Derivation of Malus' law by purely classical reasoning expressed in the language of group representation theory

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Abstract. This work is part of a reconstruction of quantum mechanics from scratch with the aim to understand what it means. In previous work we have shown that the Dirac equation can be derived by classical reasoning just using relativity and group theory. This implies that contrary to common belief quantum mechanics (QM) is not magical or radically different from classical mechanics. It is also not incompatible with the theory of relativity but completely part of it. As the experimental violations of the Bell inequalities seem to take exception with this general scheme for making sense of QM we have scrutinized the derivation of the Bell inequalities to figure out the limitations of our approach. We were able to show that the derivation contains a logical error based on wrong modelling. In the present paper we complete this investigation by showing that Malus' law can be derived by classical reasoning using group representation theory, both for electrons and photons. This is a further, be it somewhat more indirect proof that the derivation of the Bell inequalities contains an error. But more importantly it shows that the philosophy of our approach remains intact, by proving that we can also understand Malus' law with our methods and thereby validating our approach also for this physical phenomenon.

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1 Introduction - Malus' law seems to defy classical understanding

The present work is about the probabilities that intervene in the legendary photon correlation experiments by Aspect *et al.* [1,2,3] and its recent improvements [4,5]. In these experiments one measures the probabilities for the joint transmission of two correlated photons by two polarization filters with orientations a_j and b_k specified by the angles α_j and β_k:

$$p(a_j \wedge b_k) = \frac{1}{2}\cos^2(\alpha_j - \beta_k), \quad p(a_j) = \frac{1}{2}, \quad p(b_k) = \frac{1}{2}. \tag{1}$$

We can try to derive the results of Equation 1 by classical reasoning. But we seem to fail to find such a derivation. An example of such an unsuccessful classical calculation is the following. Let us assume that a source produces photon pairs that are both linearly polarized along an angle φ. Then applying Malus' law the probabilities for the transmissions of the filters at angles α_j, β_k will be $\cos^2(\alpha_j - \varphi)$ and $\cos^2(\beta_k - \varphi)$ respectively. Under the assumption that the probability of the angle φ is uniformly distributed, we obtain by integration over φ the following expressions for the result of the experiment:

$$\begin{aligned} &p(a_j \wedge b_k) = \int_0^{2\pi} \cos^2(\alpha_j - \varphi)\cos^2(\beta_k - \varphi)\, d\varphi = \frac{1}{8} + \frac{1}{4}\cos^2(\alpha_j - \beta_k), \\ &p(a_j) = \frac{1}{2}, \quad p(b_k) = \frac{1}{2}. \end{aligned} \tag{2}$$

This is not in agreement with Eq. 1 which reproduces the experimental results and is also the result predicted by quantum mechanics (QM). We may speculate that the failure of the attempts to derive Eq. 1 could be the consequence

of not choosing the right assumptions to start from. This keeps us then wondering if a classical derivation may exist or otherwise. A violation of Bell's inequality is deemed to show that the results of QM cannot be reproduced by classical reasoning based on hidden variables. It ought to put an end to any hopes or speculation about the possibility of deriving Malus' law by classical reasoning, by proving that such an alternative derivation just does not exist. Very much like Galois' proof that there is no general solution for the quintic equation by radicals, this is an impossibility proof which shows that it is pointless to keep searching further for a classical derivation. That is the importance of the inequality.

It all leads to the notion that QM is mysterious and not open to classical understanding. But if the violation of the inequalities were to prove that "Einstein was wrong", by what kind of theory should we then replace the theory of relativity, given that the violation of the inequality also invalidates the tenets of realism and/or locality, which are cornerstones of the theory of relativity?

In our efforts to reconstruct QM from scratch with the aim of obtaining a better understanding of it, we have obtained very convincing results which show that QM is not incompatible with the theory of relativity and is actually part of it. In [6,7] we derived the Dirac equation from scratch using only group representation theory and classical reasoning. We were also able to explain the Stern-Gerlach experiment [8]. These violations of the Bell inequalities were not in line with our results which indicated that QM is based on classical reasoning formulated in the language of group representation theory. As the consequences of the alleged violations of the Bell inequalities were just too damaging, we set out to prove in [9] that the derivations of the Bell inequalities are logically flawed. As a matter of fact, they have been based on wrongly translating the physics into mathematics in the modelling. We have also proved that Larsson's rebuttal of this argument [10] was still overlooking a normalization error.

Now that we have debunked the conclusions reached from the violation of the Bell inequalities, we can raise again the question if a classical derivation of Malus' law could perhaps exist. If our approach is right by claiming that QM is really just classical physics written in the language of group theory, such a derivation *must* exist. We will here provide such a classical derivation. This will be further evidence for the fact that the derivation of the Bell inequalities cannot be right. This will also illustrate the relation between the geometry and the algebra in our approach. Our calculation is completely based on the use of group representation theory and an excellent command of this subject matter is therefore an essential prerequisite for being able to follow the argument. The reader can gain the required level of understanding by reading references [6], [8], [11], which will also explain to him what is going on behind the scenes of the nitty-gritty of group representation theory.

We must warn the reader that the contents of these references are beyond guessing. It should be tell-tale enough in this respect that our work really explains what QM means by reconstructing it from scratch. It operates a reverse paradigm shift in the sense that it re-establishes the old paradigm of classical reasoning that was cherished by Einstein and challenged by QM. Our results deconstruct the traditional narrative and are flying bluntly in the face of it, but they are entirely logical and exact. They do not attack QM. On the contrary they confirm the exactitude of the formalism of QM. However, they lay bare the frivolity of the interpretations that have been dished out to us and replace them by the correct geometrical meaning of the algebra as given by the mathematics. This can then no longer be considered as an interpretation since rigorous mathematics are just not subject to interpretation. The geometrical meaning of the algebra of group representation theory is an unassailable objective mathematical reality based on an isomorphism that we can firmly establish prior to any application of it to physics and therefore is beyond any questioning by physicists. In other words, we claim that the algebra of QM is rigorously exact but that the narrative woven around it is disconcertingly tainted with undue *ad hoc* parallel interpretations of the geometrical meaning of the algebra of group representation theory. We must warn the reader that the contents of references [6], [8] and [11] are not stuff for the narrow-minded or the faint-hearted. Almost certainly it will come as a shock. It is therefore futile to want to read or comment on the contents of the present paper without consulting first these references, especially the parts of them which explain the meaning of spinors. In the following sections we are providing some additional information and clarifications. These sections contain all the preparations we need to set up the final stage for the calculation in Section 10.

2 Spinors do not form a vector space

A spinor in SU(2) is just a convenient shorthand for the full representation matrix [6,11]. In fact, a single column of an SU(2) matrix contains already the whole information about the matrix. Therefore a spinor of SU(2) just represents a group element [6,11]. The group of rotations in $\mathbb{R}^3$ form a non-abelian group. For any group $(\mathscr{G}, \circ)$, where $\circ$ is the composition law defined on the set $\mathscr{G}$, a linear combination of group elements $\sum_{j=1}^{n} c_j g_j$, where $c_j \in \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $g_j \in \mathscr{G}$, is not defined by the axioms for a group. In fact, the operations of summing and multiplying with a scalar are not part of the structure of a group. They are defined by the richer and more encompassing structure of a vector space and non-abelian groups are not vector spaces but curved manifolds.

Hence when we make linear combinations of group representation matrices $\sum_{j=1}^{n} c_j \mathbf{D}(g_j)$ this may well be defined algebraically but it remains geometrically meaningless, because $\sum_{j=1}^{n} c_j g_j$, that would be its counterpart in the iso-

morphism between g_j and $\mathbf{D}(g_j)$ is just not defined. To attribute a meaning to $\sum_{j=1}^{n} c_j g_j$, we must embed the group in a vector space $\mathscr{V}$ where the operations become allowed. By the presence of the operation $\circ$, the vector space $(\mathscr{V},\cdot,+)$ then actually becomes an algebra $(\mathscr{V},\cdot,+,\circ)$ on the set $\mathscr{V}$.

Despite the extension of $\mathscr{G}$ to the embedding vector space $\mathscr{V}$ with the calculus of its algebra, the elements of $\mathscr{V}\backslash\mathscr{G}$ will in general remain meaningless. An illuminating example of this is given by general relativity where space-time is a curved manifold which we could embed mathematically in a "flat" vector space $\mathbb{R}^5$ but where the points of $\mathbb{R}^5$ that do not belong to space-time are physically meaningless. It is for these reasons that general relativity is formulated intrinsically. In the same spirit the rotation groups SU(2) and SO(n) must be treated intrinsically without calling upon an embedding. For the rotation group SU(2) we have shown in [12] that attributing a geometrical meaning to the extrapolation of spinors from a shorthand for the curved manifold $\mathscr{G}=$ SU(2) to the obvious embedding in the vector space $\mathscr{V}=\mathbb{C}^2$ by using linear combinations of representation matrices is geometrically pointless. In fact, in SU(2) the 2×2 representation matrices $\mathbf{D}(g)$ transform a vector $\mathbf{v}\in\mathbb{R}^3$ with 2×2 representation matrix $\mathbf{V}$ "*quadratically*" into a vector $\mathbf{w}\in\mathbb{R}^3$ with representation matrix $\mathbf{W}$ according to $\mathbf{V}\rightarrow\mathbf{W}=[\,\mathbf{D}(g)\,][\,\mathbf{V}\,][\,\mathbf{D}(g)\,]^{-1}$. On the other hand, in SO(3) the 3×3 representation matrices $\mathbf{D}(g)$ transform $\mathbf{v}\in\mathbb{R}^3$ with 3×1 representation matrix $\mathbf{V}$ *linearly* into a vector $\mathbf{u}\in\mathbb{R}^3$ with 3×1 representation matrix $\mathbf{U}$ according to $\mathbf{V}\rightarrow\mathbf{U}=[\,\mathbf{D}(g)\,][\,\mathbf{V}\,]$. We have worked out an example of this in [12] where we have shown that the object g_1+g_2 that in the extrapolation would correspond to $\mathbf{D}(g_1+g_2)=\mathbf{D}(g_1)+\mathbf{D}(g_2)$ is not the same in SU(2) as in SO(3), as the calculation of $(g_1+g_2)(\mathbf{v})$ does not yield the same result in SU(2) as in SO(3). In other words: $\mathbf{u}\neq\mathbf{w}$.

3 Probabilities and the Born rule

But we cannot pretend to ignore that physics uses such linear combinations all the time with excellent results. There must therefore exist a justification for this. We have in this respect been able to show that the linear combinations can be given a meaning in terms of sets. This can be deduced from the fact that for finite groups one defines so-called all-commuting or Casimir operators:

$$\sum_{h\in\mathscr{C}}\mathbf{D}(h),\quad \text{for which:}\quad \forall g\in\mathscr{G}:\mathbf{D}(g)\,[\sum_{h\in\mathscr{C}}\mathbf{D}(h)\,]=[\sum_{h\in\mathscr{C}}\mathbf{D}(h)\,]\,\mathbf{D}(g). \tag{3}$$

Here $\mathscr{C}=\{h_1,h_2,\cdots h_k\}\subset\mathscr{G}$ is a so-called normal subgroup and the equation then just translates the definition of a normal subgroup:

$$\forall g\in\mathscr{G}: g\circ\mathscr{C}=\mathscr{C}\circ g, \tag{4}$$

into *a priori* meaningless algebra, by writing down purely formally:

$$g\circ\{h_1,h_2,\cdots h_k\}=\{h_1,h_2,\cdots h_k\}\circ g\quad\text{as:}\quad g\circ(h_1+h_2+\cdots+h_k)=(h_1+h_2+\cdots+h_k)\circ g. \tag{5}$$

Based on this observation it becomes natural to introduce the notion that the sum defines a set and can be used to represent a set of group elements. In SU(2), the linear combination $\sum_{j=1}^{n} c_j\mathbf{D}(g_j)$ describes then a set containing $|c_j|^2$ copies of g_j. These copies could be objects that are in the orientation (or the state) described by g_j. The reason for using the rule $|c_j|^2$ is that we associate each element of the set with a spinor $\boldsymbol{\xi}=[\,\xi_0,\xi_1\,]^{\top}$, and that spinors satisfy $\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}=1$. When we want to count group elements or the objects they represent, we must count the spinors associated with them. This can be done by counting the spinors in terms of the units $\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}=1$ they intrinsically carry with them in their algebraic structure. Hence, originally we must conceive that $|c_j|^2\in\mathbb{N}\cup\{0\}$. But by using frequencies instead of counts we can give also meaning to $|c_j|^2\in\mathbb{R}$. It becomes then a probability and we obtain a justification for the Born rule. The sums that correspond to the sets are not true sums for which carrying out the algebra would be defined. They must be considered as mere juxtapositions, just like elements of sets are occurring by mere juxtaposition in those sets. In other words, this procedure explains the rule for incoherent summing of probabilities. For coherent summing we have not found a group-theoretical justification, such that coherent summing will always call for special caution and treatment. All this shows how crucial it is to be aware of the fact that sums and multiplications by scalars are in principle not defined for group elements. It is an unsuspected eye-opener that unlocks the door to a probability calculus on sets, which otherwise could have remained beyond our conceptual grasps. The sets will be beams or statistical ensembles of particles represented by waves (because they have a periodic internal motion, see below).

As discussed in [6], the idea that a column matrix like a 2×1 spinor in SU(2) represents a group element is not general, but it is rigorously correct in SU(2). A very convincing counter-example occurs in the rotation group SO(5). The Clifford algebra of SO(n) uses $2^{\lfloor n/2\rfloor}\times 2^{\lfloor n/2\rfloor}$ representation matrices, while completely specifying an element of SO(n) requires $n(n-1)/2$ independent real parameters. Here $\lfloor\cdot\rfloor$ stands for the largest integer part. Hence SO(5)

uses 4×4 representation matrices but completely specifying one of its group elements requires 10 independent real parameters. Now the 4×1 column matrices can only accommodate 8 independent real parameters, which is not enough to specify the 10 independent parameters required. In reality only 7 independent real parameters are available as the column matrices must be normalized to 1 using the Hermitian norm. For SO(5) a 4×1 column matrix can therefore not represent a group element. After noticing this, one can decide to continue calling the column matrix a spinor. One can also decide that we rather consider any complete representation of a group element a spinor. However, the idea that a spinor represents a group element is also correct within the context of the Dirac equation, be it for a more complicated reason. It is then a so-called bi-spinor (a superposition state describing a set). In the following Section we will warn for some more consequences of the fact that a curved group manifold is not a vector space. We will see that the particle-wave duality in QM corresponds to the fact that the dynamics of single particles are described by group elements while sets of moving dynamical particles behave like propagating waves.

4 Eigenvalues and eigenvectors of rotations in SU(2)

Let us calculate the eigenvalues and eigenvectors of an SU(2) rotation matrix:

$$\begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} = \cos(\varphi/2)\mathbb{1} - \imath \sin(\varphi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \cos(\varphi/2)\mathbb{1} - \imath \sin(\varphi/2) \begin{bmatrix} s_z & s_x - \imath s_y \\ s_x + \imath s_y & -s_z \end{bmatrix}. \tag{6}$$

Here $\mathbb{1}$ is the 2×2 identity matrix, $\mathbf{s}$ the unit vector along the axis of rotation, φ the rotation angle and $uu^* + vv^* = 1$. The expression of the rotation matrix in terms of $\mathbf{s}$ and φ is called the Rodrigues formula. The notation $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ is not a true scalar product. It is just a convenient shorthand notation for $s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$. The Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ represent the unit basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, such that $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ is the representation matrix of the unit vector $\mathbf{s}$ (and also the reflection with respect to the plane through the origin which is perpendicular to $\mathbf{s}$).[1] The eigenvalue equation is:

$$\begin{vmatrix} u - \lambda & -v^* \\ v & u^* - \lambda \end{vmatrix} = uu^* + vv^* - \lambda(u + u^*) + \lambda^2 = 0. \tag{7}$$

From the Rodriguez formula it follows that:

$$u + u^* = 2\cos(\varphi/2). \tag{8}$$

Therefore the eigenvalues are $e^{\pm \imath\varphi/2}$. Based on considerations of purely algebraic expedience, we could also have introduced $u + u^* = 2\cos(\varphi/2)$ abstractly in order to solve the quadratic equation, without knowing the geometrical meaning of the quantity $\cos(\varphi/2)$ introduced this way. Subtracting $\cos(\varphi/2) = (u + u^*)/2$ from both sides of the Rodrigues equation we obtain:

$$\begin{bmatrix} (u - u^*)/2 & -v^* \\ v & (u^* - u)/2 \end{bmatrix} = -\imath \sin(\varphi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]. \tag{9}$$

As $\sqrt{4 - (u + u^*)^2} = 2\sin(\varphi/2)$, we have:

$$[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \frac{2\imath}{\sqrt{4 - (u + u^*)^2}} \begin{bmatrix} (u - u^*)/2 & -v^* \\ v & (u^* - u)/2 \end{bmatrix}. \tag{10}$$

To calculate the eigenvector for $\lambda = e^{\imath\varphi/2}$ we must calculate the matrix:

$$\begin{bmatrix} u - \lambda & -v^* \\ v & u^* - \lambda \end{bmatrix} = \begin{bmatrix} u - e^{\imath\varphi/2} & -v^* \\ v & u^* - e^{\imath\varphi/2} \end{bmatrix}. \tag{11}$$

But this is:

$$\mathbf{M} = \begin{bmatrix} (u - u^*)/2 - \imath \sin(\varphi/2) & -v^* \\ v & (u^* - u)/2 - \imath \sin(\varphi/2) \end{bmatrix} = -\imath \sin(\varphi/2)\,(\mathbb{1} + [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]). \tag{12}$$

[1] Hence, contrary to what physicists believe, $\boldsymbol{\sigma}$ has nothing to do with spin. In the present paper the spin vector appears in general under the form $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$. That is the correct mathematical expression for spin in SU(2) and this is not a point that would be open to discussion. Those who feel entitled to question this purely mathematical factual truth are just mistaken and should revise their maths, honouring my advice to consult the references I provided in Section 1, where this has been explained e.g. in Remark 13 of [6], pp. 3-4 of [8], and Footnote 12 of [11].

We must find a column vector $[\,w_1, w_2\,]^\top$ such that $\mathbf{M}[\,w_1, w_2\,]^\top = [0,0]^\top$. That is, we must have $(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])[\,w_1, w_2\,]^\top = [\,0,0\,]^\top$. Of course the solution is:

$$\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] = \left[\begin{array}{c} -v \\ (u-u^*)/2 - \imath\sin(\varphi/2) \end{array}\right]. \tag{13}$$

However, we will introduce a form that is more open to geometrical interpretation. It is obvious that $(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\,(\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]) = \mathbf{0}$. And as $\det(\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]) = 0$ we must be able to write:

$$\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \left[\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right] \otimes \left[\begin{array}{cc} \psi_0 & \psi_1 \end{array}\right]. \tag{14}$$

Such that we must have:

$$(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\left[\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \tag{15}$$

because this is absolutely necessary to obtain $(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\,[\,\chi_0,\chi_1\,]^\top \otimes [\psi_0,\psi_1] = \mathbf{0}$. Hence, $[\,w_0, w_1\,]^\top = [\,\chi_0,\chi_1\,]^\top$. Putting $\mathbf{s} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, we obtain:

$$\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \left[\begin{array}{cc} 1-\cos\theta & -\sin\theta e^{-\imath\phi} \\ -\sin\theta\, e^{+\imath\phi} & 1+\cos\theta \end{array}\right] = 2\left[\begin{array}{c} \sin(\theta/2) \\ -\cos(\theta/2)\; e^{+\imath\phi} \end{array}\right] \otimes \left[\begin{array}{cc} \sin(\theta/2) & -\cos(\theta/2)\; e^{-\imath\phi} \end{array}\right]. \tag{16}$$

Note that we make a distinction here between the two symbols φ (the rotation angle) and ϕ (a spherical coordinate of the rotation axis). The first eigenvector is thus:

$$\lambda = e^{\imath\varphi/2} \quad \rightarrow \quad \left[\begin{array}{c} \sin(\theta/2) \\ -\cos(\theta/2)\; e^{+\imath\phi} \end{array}\right]. \tag{17}$$

To calculate the eigenvector for $\lambda = e^{-\imath\varphi/2}$ we must calculate the matrix:

$$\left[\begin{array}{cc} u-\lambda & -v^* \\ v & u^*-\lambda \end{array}\right] = \left[\begin{array}{cc} u-e^{-\imath\varphi/2} & -v^* \\ v & u^*-e^{-\imath\varphi/2} \end{array}\right]. \tag{18}$$

But this is:

$$\mathbf{N} = \left[\begin{array}{cc} (u-u^*)/2 + \imath\sin(\varphi/2) & -v^* \\ v & (u^*-u)/2 + \imath\sin(\varphi/2) \end{array}\right] = \imath\sin(\varphi/2)(\,\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]). \tag{19}$$

We must find a column vector $[\,k_1, k_2\,]^\top$ such that $\mathbf{N}[\,k_1, k_2\,]^\top = [0,0]^\top$. That is, we must solve $(\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\,[\,k_1, k_2\,]^\top = [\,0,0\,]^\top$. The brute-force solution is:

$$\left[\begin{array}{c} k_1 \\ k_2 \end{array}\right] = \left[\begin{array}{c} v \\ -(u-u^*)/2 + \imath\sin(\varphi/2) \end{array}\right]. \tag{20}$$

But we can again use the fact that $(\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\,(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]) = \mathbf{0}$. And as $\det(\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]) = 0$ we must be able to write:

$$\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \left[\begin{array}{c} \zeta_0 \\ \zeta_1 \end{array}\right] \otimes \left[\begin{array}{cc} \eta_0 & \eta_1 \end{array}\right], \tag{21}$$

such that:

$$(\mathbb{1}-[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,])\left[\begin{array}{c} \zeta_0 \\ \zeta_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]. \tag{22}$$

Hence, $[\,k_0, k_1\,]^\top = [\,\zeta_0,\zeta_1\,]^\top$. As $\mathbf{s} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, we have:

$$\mathbb{1}+[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \left[\begin{array}{cc} 1+\cos\theta & \sin\theta e^{-\imath\phi} \\ \sin\theta\, e^{+\imath\phi} & 1-\cos\theta \end{array}\right] = 2\left[\begin{array}{c} \cos(\theta/2) \\ \sin(\theta/2)\, e^{+\imath\phi} \end{array}\right] \otimes \left[\begin{array}{cc} \cos(\theta/2) & \sin(\theta/2)\; e^{-\imath\phi} \end{array}\right]. \tag{23}$$

The second eigenvector is thus:

$$\lambda = e^{-\imath\varphi/2} \quad \rightarrow \quad \left[\begin{array}{c} \cos(\theta/2) \\ \sin(\theta/2)\; e^{+\imath\phi} \end{array}\right]. \tag{24}$$

Hence:

$$\left[\begin{array}{cc} u & -v^* \\ v & u^* \end{array}\right]\left[\begin{array}{cc} \cos(\theta/2) & -\sin(\theta/2)\,e^{-\imath\phi} \\ \sin(\theta/2)\,e^{+\imath\phi} & +\cos(\theta/2) \end{array}\right]=\left[\begin{array}{cc} \cos(\theta/2) & -\sin(\theta/2)e^{-\imath\phi} \\ \sin(\theta/2)\;e^{+\imath\phi} & +\cos(\theta/2) \end{array}\right]\left[\begin{array}{cc} e^{-\imath\varphi/2} & \\ & e^{+\imath\varphi/2} \end{array}\right], \quad (25)$$

and consequently:

$$\left[\begin{array}{cc} u & -v^* \\ v & u^* \end{array}\right]=\left[\begin{array}{cc} \cos(\theta/2) & -\sin(\theta/2)e^{-\imath\phi} \\ \sin(\theta/2)\,e^{+\imath\phi} & +\cos(\theta/2) \end{array}\right]\left[\begin{array}{cc} e^{-\imath\varphi/2} & \\ & e^{+\imath\varphi/2} \end{array}\right]\left[\begin{array}{cc} \cos(\theta/2) & \sin(\theta/2)e^{-\imath\phi} \\ -\sin(\theta/2)\;e^{+\imath\phi} & +\cos(\theta/2) \end{array}\right], \quad (26)$$

which leads to:

$$\begin{aligned}\left[\begin{array}{cc} u & -v^* \\ v & u^* \end{array}\right]&=\left[\begin{array}{c} \cos(\theta/2) \\ +\sin(\theta/2)\,e^{+\imath\phi} \end{array}\right]\otimes\left[\begin{array}{cc} +\cos(\theta/2) & +\sin(\theta/2)e^{-\imath\phi} \end{array}\right]e^{-\imath\varphi/2} \\ &+\left[\begin{array}{c} -\sin(\theta/2)e^{-\imath\phi} \\ \cos(\theta/2) \end{array}\right]\otimes\left[\begin{array}{cc} -\sin(\theta/2)\;e^{+\imath\phi} & +\cos(\theta/2) \end{array}\right]e^{+\imath\varphi/2}. \end{aligned} \quad (27)$$

We can express this fully in terms of u and v by identification with the aid of Eq. 10:

$$\begin{aligned}\frac{2\imath}{\sqrt{4-(u+u^*)^2}}\left[\begin{array}{cc} (u-u^*)/2 & -v^* \\ v & (u^*-u)/2 \end{array}\right]&=\left[\begin{array}{cc} \cos\theta & \sin\theta e^{-\imath\phi} \\ \sin\theta\,e^{+\imath\phi} & -\cos\theta \end{array}\right]=\left[\begin{array}{cc} s_z & s_x-\imath s_y \\ s_x+\imath s_y & -s_z \end{array}\right] \\ &=\left[\begin{array}{cc} 2\cos^2(\theta/2)-1 & 2\sin(\theta/2)\cos(\theta/2)\,e^{-\imath\phi} \\ 2\sin(\theta/2)\cos(\theta/2)\,e^{+\imath\phi} & 1-2\cos^2(\theta/2) \end{array}\right]. \end{aligned} \quad (28)$$

We should then up to normalization factors recover the results of the brute-force solutions in Eqs. 13 and 20, but this laborious. We may finally note that the eigenvectors we have defined this way correspond to sets. In fact, we have e.g.

$$[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\left[\begin{array}{c} \zeta_0 \\ \zeta_1 \end{array}\right]=\left[\begin{array}{c} \zeta_0 \\ \zeta_1 \end{array}\right]. \quad (29)$$

This gives the impression that the "spinor" $[\,\zeta_0,\zeta_1\,]^\top$ is an eigenstate of the reflection operator. But only sets can be eigenvectors of reflection operators because a reflection transforms a rotation into a reversal and a reversal into a rotation. Therefore, whenever we use eigenstates of reflection operators we are no longer talking about genuine spinors but about sets. Hence, when we will write below that $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\boldsymbol{\xi}_\uparrow=\boldsymbol{\xi}_\uparrow$ the quantity $\boldsymbol{\xi}_\uparrow$ will represent a set. Reasoning on Eq. 41 one may conclude that this set is identical to a spinor but this error is due to the coincidence that the first columns of $\mathbb{1}$ and σ_z are algebraically identical. Pure spinors are geometrically different from sets.

5 Eigenvalues and eigenvectors of reflections in SU(2)

Let the reflection matrix be:

$$[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]=\left[\begin{array}{cc} a_z & a_x-\imath a_y \\ a_x+\imath a_y & -a_z \end{array}\right]. \quad (30)$$

The eigenvalues are determined by:

$$\left|\begin{array}{cc} a_z-\lambda & a_x-\imath a_y \\ a_x+\imath a_y & -a_z-\lambda \end{array}\right|=\lambda^2-(a_x^2+a_y^2+a_z^2)\quad\Rightarrow\quad\lambda\in\{-1,1\}. \quad (31)$$

The first eigenvector $[\,k_1,k_2\,]^\top$, for $\lambda=1$, is determined by:

$$\left[\begin{array}{cc} a_z-1 & a_x-\imath a_y \\ a_x+\imath a_y & -a_z-1 \end{array}\right]\left[\begin{array}{c} k_1 \\ k_2 \end{array}\right]=0,\quad\text{i.e.}\quad(\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]-\mathbb{1}\,)\,[k_1,k_2\,]^\top=0. \quad (32)$$

As $(\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]-\mathbb{1}\,)(\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]+\mathbb{1}\,)=0$ according to the same argument as the one that led to Eq. 24 we will have:

$$\lambda = +1 \quad \rightarrow \quad \begin{bmatrix} \cos(\theta/2) \\ +\sin(\theta/2)\, e^{+\imath\phi} \end{bmatrix}. \tag{33}$$

The second eigenvector $[\, w_1, w_2 \,]^\top$, for $\lambda = -1$, is determined by $([\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]+\mathbb{1}\,)\,[\, w_1, w_2 \,]^\top = 0$. And as $([\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]+\mathbb{1}\,)\,([\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]-\mathbb{1}\,) = 0$, according to the same argument as the one that led to Eq. 17 we obtain:

$$\lambda = -1 \quad \rightarrow \quad \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2)\, e^{+\imath\phi} \end{bmatrix}. \tag{34}$$

We have then:

$$\begin{bmatrix} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2)e^{-\imath\phi} \\ \sin(\theta/2)\, e^{+\imath\phi} & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-\imath\phi} \\ -\sin(\theta/2)\, e^{+\imath\phi} & \cos(\theta/2) \end{bmatrix}. \tag{35}$$

This leads to:

$$\begin{aligned} \begin{bmatrix} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{bmatrix} = & \begin{bmatrix} \cos(\theta/2) \\ +\sin(\theta/2)\, e^{+\imath\phi} \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta/2) & +\sin(\theta/2)e^{-\imath\phi} \end{bmatrix} \\ & - \begin{bmatrix} -\sin(\theta/2)e^{-\imath\phi} \\ \cos(\theta/2) \end{bmatrix} \otimes \begin{bmatrix} -\sin(\theta/2)\; e^{+\imath\phi} & \cos(\theta/2) \end{bmatrix}. \end{aligned} \tag{36}$$

A rotation with an axis defined by the unit vector $\mathbf{s}$ and a reflection defined by the unit vector $\mathbf{s}$ are diagonalized by the same transfer matrix. This is obvious because the rotation is represented by $\cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ and a matrix which diagonalizes $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ will also diagonalize $\mathbb{1}$. This allows us to see immediately that the product $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,][\,\mathbf{R}(\mathbf{s},\varphi)\,][\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ corresponds to a rotation over an angle φ around the "reflected" rotation axis $\mathbf{s}'$, where (up to a sign) $\mathbf{s}'$ is the reflection of $\mathbf{s}$ with respect to the plane orthogonal to $\mathbf{a}$.

6 Spin and energy of the electron described within the framework of SU(2)

6.1 Spin-up and spin-down

By using the identities:

$$\cos x = \frac{e^{+\imath x} + e^{-\imath x}}{2}, \quad \sin x = \frac{e^{+\imath x} - e^{-\imath x}}{2\imath}, \tag{37}$$

we can rewrite the Rodrigues equation for a rotation $\mathbf{R}(\mathbf{s},\varphi)$ around an axis d defined by the unit vector $\mathbf{s} \parallel d$ and over an angle φ:

$$\mathbf{R}(\mathbf{s},\varphi) = \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,], \tag{38}$$

as:

$$\mathbf{R}(\mathbf{s},\varphi) = \frac{1}{2}\,[\,\mathbb{1} + [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]]\; e^{-\imath\varphi/2} + \frac{1}{2}\,[\,\mathbb{1} - [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]]\; e^{+\imath\varphi/2}. \tag{39}$$

This is actually nothing more than a brute-force calculation of Eq. 27. According to what we explained above, this is absolute non-sense, unless we start to interpret the formalism in terms of sets. This becomes important when we want to describe spinning electrons in QM. When we put $\varphi = \omega_0\tau$, where ω_0 is the angular frequency in the rest frame of the electron and τ its proper time, we obtain:

$$\mathbf{R}(\mathbf{s},\tau) = \frac{1}{2}\,[\,\mathbb{1} + [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]]\; e^{-\imath\omega_0\tau/2} + \frac{1}{2}\,[\,\mathbb{1} - [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]]\; e^{+\imath\omega_0\tau/2}. \tag{40}$$

This equation describes the simple intuitive dynamics of spinning as the superposition of two states, one with a frequency ω_0 and one with a frequency $-\omega_0$. As already mentioned, such a superposition describes a set. It describes thus a set of spinning things, half of them which have the frequency ω_0 and half of them the frequency $-\omega_0$. We call them "things" rather then particles because $\mathbb{1} + [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ is itself a set $\mathscr{S}_+ = \{\mathbb{1}, [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\}$. This set is an eigenstate of $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$, with eigenvalue $+1$, *viz.* $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\mathscr{S}_+ = \mathscr{S}_+$. Similarly, $\mathbb{1} - [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ is a set $\mathscr{S}_- = \{\mathbb{1}, -[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\}$ which is an eigenstate

of $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ with corresponding eigenvalue -1, *i.e.* $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\mathscr{S}_- = -\mathscr{S}_-$ As a matter of fact, a single group element can never be an eigenstate of a reflection operator $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ because a reflection transforms a rotation into a reversal and *vice versa*.

We may feel flabbergasted by the presence of two angular frequencies of opposite signs in Eq. 40 and in the matrix in Eq. 41 below. What on Earth does that come from? Did we not specify the sense of the rotation in deriving the Rodrigues formula? In principle within the group theory the negative angles correspond to counterclockwise motion. The reason for the rotation appearing as a set and the presence of the term $e^{+\imath\omega_0\tau/2}$ in Eq. 40 is that SU(2) does not only describe rotations but also reversals, i.e. group elements that are obtained as the result of an odd number of reflections. Rotations are obtained from an even number of reflections. Whereas for visualization purposes rotations can be identified with right-handed reference frames, reversals can be identified with left-handed reference frames. But in a left-handed frame the vector-product $\times$ must be calculated with the left-hand rule, while we consistently use $\wedge$ which is based on always using the right-hand rule, without bothering about the handedness of the frame wherein one makes the calculation. To correct for the error of using $\wedge$ instead of $\times$ we need a change of sign. This use of the wedge product occurs in the calculation of a rotation as the product of two reflections, i.e. the proof of the Rodrigues formula (see e.g. [6], p.11, Eq. 8).

For the moment we can observe that e.g. for the spinning motion around the z-axis:

$$\mathbf{R}(\mathbf{e}_z,\tau) = \begin{bmatrix} e^{-\imath\omega_0\tau/2} & \\ & e^{+\imath\omega_0\tau/2} \end{bmatrix}. \tag{41}$$

Let us show now that the second column of Eq. 41 is indeed a reversal.The first column of the matrix in Eq. 41 is the spinor:

$$\boldsymbol{\xi}_\uparrow = \begin{bmatrix} e^{-\imath\omega_0\tau/2} \\ 0 \end{bmatrix}. \tag{42}$$

It is also the first column of $\mathbf{R}(\mathbf{e}_z,\tau)$:

$$\mathbf{R}_\uparrow(\mathbf{e}_z,\tau) = \frac{1}{2}\,[\,\mathbb{1} + [\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]]\; e^{-\imath\omega_0/2} = \begin{bmatrix} e^{-\imath\omega_0\tau/2} & \\ & 0 \end{bmatrix}. \tag{43}$$

This matrix $\mathbf{R}_\uparrow(\mathbf{e}_z,\tau)$ does not seem to present a meaningful group element (such that it can only be given meaning in terms of sets). The second column of Eq. 41 is the conjugated spinor:

$$\boldsymbol{\xi}_\downarrow = \begin{bmatrix} 0 \\ e^{+\imath\omega_0\tau/2} \end{bmatrix}. \tag{44}$$

It is also the second column of:

$$\mathbf{R}_\downarrow(\mathbf{e}_z,\tau) = \frac{1}{2}\,[\,\mathbb{1} - [\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]]\; e^{+\imath\omega_0\tau/2} = \begin{bmatrix} 0 & \\ & e^{+\imath\omega_0\tau/2} \end{bmatrix}. \tag{45}$$

This matrix presents also difficulties of interpretation without invoking sets. But it can now be seen that the spinor in Eq. 44 corresponds to a reversal as by operating the reflection σ_x with respect to the Oyz plane on it, we obtain:

$$\sigma_x\,\boldsymbol{\xi}_\downarrow = \begin{bmatrix} e^{+\imath\omega_0\tau/2} \\ 0 \end{bmatrix}. \tag{46}$$

This is now the spinor for a clockwise rotation expressed in a right-handed frame. We have explained the change of sign in the angular frequency above. We have:

$$[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}_\uparrow(\mathbf{s},\tau) = \mathbf{R}_\uparrow(\mathbf{s},\tau), \quad [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\boldsymbol{\xi}_\uparrow = \boldsymbol{\xi}_\uparrow, \tag{47}$$

and

$$-\frac{\hbar}{\imath}\frac{d}{d\tau}\boldsymbol{\xi}_\uparrow = \frac{\hbar\omega_0}{2}\boldsymbol{\xi}_\uparrow, \tag{48}$$

while in perfect analogy:

$$[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}_\downarrow(\mathbf{s},\tau) = -\mathbf{R}_\downarrow(\mathbf{s},\tau), \quad [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\boldsymbol{\xi}_\downarrow = -\boldsymbol{\xi}_\downarrow, \tag{49}$$

and:

$$-\frac{\hbar}{\imath}\frac{d}{d\tau}\boldsymbol{\xi}_\downarrow = -\frac{\hbar\omega_0}{2}\boldsymbol{\xi}_\downarrow. \tag{50}$$

The spin-up and spin-down states are this way eigenspinors of the spin operators and of the energy operators. This shows that the introduction of sets was a very natural process. We even need it to make sense of the algebra of the group theory itself. We may finally note that vector calculus teaches us that a phase factor $e^{\imath\varphi/2}$ is not important. Our reflex is therefore to drop it. But Eqs. 42 and 44 reveal that the phase has mathematical and physical meaning. That is why we call the expressions without the phase factors rays. They are then sets containing all possible phases.

A rotation matrix of SU(2) is not a matrix in the sense we are used to it in linear algebra, because spinors do not build a vector space. In linear algebra applied to a vector space $\mathscr{V}$, a matrix $\mathbf{M}$ consists of columns that are the images of the basis vectors. The column k of matrix $\mathbf{M}$ is $\mathbf{M}(\mathbf{e}_k)$ where $\mathbf{e}_k$ is a canonical basis vector. In a vector space we can decompose linearly $\mathbf{M}(\mathbf{e}_k) = \sum_j M_{jk}\mathbf{e}_j$, but in a group such sums cannot be given the meaning they are given in vector space calculus. That is why we will be obliged to conclude below that we must take the SU(2) spin matrices apart into components with different frequencies. We may also note that the Clifford algebra introduces the square representation matrices monolithically without addressing the issue what the constituent columns may mean. That is why it so hard to figure out what spinors are. If this were all about vector calculus, the columns would be the images of the basis vectors, but groups are not about vector calculus.

6.2 Sign conventions

We see that SU(2) distinguishes four different possible group elements, which for $\varphi = \omega_0\tau$ are states:

spinor	$\begin{bmatrix} e^{-\imath\omega_0\tau/2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ e^{+\imath\omega_0\tau/2} \end{bmatrix}$	$\begin{bmatrix} e^{+\imath\omega_0\tau/2} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ e^{-\imath\omega_0\tau/2} \end{bmatrix}$
state	$\uparrow\circlearrowleft$	$\downarrow\circlearrowleft$	$\downarrow\circlearrowright$	$\uparrow\circlearrowright$
frame	right-handed	left-handed	right-handed	left-handed

Table 1: The four different states of motion with angular frequency ω_0 in SU(2). The symbols $\uparrow$ and $\downarrow$ refer to the spin states. Spin is an axial vector or pseudo-vector. The symbols $\circlearrowright$ and $\circlearrowleft$ correspond to the geometrical sense of rotational motion which relates to true vectors.

This is certainly somewhat confusing. Here $\uparrow$ and $\downarrow$ correspond to the spin-up and spin-down states of intrinsic angular momentum, while $\circlearrowleft$ and $\circlearrowright$ correspond to counterclockwise and clockwise geometrical rotation.

□ *The state* $\uparrow\circlearrowleft$. Its attribution follows from the calculation of the rotation matrix $\mathbf{R}$ in Eq. 41 for counterclockwise rotation as the product of two reflections in the derivation of the Rodrigues formula (e.g. according to [6], p.11, Eq. 8). This state $\uparrow\circlearrowleft$ will be our reference state. When we say opposite it will mean with respect to this reference.

□ *The state* $\downarrow\circlearrowleft$. Its attribution of $\downarrow\circlearrowleft$ follows from the fact that it is encountered in the the same SU(2) rotation matrix $\mathbf{R}$ as $\uparrow\circlearrowleft$. This is an SU(2) matrix for counterclockwise rotation, such that $\downarrow\circlearrowleft$ has definitely the same sense of geometrical rotation as $\uparrow\circlearrowleft$. As it appears in the right column of $\mathbf{R}$ it corresponds to a reversal, i.e. to a left-handed frame wherein the vector product $\times$ must be calculated with the left-hand rule. This results in an opposite sign of the spin, which explains the attribution $\downarrow$. That the sign of the angular frequency in $\downarrow\circlearrowleft$ is now opposite is due to the fact that the calculation is now done in a left-handed frame.

□ *The states* $\downarrow\circlearrowright$ *and* $\uparrow\circlearrowright$. We can now investigate the two other states by considering the inverse matrix $\mathbf{R}^{-1}$. The geometrical sense of rotation and the sign of the angular frequency are this way inverted with certainty. The states have then both definitely the opposite sense of rotation of the corresponding states in $\mathbf{R}$. They will also have the opposite spin as the corresponding states in $\mathbf{R}$. The calculations are still calculated in the same frames as those for the corresponding states in $\mathbf{R}$. This justifies all the attributions.

□ *Comment.* We must observe that the attributions spin-up and down have nothing to do with the positions in the top line or the bottom line of the matrix. We must dissociate these positions from the notions spin-up and spin-down. A position in the top line only indicates that the right-hand rule has been used in the calculation, just as the positions in the columns do.

□ *Cross-check.* We can also investigate the states occurring in $\mathbf{R}^{-1}$ by considering the effect of the reflection σ_x on the states occurring in $\mathbf{R}$. It transforms: $\uparrow\circlearrowleft$ into $\uparrow\circlearrowright$ and $\downarrow\circlearrowleft$ into $\downarrow\circlearrowright$. The sign of the spin is not affected by a parity transformation because the spin is an axial vector. When we apply a parity transformation to two vectors $\mathbf{a}|-\mathbf{a}$ and

$\mathbf{b}|-\mathbf{b}$ in the calculation of the Rodrigues equation, then the sign of $\mathbf{b}\wedge\mathbf{a}$ is not inverted in the calculation shown in [6], p.11, Eq. 8. The sign of the frequency is also not affected because it corresponds to an opposite motion calculated in a frame of opposite handedness. Furthermore, the reflection operator transforms right-handed frames into left-handed frames and vice versa. It changes therefore the reversal $\downarrow\circlearrowleft$ into a rotation $\downarrow\circlearrowright$, such that $\downarrow\circlearrowright$ is certainly right-handed as also its position in the first column of the 2×2 matrix indicates. We see thus that the states $\uparrow\circlearrowleft$ and $\downarrow\circlearrowright$ have the same spin, the same angular frequency but opposite senses of rotation.

6.3 Calculating the energy for more complicated motions

The two columns of the matrix in Eq. 41 correspond to a rotation and a reversal. We have proved this above for the special case when the rotation axis is the z-axis. But when we multiply the matrix in Eq. 41 with an SU(2) matrix it will contain again a rotation and a reversal. However, the two states will now be decomposed with respect to the old vector space logic of Eq. 41. They can now contain both $e^{-\imath\omega_0\tau/2}$ and $e^{\imath\omega_0\tau/2}$. This is hard to make sense of. To make sense of Eq. 41 we could speculate about using the energy operator:

$$\hat{\mathrm{E}}=-\frac{\hbar}{\imath}\begin{bmatrix}1&\\&-1\end{bmatrix}\frac{d}{d\tau}:\quad\begin{bmatrix}e^{-\imath\omega_0\tau/2}&\\&e^{+\imath\omega_0\tau/2}\end{bmatrix}\rightarrow\frac{\hbar\omega_0}{2}\begin{bmatrix}e^{-\imath\omega_0\tau/2}&\\&e^{+\imath\omega_0\tau/2}\end{bmatrix}. \tag{51}$$

It would have the full rotation matrix as its eigenstate, with the eigenvalue $E=\frac{\hbar\omega_0}{2}$. The general form would be $-\frac{\hbar}{\imath}[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\frac{d}{d\tau}$. That could make sense. When we transform $\mathbf{R}(\mathbf{e}_z,\tau)\rightarrow\mathbf{MR}(\mathbf{e}_z,\tau)$ we then must transform $\hat{\mathrm{E}}\rightarrow\mathbf{M}\hat{\mathrm{E}}\mathbf{M}^{-1}$, whereby $\mathbf{M}[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\mathbf{M}^{-1}=[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$.

That is the only way to calculate the energy of the new state correctly. In fact, there are no negative energies. Negative energies are a meaningless concept. The axioms of Euclidean geometry are devoid of any notion of anti-particles or charge, such that (over)interpreting the negative-energy states in terms of anti-particles is just not warranted by the geometrical meaning of the algebra. This geometrical meaning of the algebra is unambiguously established before any application of it to physics. An alternative parallel interpretation of it, based on "physical intuition", is not in demand. Intuition is not always right and not an argument of persuasion. There was a time when people thought that the Sun was turning around the Earth. That was also a "physical intuition", viz. the one in vogue those days. The two signs just correspond to opposite motions or handedness of frames. The spinning motions of both types of frames have positive kinetic energy. However, the problem with introducing Eq. 51 is that the energy is not a vector but a scalar. It is therefore a violation of the symmetry to write the energy operator as a vector operator.

Let us multiply $\mathbf{R}(\mathbf{e}_z,\tau)$ with some constant matrix $\mathbf{Q}\in\mathrm{SU}(2)$. Unlike in the problem-less states from which we started, the new spinors will now contain a mixture of positive and negative angular frequencies. If we want to keep the positive and negative angular frequencies as meaningful quantities we must analyze the states that contain mixtures of them in terms of sets, because that is the only way to give meaning to sums. A state will then be of the form $c_\uparrow\boldsymbol{\xi}_\uparrow+c_\downarrow\boldsymbol{\xi}_\downarrow$. If we are just worried about calculating the energy, we can use the energy operator we just introduced.

But the fact that the energy operator cannot be a vector operator manifests itself when the multiplying matrix contains itself a time dependence as e.g. $e^{-\imath\Omega\tau/2}$ in the case of precession within a magnetic field. Here $\Omega=qB/m_0$ is the cyclotron frequency. To obtain meaningful results we must then keep track of the algebraic angular frequencies and the energy operator defined in Eq. 51 will no longer make sense because both algebraic combinations $\hbar(\omega_0\pm\Omega)/2$ will be meaningful different energies. A fortiori, also the traditional energy operator $-\frac{\hbar}{\imath}\frac{\partial}{\partial t}$ does no longer make sense as discussed in [8]. The only way to deal with precession is therefore to decompose the SU(2) matrix obtained by the multiplication into a sum of matrices:

$$e^{-\imath(\omega_0+\Omega)\tau/2}\mathbf{M}_{-+}+e^{+\imath(\omega_0+\Omega)\tau/2}\mathbf{M}_{++}\quad\text{with energy:}\quad\hbar(\omega_0+\Omega)/2, \tag{52}$$

and:

$$e^{-\imath(\omega_0-\Omega)\tau/2}\mathbf{M}_{--}+e^{+\imath(\omega_0-\Omega)\tau/2}\mathbf{M}_{+-}\quad\text{with energy:}\quad\hbar(\omega_0-\Omega)/2. \tag{53}$$

The four matrices do not seem to have a simple geometrical interpretation. But after recombining them we obtain again rotation matrices (up to a normalization factor, see below). They must thus be considered as right-handed and left-handed states. We are only used to make sense of rotation matrices, which are sets containing both left- and right-handed states. The miracle that the matrices in Eq. 52 recombine to a rotation could be expected, because we have started from left-handed and right-handed representations of the same motion, and we should thus end up with left-handed and right-handed representations of the same motion. In our treatment of the Stern-Gerlach experiment we ended up with a rotation around the z-axis, because we are spinning an object (a spinning electron) around the z-axis. This describes then precession.

The spin-up an spin-down states in Eq. 40 have themselves the outlook of superposition states. As explained above, the reason for this is that a rotation cannot be the eigenvector of reflection operator. Only sets can be eigenvectors

of a reflection operator. We could take these sets further apart algebraically. But it will be equally difficult to see what they mean geometrically. Eventually, we must consider the whole calculation as a determination of states with their energies. We are taking a rotation apart and recombining the resulting pieces in the end as though the whole formalism were a Meccano game, but the pieces of the Meccano game do not all have a meaning in terms of group elements. The convenience of playing Meccano is that it permits to obtain a better overview of the calculations. In a sense it is only the end result which is meaningful again. Hence we could just perform the brute-force calculation and split it up in the end. The various parts will then not be normalized to one and the normalization factors can be used to calculate the probabilities. When we take apart and reassemble the representations of two identical spinning electrons, the parts and the final results will of course be identical.

We have this way derived how we must describe spinning motion in SU(2) and in QM. There are a lot of things in it that have never been explored because they are situated in a no-man's-land between physics and mathematics. Note that the two frequencies correspond to two spinning motions in the same sense, because we have started from a rotation matrix expressing counterclockwise motion. Hence, if we want to describe a set that contains the two opposite states of motion, like e.g. the clockwise and counterclockwise spinning of an electron, we must still add two terms, because we are describing a set of two electrons. Along the same lines, we will see that if we want to describe a set that contains the two angular-momentum states of a linearly polarized "particle" like a photon, we must add two terms states representing circular polarization to represent the set, because it is only a set of particles, not a single particle, that can be linearly polarized. Single photons can only be circularly polarized.[2] That is the only sensible way to attribute angular momentum to them [13].

7 Correlation experiments for electrons with Stern-Gerlach filters

We characterize a spin axis $\mathbf{s}$ by its spherical coordinates (θ, ϕ). Let us remind that the symbol ϕ (which defines a spherical coordinate of an axis) has been attributed a different meaning than the symbol φ which we use to define a rotation angle. For electrons we can consider the direction of a magnetic field $\mathbf{B} = B\mathbf{m}$ as analogous to the direction of the polarization of a polarization filter for photons. The direction $\mathbf{m}_a = (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a)$ of a magnetic field $\mathbf{B}_a$ is expressed by:

$$[\,\mathbf{m}_a \cdot \boldsymbol{\sigma}\,] = \begin{bmatrix} \cos\theta_a & e^{-\imath\phi_a}\sin\theta_a \\ e^{+\imath\phi_a}\sin\theta_a & -\cos\theta_a \end{bmatrix}. \tag{54}$$

We have shown above that:

$$\begin{aligned} \mathbb{1} + [\,\mathbf{m}_a \cdot \boldsymbol{\sigma}\,] = \begin{bmatrix} 1+\cos\theta_a & \sin\theta_a e^{-\imath\phi_a} \\ \sin\theta_a e^{+\imath\phi_a} & 1-\cos\theta_a \end{bmatrix} &= \begin{bmatrix} 2\cos^2(\theta_a/2) & 2\sin(\theta_a/2)\cos(\theta_a/2)e^{-\imath\phi_a} \\ 2\sin(\theta_a/2)\cos(\theta_a/2)e^{+\imath\phi_a} & 2\sin^2(\theta_a/2) \end{bmatrix} \\ &= 2\begin{bmatrix} \cos(\theta_a/2) \\ e^{+\imath\phi_a}\sin(\theta_a/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta_a/2) & e^{-\imath\phi_a}\sin(\theta_a/2) \end{bmatrix} \end{aligned} \tag{55}$$

and:

$$\begin{aligned} \mathbb{1} - [\,\mathbf{m}_a \cdot \boldsymbol{\sigma}\,] = \begin{bmatrix} 1-\cos\theta_a & -\sin\theta_a e^{-\imath\phi_a} \\ -\sin\theta_a e^{+\imath\phi_a} & 1+\cos\theta_a \end{bmatrix} &= \begin{bmatrix} 2\sin^2(\theta/2) & -2\sin(\theta/2)\cos(\theta/2)e^{-\imath\phi} \\ -2\sin(\theta/2)\cos(\theta/2)e^{+\imath\phi} & 2\cos^2(\theta/2) \end{bmatrix} \\ &= 2\begin{bmatrix} \sin(\theta_a/2) \\ -e^{+\imath\phi_a}\cos(\theta_a/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\theta_a/2) & -e^{-\imath\phi_a}\cos(\theta_a/2) \end{bmatrix} \end{aligned} \tag{56}$$

Consider now two perfectly identical electrons at rest, which are rotating around the z-axis:

$$\mathbf{R}_0 = \begin{bmatrix} e^{-\imath\omega_0\tau/2} & \\ & e^{+\imath\omega_0\tau/2} \end{bmatrix}. \tag{57}$$

We let them fly apart at velocities $\mathbf{v} = v\mathbf{u}$ and $\mathbf{v} = -v\mathbf{u}$. We assume that $\mathbf{u} = \mathbf{e}_x$. For non-relativistic velocities $v \ll c$, the angular frequency of the spinning motion will remain ω_0. The point where the electrons quit is taken as the origin of the reference frame. After some time the electrons will reach simultaneously the Stern-Gerlach magnets at $(-L, 0, 0)$ and at $(+L, 0, 0)$. Their spins will then still be in phase. In a magnetic field they will both start precessing around the

[2] We will later correct this statement in Section 10. We cannot be more precise here for reasons of clarity and didactical purposes.

magnetic field vector in the sense according to the right-hand rule. In other words, their spinning motions will take place in the same sense. We choose the magnetic fields $\mathbf{B}_a$ and $\mathbf{B}_b$ in the filters to be parallel to the Oyz plane and of the same magnitude $|\mathbf{B}_a| = |\mathbf{B}_b|$. In the left arm we have $\theta_a = \alpha$, $\phi_a = \pi/2$, $e^{+\imath\phi_a} = \imath$, $e^{-\imath\phi_a} = -\imath$. The precession is then expressed by:

$$\mathbf{R}_a = \begin{bmatrix} \cos(\Omega t/2) - \imath\sin(\Omega t/2)\cos\alpha & -\sin(\Omega t/2)\sin\alpha \\ +\sin(\Omega t/2)\sin\alpha & \cos(\Omega t/2) + \imath\sin(\Omega t/2)\cos\alpha \end{bmatrix}, \tag{58}$$

and:

$$\mathbf{R}_a = \begin{bmatrix} \cos(\alpha/2) \\ +\imath\sin(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha/2) & -\imath\sin(\alpha/2) \end{bmatrix} e^{-\imath\Omega t/2} + \begin{bmatrix} \sin(\alpha/2) \\ -\imath\cos(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\alpha/2) & +\imath\cos(\alpha/2) \end{bmatrix} e^{+\imath\Omega t/2}. \tag{59}$$

$$\mathbf{R}_a^{-1} = \begin{bmatrix} \cos(\alpha/2) \\ +\imath\sin(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha/2) & -\imath\sin(\alpha/2) \end{bmatrix} e^{+\imath\Omega t/2} + \begin{bmatrix} \sin(\alpha/2) \\ -\imath\cos(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\alpha/2) & +\imath\cos(\alpha/2) \end{bmatrix} e^{-\imath\Omega t/2}. \tag{60}$$

In the right arm we have $\theta_b = \beta$, $\phi_b = \pi/2$, $e^{+\imath\phi_b} = \imath$, $e^{-\imath\phi_b} = -\imath$ and the precession will be expressed by:

$$\mathbf{R}_b = \begin{bmatrix} \cos(\Omega t/2) - \imath\sin(\Omega t/2)\cos\beta & -\sin(\Omega t/2)\sin\beta \\ +\sin(\Omega t/2)\sin\beta & \cos(\Omega t/2) + \imath\sin(\Omega t/2)\cos\beta \end{bmatrix}. \tag{61}$$

$$\mathbf{R}_b = \begin{bmatrix} \cos(\beta/2) \\ +\imath\sin(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\beta/2) & -\imath\sin(\beta/2) \end{bmatrix} e^{-\imath\Omega t/2} + \begin{bmatrix} \sin(\beta/2) \\ -\imath\cos(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\beta/2) & +\imath\cos(\beta/2) \end{bmatrix} e^{+\imath\Omega t/2}. \tag{62}$$

We must then calculate the components of $\mathbf{R}_a\mathbf{R}_0$ and $\mathbf{R}_b\mathbf{R}_0$ and show that they occur with probabilities $\cos^2(\alpha/2)$, $\sin^2(\alpha/2)$ and $\cos^2(\beta/2)$, $\sin^2(\beta/2)$. This has been done in [8]. In fact, the two probabilities $\cos^2(\alpha/2)$ and $\cos^2(\beta/2)$ correspond to electrons which have the same energy $\hbar(\omega_0 + \Omega)/2$ and will leave the filter in a same type of beam. Similarly, the two terms in $\sin^2(\alpha/2)$ and $\sin^2(\beta/2)$ correspond to electrons which have the same energy $\hbar(\omega_0 - \Omega)/2$. They will also leave the filter in a same type of beam but one that is different from the type of beam where the energy of the electrons is $\hbar(\omega_0 + \Omega)/2$. We can really carry out a physical selection on the states in a Stern-Gerlach experiment because it functions as a beam splitter. And we can select any combination of two beams we like. If instead of being parallel to $\mathbf{e}_z$, the spin axes $\mathbf{s}$ of the incoming electrons explore all angles θ in the Oyz plane, the average probabilities for emerging in a given beam from a filter will for both beams from both Stern-Gerlach filters be $\frac{1}{2}$. The four conditional probabilities for the combined presence in a beam b_k leaving filter b if the buddy electron leaves the filter a in a beam a_j will be given by the probabilities for the components of $(\mathbf{R}_b\mathbf{R}_0)(\mathbf{R}_a\mathbf{R}_0)^{-1} = \mathbf{R}_b\mathbf{R}_a^{-1}$. The calculation is somewhat different from the ones for $\mathbf{R}_b\mathbf{R}_0$ and $\mathbf{R}_a\mathbf{R}_0$. We therefore reproduce it here. The terms in $\mathbf{R}_b\mathbf{R}_a^{-1}$ that link states in $\mathbf{R}_a$ and $\mathbf{R}_b$ of opposite handedness are:

$$(+-) \equiv e^{-\imath\Omega t} \begin{bmatrix} \cos(\beta/2) \\ +\imath\sin(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\beta/2) & -\imath\sin(\beta/2) \end{bmatrix} \begin{bmatrix} \sin(\alpha/2) \\ -\imath\cos(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\alpha/2) & +\imath\cos(\alpha/2) \end{bmatrix}, \tag{63}$$

and

$$(-+) \equiv e^{+\imath\Omega t} \begin{bmatrix} \sin(\beta/2) \\ -\imath\cos(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\beta/2) & +\imath\cos(\beta/2) \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) \\ +\imath\sin(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha/2) & -\imath\sin(\alpha/2) \end{bmatrix}. \tag{64}$$

Those that link states of the same handedness are:

$$(--) \equiv \begin{bmatrix} \cos(\beta/2) \\ +\imath\sin(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\beta/2) & -\imath\sin(\beta/2) \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) \\ +\imath\sin(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha/2) & -\imath\sin(\alpha/2) \end{bmatrix} \tag{65}$$

and:

$$(++) \equiv \begin{bmatrix} \sin(\beta/2) \\ -\imath\cos(\beta/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\beta/2) & +\imath\cos(\beta/2) \end{bmatrix} \begin{bmatrix} \sin(\alpha/2) \\ -\imath\cos(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \sin(\alpha/2) & +\imath\cos(\alpha/2) \end{bmatrix}. \tag{66}$$

Those are states which are spinning in phase. Therefore their relative frequency is zero. The other two terms are getting out of phase at a rate given by the relative angular frequency Ω. The term $(+-)$ with the relative angular frequency $-\Omega$ can be simplified to:

$$\sin[\,(\alpha-\beta)/2\,]\,e^{-\imath\Omega t}\begin{bmatrix}\cos(\beta/2)\\ \imath\sin(\beta/2)\end{bmatrix}\otimes\begin{bmatrix}\sin(\alpha/2) & \imath\cos(\alpha/2)\end{bmatrix}. \tag{67}$$

The amplitude $\sin[\,(\alpha-\beta)/2\,]$ stems from the product of the two spinors in the middle. We could consider this result as a kind of projection of a spinor $\boldsymbol{\psi}_2$ onto a spinor $\boldsymbol{\psi}_1$, whereby the "scalar product" (which can still be a complex number) is based on the Hermitian norm. For the term $(-+)$ with the relative angular frequency $+\Omega$ this becomes:

$$\sin[\,(\beta-\alpha)/2\,]\,e^{+\imath\Omega t}\begin{bmatrix}\sin(\beta/2)\\ -\imath\cos(\beta/2)\end{bmatrix}\otimes\begin{bmatrix}\cos(\alpha/2) & -\imath\sin(\alpha/2)\end{bmatrix}. \tag{68}$$

For the terms $(++)$ and $(--)$ which do not contain a dependence on Ωt because they stay in phase we obtain:

$$\begin{aligned}&\cos[\,(\alpha-\beta)/2\,]\begin{bmatrix}\cos(\beta/2)\\ +\imath\sin(\beta/2)\end{bmatrix}\otimes\begin{bmatrix}\cos(\alpha/2) & -\imath\sin(\alpha/2)\end{bmatrix}\\ &+\cos[\,(\alpha-\beta)/2\,]\begin{bmatrix}\sin(\beta/2)\\ -\imath\cos(\beta/2)\end{bmatrix}\otimes\begin{bmatrix}\sin(\alpha/2) & +\imath\cos(\alpha/2)\end{bmatrix}.\end{aligned} \tag{69}$$

Recombining these matrices which have a same probability amplitude yields:

$$\begin{aligned}&\cos[\,(\alpha-\beta)/2\,]\begin{bmatrix}\cos(\beta/2)\cos(\alpha/2) & -\imath\cos(\beta/2)\sin(\alpha/2)\\ +\imath\sin(\beta/2)\cos(\alpha/2) & \sin(\beta/2)\sin(\alpha/2)\end{bmatrix}\\ &+\cos[\,(\alpha-\beta)/2\,]\begin{bmatrix}\sin(\beta/2)\sin(\alpha/2) & +\imath\sin(\beta/2)\cos(\alpha/2)\\ -\imath\cos(\beta/2)\sin(\alpha/2) & \cos(\beta/2)\cos(\alpha/2)\end{bmatrix}.\end{aligned} \tag{70}$$

The sum yields:

$$\cos[\,(\alpha-\beta)/2\,]\begin{bmatrix}\cos[\,(\alpha-\beta)/2\,] & -\imath\sin[\,(\alpha-\beta)/2\,]\\ -\imath\sin[\,(\alpha-\beta)/2\,] & \cos[\,(\alpha-\beta)/2\,]\end{bmatrix}. \tag{71}$$

The angle $\beta-\alpha$ is the angle of a rotation around the x-axis that would turn $\mathbf{B}_a$ into $\mathbf{B}_b$. We could consider $\cos((\beta-\alpha)/2)$ just as the projection of $\mathbf{m}_a$ onto $\mathbf{m}_b$ (expressed in SU(2)), which is the rule used in QM. One may note that it is not the projection from one spin onto the other. The states in a magnetic field are not spin-up and spin-down but precession-up and precession-down as we already pointed out in [8]. We would like now also to recombine the two other terms which correspond to a difference Ω in the angular frequencies:

$$\sin[\,(\alpha-\beta)/2\,]\,e^{-\imath\Omega t}\begin{bmatrix}\cos(\beta/2)\sin(\alpha/2) & +\imath\cos(\beta/2)\cos(\alpha/2)\\ +\imath\sin(\beta/2)\sin(\alpha/2) & -\sin(\beta/2)\cos(\alpha/2)\end{bmatrix}. \tag{72}$$

$$\sin[\,(\beta-\alpha)/2\,]\,e^{+\imath\Omega t}\begin{bmatrix}\sin(\beta/2)\cos(\alpha/2) & -\imath\sin(\alpha/2)\sin(\beta/2)\\ -\imath\cos(\beta/2)\cos(\alpha/2) & -\sin(\alpha/2)\sin(\beta/2)\end{bmatrix}. \tag{73}$$

We can sum their probabilities based on the argument that they both represent the probability that the identical incoming electrons with energy $\hbar\omega_0/2$ in the two arms of the set-up have acquired different energies $\hbar(\omega_0+\Omega)/2$ and $\hbar(\omega_0-\Omega)/2$ just before they leave the filters and therefore end up in beams of different types. Of course outside the filters they will have resumed again the energy $\hbar\omega_0/2$. The same argument applies to the terms with the cosines, which represent the probability that the identical incoming electrons in the two arms leave the filters in beams of the same type.

The rest just depends on the selection criteria we choose to apply now to both beams. Selecting a combination of out-coming beams of the same type yields a cosine term, selecting a combination of out-coming beams of the opposite type a sine term. Summing the algebraic expressions in Eq. 72 and 73 is in itself meaningless. The two frequencies $\pm\Omega$ are due to the opposite sense of rotation in the magnetic field adopted by the right-handed frames and the left-handed frames. A more tidy solution which renders it meaningful to sum the algebraic expressions that contribute to the same probability $\sin^2(\alpha-\beta)$ is to consider the mixed state whereby the incoming beams consist of equal amounts of electron pairs in clockwise and counterclockwise spinning motion. There will then be 4 terms tagged with a sine and we will be able to sum them two by two. In fact, we have written the original rotation matrix for just one sense of rotation. The signs were introduced by the handedness of the frames. By considering the two senses of rotation we double the

number of states and taking the sums is then no longer artificial. We see this way that selecting analogous beams on both sides leads to a probability $\cos^2[\,(\alpha-\beta)/2\,]$. If we select only one energy, then the probability to end up in a particular beam of a particular filter will be $\frac{1}{2}$ (due to averaging). And the conditional probability that the buddy electron has the same energy will be $\cos^2[\,(\alpha-\beta)/2\,]$. This would yield then a Malus-type law $\frac{1}{2}\cos^2[\,(\alpha-\beta)/2\,]$ for the coincidence rate in the two arms when we select electrons from beams of the same type. The coincidence rate will be $\frac{1}{2}\sin^2[\,(\alpha-\beta)/2\,]$ when we select electrons from beams of opposite type. On purely formal grounds we can imagine an analogous mechanism for photons. But for photons we need a probability law $\frac{1}{2}\cos^2(\alpha-\beta)$ with a doubled angle $\alpha-\beta$ which then corresponds to the true angle between the orientations of two polarization filters.

We can learn from the development given above for electrons something important, viz. how we can calculate probabilities by group-theoretical methods. We split the representation matrix $\mathbf{M}$ into components that correspond to its eigenvalues. These components can be written as tensor products. In fact, when the $n\times n$ matrix $\mathbf{M}$ can be diagonalized as $\mathbf{M}=\mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}$, the contribution of an eigenvalue $\lambda_j\neq 0$ to the diagonal matrix $\boldsymbol{\Lambda}$ will be given by $\boldsymbol{\eta}_j^\top\otimes\boldsymbol{\eta}_j$ where $\boldsymbol{\eta}_j$ is the row matrix whose entries are all zero except in position j. From this it follows that:

$$\mathbf{M}=\sum_{j=1}^{n}\mathbf{S}\,[\,\boldsymbol{\eta}_j^\top\otimes\boldsymbol{\eta}_j\,]\,\mathbf{S}^{-1}=\sum_{j=1}^{n}\boldsymbol{\chi}_j^\top\otimes\boldsymbol{\zeta}_j,\quad\text{where:}\quad\boldsymbol{\chi}_j^\top=\mathbf{S}\,\boldsymbol{\eta}_j^\top\ \ \&\ \ \boldsymbol{\zeta}_j=\boldsymbol{\eta}_j\,\mathbf{S}^{-1}. \tag{74}$$

Products of two tensor products $\boldsymbol{\chi}_j^\top\otimes\boldsymbol{\zeta}_j$ give then rise to scalar probability amplitudes.

8 The tensor product representation $\mathbf{R}\otimes\mathbf{R}$

8.1 Preamble

The calculations in this Section will serve only for illustrative purposes. We explain how to construct tensor products of representations of rotation groups. The aim is to familiarize the reader with the notion that representations of the rotation group operating on harmonic polynomials of degree ℓ are tensor products of rank 2ℓ of the representation SU(2) which operates on spinors. Harmonic polynomials of degree ℓ can therefore be calculated from the tensor products $\boldsymbol{\xi}\otimes\boldsymbol{\xi}\otimes\cdots\otimes\boldsymbol{\xi}$ of 2ℓ spinors $\boldsymbol{\xi}$. This provides then also the insight needed to calculate Clebsch-Gordan coefficients. In this work we will only need tensor products of rank 2. We need the rank 2 because $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ leads to spin one. Since $\boldsymbol{\xi}$ describes spin-1/2, it looks as though the tensor product describes two spin-1/2 particles. This does not need to be true as we are only treating symmetries, but we will adapt this language to talk about it in order not to get lost in hyperbolic phrasings. All we achieve in this section will turn out to be more or less useless with respect to our ultimate agenda of calculating Malus' law for photons. But it was a necessary step in our quest to attain this goal. We had to make a survey of the methodologies. After observing that the more common formalisms did not work we reached the conclusion we had to develop a new variant adapted to the specific requirements of symmetry imposed by photons. This will be described in Section 9. The reader in a hurry can just get a flavour of what the present section is all about and then move on to Section 9.

8.2 Method 1 - writing out the equations

8.2.1 The matrix needed to rotate $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$

Photons have spin one and must therefore be described by a tensor product representation $\mathbf{R}\otimes\mathbf{R}$ of SU(2) × SU(2). This means that we must replace the spinors $\boldsymbol{\xi}=[\,\xi_0,\xi_1\,]^\top$ by:

$$\begin{bmatrix}\xi_0\boldsymbol{\xi}\\ \xi_1\boldsymbol{\xi}\end{bmatrix}=\begin{bmatrix}\xi_0^2\\ \xi_0\xi_1\\ \xi_1\xi_0\\ \xi_1\end{bmatrix}\rightarrow\begin{bmatrix}\xi_0^2\\ 2\xi_0\xi_1\\ \xi_1^2\end{bmatrix}=\begin{bmatrix}(x-\imath y)/2\\ -z\\ -(x+\imath y)/2\end{bmatrix}, \tag{75}$$

where we have regrouped the two identical terms $\xi_0\xi_1$ into one entry and used Eq. (31) of [6]. Let us note a general rotation matrix in SU(2) as:

$$\mathbf{R}=\begin{bmatrix}u & -v^*\\ v & u^*\end{bmatrix}. \tag{76}$$

Using this we can calculate (see [14]):

$$\xi_0^2 \to (u\xi_0 - v^*\xi_1)^2 = u^2\xi_0^2 - 2uv^*\xi_0\xi_1 + v^{*2}\xi_1^2,$$

$$2\xi_0\xi_1 \to 2(u\xi_0 - v^*\xi_1)(v\xi_0 + u^*\xi_1) = 2uv\xi_0^2 + 2(uu^* - vv^*)\xi_0\xi_1 - 2u^*v^*\xi_1^2,$$

$$\xi_1^2 \to (v\xi_0 + u^*\xi_1)^2 = v^2\xi_0^2 + 2u^*v\xi_0\xi_1 + u^{*2}\xi_1^2. \tag{77}$$

In summary:

$$\begin{bmatrix} \xi_0^2 \\ 2\xi_0\xi_1 \\ \xi_1^2 \end{bmatrix} \to \begin{bmatrix} u^2 & -uv^* & v^{*2} \\ 2uv & uu^* - vv^* & -2u^*v^* \\ v^2 & u^*v & u^{*2} \end{bmatrix} \begin{bmatrix} \xi_0^2 \\ 2\xi_0\xi_1 \\ \xi_1^2 \end{bmatrix}. \tag{78}$$

We can rewrite this as:

$$\frac{1}{\sqrt{2}}\mathbf{T}\begin{bmatrix} \xi_0^2 \\ 2\xi_0\xi_1 \\ \xi_1^2 \end{bmatrix} \to \mathbf{T}\begin{bmatrix} u^2 & -uv^* & v^{*2} \\ 2uv & uu^* - vv^* & -2u^*v^* \\ v^2 & u^*v & u^{*2} \end{bmatrix}\mathbf{T}^{-1} \cdot \mathbf{T}\begin{bmatrix} \xi_0^2 \\ 2\xi_0\xi_1 \\ \xi_1^2 \end{bmatrix}, \tag{79}$$

where:

$$\mathbf{T} = \begin{bmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \\ & & 1 \end{bmatrix}. \tag{80}$$

This implies that we work with $\sqrt{2}\xi_0\xi_1$ instead of $2\xi_0\xi_1$. The reason for this is a concern to keep the formalism symmetrical as will become obvious below in Eq. 92. We obtain then:

$$\begin{bmatrix} u^2 & -\sqrt{2}uv^* & v^{*2} \\ \sqrt{2}uv & uu^* - vv^* & -\sqrt{2}u^*v^* \\ v^2 & \sqrt{2}u^*v & u^{*2} \end{bmatrix}. \tag{81}$$

8.2.2 The rotation matrix for the isotropic vector (x, y, z)

An isotropic vector $\mathbf{e}_x + \imath\mathbf{e}_y$ contains all the information about the triad of basis vectors. We can use therefore its image $(x, y, z) \in \mathscr{I} \subset \mathbb{C}^3$ under a rotation to specify the rotation. We can decompose (x, y, z) at any time into its real and imaginary parts and use the wedge product to calculate the image of $\mathbf{e}_z$. This way (x, y, z) are rotation coordinates. They belong to the so-called isotropic cone $\mathscr{I} = \{(x, y, z) \in \mathbb{C}^3 \parallel x^2 + y^2 + z^2 = 0\}$. They are not position coordinates. The only position coordinate that belongs to the isotropic cone is $(0, 0, 0)$ but $(0, 0, 0)$ is the only element of $\mathscr{I}$ that does not represent a triad of basis vectors. We have now:

$$x = \xi_0^2 - \xi_1^2 \to (u^2 - v^2)\xi_0^2 - 2(uv^* - u^*v)\xi_0\xi_1 + (v^{*2} - u^{*2})\xi_1^2,$$

$$y = \imath(\xi_0^2 + \xi_1^2) \to \imath(u^2 + v^2)\xi_0^2 + 2\imath(u^*v - uv^*)\xi_0\xi_1 + \imath(v^{*2} + u^{*2})\xi_1^2,$$

$$z = -2\xi_0\xi_1 \to -2uv\xi_0^2 + 2(vv^* - uu^*)\xi_0\xi_1 + 2u^*v^*\xi_1^2. \tag{82}$$

To turn this into a transformation of x, y, z must now introduce the identities of Eq. 75 and recombine the terms.

$$x \to (u^2 - v^2)(x - \imath y)/2 + (uv^* - u^*v)z - \imath(v^{*2} - u^{*2})(x + \imath y)/2,$$

$$y \to \imath(u^2 + v^2)(x - \imath y)/2 - \imath(u^*v - uv^*)z + (v^{*2} + u^{*2})(x + \imath y)/2,$$

$$z \to -2uv(x - \imath y)/2 - (vv^* - uu^*)z - \imath 2u^*v^*(x + \imath y)/2. \tag{83}$$

such that:

$$x \to (u^2 - v^2 + \imath u^{*2} - \imath v^{*2})x/2 + (uv^* - u^*v)z + (v^{*2} - u^{*2} + \imath v^2 - \imath u^2)y/2, \tag{84}$$

$$y \to (v^{*2} + u^{*2} + \imath u^2 + \imath v^2)x/2 - \imath(u^* v - u v^*)z + (u^2 + v^2 + \imath v^{*2} + \imath u^{*2})y/2,$$

$$z \to -(uv + \imath u^* v^*)x - (vv^* - uu^*)z + (\imath uv + u^* v^*)y. \tag{85}$$

The quantities (x, y, z) are the coordinates of the isotropic vector $\mathbf{e}_x + \imath\mathbf{e}_y = (x_1, y_1, z_1) + \imath(x_2, y_2, z_2)$. The complex formalism is this way a formalism for rotations. But by taking the real parts of (x, y, z), we can obtain the transformed vector $\mathbf{e}_x = (x_1, y_1, z_1)$, and we can multiply this with r to obtain the transformed general real vector $\mathbf{r}$ (as all vectors transform the same way). This way we obtain a formalism for vectors and position coordinates. This explains why we can use harmonic polynomials in $(x, y, z) \in \mathscr{I}$ which are conceptually rotation coordinates with $x^2 + y^2 + z^2 = 0$ nevertheless as functions of position coordinates $(x_1, y_1, z_1) \in \mathbb{R}^3$ with $x_1^2 + y_1^2 + z_1^2 = r^2 > 0$ and then substitute $(x_1, y_1, z_1)|(x, y, z)$ in the notations as is done in QM. This naturally raises the question if QM identifies $0 \equiv r^2 > 0$ (see [7], p. 5). This question has now been answered. But supposedly, very few people were aware of this problem in the first place. Or they may have thought that it was a manifestation of quantum magic. We may note that harmonic polynomials of degree ℓ are all rank-2ℓ tensor products $\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \cdots \otimes \boldsymbol{\xi}$ of spinors $\boldsymbol{\xi}$ [15]. We see this way that depending on the type of polynomials and mathematical objects we want to treat, the further elaboration of the formalism can take several alternative roads, and for each road we will have a different (laborious) algebra.

8.3 Method 2 - Matrix formalism

8.4 Simple case - rotation around the z-axis

We can also calculate the transformation matrix by taking the tensor product $\mathbf{R} \otimes \mathbf{R}$. For Eq. 41 this yields:

$$\begin{bmatrix} e^{-\imath\omega_0\tau/2}\mathbf{R} & \\ & e^{+\imath\omega_0\tau/2}\mathbf{R} \end{bmatrix} = \begin{bmatrix} e^{-\imath\omega_0\tau} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{+\imath\omega_0\tau} \end{bmatrix}. \tag{86}$$

In stead of this we will use $\mathbf{P} = \mathbf{S}\,[\,\mathbf{R} \otimes \mathbf{R}\,]\,\mathbf{S}^{-1}$:

$$\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix} \begin{bmatrix} e^{-\imath\omega_0\tau} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{+\imath\omega_0\tau} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} e^{-\imath\omega_0\tau} & & & \\ & e^{+\imath\omega_0\tau} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{87}$$

There are two equivalent substates in this such that we can work with a three-dimensional representation:

$$\mathbf{C} = \begin{bmatrix} e^{-\imath\omega_0\tau} & & \\ & e^{+\imath\omega_0\tau} & \\ & & 1 \end{bmatrix}. \tag{88}$$

For the three independent substates in the tensor product we have then:

$$-\frac{\hbar}{\imath}\frac{d}{d\tau}\mathbf{C} = \begin{bmatrix} +\hbar\omega_0 & & \\ & -\hbar\omega_0 & \\ & & 0 \end{bmatrix} \mathbf{C}. \tag{89}$$

We will see below that this result is not general. But for a propagation parallel to the rotation axis this can be used to explain why we do not have three substates for a photon. The third component has so to say no energy. If we think of the tensor product as truly representing two spinning particles, of course the total energy of these two particles will not be zero. But if we just want to use the representation of the rotation group obtained from the mathematics to describe a supposedly indivisible elementary particle of spin 1 like a photon, then the photon would indeed have zero energy, such that the third mode would just not exist. This is an issue that is in very poorly explained in most physics textbooks. We can also argue that lengths parallel to the direction of motion of a photon are reduced to zero by the Lorentz contraction. Therefore only rotations in the plane perpendicular to the direction of motion are possible. Of course we must replace $\omega_0\tau$ by ωt now because the photon has no rest frame. This seems to indicate that we can describe a photon by a SU(2)-like formalism, provided we replace $\omega_0\tau/2$ by ωt. But it would be cavalier to just drop a line and a column in the 3×3 matrix to achieve our goals of ending up with only two substates. In principle the normalizations of the columns and the lines in the resulting matrix would then no longer be correct.

8.5 General rotation matrix

For a general rotation we have:

$$\mathbf{R}\otimes\mathbf{R}=\begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}\otimes\begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}=\begin{bmatrix} u\,\mathbf{R} & -v^*\,\mathbf{R} \\ v\,\mathbf{R} & u^*\,\mathbf{R} \end{bmatrix}=\begin{bmatrix} u^2 & -uv^* & -uv^* & v^{*2} \\ uv & uu^* & -vv^* & -v^*u^* \\ vu & -vv^* & uu^* & -u^*v^* \\ v^2 & vu^* & vu^* & u^{*2} \end{bmatrix}. \tag{90}$$

Let us define:

$$\mathbf{S}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{91}$$

We will use this to reduce the rank of the matrix from 4 to 3. But we want a form of this that is rendered symmetrical. The inverse of:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\quad\text{is:}\quad\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}. \tag{92}$$

Hence we should use:

$$\mathbf{S}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{93}$$

Then:

$$\mathbf{S}\begin{bmatrix} \xi_0^2 \\ \xi_0\xi_1 \\ \xi_1\xi_0 \\ \xi_1^2 \end{bmatrix}=\begin{bmatrix} \xi_0^2 \\ \sqrt{2}\xi_0\xi_1 \\ 0 \\ \xi_1^2 \end{bmatrix}. \tag{94}$$

We have then $\mathbf{S}^{-1}=\mathbf{S}^\dagger=\mathbf{S}$.

$$\mathbf{S}\begin{bmatrix} \xi_0^2 \\ \xi_0\xi_1 \\ \xi_1\xi_0 \\ \xi_1^2 \end{bmatrix}\rightarrow\mathbf{S}\begin{bmatrix} u^2 & -uv^* & -uv^* & v^{*2} \\ uv & uu^* & -vv^* & -v^*u^* \\ vu & -vv^* & uu^* & -u^*v^* \\ v^2 & vu^* & vu^* & u^{*2} \end{bmatrix}\mathbf{S}^{-1}\mathbf{S}\begin{bmatrix} \xi_0^2 \\ \xi_0\xi_1 \\ \xi_1\xi_0 \\ \xi_1^2 \end{bmatrix}. \tag{95}$$

After some tedious algebra this yields:

$$\begin{bmatrix} u^2 & -\sqrt{2}uv^* & 0 & v^{*2} \\ \sqrt{2}uv & uu^*-vv^* & 0 & -\sqrt{2}v^*u^* \\ 0 & 0 & (uu^*+vv^*) & 0 \\ v^2 & \sqrt{2}vu^* & 0 & u^{*2} \end{bmatrix}. \tag{96}$$

Next we swap lines 3 and 4 and columns 3 and 4:

$$\begin{bmatrix} u^2 & -\sqrt{2}uv^* & v^{*2} & 0 \\ \sqrt{2}uv & uu^*-vv^* & -\sqrt{2}v^*u^* & 0 \\ v^2 & \sqrt{2}vu^* & u^{*2} & 0 \\ 0 & 0 & 0 & (uu^*+vv^*) \end{bmatrix}. \tag{97}$$

We obtain $uu^*+vv^*=1$ in position $(4,4)$. This is logical because we are making a revertible transformation. We can reduce this then to a 3×3 matrix:

$$\begin{bmatrix} u^2 & -\sqrt{2}uv^* & v^{*2} \\ \sqrt{2}uv & uu^*-vv^* & -\sqrt{2}v^*u^* \\ v^2 & \sqrt{2}vu^* & u^{*2} \end{bmatrix}. \tag{98}$$

And after swapping lines 2 and 3 and columns 2 and 3:

$$\begin{bmatrix} u^2 & v^{*2} & -\sqrt{2}uv^* \\ v^2 & u^{*2} & +\sqrt{2}vu^* \\ \sqrt{2}vu & -\sqrt{2}u^*v^* & uu^* - vv^* \end{bmatrix}. \tag{99}$$

This is just the same matrix as in Eq. 81 and it can also be compared with Eq. 2.2.19 in [14] (who uses a different definition of the SU(2) matrix). In fact, we can derive his expressions from ours by substituting $(u, v, u^*, -v^*)|(a, -b^*, a^*, b)$. These expressions are not particularly useful. We have seen above that it is more convenient to specify the rotation by the parameters $\mathbf{s}(\theta, \phi)$ and $\varphi = \omega_0\tau$. Now $u = \cos(\omega_0\tau/2) - \imath\sin(\omega_0\tau/2)\cos(\theta)$ and $v = -\imath\sin(\omega_0\tau/2)\sin(\theta)e^{\imath\phi}$. This will contain components in $e^{\imath\omega_0\tau/2}$, $e^{-\imath\omega_0\tau/2}$ and 1, which we should all want to express in terms of tensor products.

First we give an example. For the spinning motion around the z-axis, we have $v = 0$ such that also $uu^* - vv^* = 1$ and:

$$\begin{bmatrix} u^2 & v^{*2} & -\sqrt{2}uv^* \\ v^2 & u^{*2} & +\sqrt{2}vu^* \\ \sqrt{2}vu & -\sqrt{2}u^*v^* & uu^* - vv^* \end{bmatrix} = \begin{bmatrix} e^{-\imath\omega_0\tau} & 0 & 0 \\ 0 & e^{+\imath\omega_0\tau} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{100}$$

For a general orientation of the rotation axis, the eigenvalues will still be $e^{-\imath\omega_0\tau}$, $e^{+\imath\omega_0\tau}$, and 1, because the general case is obtained from the special case by a similarity transformation. Thus there will be also an eigenvalue with zero energy after a change of basis.

□ *Note.* It is important that we have made all the transformations to eliminate lines and columns with invertible matrices such that we can get back to the original. These transformations are made to eliminate superfluous identical quantities, like $\xi_0\xi_1$ which above occurs twice. There is a method to achieve this also when the number of identical quantities is larger then 2. For n identical quantities it uses the invertible transfer matrix:

$$\mathbf{S} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 & 1 \\ 1 & e^{\imath 2\pi/n} & \cdots & e^{\imath 2k\pi/n} & \cdots & e^{\imath 2(n-2)\pi/n} & e^{\imath 2(n-1)\pi/n} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 1 & e^{\imath 2j\pi/n} & \cdots & e^{\imath 2jk\pi/n} & \cdots & e^{\imath 2j(n-2)\pi/n} & e^{\imath 2j(n-1)\pi/n} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 1 & e^{\imath 2(n-2)\pi/n} & \cdots & e^{\imath 2(n-2)k\pi/n} & \cdots & e^{\imath 2(n-2)(n-2)\pi/n} & e^{\imath 2(n-2)(n-1)\pi/n} \\ 1 & e^{\imath 2(n-1)\pi/n} & \cdots & e^{\imath 2(n-1)k\pi/n} & \cdots & e^{\imath 2(n-1)(n-2)\pi/n} & e^{\imath 2(n-1)(n-1)\pi/n} \end{bmatrix}. \tag{101}$$

The inverse is $\mathbf{S}^{-1} = \mathbf{S}^\dagger$. This can then be used to calculate Clebsch-Gordan coefficients.

8.6 Derivation of some algebraic identities

8.6.1 Arbitrary products

Let us prove that $[\mathbf{K} \otimes \mathbf{L}][\mathbf{M} \otimes \mathbf{N}] = \mathbf{KM} \otimes \mathbf{LN}$. Let:

$$\mathbf{K} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{102}$$

Then:

$$[\mathbf{K} \otimes \mathbf{L}][\mathbf{M} \otimes \mathbf{N}] = \begin{bmatrix} a\mathbf{L} & b\mathbf{L} \\ c\mathbf{L} & d\mathbf{L} \end{bmatrix} \begin{bmatrix} A\mathbf{N} & B\mathbf{N} \\ C\mathbf{N} & D\mathbf{N} \end{bmatrix} = \begin{bmatrix} (aA + bC)\mathbf{LN} & (aB + bD)\mathbf{LN} \\ (cA + dC)\mathbf{LN} & (cB + dD)\mathbf{LN} \end{bmatrix} = \mathbf{KM} \otimes \mathbf{LN}. \tag{103}$$

We have used here the theorem that matrices can be multiplied block-wise. Of course, we can also prove this if the tensor-product is carried out the other way around, by injecting $\mathbf{K}$ into $\mathbf{L}$ and $\mathbf{M}$ into $\mathbf{N}$ rather than $\mathbf{L}$ into $\mathbf{K}$ and $\mathbf{N}$ into $\mathbf{M}$. To prove this we are putting now:

$$\mathbf{L} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}. \tag{104}$$

Then:

$$[\,\mathbf{K}\otimes\mathbf{L}\,][\,\mathbf{M}\otimes\mathbf{N}\,]=\left[\begin{array}{cc}\mathbf{K}p & \mathbf{K}q\\ \mathbf{K}r & \mathbf{k}s\end{array}\right]\left[\begin{array}{cc}\mathbf{M}P & \mathbf{M}Q\\ \mathbf{M}R & \mathbf{M}S\end{array}\right]=\left[\begin{array}{cc}\mathbf{KM}\,(pP+qR) & \mathbf{KM}\,(pQ+qS)\\ \mathbf{KM}\,(rP+sR) & \mathbf{KM}\,(rQ+sS)\end{array}\right]=\mathbf{KM}\otimes\mathbf{LN}. \quad (105)$$

A special case of this is:

$$[\,\mathbf{K}\otimes\mathbf{K}\,][\,\mathbf{M}\otimes\mathbf{M}\,]=\mathbf{KM}\otimes\mathbf{KM}. \quad (106)$$

In this special case we can introduce the similarity transformation $\mathbf{S}$ to reduce the ranks of the matrices from 4 to 3, based on the fact that the tensor products will now contain duplicate lines and columns because $\mathbf{K}=\mathbf{L}$ and $\mathbf{M}=\mathbf{N}$:

$$\mathbf{S}\,[\mathbf{K}\otimes\mathbf{K}\,]\,\mathbf{S}^{-1}\cdot\mathbf{S}\,[\mathbf{M}\otimes\mathbf{M}\,]\,\mathbf{S}^{-1}=\mathbf{S}\,[\mathbf{K}\otimes\mathbf{K}\,]\,[\mathbf{M}\otimes\mathbf{M}\,]\,\mathbf{S}^{-1}=\mathbf{S}\,[\,\mathbf{KM}\otimes\mathbf{KM}\,]\,\mathbf{S}^{-1}. \quad (107)$$

We can use this also to prove that the inverse of a tensor product is the tensor product of the inverses. It is just the same proof where in addition we now also use:

$$\left[\begin{array}{cc}a & b\\ c & d\end{array}\right]\left[\begin{array}{cc}A & B\\ C & D\end{array}\right]=\mathbb{1}. \quad (108)$$

8.6.2 Inverse by complex conjugation

Furthermore, the inverse of $\mathbf{R}\otimes\mathbf{R}$ is $\mathbf{R}^\dagger\otimes\mathbf{R}^\dagger$ which is its own Hermitian conjugate. Indeed, from:

$$\mathbf{R}\mathbf{R}^\dagger=\left[\begin{array}{cc}u & -v^*\\ v & u^*\end{array}\right]\left[\begin{array}{cc}u^* & v^*\\ -v & u\end{array}\right]=\left[\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right], \quad (109)$$

it follows that:

$$\left[\begin{array}{cc}u\mathbf{R} & -v^*\mathbf{R}\\ v\mathbf{R} & u^*\mathbf{R}\end{array}\right]\left[\begin{array}{cc}u^*\mathbf{R}^\dagger & v^*\mathbf{R}^\dagger\\ -v\ \mathbf{R}^\dagger & u\ \mathbf{R}^\dagger\end{array}\right]=\left[\begin{array}{cc}\mathbb{1} & \\ & \mathbb{1}\end{array}\right]. \quad (110)$$

We have then also:

$$\mathbf{S}\left[\begin{array}{cc}u\mathbf{R} & -v^*\mathbf{R}\\ v\mathbf{R} & u^*\mathbf{R}\end{array}\right]\mathbf{S}^{-1}\cdot\mathbf{S}\left[\begin{array}{cc}u^*\mathbf{R}^\dagger & v^*\mathbf{R}^\dagger\\ -v\ \mathbf{R}^\dagger & u\ \mathbf{R}^\dagger\end{array}\right]\mathbf{S}^{-1}=\left[\begin{array}{cc}\mathbb{1} & \\ & \mathbb{1}\end{array}\right] \quad (111)$$

Because $\mathbf{S}=\mathbf{S}^{-1}=\mathbf{S}^\dagger$, the second matrix is the Hermitian conjugate of the first one. The further manipulations preserve this relation, such that:

$$\left[\begin{array}{ccc}u^2 & v^{*2} & -\sqrt{2}uv^*\\ v^2 & u^{*2} & +\sqrt{2}vu^*\\ \sqrt{2}vu & -\sqrt{2}u^*v^* & uu^*-vv^*\end{array}\right]\left[\begin{array}{ccc}u^{*2} & v^{*2} & \sqrt{2}u^*v^*\\ v^2 & u^2 & -\sqrt{2}vu\\ -\sqrt{2}vu^* & +\sqrt{2}uv^* & uu^*-vv^*\end{array}\right]=\mathbb{1}. \quad (112)$$

8.7 Reflections

We can now also calculate the representation matrix of a vector:

$$\left[\begin{array}{cc}a_z\mathbf{A} & (a_x-\imath a_y)\mathbf{A}\\ (a_x+\imath a_y)\mathbf{A} & -a_z\mathbf{A}\end{array}\right]=\left[\begin{array}{cccc}a_z^2 & a_z(a_x-\imath a_y) & a_z(a_x-\imath a_y) & (a_x-\imath a_y)^2\\ a_z(a_x+\imath a_y) & -a_z^2 & (a_x-\imath a_y)(a_x+\imath a_y) & -a_z(a_x-\imath a_y)\\ a_z(a_x+\imath a_y) & (a_x+\imath a_y)(a_x-\imath a_y) & -a_z^2 & -a_z(a_x-\imath a_y)\\ (a_x+\imath a_y)^2 & -a_z(a_x+\imath a_y) & -a_z(a_x+\imath a_y) & a_z^2\end{array}\right]. \quad (113)$$

After the same manipulations as for the rotation matrix we obtain:

$$\left[\begin{array}{ccc}a_z^2 & (a_x-\imath a_y)^2 & +\sqrt{2}a_z(a_x-\imath a_y)\\ (a_x+\imath a_y)^2 & a_z^2 & -\sqrt{2}a_z(a_x+\imath a_y)\\ \sqrt{2}a_z(a_x+\imath a_y) & -\sqrt{2}a_z(a_x-\imath a_y) & -a_z^2+a_x^2+a_y^2\end{array}\right]. \quad (114)$$

The algebra is tedious and uninspiring.

9 The crucial simplification for studying photons

9.1 Rotations, reflections and reversals in the Oxy-plane

As the representation obtained from SU(2) $\otimes$ SU(2) corresponds to spin 1, one can try to work with the three-dimensional representation matrices we derived, but one gets lost in the algebra and the problem with the eigenvalue 1 always pops up again, which is a nuisance. We will therefore introduce a treatment for the specific case where the polarization and the filters remain perpendicular to the direction of propagation of the photon. Let us choose the reflection operators for the group of two-dimensional rotations just like we did for SU(2) in [6,11]. We can then choose:

$$\sigma_x = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}. \tag{115}$$

If we were to embed these two-dimensional rotations SO(2) into SO(3), then we should take:

$$\sigma_z = \begin{bmatrix} & -\imath \\ \imath & \end{bmatrix}. \tag{116}$$

The in-plane reflection with respect to the straight-line perpendicular to the in-plane normal vector $(\cos\alpha, \sin\alpha)$ is then represented by:

$$\mathbf{G}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}. \tag{117}$$

Together with the z-axis, the straight-line defines what would be the reflection plane in $\mathbb{R}^3$. In planar geometry, a reflection and a reversal are the same thing. The product of two reflections becomes then the rotation:

$$\mathbf{R} = [\,\mathbf{G}(\beta)\,]\,[\,\mathbf{G}(\alpha)\,] = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\beta-\alpha) & -\sin(\beta-\alpha) \\ \sin(\beta-\alpha) & \cos(\beta-\alpha) \end{bmatrix}. \tag{118}$$

The angle $(\alpha-\beta) = \varphi/2$ is half the rotation angle φ. It corresponds to the full angle between the polarizer orientations. We see that the nonstandard choice we made for the Pauli matrices has the advantage to reproduce the standard form of rotation matrices in $\mathbb{R}^2$. The representation matrix of the rotation contains again a column for the spinors that represent rotations and a column for the spinors that represent reversals. We now calculate again the eigenvalues and eigenvectors of a reflection operator. The eigenvalues are given by:

$$\begin{vmatrix} \cos\alpha - \lambda & \sin\alpha \\ \sin\alpha & -\cos\alpha - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\cos^2\alpha + \lambda^2 - \sin^2\alpha = 0 \quad \Rightarrow \quad \lambda \in \{-1, 1\}. \tag{119}$$

We then obtain the eigenvectors:

$$\lambda = 1 \quad \rightarrow \quad \begin{bmatrix} -2\sin^2(\alpha/2) & 2\sin(\alpha/2)\cos(\alpha/2) \\ 2\sin(\alpha/2)\cos(\alpha/2) & -2\cos^2(\alpha/2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{120}$$

Hence:

$$\lambda = 1 \quad \rightarrow \quad \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix}. \tag{121}$$

Similarly:

$$\lambda = -1 \quad \rightarrow \quad \begin{bmatrix} 2\cos^2(\alpha/2) & 2\sin(\alpha/2)\cos(\alpha/2) \\ 2\sin(\alpha/2)\cos(\alpha/2) & 2\sin^2(\alpha/2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{122}$$

Hence:

$$\lambda = -1 \quad \rightarrow \quad \begin{bmatrix} -\sin(\alpha/2) \\ \cos(\alpha/2) \end{bmatrix}. \tag{123}$$

This leads to the diagonalization:

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) & +\sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix}. \tag{124}$$

The decomposition of the reflection matrix is therefore:

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} + \begin{bmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix}. \tag{125}$$

After simplification this leads to:

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2) \end{bmatrix} + \begin{bmatrix} -\sin(\alpha/2) \\ \cos(\alpha/2) \end{bmatrix} \otimes \begin{bmatrix} +\sin(\alpha/2) & -\cos(\alpha/2) \end{bmatrix}. \tag{126}$$

The rotation matrix:

$$\mathbf{R} = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}, \tag{127}$$

will therefore contain four components and $\mathbf{R} \otimes \mathbf{R}$ will contain 16 components.

9.2 Calculation of $\mathbf{G}(\alpha) \otimes \mathbf{G}(\alpha)$

Let us now calculate $\mathbf{G}(\alpha) \otimes \mathbf{G}(\alpha)$:

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \otimes \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & +\cos\alpha\sin\alpha & +\cos\alpha\sin\alpha & \sin^2\alpha \\ \cos\alpha\sin\alpha & -\cos^2\alpha & +\sin^2\alpha & -\sin\alpha\cos\alpha \\ \cos\alpha\sin\alpha & +\sin^2\alpha & -\cos\alpha^2 & -\sin\alpha\cos\alpha \\ \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}. \tag{128}$$

We can rewrite this as:

$$\mathbf{G}(\alpha) \otimes \mathbf{G}(\alpha) = \frac{1}{2} \begin{bmatrix} 1+\cos(2\alpha) & +\sin(2\alpha) & +\sin(2\alpha) & 1-\cos(2\alpha) \\ \sin(2\alpha) & -1-\cos(2\alpha) & +1-\cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & +1-\cos(2\alpha) & -1-\cos(2\alpha) & -\sin(2\alpha) \\ 1-\cos(2\alpha) & -\sin(2\alpha) & -\sin(2\alpha) & 1+\cos(2\alpha) \end{bmatrix}. \tag{129}$$

There are only two eigenvalues $+1$ and -1. Each of them has a two-dimensional space of eigenvectors. In these spaces we can perform a change of basis. We make the linear combinations Line 1 $\pm$ Line 4 and Line 2 $\pm$ Line 3. This change of basis is carried out by the matrix:

$$\mathbf{T}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & 1 & -1 & \\ 1 & & & -1 \end{bmatrix}. \tag{130}$$

This matrix is its own inverse and after the similarity transformation we obtain:

$$\mathbf{T}_1 \left[\mathbf{G}(\alpha) \otimes \mathbf{G}(\alpha) \right] \mathbf{T}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos(2\alpha) & 0 & \sin(2\alpha) \\ 0 & 0 & -1 & 0 \\ 0 & \sin(2\alpha) & 0 & \cos(2\alpha) \end{bmatrix}. \tag{131}$$

Finally we swap Lines 1 and 4 and Columns 1 and 4. This change of basis is carried out by the matrix:

$$\mathbf{T}_2 = \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix}. \tag{132}$$

Carrying out the similarity transformation, we obtain:

$$\mathbf{T}_2\mathbf{T}_1\,[\,\mathbf{G}(\alpha)\otimes\mathbf{G}(\alpha)\,]\,\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}\cos(2\alpha) & \sin(2\alpha) & & \\ \sin(2\alpha) & -\cos(2\alpha) & & \\ & & -1 & \\ & & & 1\end{bmatrix}=\begin{bmatrix}\mathbf{D}_1(2\alpha) & \\ & \mathbf{D}_2\end{bmatrix}. \tag{133}$$

This way, we have block-diagonalized $\mathbf{G}(\alpha)\otimes\mathbf{G}(\alpha)$. The block structure corresponds to two independent 2×2 representations generated by $\mathbf{D}_1(2\alpha)$ and $\mathbf{D}_2$. Both $\mathbf{D}_1(2\alpha)$ and $\mathbf{D}_2$ correspond to reflection operators in their spaces. We can use the 2×2 matrix $\mathbf{D}_1(2\alpha)$ as generators to build a 2×2 representation. In this reduction we obtain an SO(2) formalism whereby one replaces angles α by their doubles 2α. The set of reflection operators $\mathbf{D}_1(2\alpha)$ generates a two-dimensional rotation group. The result is that the rotation angles we obtain in this representation are no longer the halves $\varphi/2$ but the true rotation angles φ. That is because SO(2) is the pendant of SO(3), while SO(3) is quadratic in the spinors of SU(2). The matrices of the type $\mathbf{D}_2$ just generate reflections $\mathbf{D}_2$ and rotations $\mathbb{1}$. This formalism corresponds thus to the eigenvalue $\lambda=1$ of the SO(3) rotation matrices. The eigenvalues in the reflection matrix we reduce are occurring in the order $1,-1,-1,1$ while the eigenvalues in the rotation matrix are occurring $e^{-\imath\varphi},1,1,e^{+\imath\varphi}$. As the vector space that we drop corresponds to the eigenvalues -1 in the reflection matrix, we are dropping the eigenvalues 1 in the rotation matrix.

We could try to reformulate the whole proof laconically as follows. We start from a 2×2 formalism where we restrict SU(2) to operations in a plane. This formalism for the rotations in the plane is then isomorphic to U(1). The 2×2 reflection matrices we obtain in this formalism generate the 2×2 rotation matrices. These 2×2 rotation matrices will just correspond to $e^{\imath\varphi/2}$ in the isomorphic group U(1). Now we want to introduce a product representation for photons. The representations of the rotations in U(1) $\otimes$ U(1) will now be just $e^{\imath\varphi/2}\otimes e^{\imath\varphi/2}=e^{\imath\varphi}$. This is isomorphic with a 2×2 formalism but with doubling of the angles α we used to parameterize the reflections. We must therefore express the reflections with the parameters 2α and 2β whereby $2(\alpha-\beta)$ is twice the full angle between the straight-lines that define the reflections and corresponds thus to the rotation angle φ.

As the rotations will now occur in an algebra with the doubled angle $2\times\varphi/2$, what we have obtained is just the very familiar group SO(2), a restriction of SO(3) to a plane, but with doubled angles. Therefore, the calculations on a photon can be done in SO(2) with a doubling of the angles we encountered in SU(2), which will allow us to recover the true angles $\alpha-\beta$ between the polarization axes instead of $(\alpha-\beta)/2$ in the final results for the probabilities. The internal dynamics of a photon are always planar. And by making the restriction from SO(3) to SO(2) we end up with only two substates. Furthermore, the energy we obtain for the photons is then no longer $\hbar\omega/2$ but $\hbar\omega$. However, we should not be confused by the mathematics. The way we constructed it, this representation does not describe a particle that is turning twice as fast. It is more like a bag containing two particles turning at the same angular frequency. Hence it is more like two particles with spins $\uparrow\uparrow$ or $\downarrow\downarrow$. Of course we do not claim that a photon consists of two particles, even if that may perhaps look an interesting idea. For one thing, as the photon travels at the speed of light, the two particles would have to be flat. In reality, we are only addressing the symmetry of the photon by using the fact that the representations of the rotation group are based on tensor products $\mathbf{R}\otimes\mathbf{R}\otimes\cdots\otimes\mathbf{R}$ and $\boldsymbol{\xi}\otimes\boldsymbol{\xi}\otimes\cdots\otimes\boldsymbol{\xi}$ of SU(2) matrices $\mathbf{R}$ and SU(2) spinors $\boldsymbol{\xi}$.

In SO(3) we get three eigenvalues for a rotation, $e^{\imath\varphi}$, $e^{-\imath\varphi}$ and 1. But the eigenvalue 1 is associated with eigenvectors that are perpendicular to the rotation plane and in SO(2) this eigenvalue must therefore be dropped. We also have argued that we should drop the eigenvalue 1 based on considerations about the energies of the states. By operating the restrictions we avoid taking a shortcut to answering these questions in an offhand way. The photon has spin 1 but only two substates, because we are talking about spin 1 in SO(2) rather than in SO(3).

We can now also understand why the one-dimensional calculation of Malus' law did not work. The problem is two-dimensional with two polarization states. They correspond to the two columns in the SO(2) representation matrix $\mathbf{D}_1(2\alpha)$, in complete analogy with the two columns in the representation matrix in SU(2). We must perform a probability calculus on the curved manifold SO(2) (which is isomorphic to U(1)). This manifold can be visualized by the circle of radius 1 and is a sub-manifold of the curved manifold SO(3) $\subset \mathrm{L}(\mathbb{R}^3,\mathbb{R}^3)$, where $\mathrm{L}(\mathbb{R}^3,\mathbb{R}^3)$ is the space of linear mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$. This is a space of functions, which are abstract objects that are very different from the usual physical objects like scalars or vectors. The group elements require an entirely different probability calculus based on spinors (or more generally wave functions) and the Born rule, as we have derived by classical reasoning in Section 3. That also implies that confronting the traditional calculus on scalars and vectors with the Born rule as an opposition between classical mechanics and QM is flawed. This flawed reasoning can then make us believe in mysterious quantum effects.

10 The derivation of Malus' law $\frac{1}{2}\cos^2(\alpha-\beta)$ for correlated photons

10.1 Preamble

We can thus repeat the calculations we carried out for the Stern-Gerlach experiment for rotations in Section 7. Based on the analogy we can already anticipate that this will enable us to derive Malus' law by purely classical considerations. But it remains to be seen if the physical mechanism should be based on reflection or on rotation of the spinning motion of the photon. Let us point out that we should not confuse the travelling (translational) motion of the photon with its spin dynamics. We are used to associate polarization with reflection of its translational motion because photons reflected from a window pane can be polarized. But this may not apply to the spinning motion. The direction of incidence of the photons onto the polarizers is always normal such there are no phenomena of refraction to be considered.

We can now imagine that the polarizer would reflect certain states of motion, thereby removing them from the beam while it would transmit other states of motion. At this point we must raise our awareness about the fact that one can become very easily trapped in using ambiguous language. It is therefore good to keep in mind the contents of Table 1. We see there that we can speak of opposite spinning motion in two different ways: in the sense of spin or in a geometrical sense. In Section 9 we have just based the discussion on the signs of the angles and the signs of the spin. When we describe this way two states $\uparrow\circlearrowleft$ and $\uparrow\circlearrowright$ with identical spin and angular frequencies the properties $\circlearrowleft$ and $\circlearrowright$ are staying somewhat under the radar in the discussion.

Let us now argue that we should not base the selection mechanism of the polarizer on reflections of the geometrical senses of the spinning motions of the photons. Let us assume that these geometrical senses could be inverted by the polarizer. We can now visualize the phases of two photons in opposite geometrical motion by two balls travelling along a circle. Let us assume that the place where the two balls meet each other after full turns is situated at 12 o'clock. When we apply mentally a mathematical reflection $G(\alpha)$ on the two photons, the meeting point will be switched by an angle 2α. To understand this it suffices to imagine that we put a mirror at α. In the mirror we will see then the balls crossing at 2α. But this does not make sense. The reflection plane would intersect the polarizer plane along a straight-line, the physical polarization axis of the polarizer. But this physical axis would produce a linear polarization of the sets of photons that is not on the axis but at twice the angle between the incoming beam and the polarizer axis. That would be really weird and unphysical. Therefore the operation must be a rotation. We think that a linear polarization of sets of photons should coincide with the polarization axis of the polarizer. The polarizer should force the polarization of the incoming photon beam to adjust itself to the polarization axis. That can only be achieved by rotations. It remains then to be seen what kind of selection criterium we can carry out on the photons.

In summary, with the results obtained until now it should be easy to invent an ugly mathematical model that produces Malus' law, but it is much more difficult to find one that is physical. To be physical, it should reproduce all ancillary results we know about polarizers. We should be able to find the results from Malus' for several filters in succession, etc.... We will propose a model below.

10.2 The basic building brick

According to Eq. 126 a reflection matrix can be decomposed as:

$$\mathbf{G}(\alpha)=\begin{bmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{bmatrix}=\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\otimes\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}+\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}\otimes\begin{bmatrix}+\sin\alpha & -\cos\alpha\end{bmatrix}. \quad (134)$$

Let us call:

$$\psi_{A+}=\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\otimes\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix},\quad \psi_{A-}=\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}\otimes\begin{bmatrix}+\sin\alpha & -\cos\alpha\end{bmatrix}. \quad (135)$$

10.3 The rotation matrix $\mathbf{G}(\beta)\mathbf{G}(\alpha)$

A rotation matrix $\mathbf{R}=\mathbf{G}(\beta)\mathbf{G}(\alpha)$ of SO(2) has then the following components:

$$\begin{aligned}\psi_{B+}\psi_{A+}&=\begin{bmatrix}+\cos\beta\\ \sin\beta\end{bmatrix}\otimes\begin{bmatrix}+\cos\beta & +\sin\beta\end{bmatrix}\begin{bmatrix}+\cos\alpha\\ \sin\alpha\end{bmatrix}\otimes\begin{bmatrix}+\cos\alpha & +\sin\alpha\end{bmatrix}\\ &=+\cos(\beta-\alpha)\begin{bmatrix}+\cos\beta\\ \sin\beta\end{bmatrix}\otimes\begin{bmatrix}+\cos\alpha & +\sin\alpha\end{bmatrix}. \quad (136)\end{aligned}$$

$$\psi_{B+}\psi_{A-} = \begin{bmatrix} +\cos\beta \\ \sin\beta \end{bmatrix} \otimes \begin{bmatrix} +\cos\beta & +\sin\beta \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix}$$
$$= +\sin(\beta-\alpha) \begin{bmatrix} +\cos\beta \\ \sin\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix}. \quad (137)$$

$$\psi_{B-}\psi_{A+} = \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\beta & -\cos\beta \end{bmatrix} \begin{bmatrix} +\cos\alpha \\ \sin\alpha \end{bmatrix} \otimes \begin{bmatrix} +\cos\alpha & +\sin\alpha \end{bmatrix}$$
$$= +\sin(\beta-\alpha) \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \begin{bmatrix} +\cos\alpha & +\sin\alpha \end{bmatrix}. \quad (138)$$

$$\psi_{B-}\psi_{A-} = \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\beta & -\cos\beta \end{bmatrix} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix}$$
$$= -\cos(\beta-\alpha) \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix}. \quad (139)$$

10.4 The matrix AR

As pointed out in Subsection 8.1 we went through a lot of useless calculations before we struck the line of approach developed in 9. One of those was the calculation of a matrix **AR** in a scenario where we would subject both photons to reflections. This approach manifested itself as sterile by leading to difficulties in defining selection criteria for the filter operation. But we learned something from it. A rotation matrix will be described by:

$$\mathbf{R} = e^{+2\imath\omega t}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} \otimes \begin{bmatrix} 1 & +\imath \end{bmatrix}\frac{1}{\sqrt{2}} + e^{-2\imath\omega t}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ +\imath \end{bmatrix} \otimes \begin{bmatrix} 1 & -\imath \end{bmatrix}\frac{1}{\sqrt{2}}. \quad (140)$$

Let us call the first part ψ_{R+} and the second part ψ_{R-} We have then the following four components:

$$\psi_{A+}\psi_{R+} = +\frac{1}{2}e^{+\imath(2\omega t-\alpha)}\begin{bmatrix} \cos\alpha \\ +\sin\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & +\imath \end{bmatrix}.$$
$$\psi_{A+}\psi_{R-} = +\frac{1}{2}e^{-\imath(2\omega t-\alpha)}\begin{bmatrix} \cos\alpha \\ +\sin\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & -\imath \end{bmatrix}.$$
$$\psi_{A-}\psi_{R+} = +\frac{\imath}{2}e^{+\imath(2\omega t-\alpha)}\begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & +\imath \end{bmatrix}.$$
$$\psi_{A-}\psi_{R-} = -\frac{\imath}{2}e^{-\imath(2\omega t-\alpha)}\begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & -\imath \end{bmatrix}. \quad (141)$$

It is obvious that all four combinations will lead to the same probability. We have further:

$$\psi_{A+R+} + \psi_{A-R+} = \frac{1}{2}e^{+\imath(2\omega t-\alpha)}\begin{bmatrix} \cos\alpha - \imath\sin\alpha \\ \sin\alpha + \imath\cos\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & +\imath \end{bmatrix} = \frac{1}{2}e^{+\imath(2\omega t-2\alpha)}\begin{bmatrix} 1 \\ +\imath \end{bmatrix} \otimes \begin{bmatrix} 1 & +\imath \end{bmatrix}. \quad (142)$$

$$\psi_{A+R-} + \psi_{A-R-} = \frac{1}{2}e^{-\imath(2\omega t-\alpha)}\begin{bmatrix} \cos\alpha + \imath\sin\alpha \\ \sin\alpha - \imath\cos\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & -\imath \end{bmatrix} = \frac{1}{2}e^{-\imath(2\omega t-2\alpha)}\begin{bmatrix} 1 \\ -\imath \end{bmatrix} \otimes \begin{bmatrix} 1 & -\imath \end{bmatrix}. \quad (143)$$

We can also add these two terms and we obtain then:

$$\frac{1}{2}e^{+\imath(2\omega t-2\alpha)}\begin{bmatrix} +1 & +\imath \\ +\imath & -1 \end{bmatrix} + \frac{1}{2}e^{-\imath(2\omega t-2\alpha)}\begin{bmatrix} +1 & -\imath \\ -\imath & -1 \end{bmatrix} = \begin{bmatrix} \cos[\,2(\omega t-\alpha)\,] & \sin[\,2(\omega t-\alpha)\,] \\ \cos[\,2(\omega t-\alpha)\,] & -\cos[\,2(\omega t-\alpha)\,] \end{bmatrix}. \quad (144)$$

We recognize here a reflection with angle $\omega t - \alpha$. As the angle is varying it is perhaps more appropriate to describe this spinning reflection as a reversal. We discover this way that there are two possible decompositions for a reversal. In Eq. 126 we decomposed the reversal following the proper rules, *i.e.* according to the eigenvalues -1 and $+1$. The brute-force decomposition of the reversal in Eq. 144 into two components according to $e^{+2\imath\omega t}$ and $e^{-2\imath\omega t}$ given by Eqs. 142-143 is in this respect not following the proper rules. We can observe that we obtained terms which do not yield cosines and sines, such that this method does not permit us to calculate probabilities, except probabilities 1/2. This looks quite puzzling and worrisome. The reason for this is that Eq. 140 does not give us any information about the way this group element was created by successive reflections. By working with successive reflections:

$$G(\alpha_n) \circ G(\alpha_{n-1}) \circ \cdots G(\alpha_2) \circ G(\alpha_1), \tag{145}$$

we are writing the history of the group element. We see that the last probabilities are given by $\cos^2(\alpha_n - \alpha_{n-1})$ and $\sin^2(\alpha_n - \alpha_{n-1})$. But for a turning beam in Eq. 144 we do not have the information about the one but last reversal $G(\alpha_{n-1})$ that occurred before $G(\alpha_n)$. We may now remember that we can obtain a rotation $R(\varphi)$ by two reflections $G(\alpha)$ and $G(\beta)$. We have then $R(\varphi) = G(\beta) \circ G(\alpha)$. It results in $\varphi = 2(\beta - \alpha)$. We can choose the pairs of reflections $G(\alpha)$ and $G(\beta)$ at will as long as $\beta - \alpha = \varphi/2$. This means that we can choose the starting point α at will. Only the angle $\beta - \alpha$ between the two reflection planes counts. This is convenient to calculate a product of two rotations. We can always choose β in such a way that we can write the product as $[\, G(\gamma)G(\beta)\,][\, G(\beta)G(\alpha)\,] = [\, G(\gamma)G(\alpha)\,]$, whereby the two reflections $G(\beta)$ cancel out. This is especially useful for calculations on three-dimensional rotations. That is the reason why Hamilton invented the quaternions, which correspond to the Pauli matrices. We will use this expedient also in our two-dimensional calculations below.

When we specify the rotation by Eq. 140 we do not provide the information about the last angle β. Hence, as we do not know $G(\alpha_{n-1})$ at the time we carry out a new reflection $G(\alpha_n)$, we cannot make the calculation of the probability amplitude. In fact, if the beam is not polarized then this even means that there is no physical angle α_{n-1}. The beam is virgin and bears no scars of a previous reflection by an angle α_{n-1}. There exists only a mathematical angle α_{n-1} in our imagination that we can choose at will. And if the beam is not linearly polarized it will actually contain all angles α_{n-1} with respect to α_n. Therefore we may argue that we must average over the results in terms of $\alpha_n - \alpha_{n-1}$ whereby α_{n-1} runs over all possible angles. This is the reason why we only succeed recovering the trivial probability $\frac{1}{2}$. Based on what we learned from our confrontation with this riddle we decided to work henceforth only with products of reflections. This way we should no longer get stuck in our calculations of the probabilities.

10.5 The product of two rotations

The idea is now to consider two identical photons for the correlation experiment. On both polarizers we define a normal. It is better to keep life simple. These normals must be oriented the same way, even if it runs contrary to our intuition, else we will be using pairs of representation matrices that are related by a rotation over π or by a reflection within SO(3). This means that we will not consider the photons as identical based on their helicity, but just on the basis of their spinning motion. Of course other choices are possible. When both photons are identical, they will then be represented by the same element of the rotation group in both arms of the set-up. And as the photons are correlated we should only take combinations $\uparrow\uparrow$ and $\downarrow\downarrow$ of the circular polarizations of the two photons. Remember in this respect that in our philosophy based on sets, individual photons are always circularly polarized.

We calculate the product of a rotation **BA** representing a photon and another rotation **CB** representing the action of the polarizer on that photon. Note that we are using the reflection **B** in both rotations thereby exploiting the expedient outlined in Subsection 10.4. We must thus multiply all the components:

$$\begin{aligned}
\psi_{C+}\psi_{B+} &= +\cos(\gamma-\beta) \begin{bmatrix} +\cos\gamma \\ \sin\gamma \end{bmatrix} \otimes \begin{bmatrix} +\cos\beta & +\sin\beta \end{bmatrix}, \\
\psi_{C+}\psi_{B-} &= +\sin(\gamma-\beta) \begin{bmatrix} +\cos\gamma \\ \sin\gamma \end{bmatrix} \otimes \begin{bmatrix} +\sin\beta & -\cos\beta \end{bmatrix}, \\
\psi_{C-}\psi_{B+} &= +\sin(\gamma-\beta) \begin{bmatrix} -\sin\gamma \\ \cos\gamma \end{bmatrix} \otimes \begin{bmatrix} +\cos\beta & +\sin\beta \end{bmatrix}, \\
\psi_{C-}\psi_{B-} &= -\cos(\gamma-\beta) \begin{bmatrix} -\sin\gamma \\ \cos\gamma \end{bmatrix} \otimes \begin{bmatrix} +\sin\beta & -\cos\beta \end{bmatrix},
\end{aligned} \tag{146}$$

by all the components:

$$\psi_{B+}\psi_{A+} = +\cos(\beta-\alpha)\begin{bmatrix} +\cos\beta \\ \sin\beta \end{bmatrix} \otimes \begin{bmatrix} +\cos\alpha & +\sin\alpha \end{bmatrix},$$
$$\psi_{B+}\psi_{A-} = +\sin(\beta-\alpha)\begin{bmatrix} +\cos\beta \\ \sin\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix},$$
$$\psi_{B-}\psi_{A+} = +\sin(\beta-\alpha)\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \otimes \begin{bmatrix} +\cos\alpha & +\sin\alpha \end{bmatrix},$$
$$\psi_{B-}\psi_{A-} = -\cos(\beta-\alpha)\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \otimes \begin{bmatrix} +\sin\alpha & -\cos\alpha \end{bmatrix}. \tag{147}$$

It looks like an ordeal because the product will contain 16 terms. We have dropped here the normalization factors. We obtain for the product of the first term of **CB** with the first term of **BA**:

$$\begin{aligned}\psi_{C+}\psi_{B+}\psi_{B+}\psi_{A+} &= \cos(\gamma-\beta)\cos(\beta-\alpha)\begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix} \otimes \begin{bmatrix} \cos\beta & \sin\beta \end{bmatrix}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \otimes \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \\ &= \cos(\gamma-\beta)\cos(\beta-\alpha)\begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix} \otimes \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix}. \end{aligned} \tag{148}$$

The multiplication of the two terms $\psi_{B+}\psi_{B+}$ in the middle yields 1. This renders the task less tedious because there are four such terms with $\psi_{B+}\psi_{B+}$ in the middle:

$$\begin{aligned}&\psi_{C+}\psi_{B+}\psi_{B+}\psi_{A+},\\ &\psi_{C-}\psi_{B+}\psi_{B+}\psi_{A+},\\ &\psi_{C+}\psi_{B+}\psi_{B+}\psi_{A-},\\ &\psi_{C-}\psi_{B+}\psi_{B+}\psi_{A-}.\end{aligned} \tag{149}$$

They yield thus all an normalization factor 1. It is obvious that the sixteen terms can be regrouped in batches of four terms governed by the same middle terms. Lets us now also consider all the terms with $\psi_{B-}\psi_{B-}$ in the middle:

$$\begin{aligned}&\psi_{C+}\psi_{B-}\psi_{B-}\psi_{A+},\\ &\psi_{C-}\psi_{B-}\psi_{B-}\psi_{A+},\\ &\psi_{C+}\psi_{B-}\psi_{B-}\psi_{A-},\\ &\psi_{C-}\psi_{B-}\psi_{B-}\psi_{A-}.\end{aligned} \tag{150}$$

An example is:

$$\begin{aligned}\psi_{C+}\psi_{B-}\psi_{B-}\psi_{A+} &= +\sin(\gamma-\beta)\sin(\beta-\alpha)\begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix} \otimes \begin{bmatrix} +\sin\beta & -\cos\beta \end{bmatrix}\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} \\ &= -\sin(\gamma-\beta)\sin(\beta-\alpha)\begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix}. \end{aligned} \tag{151}$$

They all churn out a normalization factor -1. Next we consider the terms with $\psi_{B+}\psi_{B-}$ in the middle:

$$\begin{aligned}&\psi_{C+}\psi_{B+}\psi_{B-}\psi_{A+},\\ &\psi_{C-}\psi_{B+}\psi_{B-}\psi_{A+},\\ &\psi_{C+}\psi_{B+}\psi_{B-}\psi_{A-},\\ &\psi_{C-}\psi_{B+}\psi_{B-}\psi_{A-}.\end{aligned} \tag{152}$$

An example is:

$$\psi_{C+}\psi_{B+}\psi_{B-}\psi_{A+} = \cos(\gamma-\beta)\sin(\beta-\alpha)\begin{bmatrix}\cos\gamma\\ \sin\gamma\end{bmatrix}\otimes\begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}\begin{bmatrix}-\sin\beta\\ \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}$$
$$= 0. \tag{153}$$

They are all zero. Finally we consider all the terms with $\psi_{B-}\psi_{B+}$ in the middle:

$$\begin{aligned}&\psi_{C+}\psi_{B-}\psi_{B+}\psi_{A+},\\ &\psi_{C-}\psi_{B-}\psi_{B+}\psi_{A+},\\ &\psi_{C+}\psi_{B-}\psi_{B+}\psi_{A-},\\ &\psi_{C-}\psi_{B-}\psi_{B+}\psi_{A-}.\end{aligned} \tag{154}$$

An example is:

$$\psi_{C+}\psi_{B-}\psi_{B+}\psi_{A+} = \sin(\gamma-\beta)\cos(\beta-\alpha)\begin{bmatrix}\cos\gamma\\ \sin\gamma\end{bmatrix}\otimes\begin{bmatrix}+\sin\beta & -\cos\beta\end{bmatrix}\begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}\otimes\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}$$
$$= 0. \tag{155}$$

Also these ones are all zero. The examples we elaborated in Eqs. 148 and 151 add up. It is easy to see that there are four such pairs in the eight remaining non-zero terms. Moreover each pair grinds out a normalization factor $\cos(\gamma-\alpha)$ after summing. And the sine terms we might have reasonably expected have all gone missing! Now *that* is an astounding result. Let us be very clear. This did not germinate in my brain. There is no way I could claim that I anticipated this. It just descended from heaven. As the reader can check in Subsection 10.1, I was wondering what kind of selection criterion we would have to invent to obtain a consistent rule that eliminated all the sines and kept all the cosines, such that it could reproduce the observed probabilities. I was clueless. But the answer is that we did not have to worry. It comes all about automatically. We will explain why it happens in Subsection 10.6.

Let us now calculate the terms that accompany the term $\cos(\gamma-\alpha)$. We have:

$$\begin{aligned}\psi_{C+}\psi_{B+}\psi_{B+}\psi_{A+} &\rightarrow \begin{bmatrix}+\cos\gamma\\ \sin\gamma\end{bmatrix}\otimes\begin{bmatrix}+\cos\alpha & +\sin\alpha\end{bmatrix},\\ \psi_{C-}\psi_{B+}\psi_{B+}\psi_{A+} &\rightarrow \begin{bmatrix}-\sin\gamma\\ \cos\gamma\end{bmatrix}\otimes\begin{bmatrix}+\cos\alpha & +\sin\alpha\end{bmatrix},\\ \psi_{C+}\psi_{B+}\psi_{B+}\psi_{A-} &\rightarrow \begin{bmatrix}+\cos\gamma\\ \sin\gamma\end{bmatrix}\otimes\begin{bmatrix}+\sin\alpha & -\cos\alpha\end{bmatrix},\\ \psi_{C-}\psi_{B+}\psi_{B+}\psi_{A-} &\rightarrow \begin{bmatrix}-\sin\gamma\\ \cos\gamma\end{bmatrix}\otimes\begin{bmatrix}+\sin\alpha & -\cos\alpha\end{bmatrix}.\end{aligned} \tag{156}$$

They are the same ones as in Eqs. 136-139. When we apply in the other arm a rotation **DB** to **BA** we will have rotations **DBBA** and **CBBA**. The relative rotation will be **DC**. But we can rewrite it as **DBBC** such that it takes exactly the appearances of the case we already calculated. As **DBBC** is also a product of two rotations, it will yield a probability amplitude $\cos(\delta-\gamma)$. It has been a long journey but we have now succeeded in calculating Malus' law, because we must average the probability $\cos^2(\gamma-\alpha)$ obtained for the product **CBBA** over all possible rotations **BA**.

10.6 Where does the miracle come from?

In a rotation of SU(2) one can have left-handed and right-handed states that we can obtain one from another by reflection. We can qualify their handedness by plus and minus signs. But when we combine two identical spin 1/2 states then we cannot change the global sign of the combined state because the two minus signs cancel. We constructed the representation $\mathbf{D}_1$ such that it would operate only on spin 1 states, by allowing only for states $\uparrow\uparrow$ and $\downarrow\downarrow$. We relegated the states $\downarrow\uparrow$ and $\uparrow\downarrow$ to the representation $\mathbf{D}_2$ to get rid of them. This is the way it should be. The photon should only be allocated the states $\uparrow\uparrow$ and $\downarrow\downarrow$ because it should have spin 1 and not have a third substate. The rotation matrix allows this way for two states $\uparrow\uparrow$ and $\downarrow\downarrow$. Now a circularly polarized photon is in a pure state $\uparrow\uparrow$ or $\downarrow\downarrow$, while a linearly polarized set is represented by $\uparrow\uparrow+\downarrow\downarrow$. The combinations $\uparrow\downarrow$ and $\downarrow\uparrow$ are disallowed by construction. Therefore,

when we multiply two such states only the combinations $\uparrow\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow\downarrow$ are allowed by the group structure. That is what the mathematics are telling us.

These two states $\uparrow\uparrow$ and $\downarrow\downarrow$ are orthogonal in the mathematics. This is why we obtain this surprising result. The point is of course also that we have started with two photons in identical states. At this stage we cannot provide a physical mechanism that would account for these mathematics because one of the two rotations is exerted by the polarizer. This is not entirely satisfactory. We will give a more satisfactory physical explanation in Subsection 10.8.

In a sense, the way we constructed the photon, a linearly polarized beam is itself a kind of local Bell state of rank 2, while a beam of totally correlated photon pairs is a non-local Bell state of rank 4. We use the terminology "non-local" here in a non-standard sense, to indicate that the two photons of the correlated pair can be separated by large distances. In the calculations for the Stern-Gerlach experiment we were also applying rotations (the precession) to rotations (the spin) and the sines did not disappear, because the electrons are not Bell states. We could imagine that the polarizer accepts the $\downarrow\downarrow$ photons and rejects the $\uparrow\uparrow$ photons or vice versa. But this is not a good mechanism, since after this, the beam will not be linearly polarized.

10.7 The catastrophy

Note that we have assumed that single photons are circularly polarized to explain that they have spin. We opposed that to linear polarization. Sets of two photons with opposite circular polarization photons would then have a reference direction: the point where the two balls meet in the discussion above. We could call that direction the linear polarization of the set. And the beam could turn its polarization such as to render the polarization of this set parallel to that of the polarizer. However, the individual photons do not carry internally a reference direction within their "entrails". They just turn. As an individual circularly polarized photon does not have a built-in linear polarization we cannot understand how it could know how to act in such a way that it produces the desired linear polarization in sets of photons. It does not have an internal reference for this. Everything we did is falling apart! That is why we mentioned that we could construct a purely mathematical model to reproduce Malus' law but that it was far less obvious to find one that is also physical.

10.8 The way out

Fortunately, there is a way out of this deadlock. We should not forget that we built the photon itself as a set. That is the meaning of the tensor product $\mathbf{R} \otimes \mathbf{R}$. This leads to the idea that we should find a way to introduce the linear polarization into this set, i.e. into individual photons. We have seen in Subsection 6.2 that in SU(2) we have four possible states: Left-handed and right-handed clockwise and counterclockwise motion. But a left-handed counterclockwise motion cannot be distinguished algebraically from a right-handed clockwise motion. This leads to the idea that the set which presents the photon must have two elements of different handedness with the same spin but geometrically opposite rotational motion. That would permit them to be circularly and linearly polarized at the same time: after taking into account their different handedness their spins would have to be identical, but their geometrical spinning motions should be opposite.

We made the discussion in Subsection 6.2 precisely for the solution of the thorny problem we ran into here. We need it right at this stage to construct a set of two spin 1/2 particles which define a photon that could be simultaneously linearly and circularly polarized: circularly in terms of the spin and linearly in terms of the geometrical sense of rotation. An example of such a combination would be $\uparrow\circlearrowleft \otimes \uparrow\circlearrowright$.

As Table 1 shows, opposed senses of motion of states like $\uparrow\circlearrowleft$ and $\uparrow\circlearrowright$ cannot be revealed by inspecting the associated scalar quantities like the angular frequency ω_0 or an axial vector quantity like the spin, because the sign of axial vectors is not flipped by a parity transformation. We can only distinguish such opposite geometrical rotations by inspecting a co-rotating vector quantity like an electric field. A vector will reveal the opposite senses of rotation. The rotational states in the table are group elements of the group generated by reflections, which contains the rotation group. They are not vectors and if we want to determine their action on vectors $\mathbf{a}$ we must consider the matrix $\mathbf{R}$ they have been taken from and calculate $\mathbf{R}\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}$. This will reveal that states of opposite states of motion $\circlearrowright$ and $\circlearrowleft$ have an opposite effect on vectors.

A state $\uparrow\circlearrowleft \otimes \uparrow\circlearrowright$ of two spin 1/2 states will be circularly polarized with spin 1 because their spins add up and correspond to an identical polarization of the spin. But as the two states in the product correspond to opposite senses of geometrical rotation they will have opposite effects on an internal electric field vector $\mathbf{E}$. Consider now two vectors $\mathbf{E}_1(t)$, $\mathbf{E}_2(t)$ in the Oxy plane whereby $\mathbf{E}_1(t) = \uparrow\circlearrowleft\ E\,\mathbf{e}_y$ and $\mathbf{E}_2(t) = \uparrow\circlearrowright\ E\,\mathbf{e}_y$. Then $\mathbf{E}_1(t) + \mathbf{E}_2(t) = 2\,\cos(\omega_0\tau/2)\,E\,\mathbf{e}_y$ will correspond to a linearly polarized oscillating electric field. We can repeat that argument for an internal magnetic field that is orthogonal to the electric field. The electric field and magnetic filed of the photon exhibit then the same type of oscillatory behaviour as the electromagnetic field of an electromagnetic wave. The wave behaves like the

photons because it is a set of photons (see also the discussion in [6], Subsection 4.3, p. 37). The photons could align their polarization with the polarization axis of the polarizer due to an electric interaction.

The way we reasoned in Section 9 was that we constructed a tensor product with a state $\uparrow\circlearrowleft\uparrow\circlearrowleft$ with angular frequency $-\omega$, 2 states $\uparrow\circlearrowleft\downarrow\circlearrowleft$ with angular frequency zero, and one state $\downarrow\circlearrowleft\downarrow\circlearrowleft$ with angular frequency $+\omega$. The problem with such states is that they do not contain an internal reference direction.

But we could also have made a construction according to $\uparrow\circlearrowleft\uparrow\circlearrowright$ with angular frequency $-\omega$, 2 states $\uparrow\circlearrowleft\downarrow\circlearrowright$ and $\uparrow\circlearrowright\downarrow\circlearrowleft$ with zero frequency and $\downarrow\circlearrowleft\downarrow\circlearrowright$ with angular frequency $+\omega$. That would be based on the tensor product $\mathbf{R}\otimes\mathbf{R}^{-1}$ instead of $\mathbf{R}\otimes\mathbf{R}$. Now the reflection operators that generate $\mathbf{R}^{-1}$ also have eigenvalues 1 and -1 and the rotation matrices $\mathbf{R}^{-1}$ also have eigenvalues $e^{+\imath\varphi/2}$ and $e^{-\imath\varphi/2}$ such that the tensor products would also yield $+1,-1,-1,1$ and $e^{+\imath\varphi}$, $e^{-\imath\varphi}$, $1,1$. The difference is that the latter states can internally also be linearly polarized, which is not the case with the choice we made in Section 9. We have developed this in the Appendix where we show that the calculations within the representation $\mathbf{R}\otimes\mathbf{R}^{-1}$ are isomorphic to those we made in $\mathbf{R}\otimes\mathbf{R}$ such that there are no additional calculations to be made. We have been forced to adopt this second approach because we could otherwise not explain how the polarizer acts on the individual photons of a beam.

In fact, first imagine that we use the representation $\mathbf{R}\otimes\mathbf{R}$. All substates would then be of the type $\circlearrowleft\circlearrowleft$. We could then obtain linear polarization as a property for sets of two photons by considering $\circlearrowleft\circlearrowleft$ and $\circlearrowright\circlearrowright$ but we could not have linear polarization as a property for an individual photon. We could then not align an individual photon with the polarization axis of the polarizer. Therefore we take the representation $\mathbf{R}\otimes\mathbf{R}^{-1}$. Now the substates will be all $\circlearrowleft\circlearrowright$ such that they all have a linear polarization.

We can now repeat the argument that we should not base the selection mechanism of the polarizer on reflections of the geometrical senses of the spinning motions. But the balls should now not visualize the geometrical senses of the spinning motions of two photons but those of the two constituents of a single photon. I do not know if this proposal can be proved physically right or wrong. One can always attack it, but probably nobody has been able to go into this kind of detail in the analysis. In some texts the calculations needed to make the change from linear polarization to circular polarization and *vice versa* are presented as only requiring a mere change of basis.

We have now a much more physical explanation for the selection mechanism of the polarizer. It could be a interaction between an electric field in the polarizer and the electric field of the photon. We may note further that the angle $\alpha-\beta$ within $\cos(\alpha-\beta)$ is now the true angle between the two polarization axes such that we could consider it as a projection. We recover then again the rule used in QM. That rule provides us further with another physical explanation, which is the one often used in physics. One says that the polarizer consists of linear conducting molecules and that the electric field of the electromagnetic wave moves the electrons along the direction of the linear molecules and not along the perpendicular direction. This corresponds then also to an image of projection even if the full mechanism must still be detailed. In the dichotomy between particles and waves, it was easy to understand the behaviour of photons in terms of waves, i.e. sets of particles, but it was very difficult to understand them in terms of single particles. That is also the problem we encountered in Subsection 10.7 where the model for a single photon did still not correspond to physical reality. But with the model developed in the present subsection we can now reason on single photons in terms of waves and particles at the same time.

11 Conclusion

The whole reasoning has been classical. Because we are using group theory and superposition states corresponding to sets (i.e. beams of particles rather than single particles), it looks as though we are doing calculations on waves. The particle-wave duality just corresponds to the fact that electrons and photons are particles whose internal dynamics can be described by representations of a rotation group. Sets of electrons and photons behave then like waves because the internal motions are periodic (see also [6], pp. 36-37). The mathematics of our calculations on the sets (i.e. the beams of particles) are essentially identical to those of QM. However, they are giving much more detailed and physical explanations. Rather than using the algebra as a blackbox under the Leitmotiv "shut up and calculate!" we have established a one-to-one correspondence between the algebra and the geometry of the group representation theory. Just like in algebraic geometry, we can translate the algebra into geometry and *vice versa*, which renders it possible to completely understand what the algebra means. We were able to find the model for a photon by a permanent dialogue between the geometry and the algebra. There is a cornucopia of good textbooks about group theory in physics, but none of them has elucidated this isomorphism which is so crucial for understanding QM.

In our approach we preserve the algebra which is entirely correct such that everything remains the same as in traditional QM and the agreement with experiment remains the same because the comparison is based on the algebra. One does not need to know what it means to carry out the calculations. That is why one can "shut up and calculate". However, when we try to understand what the algebra is doing, the brainwash of the traditional narrative is throwing dust into our eyes. In our approach, the understanding is no longer subject to confusion due to undue distorted and *ad hoc* parallel interpretations, based on "physical intuition" based on vector calculus rather than group theory. As can be seen from the arguments used, the aha moments of the present work have nothing to do with religious guessing that

an electron presumably is a particle and a wave at the same time, that Schrödinger's cat is both dead and alive at the same time, that there is an infinity of parallel Universes which are incessantly splitting off from ours at a devilish rate[3] or that we should describe particles by wave packets to keep their group velocity smaller than c. The magic of group theory is of an all together different type than the alleged magic of QM. It is the mere beauty of mathematics. Group elements are not physical objects like vectors and require a completely different intuition than the one we are used to. And in the end QM is just a part of Einstein's theory of relativity expressed in the language of group representation theory. As far as I can see from my results up to now, he was right all the way.

Appendix - Calculation of $\mathbf{R} \otimes \mathbf{R}^{-1}$

We can use this to calculate $\mathbf{R} \otimes \mathbf{R}^{-1}$. Let us write a 2×2 rotation matrix of SO(2) as $\mathbf{R}(\beta - \alpha) = \mathbf{G}(\beta)\mathbf{G}(\alpha)$. We have shown this in Eq. 118. We can then show that $\mathbf{G}(-\beta)\mathbf{G}(-\alpha) = [\,\mathbf{R}(\beta - \alpha)\,]^{-1}$.

$$\mathbf{G}(-\beta)\mathbf{G}(-\alpha) = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ -\sin(\beta) & -\cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha) \end{bmatrix} = \begin{bmatrix} \cos(\beta-\alpha) & \sin(\beta-\alpha) \\ -\sin(\beta-\alpha) & \cos(\beta-\alpha) \end{bmatrix}$$
$$= \mathbf{R}(\alpha - \beta) = \begin{bmatrix} \cos(\beta-\alpha) & -\sin(\beta-\alpha) \\ \sin(\beta-\alpha) & \cos(\beta-\alpha) \end{bmatrix}^{-1} = [\,\mathbf{R}(\beta - \alpha)\,]^{-1}. \tag{157}$$

Now we use this to calculate $\mathbf{R} \otimes \mathbf{R}^{-1}$ by using Eq. 106 in the reverse sense.

$$\mathbf{R} \otimes \mathbf{R}^{-1} = \mathbf{G}(\beta)\mathbf{G}(\alpha) \otimes \mathbf{G}(-\beta)\mathbf{G}(-\alpha) = [\,\mathbf{G}(\beta) \otimes \mathbf{G}(-\beta)\,][\,\mathbf{G}(\alpha) \otimes \mathbf{G}(-\alpha)\,]. \tag{158}$$

Let us now calculate $\mathbf{G}(\alpha) \otimes \mathbf{G}(-\alpha)$.

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \otimes \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha) \end{bmatrix} = \begin{bmatrix} +\cos^2(\alpha) & -\sin\alpha\cos\alpha & +\sin\alpha\cos\alpha & -\sin^2(\alpha) \\ -\sin\alpha\cos\alpha & -\cos^2(\alpha) & -\sin^2(\alpha) & -\sin\alpha\cos\alpha \\ +\sin\alpha\cos\alpha & -\sin^2(\alpha) & -\cos^2(\alpha) & +\sin\alpha\cos\alpha \\ -\sin^2(\alpha) & -\sin\alpha\cos\alpha & +\sin\alpha\cos\alpha & +\cos^2(\alpha) \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1+\cos(2\alpha) & -\sin(2\alpha) & \sin(2\alpha) & -(1-\cos(2\alpha)) \\ -\sin(2\alpha) & -(1+\cos(2\alpha)) & -(1-\cos(2\alpha)) & -\sin(2\alpha) \\ +\sin(2\alpha) & -(1-\cos(2\alpha)) & -(1+\cos(2\alpha)) & +\sin(2\alpha) \\ -(1-\cos(2\alpha)) & -\sin(2\alpha) & \sin(2\alpha) & 1+\cos(2\alpha) \end{bmatrix}. \tag{159}$$

Now we calculate (Line 1 ± Line 4)/2 and (Line 2 ± Line 3)/2,

$$= \frac{1}{2}\begin{bmatrix} +\cos(2\alpha) & -\sin(2\alpha) & +\sin(2\alpha) & +\cos(2\alpha) \\ 0 & -1 & -1 & 0 \\ -\sin(2\alpha) & -\cos(2\alpha) & +\cos(2\alpha) & -\sin(2\alpha) \\ +1 & 0 & 0 & -1 \end{bmatrix}, \tag{160}$$

and (Column 1 ± Column 4)/2 and (Column 2 ± Column 3)/2:

$$= \frac{1}{2}\begin{bmatrix} +\cos(2\alpha) & 0 & -\sin(2\alpha) & 0 \\ 0 & -1 & 0 & 0 \\ -\sin(2\alpha) & 0 & -\cos(2\alpha) & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}, \tag{161}$$

Finally we swap Line 2 with Line 3 and Column 2 with Column 3:

$$= \frac{1}{2}\begin{bmatrix} +\cos(2\alpha) & -\sin(2\alpha) & 0 & 0 \\ -\sin(2\alpha) & -\cos(2\alpha) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_1(-2\alpha) & \\ & \mathbf{D}_2 \end{bmatrix}. \tag{162}$$

[3] Imagine a Universe splitting into two Universes. Call the total energy of the pristine Universe E. That quantity exists because the mass of the Universe exists. After the splitting, the two new Universes now have both the energy E. Where does the energy difference $2E - E$ come from?

To make the substitution $\mathbf{R}\otimes\mathbf{R}|\mathbf{R}\otimes\mathbf{R}^{-1}$ it suffices therefore to consistently work in the representation $\mathbf{D}_1(\alpha)$ instead of $\mathbf{D}_1(-\alpha)$. The calculations are therefore isomorphic. The whole calculation in the representation $\mathbf{R}\otimes\mathbf{R}$ in Subsection 10.5 will therefore remain *mutatis mutandis* the same in the representation $\mathbf{R}\otimes\mathbf{R}^{-1}$. All we have to do is to revert all rotation angles throughout. Now:

$$\mathbf{R}\otimes\mathbf{R}^{-1}=\begin{bmatrix}\uparrow\circlearrowleft & \\ & \downarrow\circlearrowleft\end{bmatrix}\otimes\begin{bmatrix}\downarrow\circlearrowleft & \\ & \uparrow\circlearrowleft\end{bmatrix}=\begin{bmatrix}\uparrow\circlearrowleft\downarrow\circlearrowleft & & & \\ & \uparrow\circlearrowleft\uparrow\circlearrowleft & & \\ & & \downarrow\circlearrowleft\downarrow\circlearrowleft & \\ & & & \downarrow\circlearrowleft\uparrow\circlearrowleft\end{bmatrix}$$
$$=\begin{bmatrix}e^{-\imath\varphi/2} & \\ & e^{+\imath\varphi/2}\end{bmatrix}\otimes\begin{bmatrix}e^{+\imath\varphi/2} & \\ & e^{-\imath\varphi/2}\end{bmatrix}=\begin{bmatrix}1 & & & \\ & e^{-\imath\varphi} & & \\ & & e^{+\imath\varphi} & \\ & & & 1\end{bmatrix}. \tag{163}$$

From this we can see that:

$$\begin{aligned}e^{+\imath\varphi}&=(\downarrow\circlearrowleft)(\downarrow\circlearrowleft)\\ e^{-\imath\varphi}&=(\uparrow\circlearrowleft)(\uparrow\circlearrowleft)\\ \cos\varphi&=\ (+\uparrow\uparrow+\downarrow\downarrow)\circlearrowleft\circlearrowleft/2\\ \sin\varphi&=\imath(+\uparrow\uparrow-\downarrow\downarrow)\circlearrowleft\circlearrowleft/2\end{aligned} \tag{164}$$

such that the states have spin ±1 and polarity.

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