Upper bounds on residues of Dedekind zeta functions of non-normal totally real cubic fields

Stéphane Louboutin

ACTA ARITHMETICA
Online First version

Upper bounds on residues of Dedekind zeta functions of non-normal totally real cubic fields

by

Stéphane R. Louboutin (Marseille)

1. Introduction. Various bounds on the absolute values of L-functions of number fields at $s = 1$ and on residues at $s = 1$ of Dedekind zeta functions of a number field $\mathbb{L}$ are known. Also, better bounds depending on the splitting behavior of given prime ideals of $\mathbb{L}$ of small norms are known. These bounds involve a term $\nu_{\mathbb{L}}$ in the series expansion at $s = 1$ of the Dedekind zeta function of $\mathbb{L}$. We explain why one should expect to have bounds on $\nu_{\mathbb{L}}$ which also depend on the splitting behavior in $\mathbb{L}$ of given prime integers. We explicitly do that for $\mathbb{L}$ a real quadratic number field. We deduce very good upper bounds on the residue at $s = 1$ of the Dedekind zeta function of a non-normal totally real cubic number field $\mathbb{K}$, bounds depending on the splitting behavior of the prime $p = 2$ in $\mathbb{K}$. We have almost reached in Theorem 1.7 an explicit bound

$$\begin{aligned}(1)\qquad \mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) &\le \frac{Q_2}{8}(\log d_{\mathbb{L}} + c_2)(\log(d_{\mathbb{K}}^2/d_{\mathbb{L}}) + c_2') \\ &\le \frac{Q_2}{8}(\log d_{\mathbb{K}} + (c_2 + c_2')/2)^2\end{aligned}$$

for the residue at $s = 1$ of the Dedekind zeta function of a non-normal totally real cubic number field $\mathbb{K}$ of discriminant of absolute value $d_{\mathbb{K}}$, where $d_{\mathbb{L}}$ is the absolute value of the discriminant of the real quadratic subfield $\mathbb{L} = \mathbb{Q}(\sqrt{d_{\mathbb{K}}})$ of the normal closure of $\mathbb{K}$. Here, $Q_2 \in \{1, 1/2, 1/3, 1/4, 1/7\}$ defined in (15) depends on the splitting behavior of the prime $p = 2$ in $\mathbb{K}$ and c_2 and c_2' would also depend explicitly on the splitting behavior of the prime $p = 2$ in $\mathbb{K}$. Unfortunately, the sizes of the error terms in Theorem 1.1

2020 *Mathematics Subject Classification*: Primary 11R42; Secondary 11M20, 11R11, 11R16.
Key words and phrases: L-function, zeta function, quadratic and cubic number fields, Euler–Kronecker constants.
Received 6 April 2020; revised 2 September 2020.
Published online *.

and Proposition 2.1 prevent us from getting such a simple bound and we only get the bound

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \frac{Q_2+o(1)}{8}(\log d_{\mathbb{L}})(\log(d_{\mathbb{K}}^2/d_{\mathbb{L}})) \le \frac{Q_2+o(1)}{8}(\log d_{\mathbb{K}})^2$$

as $d_{\mathbb{L}}$ tends to infinity and $d_{\mathbb{K}}/d_{\mathbb{L}} \ge \log^2 d_{\mathbb{K}}$. We hope somebody could deal nicely with these error terms or could find a completely different approach to get an explicit bound as close as possible to (1). Combined with the results obtained in [Lou15], this would have great consequences on the solution of the class number one problem for the non-normal sextic CM-fields (see [Lou08, Remark 10]).

1.1. Notation. Let us set the notation we will be using throughout the paper.

Let $\mathbb{K}$ be a number field, of degree $m = r_1+2r_2$, absolute discriminant $d_{\mathbb{K}}$ and Dedekind zeta function $\zeta_{\mathbb{K}}(s)$. Set

$$Z_{\mathbb{K}}(s) = \left(\frac{d_{\mathbb{K}}}{4^{r_2}\pi^m}\right)^{s/2} \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)\zeta_{\mathbb{K}}(s).$$

In particular, $Z_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $Z_{\mathbb{K}}(s)$ is a meromorphic function satisfying the functional equation $Z_{\mathbb{K}}(1-s) = Z_{\mathbb{K}}(s)$, with only two poles, at $s=0$ and $s=1$, both simple. We define (see [Lou98b, (22)] for the positivity)

$$\begin{aligned}
&0 < \lambda_{\mathbb{K}} := \mathrm{Res}_{s=1}(Z_{\mathbb{K}}(s)) = \frac{\sqrt{d_{\mathbb{K}}}}{(2\pi)^{r_2}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)),\\
&0 < \mu_{\mathbb{K}} := \lim_{s\to 1}\left\{\frac{1}{\lambda_{\mathbb{K}}}Z_{\mathbb{K}}(s) - \frac{1}{s(s-1)}\right\},\\
&0 < \nu_{\mathbb{K}} := \mu_{\mathbb{K}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)).
\end{aligned}$$

In particular, $\lambda_{\mathbb{Q}} = 1$ and

$$\mu_{\mathbb{Q}} = (2+\gamma-\log(4\pi))/2 = 0.02309\ldots. \tag{2}$$

Whereas we have the positivity of $\lambda_{\mathbb{K}}$, $\mu_{\mathbb{K}}$, $\nu_{\mathbb{K}}$ and explicit bounds (see [Lou01, Theorem 1])

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \left(\frac{e\log d_{\mathbb{K}}}{2(m-1)}\right)^{m-1} \quad\text{and}\quad \nu_{\mathbb{K}} \le \left(\frac{e\log d_{\mathbb{K}}}{2m}\right)^m, \tag{3}$$

obtaining explicit bounds on $\mu_{\mathbb{K}}$ is not easy. In fact, recalling that the *Euler–Kronecker constant* $\gamma_{\mathbb{K}}$ of $\mathbb{K}$ is defined by

$$\zeta_{\mathbb{K}}(s) = \mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s))\left(\frac{1}{s-1} + \gamma_{\mathbb{K}} + O(s-1)\right),$$

we have

$$\mu_{\mathbb{K}} = \gamma_{\mathbb{K}} + 1 + \tfrac{1}{2}(\log d_{\mathbb{K}} - m(\gamma + \log(4\pi)) + r_2 \log 4)$$

and only conditional upper bounds (under the assumption of the Generalized Riemann Hypothesis) for $\gamma_{\mathbb{K}}$ are known:

$$\gamma_{\mathbb{K}} \ll \log\log d_{\mathbb{K}},$$

where the implied constants are absolute (see [IM, Theorem 3 and Corollary 3.3.2]).

1.2. Bounds on $L(1,\chi)$. In [Lou93] (see also [Ram01], [Edd] and [PE]), we obtained the explicit bound

$$|L(1,\chi)| \le \tfrac{1}{2}\log f_\chi + \mu_{\mathbb{Q}} \tag{4}$$

for the value at $s=1$ of the L-function associated with a primitive and even Dirichlet character χ, of conductor $f_\chi > 1$ (see also [GS] and [Pin] for better non-explicit bounds).

This was considerably generalized in [Lou04, Theorem 1] and [Ram04, Corollary 1], leading to better explicit bounds taking into account the behavior of χ at given prime integers (see also [Toy] for better non-explicit bounds). Our present approach based on Lemma 3.1 will give a simpler proof of these latter bounds (see the first item of Theorem 1.3).

In [Lou00, Theorem 7], we generalized (4) to the case of L-functions associated with primitive characters on ray class groups of a given number field $\mathbb{L}$. Focussing on characters unramified at all the infinite places of $\mathbb{L}$, we obtained

$$\begin{aligned} |L(1,\chi)| &\le \mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \times \big(\tfrac{1}{2}\log f_\chi + \mu_{\mathbb{L}}\big) + o(1) \\ &= \tfrac{1}{2}\,\mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s))\log f_\chi + \nu_{\mathbb{L}} + o(1), \end{aligned} \tag{5}$$

where $f_\chi = N(\mathcal{F}_\chi)$ is the norm of the finite part $\mathcal{F}_\chi$ of the conductor of χ. It took us some time to generalize (5), i.e. to obtain better bounds taking into account the behavior of χ at given prime ideals of $\mathbb{L}$. In 2018, we eventually obtained a generalization of [Lou04, Theorem 1] and [Ram04, Corollary 1] (see Proposition 2.1 below for the size of the present error terms):

THEOREM 1.1 (see [Lou18, Theorem 2]). *Let P be a given finite set of prime ideals of a given number field $\mathbb{L}$ of degree m. Let $N(\mathcal{P}) = N_{\mathbb{L}/\mathbb{Q}}(\mathcal{P})$ be the norm of $\mathcal{P} \in P$. Let χ range over the non-trivial and primitive characters on ray class groups for $\mathbb{L}$ which are unramified at all the infinite real places of $\mathbb{L}$. Set $P_1(\chi) = \{\mathcal{P} \in P;\ \mathcal{P} \nmid \mathcal{F}_\chi\}$. Then*

$$\begin{aligned} |L(1,\chi)| \le B_P(\chi) \times \mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \times \big(\tfrac{1}{2}\log f_\chi + \mu_{\mathbb{L}} + S_P(\chi)\big) \\ + O\bigg(\frac{(\log(ed_{\mathbb{L}}))^{m-1}(\log(ed_{\mathbb{L}} f_\chi))}{\sqrt{f_\chi}}\bigg), \end{aligned}$$

where the implied constant depends only on m and the norms of the prime ideals in P and where

$$0 \le B_P(\chi) = \prod_{\mathcal{P}\in P} \left| \frac{N(\mathcal{P})-1}{N(\mathcal{P})-\chi(\mathcal{P})} \right| \le 1,$$

and

$$0 \le S_P(\chi) = \log 2^{|P_1(\chi)|} + \sum_{\mathcal{P}\in P} \frac{\log N(\mathcal{P})}{N(\mathcal{P})-1}.$$

Hence, by (3)*, whenever $f_\chi \ge (\log(ed_{\mathbb{L}}))^2$ we have*

$$|L(1,\chi)| \le B_P(\chi) \times \left(\tfrac{1}{2}\operatorname{Res}_{s=1}(\zeta_{\mathbb{L}}(s))\log f_\chi + \nu_{\mathbb{L}}\right) + O(\log^{m-1}(ed_{\mathbb{L}})). \tag{6}$$

1.3. Main results: bounds on $\nu_{\mathbb{L}}$ for real quadratic number fields and on $\operatorname{Res}_{s=1}(\zeta_{\mathbb{K}}(s))$ for non-normal totally real cubic number fields. Let $\mathbb{K}$ be a non-normal totally real cubic number field. The best explicit known bound is

$$\operatorname{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \tfrac{1}{8}(\log d_{\mathbb{K}} + \lambda_3)^2, \tag{7}$$

where $\lambda_3 := 2 + 2\gamma - 2\log(4\pi) = -1.90761\ldots$ (see [Lou11], where it is said to hold true for $d_{\mathbb{K}} \ge 146$, but according to [Coh, Appendix B4] there are only two totally real cubic number fields of discriminant less than 146, both of them being abelian). In [Lou08, Theorem 8] we obtained a better bound depending on the behavior of the prime 2 in $\mathbb{K}$:

$$\operatorname{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \frac{Q_2}{8}(\log d_{\mathbb{K}} + 5)^2, \tag{8}$$

where $Q_2 \in \{1, 1/2, 1/3, 1/4, 1/7\}$ defined in (15) below depends explicitly on the splitting behavior of the prime $p = 2$ in $\mathbb{K}$. In Theorem 1.7, we will obtain still better bounds. Let $\mathbb{N}$ be the normal closure of $\mathbb{K}$. Hence $\mathbb{N}$ is a totally real sextic field with Galois group isomorphic to the symmetric group of order 6. Let $\mathbb{L}$ be the unique (real) quadratic subfield of $\mathbb{N}$. The cubic extension $\mathbb{N}/\mathbb{L}$ is cyclic. Let $\chi_{\mathbb{N}/\mathbb{L}}$ denote any of the two complex conjugate cubic characters associated with $\mathbb{N}/\mathbb{L}$. Let $f_{\mathbb{N}/\mathbb{L}} = N_{\mathbb{L}/\mathbb{Q}}(\mathcal{F}_{\mathbb{N}/\mathbb{L}})$ be the norm of the conductor $\mathcal{F}_{\mathbb{N}/\mathbb{L}}$ of $\chi_{\mathbb{N}/\mathbb{L}}$.

We have

$$(\zeta_{\mathbb{K}}(s)/\zeta(s))^2 = \zeta_{\mathbb{N}}(s)/\zeta_{\mathbb{L}}(s) = L(s,\chi_{\mathbb{N}/\mathbb{L}})L(s,\bar{\chi}_{\mathbb{N}/\mathbb{L}}), \tag{9}$$

which gives $d_{\mathbb{K}}^2 = d_{\mathbb{N}}/d_{\mathbb{L}} = d_{\mathbb{L}}^2 f_{\mathbb{N}/\mathbb{L}}^2$, i.e.

$$f_{\mathbb{N}/\mathbb{L}} = d_{\mathbb{K}}/d_{\mathbb{L}} \quad \text{and} \quad \operatorname{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) = |L(1,\chi_{\mathbb{N}/\mathbb{L}})|. \tag{10}$$

Now, let $\mathbb{L}$ be a real quadratic number field, hence associated with a primitive even Dirichlet character $\chi_{\mathbb{L}}$ of conductor $d_{\mathbb{L}}$. By (4), we have

$$\operatorname{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) = L(1,\chi_{\mathbb{L}}) \le \tfrac{1}{2}\log d_{\mathbb{L}} + \mu_{\mathbb{Q}}. \tag{11}$$

While we have no explicit upper bounds on $\mu_{\mathbb{L}}$ in (5), we do have one on

$$\nu_{\mathbb{L}} := \mu_{\mathbb{L}} \operatorname{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \le \tfrac{1}{8}\log^2 d_{\mathbb{L}}, \tag{12}$$

by [Lou98a, Proposition 6] or [Lou01, Theorem 2], a better bound than the one in (3) for $m = 2$.

REMARK 1.2. Let $\mathbb{L}$ be a real quadratic number field of discriminant $d_{\mathbb{L}}$. Then

$$\begin{aligned}\nu_{\mathbb{L}} &= L(1,\chi_{\mathbb{L}})\left(\mu_{\mathbb{Q}} + \frac{\Lambda'(1,\chi_{\mathbb{L}})}{\Lambda(1,\chi_{\mathbb{L}})}\right)\\ &= L(1,\chi_{\mathbb{L}})\big(\tfrac{1}{2}\log d_{\mathbb{L}} + 1 - \log(4\pi)\big) + L'(1,\chi_{\mathbb{L}}),\end{aligned}$$

by (25). Starting from this formula it would be difficult to end up with the factor $1/8$ in (12), as by (11) it would be difficult to have anything better than a factor $1/4$.

Using (10), (5), (11), and (12) we obtain a bound on $\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s))$ which is better than the one in (7) if $d_{\mathbb{K}} \ge d_{\mathbb{L}}^2$ [Lou18, p. 255]. In this paper we improve upon (12) by taking into account the behavior of small prime integers in $\mathbb{L}$: see (14) and Corollary 1.5. Then, using (10), (5), (13) for $\chi = \chi_{\mathbb{L}}$, and (14) we obtain a bound on $\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s))$ which is better than the one in (8) if $d_{\mathbb{K}} \ge d_{\mathbb{L}}\log^2 d_{\mathbb{L}}$—see Theorem 1.7.

In other words, the aim of this paper is to prove Theorem 1.3 below. Not only does our present approach based on Lemma 3.1 give a simpler proof of (13) than the ones in [Lou04] and [Ram04]; it also enables us to get (13) and (14) in one stroke.

THEOREM 1.3. *Let $d_1, d_2 \ge 1$ be coprime square-free positive integers.*

(i) *Let χ range over the primitive and even Dirichlet characters of conductor $f_\chi > 1$ for which $\gcd(d_1, f_\chi) = 1$ and $d_2 \mid f_\chi$. Setting $c_1(d_1,d_2) = \mu_{\mathbb{Q}} + \omega(d_1)\log 2 + \sum_{p\mid d_1 d_2}\frac{\log p}{p-1}$, we have an effective bound with a negative error term for $\omega(d_2) \le 1$ and f_χ large enough:*

$$|L(1,\chi)| \le \frac{1}{2}\left\{\prod_{p\mid d_1 d_2}\left|\frac{p-1}{p-\chi(p)}\right|\right\} \times \{\log f_\chi + 2c_1(d_1,d_2)\} + O\left(\frac{\log f_\chi}{\sqrt{f_\chi}}\right). \tag{13}$$

(ii) *Let $\mathbb{L}$ range over the real quadratic fields for which $p \mid d_1 \Rightarrow \chi_{\mathbb{L}}(p) = -1$ and $p \mid d_2 \Rightarrow \chi_{\mathbb{L}}(p) = 0$. For $c_2(d_1,d_2) = 2\mu_{\mathbb{Q}} - 1 + \sum_{p\mid d_1}\frac{2p\log p}{p^2-1} + \sum_{p\mid d_2}\frac{\log p}{p-1}$ and $c_3(d_1,d_2)$ some explicit constant depending only on d_1 and d_2 given in Section 3.1, we have an effective bound with a negative error term for $\omega(d_2) \le 2$ and $d_{\mathbb{L}}$ large enough:*

$$\nu_{\mathbb{L}} \le \frac{1}{8}\Big\{\prod_{p\mid d_1d_2}\frac{p-1}{p-\chi_{\mathbb{L}}(p)}\Big\}\times\{(\log d_{\mathbb{L}}+2c_2(d_1,d_2))^2+4c_3(d_1,d_2)\}+O\Big(\frac{\log^2 d_{\mathbb{L}}}{\sqrt{d_{\mathbb{L}}}}\Big). \tag{14}$$

From the effectiveness of these bounds we obtain

COROLLARY 1.4 (compare with (4)). *For any real quadratic field* $\mathbb{L}$ *we have*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \le \begin{cases} (\log d_{\mathbb{L}}+2+\gamma-\log(4\pi))/2 & \text{in all cases,}\\ (\log d_{\mathbb{L}}+2+\gamma-\log\pi)/4 & \text{if 2 is ramified in } \mathbb{L},\\ (\log d_{\mathbb{L}}+2+\gamma-\log(\pi/4))/6 & \text{if 2 is inert in } \mathbb{L}.\end{cases}$$

COROLLARY 1.5 (compare with (12)). *Set* $\kappa = 4\mu_{\mathbb{Q}} - 2 + (8\log 2)/3 = -0.05922\ldots$ *and* $\kappa' = 4\kappa_2 = 19.52508$ *as defined in* (39). *For any real quadratic field* $\mathbb{L}$ *we have*

$$\nu_{\mathbb{L}} \le \begin{cases} (\log d_{\mathbb{L}})^2/8 & \text{in all cases,}\\ (\log d_{\mathbb{L}})^2/16 & \text{if 2 is ramified in } \mathbb{L},\\ ((\log d_{\mathbb{L}}+\kappa)^2+\kappa')/24 & \text{if 2 is inert in } \mathbb{L},\\ (\log d_{\mathbb{L}})^2/24 & \text{if 2 is inert in } \mathbb{L} \text{ and } d_{\mathbb{L}} \ge 5\cdot 10^{71}.\end{cases}$$

REMARK 1.6. Numerical computations of $\nu_{\mathbb{L}}$ based on [Lou98b, Section 4.2] suggest that the fourth bound (which follows from the third one) holds true whenever 2 is inert in $\mathbb{L}$ and $d_{\mathbb{L}} \ge 5$. Indeed, with a $24h30mn$ long high precision computation on a 3GHz PC using UBASIC, we checked that $\max\{24\nu_{\mathbb{L}}/\log^2 d_{\mathbb{L}};\ d_{\mathbb{L}} \equiv 5 \pmod 8 \text{ and } d_{\mathbb{L}} \le 10^5\} = 0.9055\ldots$ for $d_{\mathbb{L}} = 62269$. Notice that $\chi_{\mathbb{L}}(p) = 1$ for $p \in \{3,5,7,11,13,17,19,23,29\}$ and $d_{\mathbb{L}} = 62269$.

Let $p \ge 2$ be a given prime. Take P equal to the set of prime ideals of $\mathbb{L}$ dividing p. To simplify the presentation, we assume that $d_{\mathbb{L}}$ goes to infinity and that $f_{\mathbb{N}/\mathbb{L}} = d_{\mathbb{K}}/d_{\mathbb{L}} \gg \log^2 d_{\mathbb{L}}$, to be allowed to use (6). By (10), (6) and Theorem 1.3, we have

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le B_P(\chi_{\mathbb{N}/\mathbb{L}}) \times \big\{\tfrac{1}{2}\,\mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s))\log(d_{\mathbb{K}}/d_{\mathbb{L}}) + \nu_{\mathbb{L}}\big\} + O(\log d_{\mathbb{L}})$$

with

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \le B_{\{p\}}(\chi_{\mathbb{L}}) \times \frac{1+o(1)}{2}\log d_{\mathbb{L}}$$

and

$$\nu_{\mathbb{L}} \le B_{\{p\}}(\chi_{\mathbb{L}}) \times \frac{1+o(1)}{8}\log^2 d_{\mathbb{L}}.$$

Hence,

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \frac{Q_p+o(1)}{8}(\log d_{\mathbb{L}})(\log(d_{\mathbb{K}}^2/d_{\mathbb{L}})) + O(\log d_{\mathbb{L}}),$$

where

$$Q_p := B_{\{p\}}(\chi_{\mathbb{L}}) \times B_{\{\mathcal{P};\,\mathcal{P}|(p)\text{ in }\mathbb{L}\}}(\chi_{\mathbb{N}/\mathbb{L}}).$$

From (9) and the uniqueness of the Euler product of Dirichlet series, we have

$$\left(\left(1-\frac{1}{p^s}\right)\prod_{\mathfrak{P}|(p)\text{ in }\mathbb{K}}\left(1-\frac{1}{N_{\mathbb{K}/\mathbb{Q}}(\mathfrak{P})^s}\right)^{-1}\right)^2 = \prod_{\mathcal{P}|(p)\text{ in }\mathbb{L}}\left(1-\frac{\chi_{\mathbb{N}/\mathbb{L}}(\mathcal{P})}{N_{\mathbb{L}/\mathbb{Q}}(\mathcal{P})}\right)^{-1}\left(1-\frac{\bar{\chi}_{\mathbb{N}/\mathbb{L}}(\mathcal{P})}{N_{\mathbb{L}/\mathbb{Q}}(\mathcal{P})}\right)^{-1}.$$

Hence,

$$Q_p = \frac{\left(1-\frac{1}{p}\right)^3}{\prod_{\mathfrak{P}|(p)\text{ in }\mathbb{K}}\left(1-\frac{1}{N_{\mathbb{K}/\mathbb{Q}}(\mathfrak{P})}\right)} = \begin{cases} 1 & \text{if } (p)=\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3 \text{ splits in } \mathbb{K},\\ \frac{p-1}{p} & \text{if } (p)=\mathfrak{P}_1\mathfrak{P}_2^2 \text{ is partially ramified in } \mathbb{K},\\ \frac{p-1}{p+1} & \text{if } (p)=\mathfrak{P}_1\mathfrak{P}_2 \text{ in } \mathbb{K},\\ \frac{(p-1)^2}{p^2} & \text{if } (p)=\mathfrak{P}^3 \text{ is totally ramified in } \mathbb{K},\\ \frac{(p-1)^2}{p^2+p+1} & \text{if } (p)=\mathfrak{P} \text{ is inert in } \mathbb{K}. \end{cases} \tag{15}$$

Putting everything together and using $XY \le ((X+Y)/2)^2$ for $X, Y \ge 0$, we obtain

THEOREM 1.7 (compare with (7) and [Lou08, Theorem 8]). *Let $p \ge 2$ be a given prime integer. Let $\mathbb{L}$ be the real quadratic subfield of the normal closure $\mathbb{N}$ of a non-normal totally real cubic number field $\mathbb{K}$. Let Q_p be as in* (15)*. Then*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) \le \frac{Q_p+o(1)}{8}(\log d_{\mathbb{L}})(\log(d_{\mathbb{K}}^2/d_{\mathbb{L}})) \le \frac{Q_p+o(1)}{8}\log^2 d_{\mathbb{K}},$$

where the effective error terms tend to 0 as $d_{\mathbb{L}} \to \infty$, and $d_{\mathbb{K}}/d_{\mathbb{L}} \ge \log^2 d_{\mathbb{L}}$.

2. Proof of the size of the error term in Theorem 1.1

PROPOSITION 2.1 (see [Lou18, Proposition 9]). *Let $\mathbb{L}$ be a totally real number field of degree m. Let P_1 and P_2 be two disjoint finite sets of prime ideals of $\mathbb{L}$. For $a > 0$ and $X > 0$, set*

$$\Pi_{\mathbb{L}}(s) = \left\{\frac{a^{1-s}}{s-1}\prod_{\mathcal{P}\in P_1}\left(1-\frac{1}{N(\mathcal{P})^s}\right)+\frac{a^s}{s}\prod_{\mathcal{P}\in P_1}\left(1-\frac{2}{N(\mathcal{P})}+\frac{1}{N(\mathcal{P})^{1-s}}\right)\right\} \times\left\{\prod_{\mathcal{P}\in P_2}\left(1-\frac{1}{N(\mathcal{P})^s}\right)\right\}$$

and

$$F_{\mathbb{L}}(s,X)=\mathit{\Pi}_{\mathbb{L}}(s)X^{s-1}Z_{\mathbb{L}}(s)/\sqrt{d_{\mathbb{L}}},$$

a meromorphic functions with at most two poles, a double one at $s=1$ and an at most double one at $s=0$. Define

$$\begin{aligned}(16)\qquad I_{\mathbb{L}}(X)&:=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}F_{\mathbb{L}}(s,X)\,\mathrm{d}s\\&=\mathrm{Res}_{s=1}(F_{\mathbb{L}}(s,X))+\mathrm{Res}_{s=0}(F_{\mathbb{L}}(s,X))+R_{\mathbb{L}}(X)\end{aligned}$$

(which does not depend on $\alpha>1$), where

$$\begin{aligned}(17)\qquad R_{\mathbb{L}}(X)&:=\frac{1}{2\pi i}\int\limits_{\Re(s)=1-\alpha}F_{\mathbb{L}}(s,X)\,\mathrm{d}s=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}F_{\mathbb{L}}(1-s,X)\,\mathrm{d}s\\&=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\frac{\mathit{\Pi}_{\mathbb{L}}(1-s)Z_{\mathbb{L}}(s)}{X^s\sqrt{d_{\mathbb{L}}}}\,\mathrm{d}s=O\biggl(\frac{\log^m(ed_{\mathbb{L}})}{X}\biggr).\end{aligned}$$

Let χ be a non-trivial, primitive and character on a ray class group for $\mathbb{L}$, unramified at all the infinite places of $\mathbb{K}$. Assume that $\mathcal{P}\in P_1$ implies $\mathcal{P}\nmid\mathcal{F}_\chi$ and $\mathcal{P}\in P_2$ implies $\mathcal{P}\,|\,\mathcal{F}_\chi$. Then

$$\begin{aligned}(18)\qquad&\biggl|\biggl\{\prod_{\mathcal{P}\in P}\biggl(1-\frac{\chi(\mathcal{P})}{N(\mathcal{P})}\biggr)\biggr\}L(1,\chi)\biggr|\\&\qquad\le I_{\mathbb{L}}(\sqrt{f_\chi})=\mathrm{Res}_{s=1}(F_{\mathbb{L}}(s,X))+O\biggl(\frac{\log^m(ed_{\mathbb{L}})}{X}\biggr).\end{aligned}$$

The implied constants in (17)–(18) *depend only on a, m and the norms of the prime ideals in P.*

Proof. Only the sizes of the error terms in (17) and (18) require proof. For $1<\sigma=\Re(s)=1+\epsilon\le 4$ we have $|\zeta_{\mathbb{L}}(s)|\le\zeta_{\mathbb{L}}(\sigma)\le\zeta(\sigma)^m$,

$$\begin{aligned}\zeta(\sigma)&=\frac{1}{\sigma-1}+\sum_{n\ge1}\biggl(\frac{1}{n^\sigma}-\int\limits_n^{n+1}\frac{\mathrm{d}x}{x^\sigma}\biggr)\\&\le\frac{1}{\sigma-1}+\sum_{n\ge1}\biggl(\frac{1}{n^\sigma}-\frac{1}{(n+1)^\sigma}\biggr)=\frac{1}{\sigma-1}+1=\frac{\sigma}{\sigma-1}\le\frac{4}{\epsilon}\end{aligned}$$

and $|Z_{\mathbb{L}}(s)/(X^s\sqrt{d_{\mathbb{L}}})|\le\epsilon^{-m}d_{\mathbb{L}}^{\epsilon/2}|\Gamma(\sigma/2)|^m(4/\sqrt{\pi})^m/X$. By [Lou00, Lemma 10], taking $\epsilon=2m/\log(ed_{\mathbb{L}})$ we have $\epsilon\le3$ and (17) follows.

The last assertion follows from (16), (17) and the formula (see [Lou18, Lemma 10])

$$\mathrm{Res}_{s=0}(F_{\mathbb{L}}(s,X)) = -\frac{\mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s))}{X} \times \left\{\prod_{\mathcal{P}\in P_1}\left(1-\frac{1}{N(\mathcal{P})}\right)\right\}$$
$$\times \begin{cases} \log X - \mu_{\mathbb{L}} + 1 + \log a - a & \text{if } P_2 = \emptyset \text{ and } P_1 = \emptyset, \\ \log X - \mu_{\mathbb{L}} + 1 + \log a + \sum_{\mathcal{P}\in P} \frac{\log N(\mathcal{P})}{N(\mathcal{P})-1} & \text{if } P_2 = \emptyset \text{ and } P_1 \neq \emptyset, \\ \log N(\mathcal{P}_0) & \text{if } P_2 = \{\mathcal{P}_0\}, \\ 0 & \text{otherwise.} \end{cases}$$

which yields $\mathrm{Res}_{s=0}(F_{\mathbb{L}}(s,X)) \ll (\log^{m-1}(ed_{\mathbb{L}}))(\log(ed_{\mathbb{L}}X))/X$, by (3). Theorem 1.1 follows from the last assertion and the fact that

$$\mathrm{Res}_{s=1}(F_{\mathbb{L}}(s,X)) = \left\{\prod_{\mathcal{P}\in P}\left(1-\frac{1}{N(\mathcal{P})}\right)\right\}$$
$$\times \mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \times \left\{\log X + \mu_{\mathbb{L}} - 1 - \log a + 2^{|P_1|}a + \sum_{\mathcal{P}\in P}\frac{\log N(\mathcal{P})}{N(\mathcal{P})-1}\right\}$$

is minimal for $a = 1/2^{|P_1|}$. ∎

3. Proof of Theorem 1.3. In this section we use Lemma 3.1 below to prove Theorem 1.3.

Notice that the present proof of (13) drastically shortens the one of [Lou04, Theorem 1] and that the use of Lemma 3.1 would also drastically shorten the proof of [Lou18, Theorem 2].

First, in (24) and (25) we give integral representations on the vertical line $\Re(s) = \alpha > 1$ of $L(1,\chi)$ and $\nu_{\mathbb{L}}$, respectively. Second, in Lemma 3.2 we prove the positivity of some inverse Mellin transforms appearing in these integral representations. Third, in (29) we deduce bounds for $|L(1,\chi)|$ and $\nu_{\mathbb{L}}$ as inverse Mellin transforms of some $G_1(s) = \Pi_1(s)Z_{\mathbb{Q}}(s)$ and $G_2(s) = \Pi_2(s)Z_{\mathbb{Q}}(s)$ for some explicit factors $\Pi_1(s)$ and $\Pi_2(s)$ depending on both d_1 and d_2. Fourth, in (32) we reduce the proof of Theorem 1.3 to the computation of $\mathrm{Res}_{s=1}(G_1(s))$ and $\mathrm{Res}_{s=1}(G_2(s))$. Finally, in (33) and (34) we compute these residues.

By Stirling's formula, in any given vertical strip we have

$$\Gamma(\sigma+it) = O(e^{-\pi|t|/2}|t|^{\sigma-1/2}) \quad \text{for } \alpha_1 \le \sigma \le \alpha_2 \text{ and } |t| \ge 1. \tag{19}$$

Let χ be a primitive and even Dirichlet character of conductor $f > 1$. Then

$$\Lambda(s,\chi) = (\pi/f)^{-s/2}\Gamma(s/2)L(s,\chi)$$

is an entire function of order one satisfying

$$\Lambda(\sigma+it,\chi) = O(e^{-\pi|t|/4}|t|^{M}) \quad \text{for } \alpha_1 \le \sigma \le \alpha_2 \text{ and } |t| \ge 1, \tag{20}$$

for some $M = M(\alpha_1,\alpha_2)$ (e.g. see [Dav, Chapter 12] and [Lou01, Sec-

tion 4.1]). Recall also the functional equation

$$\Lambda(1-s,\chi) = W_\chi \Lambda(s,\bar{\chi}) \tag{21}$$

for some complex root number W_χ of absolute value equal to one (see e.g. [Dav, Chapter 9]). The bounds (19) and (20) show that all the integrals on vertical lines of the complex plane below are absolutely convergent and that we are allowed to move these lines to the left, provided that we take into account the residues at the poles encountered.

LEMMA 3.1. *Let $d_1 \geq 1$ be a square-free integer. Set*

$$P(s,\chi,d_1) = \prod_{p|d_1}\left(1-\frac{\chi(p)}{p^s}\right).$$

Let χ be a primitive and even Dirichlet character of conductor $f>1$. Assume that $\gcd(d_1,f)=1$. Set

$$L(s,\chi,d_1) := \sum_{\substack{n\geq 1\\ \gcd(n,d_1)=1}} \frac{\chi(n)}{n^s} = P(s,\chi,d_1)L(s,\chi)$$

(the L-function associated with the character modulo $d_1 f$ induced by χ) and

$$\Lambda(s,\chi,d_1) = (\pi/f)^{-s/2}\Gamma(s/2)L(s,\chi,d_1) = P(s,\chi,d_1)\Lambda(s,\chi).$$

For $a>0$ and $\alpha>1$, we have

$$\Lambda(1,\chi,d_1) = \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\left\{\frac{\Lambda(s,\chi,d_1)a^{1-s}}{s-1}+\frac{\Lambda(1-s,\chi,d_1)a^s}{s}\right\}\mathrm{d}s, \tag{22}$$

$$\Lambda'(1,\chi,d_1) = \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\left\{\frac{\Lambda(s,\chi,d_1)}{(s-1)^2}-\frac{\Lambda(1-s,\chi,d_1)}{s^2}\right\}\mathrm{d}s, \tag{23}$$

where these integrals are absolutely convergent and do not depend on α, by (20). Moreover,

$$\left\{\prod_{p|d_1}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi) = \Lambda(1,\chi,d_1)/\sqrt{f} \tag{24}$$

and for a real quadratic number field $\mathbb{L}$ we have

$$\left\{\prod_{p|d_1}\left(1-\frac{\chi_{\mathbb{L}}(p)}{p}\right)\right\}\nu_{\mathbb{L}} = \frac{\left(\mu_{\mathbb{Q}}-\sum_{p|d_1}\frac{\chi_{\mathbb{L}}(p)\log p}{p-\chi_{\mathbb{L}}(p)}\right)\Lambda(1,\chi_{\mathbb{L}},d_1)+\Lambda'(1,\chi_{\mathbb{L}},d_1)}{\sqrt{d_{\mathbb{L}}}}, \tag{25}$$

where $\chi_{\mathbb{L}}$ is the real primitive quadratic Dirichlet character of conductor $d_{\mathbb{L}}$ associated with $\mathbb{L}$.

Proof. Set

$$G(s) = \frac{\Lambda(s,\chi,d_1)a^{1-s}}{s-1} + \frac{\Lambda(1-s,\chi,d_1)a^s}{s}.$$

The bound (20) allows us to move the vertical line of integration $\Re(s)=\alpha>1$ to the left to $\Re(s)=1-\alpha<0$. Noticing that we pick up residues at $s=1$ and $s=0$, both simple and of residue $\Lambda(1,d_1,\chi)$, we obtain

$$\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} G(s)\,\mathrm{d}s = \mathrm{Res}_{s=1}(G(s)) + \mathrm{Res}_{s=0}(G(s)) + \frac{1}{2\pi i}\int\limits_{\Re(s)=1-\alpha} G(s)\,\mathrm{d}s$$
$$= 2\Lambda(1,d_1,\chi) + \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} G(1-s)\,\mathrm{d}s = 2\Lambda(1,d_1,\chi) - \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} G(s)\,\mathrm{d}s,$$

by making the change of variables $s\mapsto 1-s$ to come back to the line $\Re(s)=\alpha$ and then using $G(1-s)=-G(s)$. Identity (22) follows. In the same way, we have

$$\Lambda'(1,\chi,d_1) - (\log a)\Lambda(1,\chi,d_1)$$
$$= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\left\{\frac{\Lambda(s,\chi,d_1)a^{1-s}}{(s-1)^2} - \frac{\Lambda(1-s,\chi,d_1)a^s}{s^2}\right\}\mathrm{d}s,$$

which for $a=1$ gives (23).

Now, let $\mathbb{L}$ be a real quadratic number field . Then $W_{\mathbb{L}}=1$, $\Lambda(1-s,\chi_{\mathbb{L}})$ $=\Lambda(s,\chi_{\mathbb{L}})$,

$$\Lambda(s,\chi_{\mathbb{L}}) = Z_{\mathbb{L}}(s)/Z_{\mathbb{Q}}(s) = \frac{\lambda_{\mathbb{L}}}{\lambda_{\mathbb{Q}}}(1+(\mu_{\mathbb{L}}-\mu_{\mathbb{Q}})(s-1)+O((s-1)^2)), \tag{26}$$

and

$$\Lambda(s,d_1,\chi_{\mathbb{L}}) = P(s,\chi,d_1)\Lambda(s,\chi_{\mathbb{L}})$$
$$= \frac{\lambda_{\mathbb{L}}}{\lambda_{\mathbb{Q}}}P(1,\chi,d_1)\left\{1+\left(\mu_{\mathbb{L}}-\mu_{\mathbb{Q}}+\frac{P'(1,\chi,d_1)}{P(1,\chi,d_1)}\right)(s-1)+O((s-1)^2)\right\}.$$

Then (25) follows by using $\lambda_{\mathbb{Q}}=1$ and $\nu_{\mathbb{L}}=\lambda_{\mathbb{L}}\mu_{\mathbb{L}}/\sqrt{d_{\mathbb{L}}}$, for any totally real number field $\mathbb{L}$. ∎

Lemma 3.2. *For $X>0$, c real, $m\geq 1$ an integer and $k\in\{1,2\}$, set*

$$F_k(X,c,m) := \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} X^{-s}\Gamma^m(s/2)\,\frac{\mathrm{d}s}{(s-c)^k} \qquad (\alpha>\max(0,c)), \tag{27}$$

an absolutely convergent integral not depending on α, by (19). *Then*

$$F_k(X,c,m)>0, \qquad \frac{\partial F_1}{\partial c}(X,c,m) = F_2(X,c,m)>0, \qquad \frac{\partial F_2}{\partial c}(X,c)>0$$

and $X\mapsto F_k(X,c,m)$ is decreasing.

Proof. Clearly,

$$(t_1,\dots,t_m,t)\mapsto e^{-(t_1+\cdots+t_m)}(t_1\cdots t_m)^{(\alpha+it)/2-1}\frac{X^{-(\alpha+it)}}{(\alpha-c+it)^2}$$

is in $L^1(\mathbb{R}_+^m\times\mathbb{R},\mathbb{C})$ for $\alpha>0$ and $\alpha\neq c$. Now, for $\alpha>c$ we have

$$\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\frac{u^s}{(s-c)^2}\,\mathrm{d}s=\begin{cases}0 & \text{if } u<1,\\ u^c\log u & \text{if } u>1.\end{cases}$$

(move the line of integration $\Re(s)=\alpha$ of this absolutely convergent integral to the right to infinity for $0<u<1$ and to the left to infinity for $u>1$, in which case you pick up a residue at the pole $s=c$). Therefore, by Fubini's theorem, we have

$$F_2(X,c,m)=J_m(X,c)>0,$$

where

$$\begin{aligned}J_m(X,c)&:=\int\limits_{\substack{t_1,\dots,t_m>0\\ t_1\cdots t_m>X^2}}\cdots\int e^{-(t_1+\cdots+t_m)}\left(\frac{\sqrt{t_1\cdots t_m}}{X}\right)^c\log\left(\frac{\sqrt{t_1\cdots t_m}}{X}\right)\frac{\mathrm{d}t_1\cdots\mathrm{d}t_m}{t_1\cdots t_m}\\ &=\frac{1}{2}\int\limits_{\substack{t_1,\dots,t_m>0\\ t_1\cdots t_m>1}}\cdots\int e^{-X^{2/m}(t_1+\cdots+t_m)}(t_1\cdots t_m)^{c/2}\log(t_1\cdots t_m)\,\frac{\mathrm{d}t_1\cdots\mathrm{d}t_m}{t_1\cdots t_m}.\end{aligned}$$

Finally, fix $X>0$ and $\alpha>0$. Let c range in $(-\infty,\alpha)$. Clearly,

$$c\mapsto I_m(X,c):=\int\limits_{\substack{t_1,\dots,t_m>0\\ t_1\cdots t_m>1}}\cdots\int e^{-X^{2/m}(t_1+\cdots+t_m)}(t_1\cdots t_m)^{c/2}\,\frac{\mathrm{d}t_1\cdots\mathrm{d}t_m}{t_1\cdots t_m}$$

is differentiable on $(-\infty,\alpha)$, with derivative $c\mapsto J_m(X,c)=F_2(X,c,m)$. Using Stirling's formula (19), it is clear that $c\mapsto F_1(X,c,m)$ is differentiable on $(-\infty,\alpha)$, with derivative $c\mapsto F_2(X,c,m)$. Hence, $c\mapsto I_m(X,c)-F_1(X,c,m)$ is constant on $(-\infty,\alpha)$. Since $\lim_{c\to-\infty}I_m(X,c)=\lim_{c\to-\infty}F_1(X,c,m)=0$, by Lebesgue's dominated convergence theorem we obtain

$$F_1(X,c,m)=I_m(X,c)>0.$$

Finally, these two integral representations of $F_1(X,c,m)$ and $F_2(X,c,m)$ show that they are decreasing functions of $X>0$. ∎

Lemma 3.3 (see [Lou18, Lemma 7]). *Set $\phi(\delta)=\prod_{p|\delta}(p-1)$ and $\tilde{\phi}(\delta)=\prod_{p|\delta}(p-2)$ for $\delta\geq 1$ a square-free integer (by convention, empty products are 1). Let $\sum_{n\geq1}a_n$ be absolutely convergent. Then for $d_1\geq1$ we have*

$$\sum_{\delta|d_1}\mu(\delta)\delta\sum_{\substack{n\geq1\\ \delta|n}}a_n=\sum_{\delta|d_1}\mu(\delta)\phi(\delta)\sum_{\substack{n\geq1\\ \gcd(n,d_1)=\delta}}a_n$$

and

$$\sum_{\delta\mid d_1}\phi(\delta)\sum_{\substack{n\geq 1\\ \gcd(n,d_1)=\delta}}a_n=\sum_{\delta\mid d_1}\tilde{\phi}(\delta)\sum_{\substack{n\geq 1\\ \delta\mid n}}a_n.$$

THEOREM 3.4. *For $d\geq 1$, set*

$$P(s,d):=\prod_{p\mid d}\left(1-\frac{1}{p^s}\right)\quad\text{and}\quad Q(s,d):=\prod_{p\mid d_1}\left(1-\frac{2}{p}+\frac{1}{p^{1-s}}\right)$$

(*empty products are* 1). *For d_1 and d_2 coprime square-free positive integers, $k\in\{1,2\}$, $a>0$, $X>0$, and $\alpha>1$, set*

$$\begin{aligned}\mathit{\Pi}_k(s,a,d_1,d_2)&:=\left\{\frac{a^{1-s}}{(s-1)^k}P(s,d_1)+\frac{a^s}{s^k}Q(s,d_1)\right\}P(s,d_2),\\ G_k(s,a,d_1,d_2,X)&:=\mathit{\Pi}_k(s,a,d_1,d_2)X^{s-1}Z_{\mathbb{Q}}(s)\end{aligned}$$

and

$$I_k(a,d_1,d_2,X):=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}G_k(s,a,d_1,d_2,X)\,\mathrm{d}s. \tag{28}$$

Then, for any primitive and even Dirichlet character χ of conductor $f>1$ for which $\gcd(d_1,f)=1$ and $d_2\mid f$, we have

$$\begin{aligned}|\mathit{\Lambda}(1,\chi,d_1)|/\sqrt{f}&\leq I_1(a,d_1,d_2,\sqrt{f}),\\ |\mathit{\Lambda}'(1,\chi,d_1)|/\sqrt{f}&\leq I_2(1,d_1,d_2,\sqrt{f}).\end{aligned} \tag{29}$$

Proof. Let the notation be as in Lemma 3.1 and (27).

First, by the positivity of $F_k(X,c,1)$ (Lemma 3.2), the absolute value of the first integral in (22) and (23)

$$I_1:=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\frac{\mathit{\Lambda}(s,\chi,d_1)a^{1-s}}{(s-1)^k}\,\mathrm{d}s=\sum_{\substack{n\geq 1\\ \gcd(n,d_1d_2)=1}}\chi(n)aF_k\left(an\sqrt{\frac{\pi}{f}},1,1\right)$$

(where we have used $\chi(n)=0$ for $\gcd(n,d_2)>1$) is less than or equal to

$$\sum_{\substack{n\geq 1\\ \gcd(n,d_1d_2)=1}}aF_k\left(an\sqrt{\frac{\pi}{f}},1,1\right)=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\frac{a^{1-s}}{(s-1)^k}f^{s/2}P(s,d_1d_2)Z_{\mathbb{Q}}(s)\,\mathrm{d}s.$$

Second, let us deal with the second integral in (22) and (23):

$$I_2:=\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\frac{\mathit{\Lambda}(1-s,\chi,d_1)a^s}{s^k}\,\mathrm{d}s.$$

We have

$$\prod_{p\mid d_1}\left(1-\frac{\chi(p)}{p^s}\right)=\sum_{\delta\mid d_1}\mu(\delta)\frac{\chi(\delta)}{\delta^s}=\frac{\mu(d_1)\chi(d_1)}{d_1^s}\sum_{\delta\mid d_1}\mu(\delta)\bar{\chi}(\delta)\delta^s,$$

by making the change of variables $\delta \mapsto d_1/\delta$. Using (21), we obtain

$$\Lambda(1-s,\chi,d_1) = W_\chi \mu(d_1)\chi(d_1) d_1^{s-1} \Lambda(s,\bar{\chi}) \sum_{\delta | d_1} \mu(\delta)\bar{\chi}(\delta)\delta^{1-s}.$$

By the first assertion in Lemma 3.3, for $\Re(s) > 1$ we have

$$\begin{aligned} L(s,\bar{\chi}) \sum_{\delta|d_1} \mu(\delta)\bar{\chi}(\delta)\delta^{1-s} &= \sum_{\delta|d_1} \mu(\delta)\delta \sum_{\substack{n\geq 1\\ \delta|n}} \bar{\chi}(n) n^{-s} \\ &= \sum_{\delta|d_1} \mu(\delta)\phi(\delta) \sum_{\substack{n\geq 1\\ \gcd(n,d_1)=\delta}} \bar{\chi}(n) n^{-s}. \end{aligned}$$

By the second assertion in Lemma 3.3 the absolute value of

$$I_2 = \frac{W_\chi \mu(d_1)\chi(d_1)}{d_1} \sum_{\delta|d_1} \mu(\delta)\phi(\delta) \sum_{\substack{n\geq 1\\ \gcd(n,d_1)=\delta}} \bar{\chi}(n) F_k\left(\frac{n}{ad_1}\sqrt{\frac{\pi}{f}}, 0, 1\right)$$

is less than or equal to

$$\begin{aligned} &\frac{1}{d_1} \sum_{\delta|d_1} \phi(\delta) \sum_{\substack{n\geq 1\\ \gcd(n,d_1)=\delta\\ \gcd(n,d_2)=1}} F_k\left(\frac{n}{ad_1}\sqrt{\frac{\pi}{f}}, 0, 1\right) \\ &\qquad = \frac{1}{d_1} \sum_{\delta|d_1} \tilde{\phi}(\delta) \sum_{\substack{n\geq 1\\ \gcd(n,d_2)=1}} F_k\left(\frac{\delta n}{ad_1}\sqrt{\frac{\pi}{f}}, 0, 1\right) \\ &\qquad = \frac{1}{2\pi i} \int_{\Re(s)=\alpha} \frac{a^s}{s^k} f^{s/2} Q(s,d_1) P(s,d_2) Z_{\mathbb{Q}}(s)\, \mathrm{d}s, \end{aligned}$$

where we have used $\chi(n) = 0$ for $\gcd(n,d_2) > 1$ and the positivity of $F_k(X,0,1)$ (Lemma 3.2).

Third, the bounds in (29) with $I_k(a,d_1,d_2,X)$ as in (28) follow. ∎

Lemma 3.5. *Let $\alpha > 1$ and let the notation be as in Theorem* 3.4. *Then*

$$I_k(a,d_1,d_2,X) = \rho_k(s,a,d_1,d_2,X) + R_k(a,d_1,d_2,X), \tag{30}$$

where

$$\rho_k(s,a,d_1,d_2,X) = \mathrm{Res}_{s=1}(G_k(s,a,d_1,d_2,X)) + \mathrm{Res}_{s=0}(G_k(s,a,d_1,d_2,X))$$

and

$$R_k(a,d_1,d_2,X) := \frac{1}{2\pi i} \int_{\Re(s)=\alpha} \Pi_k(1-s,a,d_1,d_2) X^{-s} Z_{\mathbb{Q}}(s)\, \mathrm{d}s = O(X^{-\alpha}). \tag{31}$$

Since

$$\mathrm{Res}_{s=0}(G_k(s,a,d_1,d_2,X)) = O((\log^k X)/X) \quad \text{as } X \to +\infty,$$

we have

$$I_k(a,d_1,d_2,X) = \mathrm{Res}_{s=1}(G_k(s,a,d_1,d_2,X)) + O((\log^k X)/X). \tag{32}$$

Proof. Moving the line $\Re(s)=\alpha>1$ in (28) to the left to $\Re(s)=1-\alpha<0$ we pick up residues at $s=1$ and $s=0$ and using $Z_{\mathbb{Q}}(1-s)=Z_{\mathbb{Q}}(s)$, we obtain

$$\begin{aligned} R_k(a,d_1,d_2,X) &:= \frac{1}{2\pi i}\int\limits_{\Re(s)=1-\alpha} G_k(s,a,d_1,d_2,X)\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} G_k(1-s,a,d_1,d_2,X)\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} \Pi_k(1-s,a,d_1,d_2)X^{-s}Z_{\mathbb{Q}}(s)\,\mathrm{d}s, \end{aligned}$$

and (30) follows. ∎

Lemma 3.6. *Let $\omega(d) := \sum_{p|d} 1$ denote the number of distinct prime divisors of d. Set (where empty products are 1 and empty sums are 0)*

$$P(d) := \prod_{p|d}\Big(1-\frac{1}{p}\Big), \qquad S_1(d) := \sum_{p|d}\frac{\log p}{p-1}, \qquad S_2(d) := \sum_{p|d}\frac{p\log^2 p}{(p-1)^2}.$$

Let $\mu_{\mathbb{Q}}$ be as in (2). *Set*

$$\mu'_{\mathbb{Q}} := \frac{\pi^2}{8} - \gamma^2 - 2\gamma(1) = 1.04615\ldots,$$

where

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma(1)(s-1) + O((s-1)^2).$$

Let the notation be as in Theorem 3.4. *Then*

$$\begin{aligned} &\mathrm{Res}_{s=1}(G_1(s,a,d_1,d_2,X)) \\ &\quad = P(d_1d_2)\times\{\log X + \mu_{\mathbb{Q}} - 1 - \log a + 2^{\omega(d_1)}a + S_1(d_1d_2)\}, \end{aligned} \tag{33}$$

which is minimal for $a = 1/2^{\omega(d_1)}$,

$$\begin{aligned} &\mathrm{Res}_{s=1}(G_2(s,1,d_1,d_2,X)) = \frac{P(d_1d_2)}{2} \\ &\quad \times\big\{\big(\log X + \mu_{\mathbb{Q}} - 1 + S_1(d_1d_2)\big)^2 + \mu'_{\mathbb{Q}} + 2^{\omega(d_1)+1} - S_2(d_1d_2)\big\}, \end{aligned} \tag{34}$$

$$\mathrm{Res}_{s=0}(G_1(s,a,d_1,d_2,X))$$
$$= -\frac{P(d_1)}{X} \times \begin{cases} \log X - \mu_{\mathbb{Q}} + 1 + \log a - a & \text{if } d_2 = d_1 = 1, \\ \log X - \mu_{\mathbb{Q}} + 1 + \log a + S_1(d_1) & \text{if } d_2 = 1 \text{ and } d_1 > 1, \\ \log d_2 & \text{if } d_2 \text{ is prime}, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathrm{Res}_{s=0}(G_2(s,a,d_1,d_2,X)) = -\frac{P(d_1)}{2X}$$
$$\times \begin{cases} (\log X - \mu_{\mathbb{Q}} + 1 + \log a)^2 + \mu'_{\mathbb{Q}} + 2a & \text{if } d_2 = d_1 = 1, \\ (\log X - \mu_{\mathbb{Q}} + 1 + \log a + S_1(d_1))^2 + \mu'_{\mathbb{Q}} + \displaystyle\sum_{p \mid d_1} \frac{(p-2)\log^2 p}{(p-1)^2} & \\ \qquad \text{if } d_2 = 1 \text{ and } d_1 > 1, & \\ (\log d_2)(2\log X - 2\mu_{\mathbb{Q}} + 2 + 2\log a + 2S_1(d_1) - \log d_2) & \\ \qquad \text{if } d_2 \text{ is prime}, & \\ 2(\log p)(\log q) \quad \text{if } d_2 = pq, & \\ 0 \qquad \text{otherwise.} & \end{cases}$$

In particular, by (31)*, for* $k \in \{1,2\}$ *we have* $\mathrm{Res}_{s=0}(G_k(s,a,d_1,d_2,X)) + R_k(a,d_1,d_2,X) \le 0$ *and* $I_k(a,d_1,d_2,X) \le \mathrm{Res}_{s=1}(G_k(s,a,d_1,d_2,X))$ *for* $\omega(d_2) \le k$ *and* X *large enough.*

3.1. Proof of Theorem 1.3. For the first bound, use (24), (29), (32), (33) for $a = 1/2^{\omega(d_1)}$, and the last assertion of Lemma 3.6. For the second bound, using (25), (29), (32), (33) for $a = 1/2^{\omega(d_1)}$, and (34), we obtain

$$\Big\{\prod_{p \mid d_1}\Big(1+\frac{1}{p}\Big)\Big\}\nu_{\mathbb{L}} = \frac{\big(\mu_{\mathbb{Q}} + \sum_{p\mid d_1} \frac{\log p}{p+1}\big)\Lambda(1,\chi_{\mathbb{L}},d_1) + \Lambda'(1,\chi_{\mathbb{L}},d_1)}{\sqrt{d_{\mathbb{L}}}}$$
$$\le A \times I_1(2^{-\omega(d_1)}, d_1, d_2, \sqrt{f_\chi}) + I_2(1,d_1,d_2,\sqrt{d_{\mathbb{L}}})$$
$$= A \times \mathrm{Res}_{s=1}(G_1(s,2^{-\omega(d_1)},d_1,d_2,\sqrt{d_{\mathbb{L}}})) + \mathrm{Res}_{s=1}(G_2(s,1,d_1,d_2,\sqrt{d_{\mathbb{L}}}))$$
$$+ O\Big(\frac{\log^2 d_{\mathbb{L}}}{\sqrt{d_{\mathbb{L}}}}\Big)$$
$$= \frac{1}{8}\Big\{\prod_{p\mid d_1 d_2}\Big(1-\frac{1}{p}\Big)\Big\}\{(\log d_{\mathbb{L}} + 2c_2(d_1,d_2))^2 + 4c_3(d_1,d_2)\} + O\Big(\frac{\log^2 d_{\mathbb{L}}}{\sqrt{d_{\mathbb{L}}}}\Big),$$

where $A = \mu_{\mathbb{Q}} + \sum_{p\mid d_1} \frac{\log p}{p+1}$, $B = \mu_{\mathbb{Q}} + \log 2^{\omega(d_1)} + S_1(d_1 d_2)$, $C = \mu_{\mathbb{Q}} - 1 + S_1(d_1 d_2)$, $D = \mu'_{\mathbb{Q}} + 2^{\omega(d_1)+1} - S_2(d_1 d_2)$, $c_2(d_1,d_2) = A + C$ and $c_3(d_1,d_2) = D + A(2B - 2C - A)$. The error terms $O\big(\frac{\log^2 d_{\mathbb{L}}}{\sqrt{d_{\mathbb{L}}}}\big)$ are negative for $d_{\mathbb{L}}$ large enough, by the last assertion of Lemma 3.6.

4. Proof of Corollary 1.4

REMARK 4.1. Take $c \in \{0,1\}$, $k \in \{1,2\}$ and $X > 0$. Taking $\alpha = 2$ in (27) and using $|\zeta(2+it)| \le \zeta(2) = \pi^2/6$, we have

$$0 < Z_k(X,c) := \frac{1}{2\pi i} \int\limits_{\Re(s)=\alpha} X^{-s} Z_{\mathbb{Q}}(s) \frac{\mathrm{d}s}{(s-c)^k} \le \frac{M_k(c)}{12X^2},$$

where

$$M_k(c) := \int\limits_{\Re(s)=2} \left| \frac{\Gamma(s/2)}{(s-c)^k} \right| \mathrm{d}t = \int\limits_{-\infty}^{\infty} \sqrt{\frac{\pi t/2}{\sinh(\pi t/2)}} \frac{\mathrm{d}t}{((2-c)^2+t^2)^{k/2}}.$$

Moreover, $M_1(0) = 1.88826\ldots$, $M_1(1) = 2.94137\ldots$, $M_2(0) = 0.77941\ldots$ and $M_2(1) = 2.10564\ldots$. Moreover, $X \mapsto Z_k(X,c)$ is decreasing and $Z_k(X,0) \le Z_k(X,1)$, by Lemma 3.2.

THEOREM 4.2. *Let χ be a primitive even Dirichlet character of conductor $f > 1$. Then*

$$|L(1,\chi)| \le \tfrac{1}{2}\log f + \mu_{\mathbb{Q}}. \tag{35}$$

Proof. We apply (24), (29), (30), (31) and Lemma 3.6 with $a = d_1 = d_2 = 1$ and $\Pi_1(s,1,1,1) = \frac{1}{s-1} + \frac{1}{s}$. We obtain $|L(1,\chi)| \le I(\sqrt{f})$, where $I(X) = \rho(X) + R(X)$ with

$$\begin{aligned} \rho(X) &= (\log X + \mu_{\mathbb{Q}}) - (\log X - \mu_{\mathbb{Q}})/X, \\ R(X) &= -\frac{1}{2\pi i} \int\limits_{\Re(s)=\alpha} \left(\frac{1}{s} + \frac{1}{s-1} \right) X^{-s} Z_{\mathbb{Q}}(s)\,\mathrm{d}s \\ &= -Z_1(X,0) - Z_1(X,1) < 0. \end{aligned}$$

Since χ is a primitive even Dirichlet character of conductor $f > 1$, we have $X = \sqrt{f} \ge \sqrt{5} \ge \exp(\mu_{\mathbb{Q}})$. ∎

THEOREM 4.3. *Let $p \ge 2$ be a prime. Let χ be a primitive even Dirichlet character of conductor f divisible by p. Assume that $f \ge \left(\frac{5p}{12\log p}\right)^2$, which is the case for $2 \le p \le 139$. Then*

$$|L(1,\chi)| \le \left(1 - \frac{1}{p}\right)\left(\frac{1}{2}\log f + \mu_{\mathbb{Q}} + \frac{\log p}{p-1}\right). \tag{36}$$

Proof. We apply (24), (29), (30), (31) and Lemma 3.6 with $a = d_1 = 1$, $d_2 = p$ and $\Pi_1(s,1,1,p) = \left(1 - \frac{1}{p^s}\right)\left(\frac{1}{s-1} + \frac{1}{s}\right)$. We obtain $|L(1,\chi)| \le I(\sqrt{f})$, where $I(X) = \rho(X) + R(X)$ with

$$\rho(X) = \left(1 - \frac{1}{p}\right)\left(\log X + \mu_{\mathbb{Q}} + \frac{\log p}{p-1}\right) - \frac{\log p}{X},$$

$$\begin{aligned}R(X) &= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}(p^{s-1}-1)\left(\frac{1}{s}+\frac{1}{s-1}\right)X^{-s}Z_{\mathbb{Q}}(s)\,\mathrm{d}s\\ &= \frac{Z_1(X/p,0)+Z_1(X/p,1)}{p}-Z_1(X,0)-Z_1(X,1)\\ &\le \frac{Z_1(X/p,0)+Z_1(X/p,1)}{p}\le\frac{5p}{12X^2},\end{aligned}$$

by Remark 4.1. ∎

THEOREM 4.4. *Let $p\ge 2$ be a given prime. Let χ be a primitive even Dirichlet character of conductor $f\ge 5$ not divisible by p. Assume that either $2\le p\le 61$ or $f\ge\left(\frac{p}{6\log p}\right)^2$. Then*

$$|L(1,\chi)|\le\left|\frac{p-1}{p-\chi(p)}\right|\times\left(\frac{1}{2}\log f+\mu_{\mathbb{Q}}+\log 2+\frac{\log p}{p-1}\right). \tag{37}$$

Proof. We apply (24), (29), (30), (31) and Lemma 3.6 with $a=1/2$, $d_1=p$ and $d_2=1$. We obtain

$$\left|\left(1-\frac{\chi(p)}{p}\right)L(1,\chi)\right|\le I(\sqrt{f}),$$

where $I(X)=\rho(X)+R(X)$ with

$$\begin{aligned}\rho(X) &= \left(1-\frac{1}{p}\right)\left\{\left(\log X+\mu_{\mathbb{Q}}+\log 2+\frac{\log p}{p-1}\right)-\frac{\log X+\frac{\log\pi-\gamma}{2}+\frac{\log p}{p-1}}{X}\right\},\\ R(X) &= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\left\{\frac{2^{-s}}{s}(p^{s-1}-1)-\frac{2^{s-1}}{s-1}\left(1-\frac{2}{p}+\frac{1}{p^s}\right)\right\}X^{-s}Z_{\mathbb{Q}}(s)\,\mathrm{d}s\\ &= \frac{1}{p}Z_1(2X/p,0)-Z_1(2X,0)-\frac{p-2}{2p}Z_1(X/2,1)-\tfrac{1}{2}Z_1(pX/2,1)\\ &\le \frac{1}{p}Z_1(2X/p,0)\le\frac{p}{24X^2},\end{aligned}$$

by Remark 4.1. Hence,

$$\begin{aligned}-\left(1-\frac{1}{p}\right)&\frac{\log\sqrt{f}+\frac{\log\pi-\gamma}{2}+\frac{\log p}{p-1}}{\sqrt{f}}+R(\sqrt{f})\\ &\le\frac{p-12\sqrt{f}\left(1-\frac{1}{p}\right)\left(\log f+\log\pi-\gamma+\frac{2\log p}{p-1}\right)}{24f}\\ &\le\frac{p-11\sqrt{f}(\log f+\log\pi-\gamma)}{24f}\quad\text{if } p\ge 13.\end{aligned}$$

The first bound is negative for $f\ge 5$ and $2\le p\le 61$. The second bound is negative for $\sqrt{f}\log(\pi e^{-\gamma}f)\ge p/11$, hence for $f\ge\left(\frac{p}{6\log p}\right)^2$ and $p\ge 67$. ∎

5. Proof of Corollary 1.5

5.1. Proof of the first and second bounds in Corollary 1.5. First assume that $d_1 = 1$. Then $\Lambda(s,\chi,d_1) = \Lambda(s,\chi)$. Lemma 3.1 with $a = 1$ and $d_1 = 1$ gives

$$\Lambda(1,\chi_{\mathbb{L}}) = \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} \Lambda(s,\chi_{\mathbb{L}})\biggl\{\frac{1}{s-1}+\frac{1}{s}\biggr\}\,\mathrm{d}s \qquad (\alpha>1),$$

$$\Lambda'(1,\chi_{\mathbb{L}}) = \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} \Lambda(s,\chi_{\mathbb{L}})\biggl\{\frac{1}{(s-1)^2}-\frac{1}{s^2}\biggr\}\,\mathrm{d}s \quad (\alpha>1).$$

By Lemma 3.2, for $X>0$ we have

$$\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} X^{-s}\Gamma(s/2)\biggl\{\frac{1}{s-1}+\frac{1}{s}\biggr\}\,\mathrm{d}s = F_1(X,1,1)+F_1(X,0,1) > 0,$$

$$\frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} X^{-s}\Gamma(s/2)\biggl\{\frac{1}{(s-1)^2}-\frac{1}{s^2}\biggr\}\,\mathrm{d}s = F_2(X,1,1)-F_2(X,0,1) > 0.$$

Hence, by (25), we have $\nu_{\mathbb{L}} \le I(d_2,\sqrt{d_{\mathbb{L}}})$, where

$$\begin{aligned} I(d_2,X) &:= \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha} G(s,d_2,X)\,\mathrm{d}s \\ &= \mathrm{Res}_{s=1}(G(s,d_2,X)) + \mathrm{Res}_{s=0}(G(s,d_2,X)) + R(d_2,X) \end{aligned}$$

with

$$G(s,d_2,X) := \biggl\{\prod_{p\mid d_2}\Bigl(1-\frac{1}{p^s}\Bigr)\biggr\}\biggl\{\frac{1}{(s-1)^2}-\frac{1}{s^2}+\mu_{\mathbb{Q}}\Bigl(\frac{1}{s-1}+\frac{1}{s}\Bigr)\biggr\}X^{s-1}Z_{\mathbb{Q}}(s),$$

$$\mathrm{Res}_{s=1}(G(s,d_2,X)) = \frac{P(d_2)}{2}\times\bigl\{\bigl(\log X+2\mu_{\mathbb{Q}}-1+S_1(d_2)\bigr)^2-\mu''_{\mathbb{Q}}-S_2(d_2)\bigr\},$$

$$\mathrm{Res}_{s=0}(G(s,d_2,X)) = \frac{1}{2X}\times\begin{cases}(\log X-2\mu_{\mathbb{Q}}+1)^2-\mu''_{\mathbb{Q}} & \text{if } d_2=1,\\ (\log p)(2\log X-4\mu_{\mathbb{Q}}+2-\log p) & \text{if } d_2=p,\\ 2(\log p)(\log q) & \text{if } d_2=pq,\\ 0 & \text{otherwise,}\end{cases}$$

where $P(d)$, $S_1(d)$ and $S_2(d)$ are as in Lemma 3.6 and where

$$\mu''_{\mathbb{Q}} = 1+(1-\mu_{\mathbb{Q}})^2-\mu'_{\mathbb{Q}} = 1+(1-\mu_{\mathbb{Q}})^2-\pi^2/8+\gamma^2+2\gamma(1) = 0.90818\ldots$$

and (by taking $\alpha = 2$)

$$R(d_2, X) := \frac{1}{2\pi i} \int\limits_{\Re(s)=1-\alpha} G(s, d_2, X)\, \mathrm{d}s$$
$$= \frac{1}{2\pi i} \int\limits_{\Re(s)=\alpha} G(1-s, d_2, X)\, \mathrm{d}s = O(X^{-2}).$$

Hence we get a neat asymptotic and effective bound which we will make explicit in Theorems 5.2 and 5.3:

THEOREM 5.1. *Let $d_2 \geq 1$ be a square-free integer. Let $\mathbb{L}$ range over the real quadratic fields of discriminants divisible by d_2. Then for $d_{\mathbb{L}}$ effectively large enough we have*

$$\nu_{\mathbb{L}} \leq \frac{1}{8}\left\{\prod_{p|d_2}\left(1-\frac{1}{p}\right)\right\} \times \left(\log d_{\mathbb{L}} + 4\mu_{\mathbb{Q}} - 2 + 2\sum_{p|d_2}\frac{\log p}{p-1}\right)^2.$$

In particular, for $d_1 = d_2 = 1$, we have $G(1-s,1,X) = -G(s,1,1/X)/X$ and $R(1,X) \leq 0$, as in the proof of Theorem 4.2. Hence, we obtain (compare with Theorem 4.2):

THEOREM 5.2 (First bound of Corollary 1.5). *Let $\mathbb{L}$ be a real quadratic number field of discriminant $d_{\mathbb{L}}$. Then*

$$\nu_{\mathbb{L}} = \mu_{\mathbb{L}} \operatorname{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \leq \tfrac{1}{8}\log^2 d_{\mathbb{L}}$$

and $\nu_{\mathbb{L}} \leq (\log d_{\mathbb{L}} + 4\mu_{\mathbb{Q}} - 2)^2/8$ for $d_{\mathbb{L}} > 145$.

Take $d_2 = p$ a prime integer. Then

$$I(p,X) \leq \frac{1}{2}\left(1-\frac{1}{p}\right) \times \left\{\left(\log X + 2\mu_{\mathbb{Q}} - 1 + \frac{\log p}{p-1}\right)^2 - \mu''_{\mathbb{Q}} - \frac{p\log^2 p}{(p-1)^2}\right\}$$
$$+ \frac{\log p}{2X}(2\log X - 4\mu_{\mathbb{Q}} + 2 - \log p) + R(p,X),$$

where

$$R(p,X)$$
$$= \frac{1}{2\pi i} \int\limits_{\Re(s)=\alpha} (p^{s-1}-1)\left\{\frac{1}{(s-1)^2} - \frac{1}{s^2} + \mu_{\mathbb{Q}}\left(\frac{1}{s-1}+\frac{1}{s}\right)\right\} X^{-s} Z_{\mathbb{Q}}(s)\, \mathrm{d}s$$
$$\leq \frac{Z_2(X/p,1) + \mu_{\mathbb{Q}}(Z_1(X/p,1) + Z_1(X/p,0))}{p}$$
$$\leq \frac{p(M_2(1) + \mu_{\mathbb{Q}}(M_1(1) + M_1(0)))}{12X^2} \leq \frac{p}{5X^2},$$

by Remark 4.1. Hence, we get

THEOREM 5.3 (second bound of Corollary 1.5). *Let $p \geq 2$ be a prime integer. Let $\mathbb{L}$ be a real quadratic number field of discriminant $d_{\mathbb{L}}$. Assume that p divides $d_{\mathbb{L}}$. Assume that $d_{\mathbb{L}} \geq 4p$, which is the case for $p = 2$, or that $p \geq 337$. Then*

$$\nu_{\mathbb{L}} = \mu_{\mathbb{L}} \mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \leq \frac{1}{8}\left(1 - \frac{1}{p}\right) \times \left(\log d_{\mathbb{L}} + 4\mu_{\mathbb{Q}} - 2 + \frac{2\log p}{p-1}\right)^2.$$

For $p = 2$ we have $4\mu_{\mathbb{Q}} - 2 + \frac{2\log p}{p-1} = 2(1 + \gamma - \log(2\pi)) = -0.52132\ldots$.

Proof. Recall that we take $X = \sqrt{f} \geq \sqrt{p}$, since p divides $f > 1$. Consider the function

$$\begin{aligned} f(X) := & -\frac{1}{2}\left(1 - \frac{1}{p}\right) \times \left\{\mu''_{\mathbb{Q}} + \frac{p\log^2 p}{(p-1)^2}\right\} \\ & + \frac{\log p}{2X}(2\log X - 4\mu_{\mathbb{Q}} + 2 - \log p) + \frac{p}{5X^2}. \end{aligned}$$

Then

$$f'(X) = -\frac{\log p}{2X^2}(2\log X - 4\mu_{\mathbb{Q}} - \log p) - \frac{2p}{5X^3}$$

is negative for $X \geq \sqrt{p}$. Finally, $f(\sqrt{p}) \leq 0$ for $p \geq 337$ and $f(\sqrt{4p}) \leq 0$ for $p \geq 2$. ∎

5.2. Proof of the third and fourth bounds in Corollary 1.5

THEOREM 5.4. *Let $p \geq 2$ be a prime integer. Let $\kappa_p \leq 5.14084$ be as in (39) below. Let $\mathbb{L}$ be a real quadratic number field of discriminant $d_{\mathbb{L}} \geq \left(4\frac{\frac{p+21}{21}}{\log^2 \frac{p+21}{21}}\right)^2$. Assume that $\chi_{\mathbb{L}}(p) = -1$. Then*

$$\nu_{\mathbb{L}} = \mu_{\mathbb{L}} \mathrm{Res}_{s=1}(\zeta_{\mathbb{L}}(s)) \leq \frac{1}{8} \times \frac{p-1}{p+1} \times \left\{\left(\log d_{\mathbb{L}} + 4\mu_{\mathbb{Q}} - 2 + \frac{4p\log p}{p^2-1}\right)^2 + 4\kappa_p\right\}.$$

If $p = 2$, this bound holds true for $d_{\mathbb{L}} \geq 5$. Since $4\kappa_2 = 19.52507\ldots$ and $4\mu_{\mathbb{Q}} - 2 + \frac{4p\log p}{p^2-1} = -0.059224\ldots$, we get the third and fourth bounds in Corollary 1.5.

Proof. As in Section 3.1, we apply (25), (29), (30), (31) and Lemma 3.6 with $d_1 = p$, $a = 1/2$ and $d_2 = 1$. We obtain $\left(1 + \frac{1}{p}\right)\nu_{\mathbb{L}} \leq I(\sqrt{d_{\mathbb{L}}})$, where

$$I(X) = \mathrm{Res}_{s=1}(G(s,p,X)) + \mathrm{Res}_{s=0}(G(s,p,X)) + R(p,X),$$

with

$$\begin{aligned} G(s,p,X) &:= \left\{\left(\mu_{\mathbb{Q}} + \frac{\log p}{p+1}\right)\Pi_1(s,1/2) + \Pi_2(s,1)\right\} X^{s-1} Z_{\mathbb{Q}}(s), \\ \Pi_k(s,a) &:= \frac{a^{1-s}}{(s-1)^k}\left(1 - \frac{1}{p^s}\right) + \frac{a^s}{s^k}\left(1 - \frac{2}{p} + \frac{1}{p^{1-s}}\right) \end{aligned}$$

and

$$R(p,X) := \frac{1}{2\pi i}\int\limits_{\Re(s)=\alpha}\left\{\left(\mu_{\mathbb{Q}}+\frac{\log p}{p+1}\right)\Pi_1(1-s,1/2)+\Pi_2(1-s,1)\right\}X^{-s}Z_{\mathbb{Q}}(s)\,\mathrm{d}s. \tag{38}$$

Now,

$$\mathrm{Res}_{s=1}(G(s,p,X))=\frac{1}{2}\left(1-\frac{1}{p}\right)\times\left\{\left(\log X+2\mu_{\mathbb{Q}}-1+\frac{2p\log p}{p^2-1}\right)^2+\kappa_p\right\},$$

where

$$\kappa_p := -\left(\frac{1}{p-1}+\frac{1}{(p-1)^2}+\frac{1}{(p+1)^2}\right)\log^2 p + \frac{\log(16\pi)-\gamma}{p+1}\log p+6-\mu''_{\mathbb{Q}}+2\mu_{\mathbb{Q}}\log 2 \tag{39}$$

satisfies $\kappa_p\le\kappa_7=5.14083\ldots$ and $\kappa_2=4.88126\ldots$. (Notice that $2\mu_{\mathbb{Q}}-1+2p(\log p)/(p^2-1)=A+C=c_2(p,1)$ and $\kappa_p=c_3(p,1)$, with the notation of Section 3.1.) Moreover,

$$\mathrm{Res}_{s=0}(G(s,p,X))=-\frac{1}{2X}\left(1-\frac{1}{p}\right)\times\left\{\left(\log X+1+\frac{2p\log p}{p^2-1}\right)^2+\kappa'_p\right\},$$

where

$$\kappa'_p := \left(\frac{1}{p-1}-\frac{1}{(p-1)^2}-\frac{1}{(p+1)^2}\right)\log^2 p - \frac{2+\gamma-\log\pi}{p+1}\log p+\mu'_{\mathbb{Q}}-2\mu_{\mathbb{Q}}\log 2-\mu_{\mathbb{Q}}^2 \tag{40}$$

satisfies $\kappa'_p\ge\kappa'_2=0.62924\ldots$. Noticing that

$$\Pi_1(1-s,a)=\frac{2^{-s}}{s^k}(p^{s-1}-1)-\frac{2^{s-1}}{(s-1)^k}\left(1-\frac{2}{p}+\frac{1}{p^s}\right),$$
$$\Pi_2(1-s,a)=\frac{1}{s^2}(1-p^{s-1})+\frac{1}{(s-1)^k}\left(1-\frac{2}{p}+\frac{1}{p^s}\right)$$

and using (38) and Remarks 4.1, we have

$$\begin{aligned}R(p,X)&\le\frac{1}{p}\left(\mu_{\mathbb{Q}}+\frac{\log p}{p+1}\right)Z_1(2X/p,0)+Z_2(X,0)\\&\quad+\left(1-\frac{2}{p}\right)Z_2(X,1)+Z_2(pX,1)\\&\le\left(\mu_{\mathbb{Q}}+\frac{\log 3}{4}\right)\frac{pM_1(0)}{48X^2}+\frac{M_2(0)}{12X^2}+\frac{(p-1)^2M_2(1)}{12p^2X^2}\\&\le\frac{p}{84X^2}+\frac{M_2(0)+M_2(1)}{12X^2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_{s=0}(H(s,p,X)) + R(p,X) &\le -\frac{\log^2 X}{4X} + \frac{p+21}{84X^2}\\ &= -\frac{1}{4X^2}\left(X\log^2 X - \frac{p+21}{21}\right)\end{aligned}$$

is negative for $X \ge 4^{\frac{p+21}{21}}_{\log^2 \frac{p+21}{21}}$. Indeed, if $u > 1$ and $X \ge \frac{4u}{\log^2 u} = \left(\frac{\sqrt{u}}{\log\sqrt{u}}\right)^2$ then $X > 1$, as $v/\log v \ge e$ for $v > 1$, and since $X \mapsto X\log^2 X$ increases with $X > 1$, we have

$$X\log^2 X \ge \frac{4u}{\log^2 u}\log^2\left(\frac{4u}{\log^2 u}\right) = u\left(1 + 4\frac{\log\left(\frac{u^{1/4}}{2\log u^{1/4}}\right)}{\log u}\right)^2 \ge 1,$$

as $v/(2\log v) \ge e/2 \ge 1$ for $v > 1$.

Finally, for $p = 2$, we have

$$R(2,X) \le \left(\mu_{\mathbb{Q}} + \frac{\log 2}{3}\right)\frac{M_1(0)}{24X^2} + \frac{M_2(0)}{12X^2} + \frac{M_2(1)}{48X^2} \le \frac{3}{20X^2}$$

and $\mathrm{Res}_{s=0}(H(s,2,X)) + R(2,X) \le -\frac{1}{4X^2}\left(X\log^2 X - \frac{3}{5}\right)$ is negative for $X = \sqrt{d_{\mathbb{L}}} \ge \sqrt{5}$. ∎

References

[Coh] H. Cohen, *A Course in Computational Algebraic Number Theory*, Grad. Texts in Math. 138, Springer, Berlin, 1993.

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Grad. Texts in Math. 74, Springer, New York, 2000.

[Edd] S. S. Eddin, *An explicit upper bound for $|L(1,\chi)|$ when $\chi(2) = +1$ and χ is even*, Int. J. Number Theory 12 (2016), 2299–2315.

[GS] A. Granville and K. Soundararajan, *Upper bounds for $|L(1,\chi)|$*, Quart. J. Math. 53 (2002), 265–284.

[IM] Y. Ihara and V. K. Murty, *On the logarithmic derivatives of Dirichlet L-functions at $s = 1$*, Acta Arith. 137 (2009), 253–276.

[Lou93] S. Louboutin, *Majorations explicites de $|L(1,\chi)|$*, C. R. Acad. Sci. Paris 316 (1993), 11–14.

[Lou98a] S. Louboutin, *Majorations explicites du résidu au point 1 des fonctions zêta de certains corps de nombres*, J. Math. Soc. Japan 50 (1998), 57–69.

[Lou98b] S. Louboutin, *Upper bounds on $L(1,\chi)$ and applications*, Canad. J. Math. 50 (1998), 794–815.

[Lou00] S. Louboutin, *Explicit bounds for residues of Dedekind zeta functions, values of L-functions at $s = 1$ and relative class numbers*, J. Number Theory 85 (2000), 263–282.

[Lou01] S. Louboutin, *Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields*, Canad. J. Math. 53 (2001), 1194–1222.

[Lou04] S. Louboutin, *Explicit upper bounds for* $|L(1,\chi)|$ *for primitive characters* χ, Quart. J. Math. 55 (2004), 57–68.

[Lou08] S. Louboutin, *Some explicit upper bounds for residues of zeta functions of number fields taking into account the behavior of the prime 2*, Manuscripta Math. 125 (2008), 43–67.

[Lou11] S. Louboutin, *Upper bounds for residues of Dedekind zeta functions and class numbers of cubic and quartic number fields*, Math. Comp. 80 (2011), 1813–1822.

[Lou15] S. Louboutin, *Real zeros of Dedekind zeta functions*, Int. J. Number Theory 11 (2015), 843–848.

[Lou18] S. Louboutin, *Upper bounds on* $L(1,\chi)$ *taking into account a finite set of prime ideals*, Acta Arith. 182 (2018), 249–269.

[Pin] J. Pintz, *Elementary methods in the theory of L-functions. VII. Upper bounds on* $L(1,\chi)$, Acta Arith. 32 (1977), 397–406.

[PE] D. J. Platt and S. S. Eddin, *Explicit upper bounds for* $|L(1,\chi)|$ *when* $\chi(3)=0$, Colloq. Math. 133 (2013), 23–34.

[Ram01] O. Ramaré, *Approximate formulae for* $L(1,\chi)$, Acta Arith. 100 (2001), 245–266.

[Ram04] O. Ramaré, *Approximate formulae for* $L(1,\chi)$*. II*, Acta Arith. 112 (2004), 141–149.

[Toy] M. Toyoizumi, *Remarks on the upper bound for* $L(1,\chi)$, Acta Arith. 55 (1990), 137–140.

Stéphane R. Louboutin
Aix Marseille Univ, CNRS, Centrale Marseille, I2M
Marseille, France
E-mail: stephane.louboutin@univ-amu.fr

Abstract (will appear on the journal's web site only)

Various bounds on the absolute values of L-functions of number fields at $s = 1$ and on residues at $s = 1$ of Dedekind zeta functions of a number field $\mathbb{L}$ are known. Also, better bounds depending on the splitting behavior of given prime ideals of $\mathbb{L}$ of small norms are known. These bounds involve a term $\nu_{\mathbb{L}}$ in the series expansion at $s = 1$ of the Dedekind zeta function of $\mathbb{L}$. We explain why one should expect to have bounds on $\nu_{\mathbb{L}}$ which also depend on the splitting behavior in $\mathbb{L}$ of given prime integers. We explicitly do that for $\mathbb{L}$ a real quadratic number field. We deduce very good upper bounds on the residue at $s = 1$ of the Dedekind zeta function of a non-normal totally real cubic number field $\mathbb{K}$, bounds depending on the splitting behavior of the prime $p = 2$ in $\mathbb{K}$.