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# The Hierarchy of Weighted Densities and the First Digit Phenomenon

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## Abstract

The aim of this paper is to clarify some basic and useful facts about the various notions of densities commonly used in papers on the mantissa distribution of sequences of numbers. We focus on the weighted densities and their hierarchy. Our main tools are classical properties of summation methods. Equivalence classes of weighted densities are identified and their hierarchy is stated. This permits us to give clear answers to several questions about the first digit phenomenon. Moreover, however light the weights are, we exhibit an example of a sequence of positive numbers whose mantissae do not admit any distribution in the sense of the corresponding density.

**Keywords:** Benford's law, first digit phenomenon, mantissa, weighted density, hierarchy

**AMS classification:** 60B10,11B05,11K99

## 1 Introduction and definitions

Following the early works of Newcomb and Benford [17, 1] on real life numbers, many authors have studied the distribution of the first digit in base 10 of sequences  $(u_n)_n$  of positive numbers like  $u_n = 2^n$ ,  $u_n = n!$ ,  $u_n = n^n$ ,  $u_n = F_n$  where  $F_n$  is the  $n$ th Fibonacci number,  $u_n = n$  or  $u_n = p_n$  where  $p_n$  is the  $n$ th prime number and so on (see [19] for a survey). In the first four cases, they proved that, if  $D(u_n)$  denotes the first digit of  $u_n$  and  $\log_{10}$  the decimal logarithm, the natural density of  $A_k^u = \{n \in \mathbb{N}^* : D(u_n) = k\}$  is  $\log_{10} \left( \frac{k+1}{k} \right)$ , that is to say

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{A_k^u}(n) = \log_{10} \left( \frac{k+1}{k} \right) \quad (k = 1, \dots, 9)$$

(here and in the sequel,  $\mathbb{1}_B$  is the indicator function of the subset  $B$ ). In particular, about 30.1% of the  $u_n$  have first digit 1 in the sense of the above formula. This property is known as the *first digit phenomenon*. Classical applications of this phenomenon are fraud detection [18] and computer design [15, 11].

In fact, we know a more precise property which needs three other definitions to be stated: the *Benford's law* (in base 10) is the probability measure  $\mu_B$  on the interval  $[1; 10[$  defined by

$$\mu_B([1; a[) = \log_{10} a \quad (1 \leq a < 10).$$

The *mantissa* of a positive real number  $x$  is the unique number  $\mathcal{M}(x)$  in  $[1; 10[$  such that there exists an integer  $k$  verifying  $x = \mathcal{M}(x)10^k$  (there exists another definition of the mantissa, but for technical reasons we shall use this one). A sequence  $(U_n)_n$  of real numbers in  $[1; 10[$  is called *natural-Benford* if it is naturally distributed as  $\mu_B$ , that is to say if

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{[1; a[}(U_n) = \log_{10} a \quad (1 \leq a < 10).$$

The above formula means that, for each  $a \in [1; 10[$ , the set  $\{n \in \mathbb{N}^* : 1 \leq U_n < a\}$  admits a natural density and its natural density is  $\log_{10} a$  and this can be interpreted as the weak convergence of the uniform probability measure on the set  $\{U_1, \dots, U_N\}$  to  $\mu_B$  as  $N \rightarrow +\infty$ .

A sequence  $(u_n)_n$  of positive numbers is also called natural-Benford if the sequence of mantissae  $(\mathcal{M}(u_n))_n$  is natural-Benford. We can now state: the sequences  $(2^n)_n$ ,  $(n!)_n$ ,  $(n^n)_n$  and  $(F_n)_n$  are all natural-Benford. The study of the mantissa is of course more general than the study of the first digit and allows to derive easily the distribution of every digit and every string of digits of the  $u_n$ .

When  $u_n = n$  or  $u_n = p_n$  (see [8] and [24]),

$$\liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{A_1^u}(n) = \frac{1}{9} \quad \text{and} \quad \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{A_1^u}(n) = \frac{5}{9}. \quad (1)$$

So these two sequences do not verify the first digit phenomenon in the sense of the natural density. From [4], we know that they verify this phenomenon in the sense of the logarithmic density, that is to say

$$\lim_{N \rightarrow +\infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \mathbb{1}_{A_k^u}(n) = \log_{10} \left( \frac{k+1}{k} \right) \quad (k = 1, \dots, 9)$$

where  $\log$  is the natural logarithm. In a way (but not the same way as above), about 30.1% of the  $u_n$  have first digit 1. One corrects the defect in (1) by properly assigning lighter weights to larger numbers.

A more precise property is available in [6] and needs another definition to be stated. A sequence  $(U_n)_n$  of real numbers in  $[1; 10[$  is called *logarithmic-Benford* if it is logarithmically distributed as  $\mu_B$ , that is to say if

$$\lim_{N \rightarrow +\infty} \left( \sum_{n=1}^N \frac{1}{n} \right)^{-1} \sum_{n=1}^N \frac{1}{n} \mathbb{1}_{[1; a[}(U_n) = \log_{10} a \quad (1 \leq a < 10).$$

The above formula means that, for each  $a \in [1; 10[$ , the set  $\{n \in \mathbb{N}^* : 1 \leq U_n < a\}$  admits a logarithmic density and its logarithmic density is  $\log_{10} a$  and again this can

be interpreted as the weak convergence of some sequence  $(P_N)$  of discrete probability measures to  $\mu_B$  as  $N \rightarrow +\infty$ , the atoms of  $P_N$  being again  $U_1, \dots, U_N$ . A sequence  $(u_n)_n$  of positive numbers is also called logarithmic-Benford if the sequence of mantissae  $(\mathcal{M}(u_n))_n$  is logarithmic-Benford.

We can now state: the sequences  $(n)_n$  and  $(p_n)_n$  are logarithmic-Benford. Note that [8] and [23] consider the relative logarithmic density in the set of prime numbers which is strictly weaker than the logarithmic density, as we shall see below.

In the spirit of Diaconis' work about binomial coefficients [2], it is also proved in [6] that the rows of the infinite matrix  $(\mathcal{M}(u_n^m))_{m,n}$ , which are logarithmic-Benford and not natural-Benford, tend to be natural-Benford as  $m$  tends to infinity; that is to say: there exists an increasing function  $N$  from  $\mathbb{N}^*$  to  $\mathbb{N}^*$  such that

$$\lim_{m \rightarrow +\infty} \sup_{1 < a < 10} \left| \frac{1}{N(m)} \sum_{n=1}^{N(m)} \mathbb{1}_{[1;a]}(\mathcal{M}(u_n^m)) - \log_{10} a \right| = 0.$$

Of course, some sequences of mantissae of positive numbers are neither natural nor logarithmic-Benford. The examples of such sequences which come naturally to mind, like  $(\mathcal{M}(10^n))_n$ , do admit a distribution (distinct from the Benford's one) anyway.

## 1.1 Some questions

The quite strange facts described above generate many questions. Here are a few ones, that shall be answered in Section 5 below:

**Question 1:** Are there densities which are strictly weaker (see below) than the logarithmic one?

**Question 2:** Are there classical sequences of positive numbers whose mantissae do not admit any distribution in the sense of natural or logarithmic densities? If yes, what about weaker densities?

**Question 3:** Is there an interest to consider weights lighter than  $\frac{1}{n}$  or heavier than 1?

**Question 4:** Is the first digit phenomenon verified by  $u_n = 2^n$ ,  $u_n = n!$ ,  $u_n = n^n$  or  $u_n = F_n$  if we use weights heavier than 1, like  $n^\alpha$  with  $\alpha > 0$  or  $\alpha^n$  with  $\alpha > 1$  for instance? If yes, is there a maximal value for  $\alpha$ ?

**Question 5:** Is the first digit phenomenon verified by  $u_n = n$  or  $u_n = p_n$  if we use weights heavier than  $\frac{1}{n}$ , like  $\frac{1}{\sqrt{n}}$  for instance? If not, does the choice of one of these weights have an influence anyway?

**Question 6:** If  $u_n = n$  or  $u_n = p_n$ , we know from [6] that the rows of  $(\mathcal{M}(u_n^m))_{m,n}$  are logarithmic-Benford, do not admit any distribution in the sense of the natural density and tend to be natural-Benford. What will happen if we choose intermediate weights (between 1 and  $\frac{1}{n}$ )?

## 1.2 Weighted densities

This leads us, in the wake of [9] and [8] for example, to consider the general notion of weighted densities of  $A \subset \mathbb{N}^*$  where  $\mathbb{N}^*$  is the set of positive integers.

Let  $(w_n)_{n \geq 1}$  be a sequence of positive real numbers summing to infinity and, for each  $N \geq 1$ , let  $W_N = \sum_{n=1}^N w_n$ . One says that  $A \subset \mathbb{N}^*$  has a  $w_n$ -density when the sequence  $\left( \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n) \right)_N$  converges and its limit is then called the  $w_n$ -density of  $A$ . This is the limit of the weighted frequency of the elements of  $A$  among those of  $\mathbb{N}^*$ . The condition on the weights  $w_n$  is necessary to assign the density  $\frac{1}{2}$  to the set of even numbers, for example, and the density 0 to every finite subset of  $\mathbb{N}^*$ .

Another sequence  $(v_n)_{n \geq 1}$  of positive real numbers summing to infinity being given, we set  $V_N = \sum_{n=1}^N v_n$  and we say that the  $w_n$ -density is *stronger* than the  $v_n$ -density when, for every  $A \subset \mathbb{N}^*$ ,

$$\left( \left( \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n) \right)_N \text{ converges} \right) \implies \left( \left( \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_A(n) \right)_N \text{ converges} \right)$$

and when, in this case, the two limits are equal. If either density is stronger than the other one, then the two densities are said to be *equivalent*.

The most commonly used weighted densities are the two we have considered above: the 1-density called *natural* or *arithmetic density* and the  $\frac{1}{n}$ -density called *logarithmic* or *harmonic density*. In [6], the  $\frac{1}{n \log n}$ -density is also considered and called the *loglog-density*. In some papers on Benford's law ([8] and [23] for example), we also find the so-called *logarithmic density conditioned (or relative) to the prime numbers* which can be seen as the  $\frac{1}{p_n}$ -density. The other weights that come immediately to mind are  $w_n = \alpha^n$  where  $\alpha > 1$ ,  $w_n = n^\alpha$  or  $w_n = p_n^\alpha$  with  $\alpha > 0$  or  $-1 < \alpha < 0$ ,  $w_n = n^\alpha (\log n)^\beta$  or  $w_n = p_n^\alpha (\log p_n)^\beta$  with  $\alpha > -1$  and  $\beta \in \mathbb{R}$ ,  $w_n$  polynomial, and  $w_n = \frac{1}{g_q(n)}$  with  $g_0(n) = n$ ,  $g_1(n) = n \log n$ ,  $g_2(n) = n(\log n)(\log \log n)$ , and so on ( $n$  large enough).

### 1.3 Contents

It is well known ([22, page 272] for example) that the 1-density is strictly stronger than the  $\frac{1}{n}$ -density and Kuipers and Niederreiter [16, page 64] mentioned a quite surprising property: all the  $n^\alpha$ -densities with  $-1 < \alpha < 0$  and  $0 < \alpha$  are equivalent to the 1-density. In Section 2, we state three theorems found in Hardy's book [12] which give a clear view on the hierarchy between weighted densities. To the best of our knowledge, these theorems are not mentioned in papers on Benford's law. Section 3 is devoted to new results. The first one shows that equivalent weights lead to equivalent densities. The second one proves that the hierarchy between the  $\frac{1}{g_q(n)}$ -densities ( $q = 0, 1, 2, \dots$ ) is strict. The third one states that, however light the considered weights are, we can find a classical sequence of positive numbers whose mantissae do not admit any distribution in the sense of the corresponding density. Combining sections 2 and 3 enables us to identify in Section 4 an infinite number of equivalence classes of weighted densities and in Section 5 to give simple and clear answers to the questions we have listed above. We give in Section 6 a short overview on other densities used in the study of the first digit phenomenon and their connections with weighted densities. Some open problems are described in Section 7. For the sake of clarity and self-contained exposition, we give in

an Appendix the proofs of the theorems stated in Section 2, rewritten in the context of weighted densities.

We have focused on numeration in base 10, but all statements remain true in every base  $b \geq 2$  except for the sequence  $(2^n)_n$  which is not a Benford sequence in bases 2, 4, 8, and so on.

## 2 Summation methods properties

These properties are derived from Cesàro's and Toeplitz's works and are stated and proved in [12, pages 42–63] in the general context of hierarchy of summation methods applied to a sequence  $(s_n)$  of finite sums of a series. But a close look at the proofs shows that the nature of the sequence  $(s_n)$  is not important. We state them below in the context of weighted densities. So the sequence  $(\mathbf{1}_A(n))$  where  $A \subset \mathbb{N}^*$  will take the place of  $(s_n)$ . Theorem 1 below is also stated and proved in [16, pages 63–64].

Let us recall that  $(v_n)$  and  $(w_n)$  are two sequences of positive real numbers summing to infinity and that  $V_N = \sum_{n=1}^N v_n$  and  $W_N = \sum_{n=1}^N w_n$ .

**Theorem 1.** *A sufficient condition for the  $w_n$ -density to be stronger than the  $v_n$ -density is*

$$\left(\frac{v_n}{w_n}\right)_n \text{ is non-increasing} \quad (2)$$

or

$$\left(\frac{v_n}{w_n}\right)_n \text{ is non-decreasing and } \left(\frac{v_N W_N}{w_N V_N}\right)_N \text{ is bounded.} \quad (3)$$

Theorem 2 below gives a clear view on the effect of the heaviness of the weights.

**Theorem 2.** *Let  $A \subset \mathbb{N}^*$ . If the sequence  $\left(\frac{v_n}{w_n}\right)_n$  is non-increasing, then*

$$\underline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n) \leq \underline{\lim}_N \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_A(n) \leq \overline{\lim}_N \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_A(n) \leq \overline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n).$$

Theorem 3 below will permit us to prove that the exponential weights are not relevant in the context of weighted densities of a subset of  $\mathbb{N}^*$ .

**Theorem 3.** *Let  $A \subset \mathbb{N}^*$ . If the sequence  $\left(\frac{W_{N-1}}{w_N}\right)_N$  is bounded, then  $A$  cannot admit any  $w_n$ -density unless  $A$  is finite or cofinite.*

## 3 New results

Let us recall that  $(v_n)$  and  $(w_n)$  are two sequences of positive real numbers summing to infinity and that  $V_N = \sum_{n=1}^N v_n$  and  $W_N = \sum_{n=1}^N w_n$ .

**Proposition 1.** *If  $v_n$  and  $w_n$  are equivalent as  $n \rightarrow +\infty$ , then the  $v_n$ -density and the  $w_n$ -density are equivalent too.*

*Proof.* It is well known that  $V_N$  and  $W_N$  are equivalent as  $N \rightarrow +\infty$  since the  $v_n$  and the  $w_n$  sum to infinity. So we can write  $w_n = v_n + v_n\theta_1(n)$  with  $\lim_n \theta_1(n) = 0$  and  $W_N = V_N + V_N\theta_2(N)$  with  $\lim_N \theta_2(N) = 0$  and

$$\theta_2(N) = \sum_{n=1}^N \frac{v_n}{V_N} \theta_1(n).$$

Consider now  $A \subset \mathbb{N}^*$ . For every  $N \geq 1$ ,

$$\begin{aligned} \left| \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n) - \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_A(n) \right| &\leq \sum_{n=1}^N \frac{v_n |V_N + V_N\theta_1(n) - W_N|}{W_N V_N} \\ &\leq \frac{|V_N - W_N|}{W_N} + \left| \sum_{n=1}^N \frac{v_n}{W_N} \theta_1(n) \right| \\ &\leq 2 \frac{V_N}{W_N} |\theta_2(N)|. \end{aligned}$$

So  $A$  admits a  $w_n$ -density if and only if it admits a  $v_n$ -density and, in this case, the two densities are equal.

**Remark.** In fact the above calculations imply a deeper property: Define the lower and the upper  $w_n$ -densities of  $A$  by  $\underline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n)$  and  $\overline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_A(n)$ . If  $v_n$  and  $w_n$  are equivalent, then the lower (respectively upper)  $w_n$ -density and the lower (respectively upper)  $v_n$ -density of any  $A \subset \mathbb{N}^*$  are equal. And it is easy to verify that if two weighted densities are equivalent, the corresponding lower (or upper) densities of any  $A \subset \mathbb{N}^*$  are not necessarily equal. For example, the upper 1-density and the upper  $n$ -density of the set of positive integers whose first digit is 1 are respectively  $\frac{5}{9}$  and  $\frac{3}{4}$ , although the 1-density and the  $n$ -density are equivalent as we shall see below.

The classical example of a subset of  $\mathbb{N}^*$  which admits a logarithmic density but does not admit any natural density is  $B = \bigcup_{m=0}^{+\infty} \{n : e^m \leq n < 2e^m\}$ . We now generalize this example to densities lighter than the natural and the logarithmic ones.

Set  $\log^{(1)} = \log$ ,  $\exp^{(1)} = \exp$  and, for  $q \geq 1$ ,  $\log^{(q+1)} = \log^{(q)} \circ \log$ ,  $\exp^{(q+1)} = \exp^{(q)} \circ \exp$  and

$$B_q = \bigcup_{m=0}^{+\infty} \{n : \exp^{(q)}(e^m) \leq n < \exp^{(q)}(2e^m)\}.$$

With these notations, the numbers  $g_q(n)$  (see the introduction for definition) can be defined by  $g_0(n) = n$  and  $g_q(n) = g_{q-1}(n) \log^{(q)} n$  ( $q \geq 1$  and  $n$  large enough).

**Proposition 2.** For  $q \geq 1$ ,  $B_q$  admits a  $\frac{1}{g_q(n)}$ -density, but does not admit any  $\frac{1}{g_{q-1}(n)}$ -density.

*Proof.* We shall use techniques like those of Fuchs and Letta in [4]. Let  $q \geq 1$ ,  $w_n = \frac{1}{g_{q-1}(n)}$ ,  $v_n = \frac{1}{g_q(n)}$ ,  $W_N = \sum_1^N w_n$ ,  $V_N = \sum_1^N v_n$ ,  $a_m = \exp^{(q)}(e^m)$ ,  $b_m = \exp^{(q)}(2e^m)$ ,

$C_m = \{n : a_m \leq n < b_m\}$ ,  $D_m = \{n : a_{m-1} \leq n < a_m\}$  and  $E_m = \{n : b_{m-1} \leq n < b_m\}$ .

Then

$$\underline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_{B_q}(n) = \lim_M \frac{\sum_{m=1}^M \sum_{n \in C_{m-1}} w_n}{\sum_{m=0}^M \sum_{n \in D_m} w_n} = \lim_m \frac{\sum_{n \in C_{m-1}} w_n}{\sum_{n \in D_m} w_n}$$

because the numbers in the numerators sum to infinity and so do the ones in the denominators. For the same reasons,

$$\overline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_{B_q}(n) = \lim_m \frac{\sum_{n \in C_m} w_n}{\sum_{n \in E_m} w_n}.$$

Using classical integral calculations, we get  $\sum_{n \in C_m} w_n \sim e^m$ ,  $\sum_{n \in D_m} w_n \sim e^m - e^{m-1}$  and  $\sum_{n \in E_m} w_n \sim 2(e^m - e^{m-1})$  as  $m \rightarrow +\infty$ . Hence

$$\underline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_{B_q}(n) = \frac{1}{e-1} \quad \text{and} \quad \overline{\lim}_N \sum_{n=1}^N \frac{w_n}{W_N} \mathbf{1}_{B_q}(n) = \frac{e}{2(e-1)}.$$

Since  $\lim_m \sum_{n \in C_m} v_n = \log 2$  and  $\lim_m \sum_{n \in D_m} v_n = \lim_m \sum_{n \in E_m} v_n = 1$ , the same arguments give

$$\underline{\lim}_N \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_{B_q}(n) = \overline{\lim}_N \sum_{n=1}^N \frac{v_n}{V_N} \mathbf{1}_{B_q}(n) = \log 2.$$

The following theorem gives a general view of the  $\frac{1}{g_q(n)}$ -densities ( $q = 1, 2, \dots$ ) in connection with the first digit phenomenon. For any real  $x$ ,  $\{x\}$  denotes the fractional part of  $x$ , that is to say  $\{x\} = x - [x]$  where  $[x]$  is the greatest integer smaller than  $x$  and  $\delta_x$  is the Dirac measure at  $x$ . For  $x > 0$ , we set  $f(x) = \{\log_{10} x\}$ .

**Theorem 4.** *Let  $q \geq 0$ . The sequence  $\left(\mathcal{M}(\log^{(q+2)} n)\right)_n$  does not admit any distribution in the sense of the  $\frac{1}{g_q(n)}$ -density.*

*Proof.* In the sequel,  $n_0$  is any integer greater than  $\exp^{(q+2)}(1)$ . For  $n \geq n_0$ , let  $w_n = \frac{1}{g_q(n)}$ ,  $x_n = \mathcal{M}(\log^{(q+2)} n)$  and  $y_n = \log_{10}(\log^{(q+2)} n)$ . For  $N \geq n_0$ , let

$$P_N = \sum_{n=n_0}^N \frac{w_n}{W_N} \delta_{x_n} \quad \text{and} \quad Q_N = \sum_{n=n_0}^N \frac{w_n}{W_N} \delta_{\{y_n\}}$$

where  $W_N = \sum_{n=n_0}^N w_n$ , and let

$$G_N(t) = \sum_{n=n_0}^N \frac{w_n}{W_N} e^{it\{y_n\}} \quad (t \text{ real}).$$

Note that  $W_N$  is equivalent to  $\log^{(q+1)} N$  as  $N \rightarrow +\infty$  and that  $G_N$  is the Fourier transform of  $Q_N$ . Since  $f(x) = f(\mathcal{M}(x))$ , we get  $Q_N = P_N f^{-1}$ , that is to say,  $Q_N(I) = P_N(f^{-1}(I))$  for every interval  $I \subset [0, 1[$ . It is easy to verify that the weak convergence of the sequence  $(P_N)_N$  to some probability measure  $\mu$  is equivalent to the weak convergence of  $(Q_N)_N$  to  $\mu f^{-1}$  and, by Lévy's theorem on weak convergence, this



is equivalent to the pointwise convergence of  $(G_N)_N$  to the Fourier transform of  $\mu f^{-1}$ . So, to prove our theorem, it suffices to verify that the sequence  $(G_N(2\pi))_N$  diverges.

Fix  $N > n_0$ . Since  $e^{2i\pi\{y_n\}} = e^{2i\pi y_n}$ , the classical Abel transform of a sum gives

$$G_N(2\pi) = e^{2i\pi y_N} + \frac{1}{W_N} \sum_{n=n_0}^{N-1} W_n (e^{2i\pi y_n} - e^{2i\pi y_{n+1}}).$$

Now, the mean value inequality gives

$$|e^{2i\pi y_n} - e^{2i\pi y_{n+1}}| \leq \frac{2\pi(\log 10)^{-1}}{g_{q+2}(n)} \quad (n = n_0, \dots, N-1).$$

Moreover,

$$\frac{W_n}{g_{q+2}(n)} \sim \frac{w_n}{\log^{(q+2)} n} \quad (n \rightarrow +\infty).$$

Since the numbers above sum to infinity,

$$\sum_{n=n_0}^{N-1} \frac{W_n}{g_{q+2}(n)} \sim \sum_{n=n_0}^{N-1} \frac{w_n}{\log^{(q+2)} n} \quad (N \rightarrow +\infty).$$

The classical generalizations of Cesàro's theorem (see [12, page 43] or the lemma in Section 8 of the present paper) show that

$$\lim_{N \rightarrow +\infty} \sum_{n=n_0}^{N-1} \frac{w_n}{W_N \log^{(q+2)} n} = 0$$

because  $\lim_n \frac{1}{\log^{(q+2)} n} = 0$ . So  $(G_N(2\pi))_N$  diverges since  $(e^{2i\pi y_N})_N$  diverges and

$$\lim_{N \rightarrow +\infty} \frac{1}{W_N} \sum_{n=n_0}^{N-1} W_n (e^{2i\pi y_n} - e^{2i\pi y_{n+1}}) = 0.$$

## 4 Hierarchy of weighted densities

Combining sections 2 and 3 gives a clear vision of the hierarchy between weighted densities.

**General principle:** Theorem 2 shows clearly that the heavier the weights  $w_n$  are, the rarer are the subsets  $A$  of  $\mathbb{N}^*$  which admit a  $w_n$ -density. Moreover, when  $A$  does not admit any density, the  $w_n$ 's heaviness affects the lower and the upper  $w_n$ -densities of  $A$  (see the above remark).

On this subject, we must mention [9] which contains a study of the continuity of the function  $\alpha \mapsto (\underline{d}_\alpha(A), \bar{d}_\alpha(A))$  where  $A \subset \mathbb{N}^*$  is fixed,  $\alpha$  is varying in  $[-1; +\infty]$  and  $\underline{d}_\alpha(A)$  and  $\bar{d}_\alpha(A)$  are respectively the lower and the upper  $n^\alpha$ -densities. For example, the set  $A$  of positive integers whose first digit is 1 does not admit any  $n^\alpha$ -density for  $\alpha > -1$ , but, since it verifies the conditions of theorem 2 in [9], the function

$\alpha \mapsto (\underline{d}_\alpha(A), \bar{d}_\alpha(A))$  is continuous at point  $-1$  and so  $\underline{d}_\alpha(A)$  and  $\bar{d}_\alpha(A)$  monotonically tend to  $\log_{10} 2$  as  $\alpha \rightarrow -1^+$ .

**Exponential weights:** The weights  $w_n = \alpha^n$  ( $\alpha \neq 1$ ) are not relevant because the condition  $\sum_n w_n = +\infty$  implies  $\alpha > 1$  and then, by theorem 3, the only subsets of  $\mathbb{N}^*$  which admit a  $w_n$ -density are the finite and the cofinite ones. The densities of these kinds of subsets of  $\mathbb{N}^*$  are respectively equal to 0 and 1, whatever the value of  $\alpha > 1$  is.

**Strict hierarchy:** It is evident (see straight above) that the  $\alpha^n$ -densities with  $\alpha > 1$  are strictly stronger than the 1-density and it is well known that the 1-density is strictly stronger than the  $\frac{1}{n}$ -density ([22, page 272] for example). We can now state: the  $\frac{1}{n}$ -density is strictly stronger than the  $\frac{1}{n \log n}$ -density which is strictly stronger than the  $\frac{1}{n(\log n)(\log \log n)}$ -density and more generally, for  $q \geq 0$ , the  $\frac{1}{g_q(n)}$ -density is strictly stronger than the  $\frac{1}{g_{q+1}(n)}$ -density. Indeed, the  $\frac{1}{g_q(n)}$ -density is stronger than the  $\frac{1}{g_{q+1}(n)}$ -density by the condition (2) in theorem 1 and the  $\frac{1}{g_{q+1}(n)}$ -density is not stronger than the  $\frac{1}{g_q(n)}$ -density as proposition 2 shows.

**Natural density equivalence class:** Most of the weights which come naturally to mind lead to densities which are equivalent to the 1-density. Firstly, as Kuipers and Niederreiter noticed [16, page 64], all the  $n^\alpha$ -densities with  $-1 < \alpha$  are equivalent. Indeed, if  $-1 < \alpha_1 < \alpha_2$ , condition (2) of theorem 1 proves that the  $n^{\alpha_2}$ -density is stronger than the  $n^{\alpha_1}$ -density and condition (3) shows the converse. Secondly, for  $-1 < \alpha$  and  $\beta \in \mathbb{R}$ , the  $n^\alpha(\log n)^\beta$ -density is equivalent to the  $n^\alpha$ -density and then to the 1-density because the condition (2) proves that it is stronger than the  $\sqrt{n^{\alpha-1}}$ -density and weaker than the  $n^{\alpha+1}$ -density. Moreover, proposition 1 shows that all the  $P(n)Q(\log n)$ -densities ( $P$  and  $Q$  polynomials) belong to this equivalence class too and so do the  $p_n^\alpha$ -densities with  $\alpha > -1$  because  $p_n \sim n \log n$  as  $n \rightarrow +\infty$ .

**Logarithmic density equivalence class:** Somehow, this equivalence class is smaller than the previous one (although it possesses infinitely many elements). Condition (3) of theorem 1 shows that, if  $w_n = \frac{(\log n)^\alpha}{n}$  with  $\alpha > -1$ , then the  $w_n$ -density is equivalent to the  $\frac{1}{n}$ -density. This and proposition 1 prove that the  $\frac{P(\log n)}{n}$  ( $P$  polynomial) and the  $\frac{\log p_n}{p_n}$ -density (utilized in [14]) belong to this equivalence class too.

**Weaker densities equivalence classes:** When  $w_n = \frac{(\log^{(q+1)} n)^\alpha}{g_q(n)}$  ( $q \geq 1$ ) with  $\alpha > -1$ , the same arguments show that the  $w_n$ -density is equivalent to the  $\frac{1}{g_q(n)}$ -density and again the only simple way to construct other equivalent densities seems to use proposition 1. For example, the  $\frac{1}{p_n}$ -density (utilized in [8] and [23]) and the  $\frac{1}{n \log n}$ -density are equivalent (case  $q = 1$ ).

**Maybe the weakest density:** Let  $w_n^{(q)} = \frac{1}{g_q(n)}$ . Then, by theorem 2, the limits below exist and we can set

$$\underline{W}_\infty(A) = \lim_{q \rightarrow +\infty} \left( \liminf_N \sum_{n=1}^N \frac{w_n^{(q)}}{W_N^{(q)}} \mathbb{1}_A(n) \right) \quad \text{and} \quad \overline{W}_\infty(A) = \lim_{q \rightarrow +\infty} \left( \overline{\lim}_N \sum_{n=1}^N \frac{w_n^{(q)}}{W_N^{(q)}} \mathbb{1}_A(n) \right).$$

And so we have defined a new lower density and a new upper density which define a density when they are equal. This new density can be named the  $W_\infty$ -density. It is not,

apparently, a weighted density and is weaker than all the densities we have considered in the present paper.

## 5 Consequences for Benford sequences

We give here some answers to the questions we have listed in Section 1.

**Answer to question 1:** The  $\frac{1}{g_q(n)}$ -densities for  $q \geq 1$  are all strictly weaker than the logarithmic one.

**Answer to question 2:** Assuming that everybody agrees to consider  $(\log \log n)_n$  as a classical sequence, theorem 4 shows that the correct answer to question 2 is yes. It also shows that, however light the weights are, we can find a sequence of positive numbers whose mantissae do not admit any distribution in the sense of the corresponding density.

**Answer to question 3:** Proposition 2 and theorem 4 show that there is an interest to consider densities which are strictly weaker than the logarithmic one. The question of the interest of densities strictly stronger than the natural one remains open because of the second point of Section 4.

**Answer to question 4:** Yes, the first digit phenomenon is verified by  $u_n = 2^n$ ,  $u_n = n!$ ,  $u_n = n^n$  or  $u_n = F_n$  if we choose heavier weights like  $n^\alpha$  with  $\alpha > 0$  and no, there is no maximal value for  $\alpha$ . The first digit phenomenon is not verified if we choose weights like  $\alpha^n$  with  $\alpha > 1$ .

**Answer to question 5:** No, the first digit phenomenon is not verified by  $u_n = n$  or  $u_n = p_n$  if we choose weights like  $\frac{1}{\sqrt{n}}$  because, if that was the case, it would be verified in the sense of the natural density. However, theorem 2 shows that, if we choose weights like  $n^\alpha$  with  $-1 < \alpha < 0$ , that will bring the upper and the lower densities (see the remark in 3.1) together.

**Answer to question 6:** No line of  $(\mathcal{M}(u_n^m))_{m,n}$  admits a distribution in the sense of the natural density and then in the sense of the  $n^\alpha$ -density for  $-1 < \alpha < 0$ . But these rows tend (as  $m \rightarrow +\infty$ ) to be distributed as  $\mu_B$  in the sense of the natural density [6]. Somehow, there is a quantum leap from  $\alpha = -1$  to  $\alpha = 0$  (and even  $\alpha = 1$ , and so on) as  $m \rightarrow +\infty$ . Of course, the last sentence of answer to question 5 is still true here.

## 6 Other densities

Some authors [8], [14] or [23] have introduced the notion of *conditional densities relative to the set of prime numbers*. The natural conditional density [8] is in fact the 1-density, the logarithmic conditional density [8] and [14] is the  $\frac{1}{p_n}$ -density and is equivalent to the  $\frac{1}{n \log n}$ -density by proposition 1 and the  $\frac{\log p_n}{p_n}$ -density [23] is equivalent to the  $\frac{1}{n}$ -density.

In [21], Serre claimed that Bombieri proved the following result: the *analytic density* of  $A = \{n : D(p_n) = 1\}$  is  $\log_{10} 2$ , that is to say

$$\lim_{\sigma \rightarrow 1^+} \zeta(\sigma)^{-1} \sum_{n \in A} n^{-\sigma} = \lim_{\sigma \rightarrow 1^+} (\sigma - 1) \sum_{n \in A} n^{-\sigma} = \log_{10} 2$$

where  $\zeta$  is the Riemann zeta function. This density is sometimes called *Dirichlet density* or  $\zeta$ -*density* and is equivalent to the logarithmic density [22, page 274]. Moreover, the *analytic density relative to prime numbers* of  $B \subset \mathbb{N}^*$  is defined by

$$\lim_{\sigma \rightarrow 1^+} (-\log(\sigma - 1))^{-1} \sum_{p_n \in B} p_n^{-\sigma}$$

(Dirichlet used it to prove his theorem on arithmetic progressions), but we have not found it in papers on the first digit phenomenon.

In [7], [13], [3], [20] and [5] for instance, the  $H_\infty$ -*density* is defined in the following manner. A subset  $A \subset \mathbb{N}^*$  being given, set  $H_{0,n} = \mathbb{1}_A(n)$  ( $n \geq 1$ ) and for  $m \geq 1$ ,

$$H_{m,n} = \frac{1}{n} \sum_{j=1}^{j=n} H_{m-1,j} \quad (n \geq 1).$$

When the sequence  $(H_{m,n})_n$  converges, its limit is called the  $H_m$ -density of  $A$ . One says that  $A$  admits a  $H_\infty$ -density when

$$\lim_{m \rightarrow +\infty} \underline{\lim}_n H_{m,n} = \lim_{m \rightarrow +\infty} \overline{\lim}_n H_{m,n}$$

and then its  $H_\infty$ -density is the common value of these two limits. Since the sequences  $(\mathbb{1}_A(n))_n$  are bounded, all the  $H_m$ -densities are equivalent to the natural density [12, page 62]. By Cesàro's theorem, the natural density is stronger than the  $H_\infty$ -density. In [5], Duran shows that the  $H_\infty$ -density is stronger than the logarithmic density. In [3], Diaconis exhibits examples which prove that the converses of these two properties are false.

In short, each conditional weighted density can be viewed as a classical weighted density and belongs to one of the equivalence classes we have listed above and the natural density is strictly stronger than the  $H_\infty$ -density which is strictly stronger than the logarithmic density which is equivalent to the analytic density.

We have focused our attention on densities employed in papers about the first digit phenomenon papers, but the Analytic Number Theory specialists use to consider other densities. Many of them are listed in [10].

## 7 Conclusion

We list in this Section a few open (as far as we know) questions about weighted densities and their hierarchy.

**Open question 1:** What is the exact influence of the weights  $w_n$  over the discrepancy

$$\sup_{1 < a < 10} \left| \left( \sum_{n=1}^N \frac{w_n}{W_N} \mathbb{1}_{[1; a[}(\{\mathcal{M}(u_n)\}) \right) - \log_{10} a \right| ?$$

**Open question 2:** Is the natural density the strongest weighted density among those which are relevant in the study of mantissae distributions?

**Open question 3:** Can we find two weighted densities such that none of them is stronger than the other? If yes, can we find a subset of  $\mathbb{N}^*$  admitting two distinct weighted densities?

**Open question 4:** We can replace  $w_n = 1$  by  $w_n = n^\alpha$  with  $-1 < \alpha < 0$  in the study of the rows of the infinite matrix  $(\mathcal{M}(u_n^m))_{m,n}$  where  $u_n = n$  or  $u_n = p_n$ . How does the choice of  $\alpha$  influence over the choice of the function  $N$  in the formula

$$\lim_{m \rightarrow +\infty} \sup_{1 < a < 10} \left| \sum_{n=1}^{N(m)} \frac{w_n}{W_{N(m)}} \mathbb{1}_{[1; a[}(\mathcal{M}(u_n^m)) - \log_{10} a \right| = 0$$

and over the convergence rate?

**Open question 5:** Is the analytic density relative to prime numbers equivalent to the  $\frac{1}{p_n}$ -density and then equivalent to the  $\frac{1}{n \log n}$ -density?

**Open question 6:** Is the  $W_\infty$ -density (see the last paragraph of Section 4) strictly weaker than any  $\frac{1}{g_q(n)}$ -density?

**Open question 7:** Does the sequence  $(\mathcal{M}(p_n))$  admit a distribution in the sense of the  $H_\infty$ -density?

## 8 Appendix

In the sake of clarity and self-contained presentation, here are the proofs of the three theorems stated in Section 2, rewritten in the context of weighted densities.

**Lemma.** *Let  $(S_n)_{n \geq 1}$  be a convergent sequence of real numbers and  $(C_{N,n})_{N,n}$  be a triangular array ( $N \geq 1, 1 \leq n \leq N$ ) of real numbers verifying*

$$\text{the sequence } \left( \sum_{n=1}^N |C_{N,n}| \right)_N \text{ is bounded,} \quad (4)$$

$$\forall n \geq 1, \lim_{N \rightarrow +\infty} C_{N,n} = 0 \quad (5)$$

and

$$\forall N \geq 1, \sum_{n=1}^N C_{N,n} = 1. \quad (6)$$

For  $N \geq 1$ , we set

$$T_N = \sum_{n=1}^N S_n C_{N,n}.$$

Then the sequence  $(T_N)_{N \geq 1}$  converges and  $(S_n)_{n \geq 1}$  and  $(T_N)_{N \geq 1}$  have the same limit.

*Proof.* Let  $K > 0$  be a bound evoked in (4). Let  $\varepsilon > 0$ ,  $N_0$  such that, for every  $n > N_0$

and every  $N > N_0$ ,  $|S_n - S_N| \leq \varepsilon K^{-1}$  and let  $N > N_0$ . Then

$$\begin{aligned} |T_N - S_N| &= \left| \sum_{n=1}^N S_n C_{N,n} - S_N \sum_{n=1}^N C_{N,n} \right| \\ &= \left| \sum_{n=1}^{N_0} (S_n - S_N) C_{N,n} + \sum_{n=N_0+1}^N (S_n - S_N) C_{N,n} \right| \\ &\leq \left| \sum_{n=1}^{N_0} (S_n - S_N) C_{N,n} \right| + \varepsilon \end{aligned}$$

by (6) and the definition of  $N_0$ . It remains to remark that, by (5),

$$\lim_{N \rightarrow +\infty} \left| \sum_{n=1}^{N_0} (S_n - S_N) C_{N,n} \right| = 0$$

since all the sequences  $(S_n - S_N)_{N \geq 1}$  ( $n = 1, \dots, N_0$ ) converge.

## 8.1 Proof of theorem 1.

Let  $A \subset \mathbb{N}^*$  and, for every  $N \geq 1$ ,

$$S_N = \sum_{n=1}^N \frac{w_n}{W_N} \mathbb{1}_A(n) \quad \text{and} \quad T_N = \sum_{n=1}^N \frac{v_n}{V_N} \mathbb{1}_A(n).$$

Then  $w_1 s_1 = W_1 S_1$  and  $w_N s_N = W_N S_N - W_{N-1} S_{N-1}$  ( $N = 2, \dots$ ) and then, for every  $N \geq 1$ ,

$$T_N V_N = \frac{v_1}{w_1} W_1 S_1 + \frac{v_2}{w_2} (W_2 S_2 - W_1 S_1) + \dots + \frac{v_N}{w_N} (W_N S_N - W_{N-1} S_{N-1}),$$

that is to say

$$T_N = \sum_{n=1}^N S_n C_{N,n}$$

with

$$C_{N,N} = \frac{v_N}{w_N} \frac{W_N}{V_N}$$

and, for  $n = 1, \dots, N-1$ ,

$$C_{N,n} = \left( \frac{v_n}{w_n} - \frac{v_{n+1}}{w_{n+1}} \right) \frac{W_n}{V_N}.$$

So the condition (5) of the lemma is verified. The condition (6) too since, if the  $\mathbb{1}_A(n)$  are all equal to 1, so are the  $S_N$  and the  $T_N$ .

If (2) is verified, then the  $C_{N,n}$  are nonnegative and so the sequence  $\left( \sum_{n=1}^N |C_{N,n}| \right)_N$  is constant and this gives the lemma's condition (4).

If (3) is verified, then  $C_{N,N}$  is nonnegative and  $C_{N,n}$  is negative for  $n = 1, \dots, N-1$ . Then

$$\sum_{n=1}^N |C_{N,n}| = C_{N,N} - \sum_{n=1}^{N-1} C_{N,n}.$$

Since, as we have seen above,

$$C_{N,N} + \sum_{n=1}^{N-1} C_{N,n} = 1,$$

this and the second part of second condition of theorem 1 prove that

$$\sum_{n=1}^N |C_{N,n}| = 2C_{N,N} - 1 = 2 \frac{v_N}{w_N} \frac{W_N}{V_N} - 1$$

is bounded. Condition (4) of the lemma is verified.

## 8.2 Proof of theorem 2.

With the same notations as in theorem 1, the coefficients  $C_{N,n}$  in the formula

$$T_N = \sum_{n=1}^N S_n C_{N,n}$$

are nonnegative (see the calculations above) if the condition of theorem 2 is verified.

Let  $\varepsilon > 0$ ,  $I = \underline{\lim}_n S_n$  and  $N_0$  an integer such that

$$\forall n > N_0, S_n > I - \varepsilon.$$

Then, for each  $N > N_0$ ,

$$T_N > \sum_{n=1}^{N_0} S_n C_{N,n} + (I - \varepsilon) \sum_{n=N_0+1}^N C_{N,n}.$$

This implies  $\underline{\lim}_N T_N \geq I - \varepsilon$  because

$$\lim_{N \rightarrow +\infty} C_{N,n} = 0 \quad (n = 1, \dots, N_0)$$

and so

$$\lim_{N \rightarrow +\infty} \sum_{n=N_0+1}^N C_{N,n} = 1$$

(recall that the  $C_{N,n}$  verify the lemma's condition (6)).

The superior limits can be investigated in the same way.

## 8.3 Proof of theorem 3.

Let again  $A \subset \mathbb{N}^*$  and

$$S_N = \sum_{n=1}^N \frac{w_n}{W_N} \mathbb{1}_A(n).$$

Then, for each  $N$ ,

$$\begin{aligned} \mathbb{1}_A(N) - S_N &= \frac{W_N S_N - W_{N-1} S_{N-1}}{w_N} - \frac{W_N - W_{N-1}}{w_N} S_N \\ &= \frac{W_{N-1}}{w_N} (S_N - S_{N-1}). \end{aligned}$$

We can conclude due to Cauchy's criteria since the sequence  $\left(\frac{W_{N-1}}{w_N}\right)_N$  is bounded and the sequence  $(\mathbb{1}_A(N))_N$  cannot converge unless  $A$  is finite or cofinite.

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