}(\infty,x) = x$ (uniform margins) and $\mathbf{C}(0,x) = \mathbf{C}(x,0) = 0$ for all $x \in [0,+\infty]$ (grounded function). Sklar's Theorem for Lévy copulas says that, starting from a Lévy process $\mathbf{Y} = (\mathbf{Y}(t))_{t \geq 0}$, there exists a Lévy-copula $\mathbf{C}$ such that

$$\mathbf{U}(y_1, y_2) = \mathbf{C}(U_1(y_1), U_2(y_2)) \tag{27}$$

for all $(y_1,y_2)\in[0,\infty]^2$, where, for $i=1,2$, the function U_i is the tail integral function of the Lévy measure ρ_i of $(Y_i(t))_{t\geq 0}$ defined in (15) and where $\mathbf{U}$ is the tail integral function of ρ, with

$$\mathbf{U}(\mathbf{y})=\mathbf{U}(y_1,y_2)=\int_{(y_1,+\infty)\times(y_2,+\infty)}\rho(d\mathbf{y})$$

for all $y=(y_1,y_2)\in\mathbb{R}_+^2\setminus\{(0,0)\}$, $\mathbf{U}(y_1,\infty)=\mathbf{U}(\infty,y_2)=0$ and $\mathbf{U}(0,0)=+\infty$. Recall also that the Lévy copula is unique as soon as the tail integral functions $U_i(y_i)$, $i=1,2$ are continuous.

In case of a Lévy process with dependence between margins modelled by a Lévy copula, we need to express the survival function with respect to the tail integral function $\mathbf{U}$, in order to be able to compute it. This is the main object of the following theorem.

Theorem 3 *Let $\mathbf{Y}=(\mathbf{Y}(t))_{t\geq 0}$ be a bivariate non negative Lévy process fulfilling Assumption 1 with $\mathbf{C}$ as Lévy copula $\mathbf{C}$ and U_i as univariate tail integral function of ρ_i, $i=1,2$. Then*

$$\bar{F}_{\tau}(\mathbf{v})=\exp\left[\int_{\mathbb{R}_+^2}\mathbf{C}(U_1(u_1),U_2(u_2))\ \gamma_{\mathbf{v}}(\mathbf{u})\ du_1\,du_2\right]\bar{F}_{\tau_1}(v_1)\bar{F}_{\tau_2}(v_2) \tag{28}$$

with

$$\begin{aligned}\gamma_{\mathbf{v}}(\mathbf{u})&=\int_0^{\min(v_1,v_2)}(v_1-s)(v_2-s)\,e^{-(v_1-s)u_1-(v_2-s)u_2}ds \\ &=\left(|v_2-v_1|+\frac{2}{u_1+u_2}\right)\frac{e^{-u_1(v_1-v_2)^+-u_2(v_2-v_1)^+}}{(u_1+u_2)^2} \\ &\quad-\left(v_1v_2+\frac{v_1+v_2}{u_1+u_2}+\frac{2}{(u_1+u_2)^2}\right)\frac{e^{-(u_1v_1+u_2v_2)}}{u_1+u_2}\end{aligned} \tag{29}$$

for all $\mathbf{u},\ \mathbf{v}\in\mathbb{R}_+^2$.

Proof Starting again from (11), the point is to compute $K(\mathbf{v})$ with

$$K(\mathbf{v})=\int_F\left(1-e^{-\sum_{i=1}^2(v_i-s)^+\,y_i}\right)ds\,\rho(d\mathbf{y})$$

where $\mathbf{v}=(v_1,v_2)\in\mathbb{R}_+^2$. Assume first that $v_1\leq v_2$.

Then, we have

$$\begin{aligned}K(\mathbf{v})&=\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(\int_0^{v_1}\left(1-e^{-(v_1-s)\,y_1-(v_2-s)\,y_2}\right)+\int_{v_1}^{v_2}\left(1-e^{-(v_2-s)\,y_2}\right)\right)ds\,\rho(d\mathbf{y}) \\ &=\int_0^{v_1}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_1-s)\,y_1-(v_2-s)\,y_2}\right)\rho(d\mathbf{y})\right)ds \\ &\quad+\int_{v_1}^{v_2}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_2-s)\,y_2}\right)\rho(d\mathbf{y})\right)ds.\end{aligned} \tag{30}$$

Let us write

$$1-e^{-(v_1-s)\ y_1-(v_2-s)\ y_2} = -\left(1-e^{-(v_1-s)y_1}\right)\left(1-e^{-(v_2-s)y_2}\right) + \left(1-e^{-(v_2-s)y_2}\right)+\left(1-e^{-(v_1-s)y_1}\right)$$

in the first integral.

We obtain:

$$\begin{aligned}
K(\mathbf{v}) =& -\int_0^{v_1}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_1-s)y_1}\right)\left(1-e^{-(v_2-s)y_2}\right)\rho(d\mathbf{y})\right)ds\\
&+\int_0^{v_1}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_2-s)y_2}\right)\rho(d\mathbf{y})\right)ds\\
&+\int_0^{v_1}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_1-s)y_1}\right)\rho(d\mathbf{y})\right)ds\\
&+\int_{v_1}^{v_2}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_2-s)\ y_2}\right)\rho(d\mathbf{y})\right)ds\\
=& -\int_0^{v_1}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_1-s)y_1}\right)\left(1-e^{-(v_2-s)y_2}\right)\rho(d\mathbf{y})\right)ds \qquad (31)\\
&+\int_0^{v_1}\left(\int_{\mathbb{R}_+^*}\left(1-e^{-(v_1-s)y_1}\right)\rho_1(dy_1)\right)ds\\
&+\int_0^{v_2}\left(\int_{\mathbb{R}_+^*}\left(1-e^{-(v_2-s)\ y_2}\right)\rho_2(dy_2)\right)ds
\end{aligned}$$

Looking at the case $v_2 \geq v_1$ provides a similar result, where the upper bound v_1 in the integral in (31) is substituted by v_2.

Hence, for all $v_1, v_2 > 0$, we have

$$K(\mathbf{v}) = -J(\mathbf{v}) + J_1(v_1) + J_2(v_2)$$

where

$$J_i(v_i) = \int_0^{v_i}\left(\int_{\mathbb{R}_+^*}\left(1-e^{-(v_i-s)\ y_i}\right)\rho_i(dy_i)\right)ds,\ i=1,2,$$

have already been computed in the univariate case and where

$$J(\mathbf{v}) = \int_0^{\min(v_1,v_2)}\left(\int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}}\left(1-e^{-(v_1-s)y_1}\right)\left(1-e^{-(v_2-s)y_2}\right)\rho(d\mathbf{y})\right)ds.$$

Writing

$$1-e^{-(v_i-s)y_i} = \int_0^{y_i}(v_i-s)\,e^{-(v_i-s)u_i}du_i$$

for $i=1,2$, we get

$$\begin{aligned}J(\mathbf{v}) &= \int_0^{\min(v_1,v_2)} \int_{\mathbb{R}_+^2\setminus\{\mathbf{0}\}} \left(\int_0^{v_1} (v_1-s)\, e^{-(v_1-s)u_1} du_1\right) \\ &\qquad\qquad \times \left(\int_0^{v_2} (v_2-s)\, e^{-(v_2-s)u_2} du_2\right) \rho(d\mathbf{y})\, ds \\ &= \int_0^{\min(v_1,v_2)} \int_{\mathbb{R}_+^2} \left(\int_{(u_1,+\infty)\times(u_2,+\infty)} \rho(d\mathbf{y})\right) (v_1-s)(v_2-s)\, e^{-(v_1-s)u_1-(v_2-s)u_2} ds\, du_1\, du_2 \\ &= \int_{\mathbb{R}_+^2} \mathbf{U}(\mathbf{u})\, \gamma_{\mathbf{v}}(\mathbf{u})\, du_1\, du_2\end{aligned}$$

with

$$\begin{aligned}\gamma_{\mathbf{v}}(\mathbf{u}) &= \int_0^{\min(v_1,v_2)} (v_1-s)(v_2-s)\, e^{-(v_1-s)u_1-(v_2-s)u_2} ds \\ &= e^{-u_1v_1-u_2v_2} \int_0^{\min(v_1,v_2)} (v_1-s)(v_2-s)\, e^{s(u_1+u_2)} ds.\end{aligned}$$

A double integration by parts provides

$$\begin{aligned}&\gamma_{\mathbf{v}}(\mathbf{u}) \\ &= \frac{e^{-u_1v_1-u_2v_2}}{u_1+u_2}\left[e^{(u_1+u_2)\min(v_1,v_2)}\left(\frac{1}{u_1+u_2}(v_1+v_2-2\min(v_1,v_2))+\frac{2}{(u_1+u_2)^2}\right)\right. \\ &\qquad\qquad \left. -v_1v_2-\frac{v_1+v_2}{u_1+u_2}-\frac{2}{(u_1+u_2)^2}\right] \\ &= \left(v_1+v_2-2\min(v_1,v_2)+\frac{2}{u_1+u_2}\right)\frac{e^{-u_1v_1-u_2v_2+(u_1+u_2)\min(v_1,v_2)}}{(u_1+u_2)^2} \\ &\qquad\qquad -\left(v_1v_2+\frac{v_1+v_2}{u_1+u_2}+\frac{2}{(u_1+u_2)^2}\right)\frac{e^{-(u_1v_1+u_2v_2)}}{u_1+u_2} \\ &= \left(|v_2-v_1|+\frac{2}{u_1+u_2}\right)\frac{e^{-u_1(v_1-v_2)^+-u_2(v_2-v_1)^+}}{(u_1+u_2)^2}-\left(v_1v_2+\frac{v_1+v_2}{u_1+u_2}+\frac{2}{(u_1+u_2)^2}\right)\frac{e^{-(u_1v_1+u_2v_2)}}{u_1+u_2}\end{aligned}$$

Using Proposition 2 for $J_1(v_1)$ and $J_2(v_2)$, we obtain

$$K(\mathbf{v}) = -\int_{\mathbb{R}_+^2} \mathbf{U}(\mathbf{u})\, \gamma_{\mathbf{v}}(\mathbf{u})\, du_1\, du_2 + \int_{\mathbb{R}_+^*} U_1(u_1)\, \gamma_{v_1}(u_1)\, du_1 + \int_{\mathbb{R}_+^*} U_2(u_2)\, \gamma_{v_2}(u_2)\, du_2,$$

which allows to conclude, due to (27). □

Remark 5 Based on (29), it is clear that $\gamma_{\mathbf{v}}(\mathbf{u}) \geq 0$ for all $\mathbf{u}$, and hence we can see that

$$\bar{F}_\tau(\mathbf{v}) \geq \bar{F}_{\tau_1}(v_1)\, \bar{F}_{\tau_2}(v_2)$$

from (28). This shows that τ has the positive quadrant dependence property, which entails that higher values for τ_1 leads to higher values for τ_2 (and conversely), see e.g. [24] for more details on this notion. This behaviour is coherent with what could be expected in such a context (see e.g. [23]).

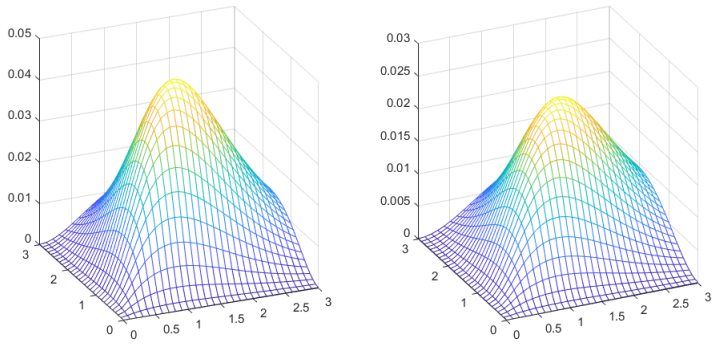

Fig. 2 The functions $\bar{F}_{\tau} - \bar{F}_{\tau^{\perp}}$ (left) and $\bar{F}_{\tau^{\parallel}} - \bar{F}_{\tau}$ (right). Example 1.

Example 1 We here consider a Clayton-Lévy copula

$$\mathbf{C}_{\Theta}(v_1, v_2) = \left(v_1^{-\theta} + v_2^{-\theta}\right)^{-\frac{1}{\theta}}$$

with $\theta > 0$, where we recall that the dependence increases with θ (independence: $\theta \to 0^{+}$; complete dependence: $\theta \to \infty$), see [9] for more details.

We also take $Y_i(t) \sim \Gamma(t, 1)$ for $i = 1, 2$, with

$$U_i(u_i) = \int_{u_i}^{\infty} \frac{e^{-x}}{x} dx = \mathrm{Ei}(x), \ \forall u_i > 0,$$

where Ei is the exponential integral function. We set $\bar{F}_{\tau^{\perp}}$ and $\bar{F}_{\tau^{\parallel}}$ to be the survival functions when $(Y_1(t))_{t \geq 0}$ and $(Y_2(t))_{t \geq 0}$ are independent and completely dependent, respectively. The functions $\bar{F}_{\tau} - \bar{F}_{\tau^{\perp}}$ and $\bar{F}_{\tau^{\parallel}} - \bar{F}_{\tau}$ are plotted in Figure 2 (left and right plots, respectively) for $\theta = 1$. We can see that both functions remain non negative, which is coherent with the previous remark.

5 Other additive processes as conditional hazard rate increment processes

5.1 Time-scaled Lévy process

We begin with a first extension of the previous section considering time-scaled Lévy processes, which are known from [26], to be additive processes.

Definition 1 Let $\mathbf{Y} = (\mathbf{Y}(t))_{t \geq 0}$ be a multivariate non negative Lévy process and let ρ be its Lévy measure. Let $\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous function such that $\lambda(t) > 0$ for all $t > 0$ and $\Lambda(t) = \int_0^t \lambda(s)\ ds < +\infty$ for all $t > 0$, and $\lim_{t \to \infty} \Lambda(t) = \infty$. The process $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ with

$$\mathbf{X}(t) = \mathbf{Y}(\Lambda(t)) \text{ for all } t \geq 0 \tag{32}$$

is called a time-scaled Lévy process with time-scaling function Λ and Lévy measure ρ.

We first look at the intensity measure of $\mathbf{X}$ (for which we could not find any reference so that a proof is provided).

Lemma 2 *The intensity measure of the time-scaled Lévy process* $\mathbf{X}$ *with time-scaling function* Λ *and Lévy measure* ρ *is:*

$$\tilde{\rho}(ds,d\mathbf{u}) = \lambda(s)\,ds\,\rho(d\mathbf{u}).$$

Proof Let $J_{\mathbf{X}}$ and $J_{\mathbf{Y}}$ be the Poisson random measures with intensity measures $\tilde{\rho}(ds,d\mathbf{u})$ and $\tilde{\rho}_{\mathbf{Y}}(ds,d\mathbf{u})$, respectively.

For each $B \in \mathscr{B}\left(\mathbb{R}_+^d \setminus \{\mathbf{0}\}\right)$ and each $t>0$, we have:

$$\begin{aligned}\tilde{\rho}((0,t]\times B) &= \mathbb{E}\left[J_{\mathbf{X}}((0,t]\times B)\right] \\ &= \mathbb{E}\left[\#\{s>0 : s\in(0,t] \text{ and } \Delta\mathbf{X}(s)\in B\}\right]\end{aligned}$$

where $\Delta\mathbf{X}(s) = \mathbf{X}(s) - \mathbf{X}(s^-) = \mathbf{Y}(\Lambda(s)) - \mathbf{Y}\left(\Lambda(s)^-\right) = \mathbf{Y}(\Lambda(s)) - \mathbf{Y}(\Lambda(s^-)) = \Delta\mathbf{Y}(\Lambda(s))$ (by continuity of $\Lambda(\cdot)$).

As $\Lambda(\cdot)$ is a one-to-one continuous function from $(0,t]$ to $(0,\Lambda(t)]$, we now have:

$$\begin{aligned}\tilde{\rho}((0,t]\times B) &= \mathbb{E}\left[\#\{\Lambda(s)>0 : \Lambda(s)\in(0,\Lambda(t)] \text{ and } \Delta\mathbf{Y}(\Lambda(s))\in B\}\right] \\ &= \mathbb{E}\left[\#\{u>0 : u\in(0,\Lambda(t)] \text{ and } \Delta\mathbf{Y}(u)\in B\}\right]\end{aligned}$$

setting $u=\Lambda(s)$. Hence:

$$\begin{aligned}\tilde{\rho}((0,t]\times B) &= \mathbb{E}\left[J_{\mathbf{Y}}((0,\Lambda(t)]\times B)\right] \\ &= \tilde{\rho}_{\mathbf{Y}}((0,\Lambda(t)]\times B) \\ &= \Lambda(t)\,\rho_{\mathbf{Y}}(B),\end{aligned}$$

because $\tilde{\rho}_{\mathbf{Y}}(ds,d\mathbf{u}) = ds\,\rho_{\mathbf{Y}}(d\mathbf{u})$, which provides the result. □

Based on Theorem 2, and using similar arguments as for Proposition 3, Corollary 3 and Theorem 3, it is now straightforward to derive the following results in case of a time-changed Lévy process.

Corollary 5 *Let* $\mathbf{X}$ *be a time-scaled Lévy process with time-scaling function* Λ *and Lévy measure* ρ*, such that the underlying homogeneous Lévy process* $\mathbf{Y}$ *(see* (32)*) fulfills Assumption 1. Then we have the following results:*

1. *The process* $\mathbf{X}$ *fulfills Assumption 1 and*

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s})-K(\mathbf{s})}$$

with

$$\begin{aligned}K(\mathbf{s}) &= \int_F \left(1-e^{-\sum_{i=1}^d (s_i-t)^+ x_i}\right)\lambda(t)\,dt\,\rho(d\mathbf{x}) \\ &= -\int_0^\infty \ln\left(\mathscr{L}_{\mathbf{Y}(1)}\left((s_1-t)^+,\cdots,(s_d-t)^+\right)\right)\lambda(t)\,dt\end{aligned}$$

for all $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{R}_+^d$.

2. *In the bivariate case, if* $\mathbf{Y}$ *is constructed through superposition as in Corollary 3, we have*

$$K(\mathbf{s}) = -\int_0^{s_1} \ln\left(\mathscr{L}_{Z_1(1)}(s_1-t)\right)\lambda(t)\,dt - \int_0^{s_2} \ln\left(\mathscr{L}_{Z_2(1)}(s_2-t)\right)\lambda(t)\,dt$$
$$-\int_0^{\max(s_1,s_2)} \ln\mathscr{L}_{Z_3(1)}\left((s_1-t)^+ + (s_2-t)^+\right)\lambda(t)\,dt$$

3. *Assuming the dependence in* $\mathbf{Y}$ *to be modelled by a Lévy copula* $\mathbf{C}$ *with* $U_i(s_i)$ *the tail integral functions of* ρ_i, $i = 1,2$, *we have*

$$\bar{F}_\tau(\mathbf{s}) = \exp\left[\int_{\mathbb{R}_+^2} \mathbf{C}\left(U_1(s_1), U_2(s_2)\right)\ \gamma_{\mathbf{v},\lambda}(\mathbf{s})\ ds_1\, ds_2\right]\bar{F}_{\tau_1}(v_1)\,\bar{F}_{\tau_2}(v_2)$$

with

$$\gamma_{\mathbf{v},\lambda}(\mathbf{s}) = \int_0^{\min(v_1,v_2)} (v_1-t)(v_2-t)\,e^{-(v_1-t)s_1-(v_2-t)s_2}\ \lambda(t)\ dt$$

for all $\mathbf{s} = (s_1,\cdots,s_d) \in \mathbb{R}_+^d$.

Remark 6 Remark 5 remains valid here and τ exhibits the positive quadrant dependence property.

As an example, we now look at the case of a univariate non-homogeneous gamma process.

Example 2 Let $(Y(t))_{t\geq 0}$ be a univariate homogeneous gamma process with $Y(t) \sim \Gamma(t,b)$, where $b > 0$. Let $X(t) = Y(\Lambda(t))$ for all $t > 0$. Then, $(X(t))_{t\geq 0}$ is a univariate non-homogeneous gamma process with $X(t) \sim \Gamma(\Lambda(t), b)$ and

$$\bar{F}_\tau(s) = e^{-H(s)} \exp\left(-\int_0^s \lambda(t) \ln\left(\frac{b+s-t}{b}\right)\ dt\right),$$
$$h_\tau(s) = h(s) + a\int_0^s \frac{\lambda(t)}{b+s-t}dt.$$

The increment in the unconditional hazard rate is plotted in Figure 3 for the case of a non homogenous gamma conditional increment process with $a = b = 1$ and $\lambda(s) = s^\beta$ with $\beta = -0.5$, $\beta = 0$ and $\beta = 1$, which leads to $\Lambda(s) = s^{0.5}/0.5$ (concave function), $\Lambda(s) = s$ and $\Lambda(s) = s^2/2$ (convex function), and for $\lambda(s) = e^s$. We can see that for small s, the largest influence of the stressing environment is obtained in the concave case whereas for large s, it increases with β and the largest hazard rate is obtained for $\lambda(s) = e^s$.

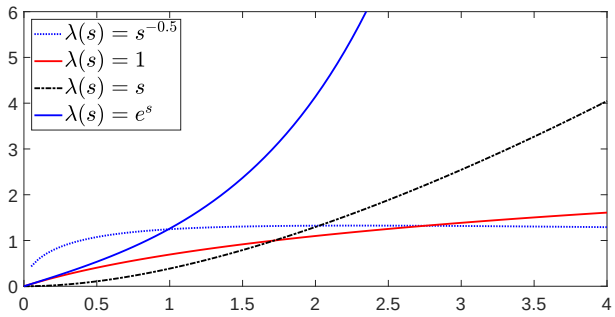

Fig. 3 The increment in the unconditional failure rate for a non-homogeneous gamma increment process, with $a = b = 1$ and $\lambda(s) = s^{\beta}$ with $\beta = -0.5$, $\beta = 0$ and $\beta = 1$, and $\lambda(s) = e^s$.

5.2 Extended time-scaled Lévy process

Let $\mathbf{Y} = (\mathbf{Y}(t))_{t\geq 0}$ be a multivariate time-scaled Lévy process with time-scaling function Λ and Lévy measure ρ as in Subsection 4. We here consider a process $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ which is constructed as a stochastic integral with respect to $\mathbf{Y}$, with

$$\mathbf{X}(t) = \int_{(0,t]} \mathbf{b}(u)\, d\mathbf{Y}(u) = \left(\int_{(0,t]} b_1(u)\, dY_1(u), \cdots, \int_{(0,t]} b_d(u)\, dY_d(u) \right) \tag{33}$$

for all $t \geq 0$, where $\mathbf{b}(u) = (b_1(u), \cdots, b_d(u))$ for all $u \geq 0$ is a componentwise positive and continuous function on $\left(\mathbb{R}_+^*\right)^d$.

Definition 2 The process $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ constructed through (33) is said to be an extended time-scaled Lévy process with parameter $(\mathbf{b}, \rho, \Lambda)$.

Remark 7 The name "extended Lévy process" comes from the extended gamma process, which is defined as an integral with respect to a gamma process in several papers such as [2, 11, 14]. Note that in [7], such processes are called local gamma processes.

Lemma 3 *Let $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ be a multivariate extended time-scaled Lévy process with parameter $(\mathbf{b}, \rho, \Lambda)$ such that the underlying Lévy process $\mathbf{Y}$ (as in (33)) fulfills Assumption 1 and such that*

$$\int_{(0,t]} b_i(u)\, \lambda(u)\, du < \infty \tag{34}$$

for all $i \in \{1, \cdots, d\}$ and all $t > 0$. Then $\mathbf{X}$ is an additive process wich fulfills Assumption 1, and its intensity measure is

$$\tilde{\rho}(ds, d\mathbf{u}) = \lambda(s)\, ds \left(\rho \varphi_{\mathbf{b}(s)}^{-1} \right)(d\mathbf{u})$$

where

$$\varphi_{\mathbf{b}(s)}(\mathbf{x}) = (b_1(s) x_1, \cdots, b_d(s) x_d) \tag{35}$$

for all $s>0$ and all $\mathbf{x}\in\mathbb{R}_+^d$, and where $\rho\varphi_{\mathbf{b}(s)}^{-1}$ stands for the push-forward measure of ρ by the function $\varphi_{\mathbf{b}(s)}$.

In the specific case where $\rho(d\mathbf{x})=\rho(\mathbf{x})d\mathbf{x}$, then

$$\tilde{\rho}(ds,d\mathbf{u})=\lambda(s)ds\,\rho\left(\frac{u_1}{b_1(s)},\cdots,\frac{u_d}{b_d(s)}\right)\frac{1}{b_1(s)\times\cdots\times b_d(s)}d\mathbf{u}.$$

Proof Remembering that

$$\mathbf{Y}(t)=\int_{(0,t]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{y}\,J_{\mathbf{Y}}(ds,d\mathbf{y}),$$

where $J_{\mathbf{Y}}$ is a Poisson random measure on F with intensity measure $\tilde{\rho}_{\mathbf{Y}}(ds,d\mathbf{u})=\lambda(s)ds\,\rho(d\mathbf{u})$, we can write:

$$\mathbf{X}(t)=\int_{\mathbb{R}_+^*\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{1}_{(0,t]}(s)\left(b_1(s)y_1,\cdots,b_d(s)y_d\right)\,J_{\mathbf{Y}}(ds,d\mathbf{y})$$

for all $t\geq 0$.

For $m\geq 1$, $0\leq t_1<\cdots<t_m$ and $1\leq i\leq m-1$, we have

$$\mathbf{X}(t_{i+1})-\mathbf{X}(t_i)=\int_{\mathbb{R}_+^*\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{1}_{(t_i,t_{i+1}]}(s)\left(b_1(s)y_1,\cdots,b_d(s)y_d\right)\,J_{\mathbf{Y}}(ds,d\mathbf{y}),$$

which shows that the random vectors $\mathbf{X}(t_{i+1})-\mathbf{X}(t_i)$, $i=1,\cdots,m-1$ are independent, because the Borel sets $(t_i,t_{i+1}]\times\mathbb{R}_+^k$ are disjoint. Hence, $(\mathbf{X}_t)_{t\geq 0}$ is a process with independent increments.

For each $1\leq j\leq d$, $t>0$, $0\leq s\leq 1$, we have

$$X_j(t+s)-X_j(t)=\int_{(t,t+s]}b_j(u)\,dY_j(u)\leq\sup_{u\in[t,t+1]}b_j(u)\times\left(Y_j(t+s)-Y_j(t)\right).$$

Based on the continuity of b_j, the supremum in the right side is finite. The stochastic continuity of $(X_j(t))_{t\geq 0}$ for each $1\leq j\leq d$ now is a direct consequence of the same property for $\mathbf{Y}$, and hence, $\mathbf{X}$ is stochastically continuous. Finally, $(\mathbf{X}_t)_{t\geq 0}$ is a right-continuous process with left-side limits by construction (and $\mathbf{X}_0=0$ a.s.). Hence $(\mathbf{X}_t)_{t\geq 0}$ is an additive process.

Now, our aim is to use Proposition 19.5 from [26] to derive the Laplace transform of $\mathbf{X}(t)$. For that, we have to boil down to a Poisson random measure with a finite intensity measure. For $\varepsilon>0$ and $t>0$, we set $J_{\mathbf{Y}}^{t,\varepsilon}(ds,d\mathbf{y})$ to be the trace of $J_{\mathbf{Y}}(ds,d\mathbf{y})$ on $(0,t]\times\left\{(\varepsilon,\infty)^d\setminus\{\mathbf{0}\}\right\}$, which is known to be a Poisson random measure with intensity measure

$$\tilde{\rho}_{\mathbf{Y}}^{t,\varepsilon}(ds,d\mathbf{u})=\mathbf{1}_{(0,t]}(s)\mathbf{1}_{(\varepsilon,\infty)^d\setminus\{\mathbf{0}\}}(\mathbf{u})\lambda(s)ds\,\rho(d\mathbf{u}).$$

Based on the bounded variations of $\mathbf{Y}$ (that is (1) for $\mathbf{Y}$) and the assumption on λ, we have $\tilde{\rho}_{\mathbf{Y}}^{t,\varepsilon}\left((0,t]\times\left\{(\varepsilon,\infty)^d\right\}\right)<\infty$.

We set

$$\begin{aligned}\mathbf{X}^{\varepsilon}(t) &= \int_{\mathbb{R}_+^*\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \mathbf{1}_{(0,t]}(s)\,(b_1(s)y_1,\cdots,b_d(s)y_d)\; J_{\mathbf{Y}}^{t,\varepsilon}(ds,d\mathbf{y}) \\ &= \int_{\mathbb{R}_+^*\times\mathbb{R}_+^d\setminus\{\mathbf{0}\}} \mathbf{1}_{(0,t]}(s)\,\varphi_{\mathbf{b}(s)}(\mathbf{y})\; J_{\mathbf{Y}}^{t,\varepsilon}(ds,d\mathbf{y}),\end{aligned}$$

where $\varphi_{\mathbf{b}(s)}$ is defined in (35).

Based on Proposition 19.5 from [26] (written for Fourier transform), the Laplace transform of $\mathbf{X}^{\varepsilon}(t)$ is

$$\mathbb{E}\left(e^{-\langle\mathbf{u},\mathbf{X}^{\varepsilon}(t)\rangle}\right) = \exp\left(-\int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\left(1-e^{-\langle\mathbf{u},\mathbf{x}\rangle}\right)\left(\tilde{\rho}_{\mathbf{Y}}^{t,\varepsilon}\varphi_{t,\mathbf{b}}^{-1}\right)(d\mathbf{x})\right)$$

for all $\mathbf{u}\in\mathbb{R}_+^d$ and $t>0$, where $\varphi_{t,\mathbf{b}}(s,\mathbf{x}) = \mathbf{1}_{(0,t]}(s)\,\varphi_{\mathbf{b}(s)}(\mathbf{x})$. This means that the Lévy mesure of the random vector $\mathbf{X}^{\varepsilon}(t)$ is $\tilde{\rho}_{\mathbf{Y}}^{t,\varepsilon}\varphi_{t,\mathbf{b}}^{-1}$.

Then, the intensity measure $\tilde{\rho}$ of $\mathbf{X}$ is such that

$$\begin{aligned}\tilde{\rho}((0,t]\times B) &= \lim_{\varepsilon\to 0^+}\left(\tilde{\rho}_{\mathbf{Y}}^{t,\varepsilon}\varphi_{t,\mathbf{b}}^{-1}\right)((0,t]\times B) \\ &= \lim_{\varepsilon\to 0^+}\tilde{\rho}_{\mathbf{Y}}\left(\left\{(s,\mathbf{x})\in(0,t]\times\left\{(\varepsilon,\infty)^d\setminus\{\mathbf{0}\}\right\} : (b_1(s)x_1,\cdots,b_d(s)x_d)\in B\right\}\right) \\ &= \tilde{\rho}_{\mathbf{Y}}\left(\left\{(s,\mathbf{x})\in(0,t]\times\mathbb{R}_+^d\setminus\{\mathbf{0}\} : (b_1(s)x_1,\cdots,b_d(s)x_d)\in B\right\}\right) \\ &= \int_{(0,t]}\rho\left(\left\{\mathbf{x}\in\mathbb{R}_+^d\setminus\{\mathbf{0}\} : \varphi_{\mathbf{b}(s)}(\mathbf{x})\in B\right\}\right)\lambda(s)\;ds \\ &= \int_{(0,t]}\left(\rho\varphi_{\mathbf{b}(s)}^{-1}\right)(\mathbf{B})\,\lambda(s)\;ds\end{aligned}$$

for all $B\in\mathscr{B}\left(\mathbb{R}_+^d\setminus\{\mathbf{0}\}\right)$. This provides the expression for $\tilde{\rho}(ds,d\mathbf{u})$.

In the specific case where $\rho(d\mathbf{x}) = \rho(x)\,d\mathbf{x}$, then

$$\begin{aligned}\left(\rho\varphi_{\mathbf{b}(s)}^{-1}\right)(\mathbf{B}) &= \int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{1}_{\mathbf{B}}\left((b_1(s)x_1,\cdots,b_d(s)x_d)\right)\rho(\mathbf{x})\,d\mathbf{x} \\ &= \int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\mathbf{1}_{\mathbf{B}}(\mathbf{u})\,\rho\left(\frac{u_1}{b_1(s)},\cdots,\frac{u_d}{b_d(s)}\right)\frac{1}{b_1(s)\times\cdots\times b_d(s)}d\mathbf{u}\end{aligned}$$

with $\mathbf{u} = (b_1(s)x_1,\cdots,b_2(s)x_d)$.

We finally check that $\mathbf{X}$ fulfills Assumption 1, where the only thing remaining to prove is that (1) is true. We have:

$$\begin{aligned}\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho_t(d\mathbf{u}) &= \int_{(0,t]\times\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\tilde{\rho}(dt,d\mathbf{u}) \\ &= \int_{(0,t]\times\mathbb{R}_+^d}\min\left(\left\|\varphi_{\mathbf{b}(t)}(\mathbf{u})\right\|_\infty,1\right)\lambda(t)\;dt\;\rho(d\mathbf{u})\end{aligned}$$

with

$$\left\|\varphi_{\mathbf{b}(t)}(\mathbf{u})\right\|_\infty = \|(b_1(t)u_1,\cdots,b_d(u)u_d)\|_\infty \le \|\mathbf{b}(t)\|_\infty\|\mathbf{u}\|_\infty$$

and

$$\min\left(\left\|\varphi_{\mathbf{b}(t)}(\mathbf{u})\right\|_\infty,1\right)\leq\min\left(\|\mathbf{b}(t)\|_\infty\|\mathbf{u}\|_\infty,1\right)\leq\max\left(\|\mathbf{b}(t)\|_\infty,1\right)\min\left(\|\mathbf{u}\|_\infty,1\right).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho_t(d\mathbf{u})&\leq\int_{(0,t]}\max\left(\|\mathbf{b}(t)\|_\infty,1\right)\lambda(t)\;dt\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho(d\mathbf{u})\\&\leq\int_{(0,t]}\left(1+\sum_{i=1}^d b_i(t)\right)\lambda(t)\;dt\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho(d\mathbf{u})\\&=\left(\Lambda(t)+\sum_{i=1}^d\int_{(0,t]}b_i(t)\lambda(t)\;dt\right)\int_{\mathbb{R}_+^d}\min\left(\|\mathbf{u}\|_\infty,1\right)\rho(d\mathbf{u})\\&<+\infty\end{aligned}$$

based on (34) and Assumption (1) for $\mathbf{Y}$, which achives this proof. □

We next provide several expressions for $\bar{F}_\tau(\mathbf{s})$ in the following corollary. The first expression is a direct consequence of Lemma 3 and Theorem 2, as

$$\int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\phi(\mathbf{u})\left(\rho\varphi_{\mathbf{b}(s)}^{-1}\right)(d\mathbf{u})=\int_{\mathbb{R}_+^d\setminus\{\mathbf{0}\}}\phi\left(\varphi_{\mathbf{b}(s)}(\mathbf{u})\right)\rho(d\mathbf{u})$$

for any non negative Borel function ϕ. Similar arguments as for Proposition 3, Corollary 3 and Theorem 3 can be used for the other expressions.

Corollary 6 *Let $\mathbf{X}=(\mathbf{X}(t))_{t\geq0}$ be an extended time-scaled Lévy process with parameter $(\mathbf{b},\rho,\Lambda)$ such that the underlying Lévy process $\mathbf{Y}$ (see (33)) fulfills Assumption 1 and (34). Considering $\mathbf{X}$ as conditional hazard rate increment process, we get the following results:*

1. *We have*
$$\bar{F}_\tau(\mathbf{s})=e^{-H(\mathbf{s})-K(\mathbf{s})}\tag{36}$$
with
$$\begin{aligned}K(\mathbf{s})&=\int_F\left(1-e^{-\sum_{i=1}^d(s_i-t)^+b_i(t)x_i}\right)\lambda(t)dt\;\rho(d\mathbf{x})\\&=-\int_0^\infty\ln\left(\mathscr{L}_{\mathbf{Y}(1)}\left((s_1-t)^+b_1(t),\cdots,(s_d-t)^+b_d(t)\right)\right)\lambda(t)dt\end{aligned}$$
for all $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{R}_+^d$ where $H(s)$ is defined in (5).
2. *In the bivariate case, assuming $\mathbf{Y}$ to be constructed through superposition as in Corollary 3, we have*
$$\begin{aligned}K(\mathbf{s})=&-\int_0^{s_1}\ln\left(\mathscr{L}_{Z_1(1)}\left((s_1-t)b_1(t)\right)\right)\lambda(t)dt\\&-\int_0^{s_2}\ln\left(\mathscr{L}_{Z_2(1)}\left((s_2-t)b_2(t)\right)\right)\lambda(t)dt\\&-\int_0^{\max(s_1,s_2)}\ln\mathscr{L}_{Z_3(1)}\left((s_1-t)^+b_1(t)+(s_2-t)^+b_2(t)\right)\lambda(t)dt.\end{aligned}$$

3. *Finally, assuming the dependence in* $\mathbf{Y}$ *to be modelled by a Lévy copula* $\mathbf{C}$ *with* $U_i(s_i)$ *the tail integral functions of* ρ_i, $i=1,2$, *we have*

$$\bar{F}_\tau(\mathbf{s}) = \exp\left[\int_{\mathbb{R}_+^2} \mathbf{C}(U_1(s_1), U_2(s_2))\ \gamma_{\mathbf{v},\lambda,\mathbf{b}}(\mathbf{s})\ ds_1\ ds_2\right] \bar{F}_{\tau_1}(v_1)\,\bar{F}_{\tau_2}(v_2)$$

with

$$\gamma_{\mathbf{v},\lambda,\mathbf{b}}(\mathbf{s}) = \int_0^{\min(v_1,v_2)} b_1(t)\,b_2(t)\,(v_1-t)\,(v_2-t)\,e^{-(v_1-t)b_1(t)s_1-(v_2-t)b_2(t)s_2}\,\lambda(t)\,dt$$

for all $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{R}_+^d$.

Remark 8 Remark 5 remains valid and τ exhibits the positive quadrant dependence property.

As an example, we now look at the case of a bivariate extended time-scaled gamma process constructed though superposition.

Example 3 Let $Z_i(t)\sim\Gamma(a_i t,1)$ with $a_i>0$, $i=1,2,3$, and let $(Y_i(t))_{t\geq 0}$, $i=1,2$ be constructed through (25). Then

$$\bar{F}_{\tau_i}(s) = \exp\left(-H_i(s) - (a_i+a_3)\int_0^s \lambda(t)\ln(1+(s-t)\,b_i(t))\,dt\right)$$

for $i=1,2$ and $s>0$, and

$$\begin{aligned} K(\mathbf{s}) &= a_1\int_0^{s_1}\lambda(t)\ln(1+(s_1-t)\,b_1(t))\,dt + a_2\int_0^{s_2}\lambda(t)\ln(1+(s_2-t)\,b_2(t))\,dt \\ &\quad + a_3\int_0^{\max(s_1,s_2)}\lambda(t)\ln\left(1+(s_1-t)^+\,b_1(t)+(s_2-t)^+\,b_2(t)\right)dt \end{aligned}$$

for all $\mathbf{s}=(s_1,s_2)\in\mathbb{R}_+^2$, which provides the bivariate survival function through (36).

Remark 9 Considering $H(\mathbf{s})=0$ for all $\mathbf{s}\in\mathbb{R}_+^2$, $a_1=0=a_2=0$ and $a_3=1$, we get

$$\begin{aligned} \bar{F}_\tau(\mathbf{s}) &= \exp\left(-\int_0^{\max(s_1,s_2)}\lambda(t)\ln\left(1+(s_1-t)^+\,b_1(t)+(s_2-t)^+\,b_2(t)\right)dt\right) \\ &= \exp\left(\int_0^{s_1}\lambda(t)\ln(1+(s_1-t)\,b_1(t)+(s_2-t)\,b_2(t))\,dt\right) \\ &\quad\times\exp\left(\int_{s_1}^{s_2}\lambda(t)\ln(1+(s_2-t)\,b_2(t))\,dt\right) \end{aligned}$$

Comparing such results with Equations (2.2) and (2.3) from [27], even if it is not exactly the same model, it seems strange that s_1-t and s_2-t do not intervene in these two equations (just as $s-t$ intervenes in [17, Theorem 3.1] for instance, providing a similar result as in the present paper).

5.3 Shock models

We finally explore the case where the impact of the stressing environment arrives through shocks according to a non homogenous Poisson process, and may have different impacts on the components. We begin with a general result, that we next decline through different cases.

Proposition 4 *Let $(N_t)_{t\geq 0}$ be a Poisson process on $\mathbb{R}_+$ with intensity function $\lambda(\cdot)$ and points $\{T_n, n\in\mathbb{N}^*\}$, and let $Q(t,d\mathbf{u})$ be a transition probability kernel from $(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+))$ to $\left(\mathbb{R}_+^k,\mathscr{B}\left(\mathbb{R}_+^k\right)\right)$. We set $(\mathbf{U}_n)_{n\in\mathbb{N}^*}$ to be a sequence of non negative $k-$dimensional random vectors such that given $\{T_n, n\in\mathbb{N}^*\}$, the $\mathbf{U}_n$'s, $n=1,2,\cdots$ are conditionaly independent with conditional distribution $Q(T_n,d\mathbf{u})$. Finally let $\mathbf{h}:\left(\mathbb{R}_+\times\mathbb{R}_+^k,\mathscr{B}\left(\mathbb{R}_+\times\mathbb{R}_+^k\right)\right)\longrightarrow\left(\mathbb{R}_+^d,\mathscr{B}\left(\mathbb{R}_+^d\right)\right)$ be a measurable function with*

$$\mathbf{h}(t,\mathbf{u})=(h_1(t,\mathbf{u}),\cdots,h_d(t,\mathbf{u}))$$

for all $t\geq 0$ and $\mathbf{u}\in\mathbb{R}_+^k$, such that the function

$$t\longmapsto\int_{(0,t]\times\mathbb{R}_+^k}\|\mathbf{h}(v,\mathbf{u})\|_\infty\,\lambda(v)\,dv\,Q(v,d\mathbf{u}) \tag{37}$$

is continuous on $\mathbb{R}_+^*$ and such that

$$\int_{(0,t]\times\mathbb{R}_+^k}\min\left(\|\mathbf{h}(v,\mathbf{u})\|_\infty,1\right)\lambda(v)\,dv\,Q(v,d\mathbf{u})<\infty \tag{38}$$

for all $t>0$.

Let us set:

$$\mathbf{X}(t)=\sum_{n=1}^{N_t}\mathbf{h}(T_n,\mathbf{U}_n)=\sum_{n=1}^{\infty}\mathbf{1}_{(0,t]}(T_n)\;\mathbf{h}(T_n,\mathbf{U}_n)\;\;\textit{for all } t>0.$$

Then $(\mathbf{X}_t)_{t\geq 0}$ is an additive process with intensity measure

$$\tilde{\rho}(ds,d\mathbf{x})=\lambda(s)\,ds\,Q_{\mathbf{h}(s,\cdot)}(s,d\mathbf{x}),$$

where, for each s, $Q_{\mathbf{h}(s,\cdot)}(s,d\mathbf{x})$ stands for the push-forward measure of $Q(s,d\mathbf{u})$ by the function $\mathbf{h}(s,\cdot)$.

Taking $(\mathbf{X}_t)_{t\geq 0}$ as conditional hazard rate increment process, we get

$$\bar{F}_\tau(\mathbf{v})=e^{-H(\mathbf{v})}\exp\left[-\int_F\left(1-e^{-\sum_{i=1}^d(v_i-s)^+h_i(t,\mathbf{x})}\right)\lambda(s)\,ds\,Q(s,d\mathbf{x})\right]$$

for all $\mathbf{v}=(v_1,\cdots,v_d)\in\mathbb{R}_+^d$.

Proof Based on [8, Theorem 3.2 page 264], we know that $\{(T_n,\mathbf{U}_n),n\in\mathbb{N}^*\}$ are the points of a Poisson random measure $\hat{J}(ds,d\mathbf{u})$ with intensity measure $\hat{\rho}(ds,d\mathbf{u})=\lambda(s)\,ds\,Q(s,d\mathbf{u})$. Then, we may write

$$\mathbf{X}(t)=\sum_{n=1}^{\infty}\mathbf{1}_{(0,t]}(T_n)\;\mathbf{h}(T_n,\mathbf{U}_n)=\int_{\mathbb{R}_+\times\mathbb{R}_+^k}\mathbf{1}_{(0,t]}(s)\mathbf{h}(s,\mathbf{u})\hat{J}(ds,d\mathbf{u})$$

for all $t \geq 0$.

The fact that $\mathbf{X}$ has independent increments can be proved in a similar way as in Lemma 3. Also, $(\mathbf{X}(t))_{t\geq 0}$ is a right-continuous process with left-side limits by construction (and $\mathbf{X}(0) = 0$ a.s.) and the stochastic continuity of $(\mathbf{X}(t))_{t\geq 0}$ is a direct consequence of (37), as

$$\begin{aligned}\mathbb{P}\left(\|\mathbf{X}(v+s)-\mathbf{X}(v)\|_\infty > \varepsilon\right) &\leq \frac{1}{\varepsilon}\mathbb{E}\left(\|\mathbf{X}(v+s)-\mathbf{X}(v)\|_\infty\right)\\ &\leq \frac{1}{\varepsilon}\int_{(v,v+s]\times\mathbb{R}_+^k} \|\mathbf{h}(t,\mathbf{u})\|_\infty \lambda(t)\,dt\; Q(t,d\mathbf{u})\end{aligned}$$

for all $\varepsilon, v, s > 0$. Hence $(\mathbf{X}(t))_{t\geq 0}$ is an additive process.

Besides, we have

$$\begin{aligned}\mathbb{E}\left(e^{-\langle \mathbf{y},\mathbf{X}(t)\rangle}\right) &= \mathbb{E}\left[e^{-\left\langle \mathbf{y},\int_{\mathbb{R}_+\times\mathbb{R}_+^k} \mathbf{1}_{(0,t]}(s)\mathbf{h}(s,\mathbf{u})\tilde{J}(ds,d\mathbf{u})\right\rangle}\right]\\ &= \exp\left[-\int_{\mathbb{R}_+\times\mathbb{R}_+^k}\left(1-e^{-\left\langle \mathbf{y},\mathbf{1}_{(0,t]}(s)\mathbf{h}(s,\mathbf{u})\right\rangle}\right)\lambda(s)\,ds\; Q(s,d\mathbf{u})\right]\\ &= \exp\left[-\int_0^t\left(\int_{\mathbb{R}_+^k}\left(1-e^{-\langle \mathbf{y},\mathbf{x}\rangle}\right)\; Q_{\mathbf{h}(\mathbf{s},\cdot)}(s,d\mathbf{x})\right)\lambda(s)\,ds\right]\end{aligned}$$

for all $\mathbf{y}\in\mathbb{R}_+^d$, where $Q_{\mathbf{h}(\mathbf{s},\cdot)}(s,d\mathbf{x})$ is defined in the proposition.

This can be written

$$\mathbb{E}\left(e^{-\langle \mathbf{y},\mathbf{X}(t)\rangle}\right) = \exp\left[\int_{\mathbb{R}_+^d}\left(1-e^{-\langle \mathbf{y},\mathbf{x}\rangle}\right)\rho_t(d\mathbf{x})\right]$$

with

$$\rho_t(d\mathbf{x}) = \int_0^t Q_{\mathbf{h}(\mathbf{s},\cdot)}(s,d\mathbf{x})\,\lambda(s)\,ds = \tilde{\rho}\left((0,t],d\mathbf{x}\right),$$

which provides

$$\tilde{\rho}(ds,d\mathbf{x}) = \lambda(s)\,ds\; Q_{\mathbf{h}(s,\cdot)}(s,d\mathbf{x})$$

and next the expression for $\bar{F}_\tau(\mathbf{v})$, based on Theorem 2.

Finally, Assumption (1) for $\mathbf{X}$ is a direct consequence of (38), which ends this proof. □

In the specific case where the shocks induce i.i.d. (multivariate) increments $\mathbf{U}_j$, $j = 1,2,\cdots$ in the hazard rates of the components, we obtain the following result.

Corollary 7 (Non-homogeneous compound Poisson process) *Let $\mathbf{X}_t$ be a multivariate compound Poisson process (CPP) of the shape*

$$\mathbf{X}_t = \sum_{j=1}^{N_t}\mathbf{U}_j,$$

where $(N_t)_{t\geq 0}$ *is a Poisson process with intensity function* $\lambda(\cdot)$*, independent of the* $\mathbf{U}_j$*'s, which are i.i.d.* $d-$*dimensional non negative random variables such that* $\mathbb{E}(\mathbf{U}_1)$ *is finite with* $\mu(d\mathbf{u})$ *as common distribution. Then*

$$\bar{F}_\tau(\mathbf{v}) = e^{-H(\mathbf{v})} \exp\left[-\int_F \left(1 - e^{-\sum_{i=1}^d (v_i - s)^+ x^{(i)}}\right) \lambda(s)\, ds\, \mu(d\mathbf{u}) \right],$$

for all $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}_+^d$.

Proof With the notations of Proposition 4, we have

$$\begin{aligned} \mathbf{h}(t, \mathbf{u}) &= \mathbf{u}, \\ Q(t, d\mathbf{u}) &= \mu(d\mathbf{u}). \end{aligned}$$

Using the fact that $\mathbb{E}(\mathbf{U}_1)$ is finite, it is easy to check that (37) and (38) are true, and next derive the expression for $\bar{F}_\tau(\mathbf{v})$. □

In Proposition 4, all the dependence between the τ_i's, $i = 1, \cdots, d$, comes from the dependence between the $U_j^{(i)}$'s with

$$\mathbf{U}_j = \left(U_j^{(1)}, \cdots, U_j^{(d)}\right)$$

for all $j \geq 1$. We next envision possible additional dependence between components due to possibly simultaneous fatal shocks, with a time-dependent impact of the shocks on the components. A fatal shock on a component is modelled through an infinite increment in its hazard rate at the time of the shock. More specifically, we assume that by a shock at time T_n and for each subset of components $C \subset \{1, \cdots, d\}$, there is a probability $a_C(T_n)$ that:

- the shock is fatal to all components in C,
- the shock induces an increment in the hazard rate function of all other components (out of C).

Corollary 8 (Possibly fatal shocks and time-dependent impact of a shock) *Let* $(N_t)_{t\geq 0}$ *be a Poisson process on* $\mathbb{R}_+$ *with intensity function* $\lambda(\cdot)$ *and points* $\{T_n, n \in \mathbb{N}^*\}$*, and let* $Q(t, d\mathbf{u})$ *be a transition probability kernel from* $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ *to* $\left(\bar{\mathbb{R}}_+^d, \mathscr{B}\left(\bar{\mathbb{R}}_+^d\right)\right)$*, with* $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$*. We set* $(\mathbf{V}_n)_{n\in\mathbb{N}^*}$ *to be a sequence of non negative* $d-$*dimensional random vectors such that given* $\{T_n, n \in \mathbb{N}^*\}$*, the* $\mathbf{V}_n$*'s,* $n = 1, 2, \cdots$ *are conditionaly independent with conditional distribution* $Q(T_n, d\mathbf{u})$ *and*

$$Q(t, d\mathbf{u}) = \sum_{C \subset \{1, \cdots, d\}} a_C(t)\, \nu_C(d\mathbf{u})$$

where:

- *the summation is taken over all subsets* C *of* $\{1, \cdots, d\}$*;*
- *for each* $t > 0$*,* $\{a_C(t), C \subset \{1, \cdots, d\}\}$ *is a set of non negative real numbers such that* $\sum_{C \subset \{1, \cdots, d\}} a_C(t) = 1$*;*

– $\nu_C(d\mathbf{u}) = (\prod_{i\in C}\delta_\infty(du_i)) \times \mu_{\bar{C}}(d\mathbf{u}_{\bar{C}})$ *where*

$$\mu(d\mathbf{u}) = (\mu_1(du_1), \cdots, \mu_d(du_d))$$

is a probability measure on $\mathbb{R}_+^d$ *and where* $\mu_{\bar{C}}(d\mathbf{u}_{\bar{C}})$ *stands for the marginal distribution of* μ *corresponding to indexes* $i \in \bar{C}$.

Taking

$$\mathbf{X}_t = \sum_{j=1}^{N_t} \mathbf{V}_j,\ \forall t \geq 0$$

as conditional hazard rate increment process, we get

$$\begin{aligned}
&\bar{F}_\tau(\mathbf{v}) \\
&= e^{-H(\mathbf{v})-\Lambda(v_d)} \\
&\times \exp\left(\sum_{j=1}^{d} \sum_{\substack{C\subset\{1,\cdots,d\} \text{ such that} \\ \{j,\cdots,d\}\cap C=\varnothing}} \int_{v_{j-1}}^{v} \mathscr{L}_{\mu_{j\cdots d}}(v_j - s, \cdots v_d - s)\, a_C(s)\, \lambda(s)\, ds\right)
\end{aligned}$$

for all $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}_+^d$ *such that* $v_1 < \cdots < v_d$.

Proof Note firstly that the process $\mathbf{X}$ is not of bounded variation as components of $\mathbf{V}_j$'s may be infinite, so that we are not in the settings of the previous results. However, it can be seen that Lemma 1 still holds considering $\mathbf{X}$ as conditional hazard rate increment process. Indeed, looking at the details of the proof, the only point to check is the fact that the conditional independence assumption between components lifetimes (given $\mathbf{X}$) still holds. As a matter of fact, when $\mathbf{X}$ is known, the set $F \subseteq \{1, \dots d\}$ of components which are subject to an infinite increment in their hazard rate is known, together with the corresponding shock times. For each $i \in F$, let S_i be the smallest among these shock times, which is called possibly fatal shock time in the following. Note that these possibly fatal shock times can coincide among components. Note also that a possibly fatal shock on a component may occur after an intrinsic failure of the component (that is a failure due to its failure rate) and then nothing happens at the corresponding shock time. Then, for $i \in F$, the lifetime of component i is the minimum between its possibly fatal shock time S_i and its intrisic lifetime (say U_i).

With this in mind, we can write

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{d} \mathbf{1}_{\{\tau_i > v_i\}} \middle| \mathbf{X}\right) &= \mathbb{E}\left(\prod_{i\in F} \mathbf{1}_{\{\min(S_i,U_i)>v_i\}} \prod_{i\notin F} \mathbf{1}_{\{\tau_i>v_i\}} \middle| \mathbf{X}\right) \\
&= \mathbb{E}\left(\prod_{i\in F} \mathbf{1}_{\{S_i>v_i\}} \prod_{i\in F} \mathbf{1}_{\{U_i>v_i\}} \prod_{i\notin F} \mathbf{1}_{\{\tau_i>v_i\}} \middle| \mathbf{X}\right) \\
&= \prod_{i\in F} \mathbf{1}_{\{S_i>v_i\}} \mathbb{E}\left(\prod_{i\in F} \mathbf{1}_{\{U_i>v_i\}} \prod_{i\notin F} \mathbf{1}_{\{\tau_i>v_i\}} \middle| \mathbf{X}\right)
\end{aligned}$$

because the S_i's are known when $\mathbf{X}$ is known.

Using the conditional independence for the intrinsic lifetimes (that is the U_i's when $i \in F$ and the τ_i's when $i \notin F$), we get

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^{d}\mathbf{1}_{\{\tau_i>v_i\}}\middle|\mathbf{X}\right) &= \prod_{i\in F}\mathbf{1}_{\{S_i>v_i\}}\prod_{i\in F}\mathbb{E}\left(\mathbf{1}_{\{U_i>v_i\}}\middle|\mathbf{X}\right)\prod_{i\notin F}\mathbb{E}\left(\mathbf{1}_{\{\tau_i>v_i\}}\middle|\mathbf{X}\right)\\ &= \prod_{i\in F}\mathbb{E}\left(\mathbf{1}_{\{S_i>v_i\}}\mathbf{1}_{\{U_i>v_i\}}\middle|\mathbf{X}\right)\prod_{i\notin F}\mathbb{E}\left(\mathbf{1}_{\{\tau_i>v_i\}}\middle|\mathbf{X}\right)\\ &= \prod_{i=1}^{d}\mathbb{E}\left(\mathbf{1}_{\{\tau_i>v_i\}}\middle|\mathbf{X}\right)\end{aligned}$$

for all $\mathbf{v}\in\mathbb{R}_+^d$.

Hence, the conditional independence still holds and we can use Lemma 1 for the process $\mathbf{X}$. This provides

$$\bar{F}_\tau(\mathbf{v}) = e^{-H(\mathbf{v})}\mathbb{E}\left[e^{-\Sigma_{i=1}^{d}\int_0^{v_i}X_i(t)\,dt}\right].$$

Now, let us introduce the event $A_{\mathbf{v}}$ in which no fatal shock is induced in any component i during $[0,v_i]$, with $i=1,\ldots,d$. Note that if a fatal shock is induced in a component i for a given $T_n < v_i$, then $\int_0^{v_i}X_i(t)\,dt=\infty$, so that

$$\mathbb{E}\left[e^{-\Sigma_{i=1}^{d}\int_0^{v_i}X_i(t)dt}\mathbf{1}_{\bar{A}_{\mathbf{v}}}\right]=0.$$

Hence

$$\bar{F}_\tau(\mathbf{v}) = e^{-H(\mathbf{v})}\mathbb{E}\left[e^{-\Sigma_{i=1}^{d}\int_0^{v_i}X_i(t)dt}\mathbf{1}_{A_{\mathbf{v}}}\right]. \tag{39}$$

Our aim now is to compute this expression using an approximation procedure. To this end, for a given $M>0$, let us define an approximate process $(\mathbf{X}_t^M)_{t\geq 0}$ in which the measure ν_C is changed into

$$\nu_C^M(d\mathbf{u}) = \left(\prod_{i\in C}\delta_M(du_i)\right)\times\mu_{\bar{C}}(d\mathbf{u}_{\bar{C}}),$$

keeping the remaining elements. That is, each time a fatal shock is induced in the original process $\mathbf{X}$, its hazard rate is increased of an amount M in $\mathbf{X}^M$ (where M can be arbitrarily big).

Let $\bar{F}_\tau^M$ be the multivariate survival function of the approximate process. Our aim is to show that

$$\lim_{M\to\infty}\bar{F}_\tau^M = \bar{F}_\tau. \tag{40}$$

Now, for each $t>0$, let $R_t = t - T_{N_t}$ be the time elapsed at time t since the last arrival in the non-homogeneous Poisson process. Observe that on the set $\bar{A}_{\mathbf{v}}$ and provided that component i has a fatal shock before v_i, we have

$$\int_0^{v_i}X_i^M(t)\,dt \geq MR_{v_i}$$

and therefore

$$\lim_{M\to\infty} \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i^M(t)dt} \mathbf{1}_{\bar{A}_\mathbf{v}}\right] \leq \lim_{M\to\infty} \mathbb{E}\left[e^{-MR_{v_i}} \mathbf{1}_{\bar{A}_\mathbf{v}}\right] = 0.$$

Thus, taking into account the fact that on the set A we have $X_i^M(t) = X_i(t),\ 0 \leq t \leq v_i$, we now have

$$\lim_{M\to\infty} \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i^M(t)dt}\right] = \lim_{M\to\infty} \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i^M(t)dt} \mathbf{1}_{A_\mathbf{v}}\right] = \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i(t)dt} \mathbf{1}_{A_\mathbf{v}}\right].$$

Using Lemma 1 for the approximate process $\mathbf{X}^M$ and (39), we obtain that

$$\lim_{M\to\infty} \bar{F}_\tau^M(\mathbf{v}) = \lim_{M\to\infty} e^{-H(\mathbf{v})} \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i^M(t)dt}\right] = e^{-H(\mathbf{v})} \mathbb{E}\left[e^{-\Sigma_{i=1}^d \int_0^{v_i} X_i(t)dt} \mathbf{1}_{A_\mathbf{v}}\right] = \bar{F}_\tau(\mathbf{v}),$$

thus showing (40).

Now, applying Proposition 4 to the approximate process, we have

$$\bar{F}_\tau^M(\mathbf{v}) = e^{-H(\mathbf{v}) - K^M(\mathbf{v})}, \tag{41}$$

with

$$K^M(\mathbf{v}) = \int_F \left(1 - e^{-\Sigma_{i=1}^d (v_i - s)^+ x_i}\right) \lambda(s)\, ds\, Q_M(s, d\mathbf{x})$$

for all $\mathbf{v} = (v_1, \cdots, v_d) \in \mathbb{R}_+^d$ such that $v_0 = 0 < v_1 < \cdots < v_d$.

Noting that the integrand is null when $s > v_d$, we get

$$\begin{aligned} K^M(\mathbf{v}) &= \int_0^{v_d} \left(\int_{\mathbb{R}_+^d} Q_M(s, d\mathbf{x})\right) \lambda(s)\, ds - \int_0^{v_d} \left(\int_{\mathbb{R}_+^d} e^{-\Sigma_{i=1}^d (v_i - s)^+ x_i} Q_M(s, d\mathbf{x})\right) \lambda(s)\, ds \\ &= \Lambda(v_d) - \sum_{j=1}^d K_j^M(\mathbf{v}), \end{aligned} \tag{42}$$

where

$$\begin{aligned} K_j^M(\mathbf{v}) &= \int_{v_{j-1}}^{v_j} \left(\int_{\mathbb{R}_+^d} e^{-\Sigma_{i=1}^d (v_i - s)^+ x_i}\ Q_M(s, d\mathbf{x})\right) \lambda(s)\, ds \\ &= \int_{v_{j-1}}^{v_j} \left(\int_{\mathbb{R}_+^d} e^{-\Sigma_{i=j}^d (v_i - s) x_i}\ Q_M(s, d\mathbf{x})\right) \lambda(s)\, ds \\ &= \sum_{C \subset \{1,\cdots,d\}} \int_{v_{j-1}}^{v_j} \left(\int_{\mathbb{R}_+^d} e^{-\Sigma_{i=j}^d (v_i - s) x_i}\ \nu_C^M(d\mathbf{x})\right) a_C(s)\, \lambda(s)\, ds \end{aligned}$$

for $1 \leq j \leq d$.

In the case where $\{j, \cdots, d\} \cap C \neq \varnothing$, let $i_C \in \{j, \cdots, d\} \cap C$. Remembering that the $i-$th margin of ν_C^M is δ_M, we have:

$$\begin{aligned} &\lim_{M\to\infty} \sum_{\substack{C \subset \{1,\cdots,d\} \text{ such that} \\ \{j,\cdots,d\} \cap C \neq \varnothing}} \int_{v_{j-1}}^{v_j} \left(\int_{\mathbb{R}_+^d} e^{-\Sigma_{i=j}^d (v_i - s) x_i}\ \nu_C^M(d\mathbf{x})\right) a_C(s)\, \lambda(s)\, ds \\ &\leq \lim_{M\to\infty} \sum_{\substack{C \subset \{1,\cdots,d\} \text{ such that} \\ \{j,\cdots,d\} \cap C \neq \varnothing}} \int_{v_{j-1}}^{v_j} e^{-\left(v_{i_C} - s\right) M}\ a_C(s)\, \lambda(s)\, ds = 0. \end{aligned} \tag{43}$$

Hence, we only need to consider the case where $\{j,\cdots,d\}\cap C=\varnothing$. Note that in that case, components j, $j+1$, $\cdots$, d do not fail, so that the corresponding margin in ν_C^M coincides with the corresponding margin $\mu_{j\cdots d}\left(d\mathbf{x}_{j\cdots d}\right)$ in the original process. Hence

$$\int_{(x_1,\ldots x_{j-1})\in\mathbb{R}_+^{j-1}}\nu_C^M(d\mathbf{x})=\mu_{j\cdots d}\left(d\mathbf{x}_{j\cdots d}\right)$$

and

$$\begin{aligned}
&\sum_{\substack{C\subset\{1,\cdots,d\}\text{ such that}\\ \{j,\cdots,d\}\cap C=\varnothing}}\int_{v_{j-1}}^{v_j}\left(\int_{\mathbb{R}_+^{j-1}\times\mathbb{R}_+^{d-j+1}}e^{-\sum_{i=j}^d(v_i-s)x_i}\,\nu_C^M(d\mathbf{x})\right)\times a_C(s)\,\lambda(s)\,ds\\
&=\sum_{\substack{C\subset\{1,\cdots,d\}\text{ such that}\\ \{j,\cdots,d\}\cap C=\varnothing}}\int_{v_{j-1}}^{v_j}\left(\int_{\mathbb{R}_+^{d-j+1}}e^{-\sum_{i=j}^d(v_i-s)x_i}\,\mu_{j\cdots d}\left(d\mathbf{x}_{j\cdots d}\right)\right)a_C(s)\,\lambda(s)\,ds\\
&=\sum_{\substack{C\subset\{1,\cdots,d\}\text{ such that}\\ \{j,\cdots,d\}\cap C=\varnothing}}\int_{v_{j-1}}^{v_j}\mathscr{L}_{\mu_{j\cdots d}}(v_j-s,\cdots v_d-s)\,a_C(s)\,\lambda(s)\,ds. \qquad (44)
\end{aligned}$$

Taking into account (43) and (44), we conclude that

$$\lim_{M\to\infty}K_j^M(\mathbf{v})=\sum_{\substack{C\subset\{1,\cdots,d\}\text{ such that}\\ \{j,\cdots,d\}\cap C=\varnothing}}\int_{v_{j-1}}^{v_j}\mathscr{L}_{\mu_{j\cdots d}}(v_j-s,\cdots v_d-s)\,a_C(s)\,\lambda(s)\,ds,$$

which, together with (41), (42) and (40) ends this proof. □

Remark 10 The previous result is given in the case where $v_1<\cdots<v_d$ but all other cases can be easily derived by symmetry.

Example 4 (Case $d=2$) In the specific case where $d=2$, the previous corollary provides

$$\begin{aligned}
\bar{F}_\tau(\mathbf{v})=e^{-H(\mathbf{v})-\Lambda(v_2)}&\exp\left(\int_0^{v_1}\mathscr{L}_\mu(v_1-s,\cdots v_2-s)\,a_\varnothing(s)\,\lambda(s)\,ds\right)\\
&\times\exp\left(\int_{v_1}^{v_2}\mathscr{L}_{\mu_2}(v_2-s)\,(a_\varnothing(s)+a_1(s))\,\lambda(s)\,ds\right)
\end{aligned}$$

for all $\mathbf{v}=(v_1,v_2)\in\mathbb{R}_+^2$ such that $v_1<v_2$. This is the result of [23, Proposition 1], which already enlarged [6, Theorem 1]. Hence the previous corollary enlarges several results from the previous literature.

6 Concluding remarks and perspectives

We have here provided a new model for a multivariate lifetime, which encompasses many models from the previous literature and allows several types of dependence between components: simultaneous failures, simultaneous increase of the individual

failure rates of the components and some dependence between the simultaneous individual failure rates increments.

Even if the results have been provided separately for the different hazard rate increment processes envisioned in the paper (Lévy processes with extensions and shock processes), the results could easily be extended to the case of an additive process of the shape

$$\mathbf{Z}_t = \mathbf{X}_t + \mathbf{Y}_t, \forall t \geq 0,$$

where $(\mathbf{X}_t)_{t\geq 0}$ and $(\mathbf{Y}_t)_{t\geq 0}$ are independent additive processes studied in the paper, which allows to still enlarge the results.

In the same way, most results have been provided under the assumption that the additive process has bounded variations, but, as we have seen in the last model, it is possible to relax this assumption by means of an approximation procedure, which allows to enlarge the possible scope in another way.

Finally, note that for application purpose, statistical estimation procedures remain to be developed, which could be the study of a future work. Typically, such procedures would highly depend on the available data and specifically on the possibility to monitor the additive process or not, in addition to the observation of lifetimes.

As noted in the introduction, the model developped in the paper mostly leads to lifetimes with the Increasing Failure Rate (IFR) property, that is with a lasting effect of an adverse environment on the failure rate. This is for instance well adapted to the reliability field (and also to some biostatistics models). However, it could be questionable in other contexts. In that case, as suggested by a first referee, it could be interesting to consider the impact of the additive process on the cumulated hazard rate function instead of an impact on the hazard rate function. This would provide a very interesting alternate model, which could allow for non monotonic hazard rates. A second referee pointed out another possibility with a similar purpose by considering an affine Markov process as conditional hazard rate increment process. This would allow to enlarge the results of the paper as additive processes are specific affine Markov processes, please see [10, 12, 13] for more details on affine Markov processes.

These two pists provide very promising alternate models, which could allow for a non monotonic conditional hazard rate increment process. Their mathematical tractability remains to be studied.

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