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An elementary proof of the compactness of the trace operator on a Lipschitz domain

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Abstract

In the setting of bounded strongly Lipschitz domains, we present a short and simple proof of the compactness of the trace operator acting on square integrable vector fields with square integrable divergence and curl, with a boundary condition. We rely on earlier trace estimates established in a similar setting.

1 Introduction and main theorem

The compactness of the trace operator on a bounded strongly Lipschitz domain Ω is well known in the scalar case. In fact it is continuous from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$, and the compactness of the injection $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ ensures that $\text{Tr}_{|\partial\Omega}$ maps compactly $H^1(\Omega)$ to $L^2(\partial\Omega)$. By looking coordinate by coordinate the scalar case implies that vector fields in $H^1(\Omega; \mathbb{R}^d)$ are also compact. However for less regular vector fields the situation is slightly different. In the study of Maxwell and Navier-Stokes equations, it is interesting to study vector fields $u \in L^2(\Omega; \mathbb{R}^d)$ with $\text{div } u \in L^2(\Omega)$ and $\text{curl } u \in L^2(\Omega; \mathcal{M}_d(\mathbb{R}))$, with a boundary condition on either $\nu \cdot u$ or $\nu \wedge u$. A natural boundary condition is $\nu \cdot u = 0$, and in this case the trace operator is still compact. In this paper we investigate a more general boundary condition where $\nu \cdot u$ is only in some L^p spaces ($p > 2$).

Let $\delta > 0$ and let

$$X_T^\delta = \{u \in L^2(\Omega; \mathbb{R}^d); \text{div } u \in L^2(\Omega), \text{curl } u \in L^2(\Omega; \mathbb{R}^d) \text{ and } \nu \cdot u \in L^{2+\delta}(\partial\Omega)\}$$

and

$$X_N^\delta = \{u \in L^2(\Omega; \mathbb{R}^d); \text{div } u \in L^2(\Omega), \text{curl } u \in L^2(\Omega; \mathbb{R}^d) \text{ and } \nu \wedge u \in L^{2+\delta}(\partial\Omega; \mathcal{M}_d(\mathbb{R}))\}.$$

where Ω is a domain of \mathbb{R}^d .

X_T^δ and X_N^δ are Banach spaces when equipped with the natural norms

$$\|u\|_{X_T^\delta} = \|u\|_{L^2(\Omega; \mathbb{R}^d)} + \|\text{div } u\|_{L^2(\Omega)} + \|\text{curl } u\|_{L^2(\Omega; \mathcal{M}_d(\mathbb{R}))} + \|\nu \cdot u\|_{L^{2+\delta}(\partial\Omega)} \quad (1)$$

$$\|u\|_{X_N^\delta} = \|u\|_{L^2(\Omega; \mathbb{R}^d)} + \|\text{div } u\|_{L^2(\Omega)} + \|\text{curl } u\|_{L^2(\Omega; \mathcal{M}_d(\mathbb{R}))} + \|\nu \wedge u\|_{L^{2+\delta}(\partial\Omega; \mathcal{M}_d(\mathbb{R}))}. \quad (2)$$

We also denote by X_T and X_N the subspaces of X_T^δ and X_N^δ with (respectively) $\nu \cdot u = 0$ and $\nu \wedge u = 0$ on $\partial\Omega$.

Our aim is to prove the following theorem:

Theorem 1.1. *Let $\delta > 0$, let $\Omega \subset \mathbb{R}^d$ be a strongly Lipschitz domain. Then the trace operator $\text{Tr}_{|\partial\Omega}$ maps compactly X_T^δ (or X_N^δ) to $L^2(\partial\Omega; \mathbb{R}^d)$.*

The trace operators on the spaces $X_{T,N}$ are well understood for domains with regular boundaries. When Ω has a regular enough boundary, the space X_T and X_N are in fact included in $H^1(\Omega; \mathbb{R}^d)$: for example if Ω is $C^{\frac{3}{2}+\varepsilon}$ [6], if Ω is $C^{1,1}$ [1, Theorems 2.9, 2.12], or even if Ω is convex [9]. If Ω is piece-wise C^∞ and C^2 -diffeomorphic to a polyhedron, there exists $\varepsilon > 0$ such that X_T and X_N are subsets of $H^{\frac{1}{2}+\varepsilon}(\Omega; \mathbb{R}^3)$, as in [5]. In all those cases, the compactness of the trace operator derives from the scalar case by applying scalar estimates coordinate by coordinate.

If $\nu \cdot u$ is non-zero but in $L^2(\partial\Omega)$, the trace operator is continuous - see for example [7, Theorem 11.2] for a general results for L^p spaces on manifolds. The compactness of the trace operator, however, is not known. In order to obtain compactness, one possibility is to add some regularity to $\nu \cdot u$: consider for example the space $\{u \in L^2(\Omega; \mathbb{R}^d); \text{div } u \in L^2(\Omega), \text{curl } u \in L^2(\Omega; \mathcal{M}_d(\mathbb{R})) \text{ and } \nu \cdot u \in H^{\frac{1}{2}}(\partial\Omega)\}$. It is a subspace of X_T^δ for some $\delta >$

0, and in the case of a $C^{1,1}$ domain in dimension 3, Amrouche and Seloula proved in [2, Theorem 3.5] that it is in fact a subspace of $H^1(\Omega, \mathbb{R}^3)$.

On a Lipschitz domain however, this method fails: ν is only in $L^\infty(\partial\Omega, \mathbb{R}^d)$ so $\nu \cdot u$ fails to be in $H^{\frac{1}{2}}(\partial\Omega)$ even when $u \in H^1(\Omega, \mathbb{R}^d)$. Both X_T and X_N are still subspaces of $H^{\frac{1}{2}}(\Omega, \mathbb{R}^d)$ (see [3] for dimension 2 and 3 and [7, Theorem 11.2] for the general result). However Costabel recently proved in [4] that there exist C^1 domains such that X_T and X_N are in none of the $H^{\frac{1}{2}+\varepsilon}(\Omega, \mathbb{R}^d)$ spaces. It is consequently impossible to operate coordinate by coordinate, but we can still prove that the trace operator is compact as soon as the integrability of the boundary condition is better than L^2 . Our main tool is an estimate established in [8] ; it can be adapted in the X_T^δ or X_N^δ settings, allowing us to prove first the continuity, then the compactness of the trace operator.

2 Notations

In this paper we will denote by Ω a bounded strongly Lipschitz domain, i.e. an open connected subset of \mathbb{R}^d , whose boundary can locally be written as the graph of a Lipschitz function and which is only on one side of its boundary.

Definition 2.1. We recall that Ω is a special Lipschitz domain if there exist a basis of \mathbb{R}^d and a Lipschitz function ω from \mathbb{R}^{d-1} to \mathbb{R} such that

$$\Omega = \{x = (x_h, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} ; x_d > \omega(x_h)\}. \quad (3)$$

Then the outward normal ν at $x = (x_h, \omega(x_h)) \in \partial\Omega$ is defined almost everywhere by:

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} (\nabla_h \omega(x_h), -1), \quad (4)$$

where ∇_h is the horizontal gradient acting on the horizontal variable x_h .

Since ω is Lipschitz, its gradient is bounded. Thus there is a constant $c > 0$ such that $\frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \geq c$ for almost every $x_h \in \mathbb{R}^{d-1}$. We can then denote by $\theta \in [0, \frac{\pi}{2}]$ the angle between the vertical direction and the outward normal:

$$\theta = \arccos \left(\inf_{x_h \in \mathbb{R}^{d-1}} \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \right). \quad (5)$$

And writing $e = (0_{\mathbb{R}^{d-1}}, -1)$ we get for almost every $x_h \in \mathbb{R}^{d-1}$:

$$0 < \cos \theta \leq \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} = e \cdot \nu(x_h, \omega(x_h)). \quad (6)$$

In order to get a compactness result on the trace operator, we of course need a bounded domain. We can make use of the special Lipschitz case through a finite covering:

Definition 2.2. We recall that $\Omega \subset \mathbb{R}^d$ is a bounded strongly Lipschitz domain if Ω is open, connected, bounded, and there exists a finite cover $(\Omega_k)_{1 \leq k \leq N}$ of $\partial\Omega$ such that for every k , $\Omega \cap \Omega_k$ coincide with a special Lipschitz domain.

Since $\Omega_k \cap \Omega$ coincides with a special Lipschitz domain, it is possible to define a minimal angle between the outward normal and the local vertical direction on each $\partial\Omega \cap \Omega_k$, as in (5): there exist a vector e_k and an angle θ_k such that k , for every $x \in \partial\Omega \cap \Omega$, $\cos \theta_k \leq e_k \cdot \nu(x)$. Since there is a finite number of $\cos \theta_k$, we can define γ as:

$$\gamma = \min(\cos \theta_k ; 1 \leq k \leq N). \quad (7)$$

γ only depends on $\partial\Omega$ and the choice of the covering $(\Omega_k)_{k \in \llbracket 0, N \rrbracket}$. In every case we have $\gamma > 0$.

Finally let us introduce some notations for vectorial analysis: let Ω be either a special Lipschitz or bounded strongly Lipschitz domain, and let $u \in L^2(\Omega, \mathbb{R}^d)$. We denote by $\text{curl } u \in \mathcal{M}_d(\mathbb{R})$ the antisymmetric part of the Jacobian matrix of first order partial derivatives in the sense of distributions:

$$(\text{curl } u)_{i,j} = \frac{1}{\sqrt{2}} (\partial_i u_j - \partial_j u_i) = \frac{1}{\sqrt{2}} (\nabla u - (\nabla u)^T)_{i,j}, \quad 1 \leq i, j \leq d. \quad (8)$$

We use on $\mathcal{M}_d(\mathbb{R})$ the scalar product $\langle v, w \rangle = \sum_{i,j=1}^d v_{i,j} w_{i,j}$ for $v = (v_{i,j})_{1 \leq i,j \leq d}$, $w = (w_{i,j})_{1 \leq i,j \leq d} \in \mathcal{M}_d(\mathbb{R})$, and we denote by $|\cdot|$ the associated norm.

We also define the wedge product of two vectors e and f of \mathbb{R}^d by $e \wedge f = \frac{1}{\sqrt{2}} (e_i f_j - e_j f_i)_{1 \leq i,j \leq d} \in \mathcal{M}_d(\mathbb{R})$.

Finally for any $u \in L^2(\Omega, \mathbb{R}^d)$ we can make sense of $\nu \cdot u$ and $\nu \wedge u$ as distributions on $\partial\Omega$, as in [8].

3 Proof of Theorem 1.1

In this section we prove the continuity and compactness of the trace operator. We will only write it in the X_T^δ case, as the proof in for X_N^δ is exactly similar.

- First we prove the continuity of the operator

$$\text{Tr}_{|\partial\Omega} : X_T^\delta \rightarrow L^2(\partial\Omega, \mathbb{R}^d). \quad (9)$$

Let $u \in X_T^\delta$ be a vector field. Applying estimate (4.12) in [8], there exists a constant $C(\Omega)$ such that:

$$\begin{aligned} \|\text{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega, \mathbb{R}^d)}^2 &\leq \frac{2}{\gamma} \|u\|_{L^2(\Omega, \mathbb{R}^d)} \left(2 \|\text{curl } u\|_{L^2(\Omega, \mathcal{M}_d(\mathbb{R}))} + 2 \|\text{div } u\|_{L^2(\Omega)} + C(\Omega) \|u\|_{L^2(\Omega, \mathbb{R}^d)} \right) \\ &+ \left(\left(\frac{4}{\gamma^2} + \frac{2}{\gamma} + 1 \right) \min \left\{ \|\nu \wedge u\|_{L^2(\partial\Omega, \mathcal{M}_d(\mathbb{R}))}^2, \|\nu \cdot u\|_{L^2(\partial\Omega)}^2 \right\} \right). \end{aligned} \quad (10)$$

Since Ω is bounded, $L^{2+\delta}(\partial\Omega) \subset L^2(\partial\Omega)$ and there exists a new constant $C(\Omega)$ such that:

$$\begin{aligned} \|\text{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega, \mathbb{R}^d)}^2 &\leq C(\Omega) \|u\|_{L^2(\Omega, \mathbb{R}^d)} \left(\|\text{curl } u\|_{L^2(\Omega, \mathcal{M}_d(\mathbb{R}))} + \|\text{div } u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega, \mathbb{R}^d)} \right) \\ &+ C(\Omega) \min \left\{ \|\nu \wedge u\|_{L^{2+\delta}(\partial\Omega, \mathcal{M}_d(\mathbb{R}))}^2, \|\nu \cdot u\|_{L^{2+\delta}(\partial\Omega)}^2 \right\}. \end{aligned} \quad (11)$$

Finally we get $\|\text{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega, \mathbb{R}^d)}^2 \leq C(\Omega) \|u\|_{X_T^\delta}^2$, so $\text{Tr}_{|\partial\Omega}$ is bounded from X_T^δ to $L^2(\partial\Omega, \mathbb{R}^d)$.

- This estimate on the trace actually allows us to prove the compactness as follow:

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in X_T^δ , bounded by a constant $K > 0$.

The embedding $X_T^\delta \hookrightarrow L^2(\Omega, \mathbb{R}^d)$ is compact, because $X_T^0 \hookrightarrow L^2(\Omega, \mathbb{R}^d)$ is compact (see the third remark after Theorem 11.2 in [7]). We can extract a subsequence, again denoted u , such that:

$$u_n \xrightarrow[n \rightarrow \infty]{} u \text{ in } L^2(\Omega, \mathbb{R}^d).$$

Besides the injection $L^{2+\delta}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is also compact, so for any sequence $\nu \cdot u \in L^{2+\delta}(\partial\Omega)$ and for any $\nu \wedge u \in L^{2+\delta}(\partial\Omega, \mathcal{M}_d(\mathbb{R}))$, we can also extract subsequences converging in L^2 .

Let ℓ, k be two integers. Applying estimate (11) to $u_\ell - u_k$ we get:

$$\begin{aligned} \|\text{Tr}_{|\partial\Omega}(u_\ell - u_k)\|_{L^2(\partial\Omega, \mathbb{R}^d)}^2 &\leq C(\Omega) \|u_\ell - u_k\|_{L^2(\Omega, \mathbb{R}^d)} \|u_\ell - u_k\|_{X_T^\delta} \\ &+ C(\Omega) \min \left\{ \|\nu \wedge (u_\ell - u_k)\|_{L^2(\partial\Omega, \mathcal{M}_d(\mathbb{R}))}^2, \|\nu \cdot (u_\ell - u_k)\|_{L^2(\partial\Omega)}^2 \right\}. \end{aligned}$$

Since the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded by K , $\|u_\ell - u_k\|_{X_T^\delta} \leq 2K$. Moreover $(u_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega, \mathbb{R}^d)$, and both $(\nu \wedge u)_{n \in \mathbb{N}}$ and $(\nu \cdot u)_{n \in \mathbb{N}}$ converge in L^2 norms, so $(\text{Tr}_{|\partial\Omega} u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\partial\Omega, \mathbb{R}^d)$. This proves that if $(u_n)_{n \in \mathbb{N}}$ is bounded in X_T^δ , then there exists a subsequence of $(u_n)_{n \in \mathbb{N}}$ such that $(\text{Tr}_{|\partial\Omega} u_n)_{n \in \mathbb{N}}$ converges in $L^2(\partial\Omega, \mathbb{R}^d)$, which proves that the trace operator is compact.

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