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Random ergodic theorems and regularizing random weights.

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Abstract

We study the convergence of pointwise ergodic means for random subsequences, in a universal framework, together with ergodic means which are modulated by random weights. The methods used in this work involve, mainly, Gaussian tools, transference principles and new results on oscillation functions.

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1 Introduction and main results

We introduce, in this work, different and complementary approaches in the study of convergence of pointwise ergodic averages for random subsequences, in a universal framework. We study also the pointwise convergence of ergodic averages which are modulated by random weights, for any subsequence.

The different strategies will consist, for some cases, in using a transference method on \mathbb{Z}^d with the shift, and in showing maximal inequalities (see Section 3). For other cases, we shall work directly, for instance, on the dynamical system, to show maximal inequalities, or to calculate explicit bounds of the oscillation function associated with the studied transformation (see Section 4). Finally, we study, in Section 5, weighted ergodic theorems and we show the regularizing effect of some random weights. In all these cases, the basic tools (classic and less known) will be the spectral Lemma and the dilation Theorem. We shall also use results of R. Jones about oscillation in ergodic theory, together with tools which come from the Fourier analysis and the study of regularity of Gaussian random functions' trajectories, and which are already known in this domain (see [25], [26], [27]). Section 2 is devoted to these tools.

The first purpose of this work is to give conditions on the process $\{X_k, k \in \mathbb{N}\}$ define on a complete probability space (Ω, \mathcal{B}, P) that we assume to be integer-valued, to be able to construct a positive numerical sequence (a_N) and a universal measurable set Ω_0 such that $P(\Omega_0) = 1$ in order that there exists a positive random variable C such that for all $\omega \in \Omega_0$, $C(\omega) < \infty$, for any measurable dynamical system (Y, \mathcal{A}, μ, T) (that is to say, for any probability space (Y, \mathcal{A}, μ) and any μ -preserving bijective transform $T : Y \rightarrow Y$), for all $f \in L^2(\mu)$, we have, on Ω_0 ,

$$\left\| \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k} - \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k} \right\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)} a_N, \quad (1)$$

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with $\lim_{N \rightarrow +\infty} a_N = 0$.

Hence, by studying the convergence properties of the sequence of deterministic ergodic averages below

$$\left\{ \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k}, N \geq 1 \right\}$$

we shall obtain quantitative and universal results for the convergence of the following random ergodic averages :

$$\left\{ \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)}, N \geq 1 \right\}.$$

First, let us specify the notion of \mathbb{Z}^d -action.

Definition 1.1 Suppose that $\{T_j : j \in \mathbb{Z}^d\}$ is a collection of invertible transformations of the measurable space (Y, \mathcal{A}, μ) .

1) We say that $T := \{T_j : j \in \mathbb{Z}^d\}$ is a nonsingular action of \mathbb{Z}^d on (Y, \mathcal{A}, μ) if

(i) $\exists X \in \mathcal{A}$ such that $\mu(Y - X) = 0$ and $\forall j \in \mathbb{Z}^d, T_j : X \rightarrow X$.

(ii) $(i, y) \rightarrow T_i(y)$ is measurable $(\mathbb{Z}^d, X \rightarrow X)$,

(iii) $\forall y \in X, \forall (i, j) \in \mathbb{Z}^d \times \mathbb{Z}^d, T_i \circ T_j(y) = T_{i+j}(y)$.

(iv) $T_0(y) = y, \forall y \in X, 0$ being the neutral element of \mathbb{Z}^d .

2) Moreover, a nonsingular action of \mathbb{Z}^d is called measure preserving if $\mu \circ T_j = \mu, \forall j \in \mathbb{Z}^d$. We say that the quadruplet (Y, \mathcal{A}, μ, T) a dynamical system. Thus an automorphism naturally induces a unitary operator on $L^2(\mu)$.

Example: if S_1, S_2, \dots, S_d are d commuting automorphisms on Y , and if $T = S_1 \circ S_2 \circ \dots \circ S_d$, a typical example of \mathbb{Z}^d -actions is obtained by considering for any $j = (j_1, \dots, j_d) \in \mathbb{Z}^d, T^j = S_1^{j_1} \circ S_2^{j_2} \circ \dots \circ S_d^{j_d}$.

In this paper we introduce several approaches to obtain pointwise ergodic theorems for random processes. A first way consists in transferring the problem from Y to \mathbb{Z}^d (replacing T by the shift), and to show maximal inequalities.

The second approach is directly located on the studied dynamical system using the spectral Lemma (see section 2.1, Technical Lemmas). According to the general form of the process $\{X_k, k \in \mathbb{N}\}$, it is sometimes more convenient to use the first method or the second one.

Let us recall that a \mathbb{N}^d -valued sequence $(u_n)_{n \in \mathbb{N}}$ is said to be L^2 -good.

Definition 1.2 A \mathbb{N}^d -valued sequence $(u_n)_{n \in \mathbb{N}}$ is said to be L^2 -good if and only if, for all dynamical system (Y, \mathcal{A}, μ, T) and for all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{u_k} \text{ exists } \mu\text{-almost everywhere.}$$

More generally, we shall say that a sequence is L^2 -good for positive isometries and L^2 -good for positive contractions, if the average above converges also for, respectively, all positive isometry and all positive contraction T .

In order to prove that a sequence is L^2 -good, it is often convenient to show, in a first time, that there exists a positive constant C such that, for all $\rho > 1$,

$$\forall v \in l^2(\mathbb{Z}^d), \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)}, \quad (2)$$

where

$$\mathcal{N}_\rho := \{[\rho^k], k \in \mathbb{N}\}, \quad (3)$$

and, using Definition 1.3 below, $\sum_{j=1}^J \|\tilde{M}_j^\rho v\|_{l^2(\mathbb{Z}^d)} / (J\|v\|_{l^2(\mathbb{Z}^d)})$ tends to 0, when J tends to ∞ , at a speed which depends only on ρ .

Definition 1.3 For a sequence of integers $(N_j)_{j \geq 1}$ such that $N_{j+1} \geq 2N_j$, define

$$\forall v \in l^2(\mathbb{Z}^d), \quad \tilde{M}_j^\rho v(n) := \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N v(n + u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(n + u_k) \right|, \quad (4)$$

where $\mathcal{N}_\rho^j := \{[\rho^k], k \in \mathbb{N}, N_j \leq [\rho^k] < N_{j+1}\}$, for $\rho > 1$.

We shall therefore restrict ourselves to the sequences which satisfy these two properties. We shall call these sequences *good for maximal and oscillation inequalities* or *MOI-good*. These properties are just known to be sufficient conditions for a sequence to be L^2 -good.

Examples: For instance when $d = 1$, the sequences studied in [3],[28] and [30], $u_N = N$, $u_N = p_N$, the N^{th} primes number, and $u_N = [P(N)]$, the integer part of a polynomial $P(N)$ with real coefficients, are MOI-good sequences.

When $d \geq 2$, we can also choose $(u_N) \in \mathbb{N}^d$ as d polynomials with integer coefficients in the situation where the \mathbb{Z}^d -action is the same as in the example following the definition 1.1 ([28] th. 6.2, [3]).

Furthermore, from the paper of R. Jones, J. Olsen and M. Wierdl (see [13]) it provides immediately an extension of any theorem in the present paper from measure preserving transformations to positive contractions of L^2 .

Our purpose, here, is to study the behavior of MOI-good sequences, when they are perturbed by a multiplicative, independent, identically distributed (i.i.d) random variable. More precisely, if we denote by $\|\cdot\|$ a norm on \mathbb{R}^d , we show the following.

Theorem 1.1 Let $\{\theta_k : k \in \mathbb{N}\}$ and $\{\tilde{\theta}_k : k \in \mathbb{N}\}$ be two independent families of \mathbb{N}^d -valued i.i.d. random variables such that $\mathbb{E} \left| \prod_{i=1}^d \theta_0^{(i)} \right|^\delta < \infty$ and $\mathbb{E} \left| \prod_{i=1}^d \tilde{\theta}_0^{(i)} \right|^\delta < \infty$ for some $\delta > 0$. Let $(u_k)_{k \in \mathbb{N}}$ be a \mathbb{N}^d -valued sequence such that $\|u_k\| = O(2^{k^\beta})$ for some $\beta \in]0, 1[$.

Then we have:

1) Assume $(u_k)_{k \in \mathbb{N}}$ be a MOI-good sequence then there exists an absolute measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, the sequence $\{\theta_k(\omega) \cdot u_k + \tilde{\theta}_k(\omega), k \in \mathbb{N}\}$ is MOI-good.

2) Assume $(u_k)_{k \in \mathbb{N}}$ is an L^2 -good sequence and the maximal function associated to $(u_k)_{k \in \mathbb{N}}$ is of type (2,2) i.e. the maximal inequality (2) is hold then there exists an absolute measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ for all measurable dynamical system (Y, \mathcal{A}, μ, T) (T being a \mathbb{Z}^d -action) and all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{\theta_k(\omega) \cdot u_k + \tilde{\theta}_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

Remark:

We define $\theta_k \cdot u_k$ as an element of \mathbb{N}^d by the multiplication margin by margin (see section 3.2).

Using the second approach, we shall prove the following result in the one-dimensional case ($d = 1$).

Theorem 1.2 Let (Ω, \mathcal{B}, P) and $(\Omega', \mathcal{B}', P')$ be two independent probability spaces. Let S_k be a random walk generated by a sequence of non-centered i.i.d. and integer-valued random variables which have a moment of order two on $(\Omega', \mathcal{B}', P')$. Fix the walk and consider a sequence of integer-valued independent random variables $\{X_k, k \in \mathbb{N}\}$ which are defined on (Ω, \mathcal{B}, P) .

Moreover, assume that the law of the X_k 's is generated by the convolution of a given law, that is to say, there exists an integrable random variable Y such that

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(S_k)}. \quad (5)$$

Then there exists a measurable set $\Omega_0 \subset \Omega \times \Omega'$, with $P \times P'(\Omega_0) = 1$, such that, for all fixed $(\omega, \omega') \in \Omega_0$, we have:

For all measurable dynamical system (X, \mathcal{A}, μ, T) , and for all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} = \mathbb{E}(f|\mathcal{F}), \mu\text{-a.e.}$$

where \mathcal{F} denotes the σ -field generated by T -invariant functions of $L^2(\mu)$.

Let us explain the meaning of the equation (5). When S_k is fixed, the process $\{X_k, k \in \mathbb{N}\}$ is independent and the law of each X_k is the same as $Y_1 + Y_2 + \dots + Y_{S_k}$, where the Y_i 's are i.i.d. and have the same law as Y . For instance, if Y is a Poisson law of parameter 1, X_k is a Poisson law of parameter S_k . However, on $\Omega \times \Omega'$, the process $\{X_k, k \in \mathbb{N}\}$ is not independent.

We shall show more precisely that the oscillation function $O_{L^2(\mu)}(f, T^{X_k}, (N_j), \mathcal{N}_\rho)$ (see Section 2) associated with the sequence of ergodic averages above is bounded by $K(\omega, \omega') \|f\|_{L^2(\mu)}$ where $P \times P'(K(\omega, \omega') < \infty) = 1$ where K is independent of the dynamical system.

Here is another result that we shall prove using a similar technique.

Theorem 1.3 *Let (Ω, \mathcal{B}, P) be a probability space on which is defined a sequence of integer-valued independent random variables $\{X_k, k \geq 1\}$. Moreover, assume that the law of the X_k 's is generated by the convolution of a given law, that is to say, there exists an integrable random variable Y such that*

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(u_k)}.$$

where (u_k) is a sequence of integers such that there exists $\beta \in]0, 1[$ with $u_k = O(2^{k^\beta})$.

1) If (u_k) is assumed to be *MOI-good*, then there exists a measurable set Ω_0 , with $P(\Omega_0) = 1$, such that, for all fixed $\omega \in \Omega_0$ the sequence $\{X_k(\omega), k \geq 1\}$ is *MOI-good*.

2) If (u_k) is assumed to be *L²-good* for all non-negative contraction, then there exists a measurable set Ω_0 , with $P(\Omega_0) = 1$, such that, for all fixed $\omega \in \Omega_0$, we have:

For all measurable dynamical system (X, \mathcal{A}, μ, T) and all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} \text{ exists } \mu\text{-a.e.}$$

The approach we shall use to prove the preceding results (in particular, Gaussian tools) can also be used in the study of convergence of ergodic averages which are modulated by random weights. More precisely, we shall highlight the regularizing effect of the random weights generated by centered i.i.d. random variables that have a moment of order 2, in the pointwise ergodic theorem, for any subsequence.

Theorem 1.4 *Let (Ω, \mathcal{B}, P) be a probability space on which is defined a sequence of centered independent and identically distributed random variables $\{X_k, k \in \mathbb{N}\}$ that satisfies $\mathbb{E}|X_1|^2 < \infty$. Let $(u_k, k \geq 1)$ be an strictly increasing sequence of integer such that there exists $\beta \in (0, 1)$ with $u_k = O(2^{k^\beta})$. Then there exists a set Ω_0 , \mathcal{B} -measurable, with $P(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, for all measurable dynamical system (X, \mathcal{A}, μ, T)*

$$\forall f \in L^2(\mu), \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N X_k(\omega) f \circ T^{u_k} = 0 \text{ } \mu\text{-a.e.}$$

Remark that, when the sequence $(u_k)_{k \geq 1}$ is universally bad (see [20] for the definition), one can find a function $f \in L^2(\mu)$ which satisfies the property of strong sweeping out, that is to say,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(T^{u_k} x) = 1, x\text{-}\mu\text{-a.e.}$$

and

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(T^{u_k} x) = 0, x\text{-}\mu\text{-a.e.}$$

In that extreme case (and if $u_k = O(2^{k^\beta})$), we know that, for almost every realization of universal sequence of centered random variables (X_k) with a moment of order 2, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N X_k f(T^{u_k} x) = 0, \quad x\text{-}\mu\text{-a.e.}$$

In the sequel, \mathbb{E} will always denote the symbol of integration on the space (Ω, \mathcal{B}, P) . Moreover for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and all $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, we shall denote $\langle \alpha, \beta \rangle = \sum_{j=1}^d \alpha_j \beta_j$ and $\alpha \cdot \beta = (\alpha_1 \beta_1, \dots, \alpha_d \beta_d)$. In order to study (1) we shall use essentially spectral tools together with regularity properties of trajectories of Gaussian random functions. More precisely, we shall show the following result which is an extension of Salem-Zygmund inequalities (1954, see [21]). Indeed, the latter give asymptotical estimates of the uniform magnitude of random trigonometric polynomials for an independent process $\{X_k, k \in \mathbb{N}\}$ which take a finite number of integer values (see [20]). In this case, their proofs are based on Bernstein inequality for polynomials. We shall give a generalization of these inequalities; Gaussian tools will replace Bernstein inequality. Hence, we will be able to obtain uniform asymptotical estimates of trigonometric polynomials for \mathbb{R}^d -valued random processes $\{X_k, k \in \mathbb{N}\}$ which satisfy a weak moment condition. Of course, for applications in ergodic theory, the random process will be \mathbb{N}^d -valued.

Theorem 1.5 *Let $\{X_k, k \in \mathbb{N}\}$ be independent random process with values in \mathbb{R}^d ($d \geq 1$) and defined on a complete probability space (Ω, \mathcal{B}, P) . Let $\mathcal{N} \subset \mathbb{N}$ and assume that there exists an increasing positive sequence $(q_N)_{N \in \mathbb{N}}$ such that $(\sqrt{N}/q_N)_{N \geq 1}$ is non increasing sequence and*

$$\frac{1}{q_N} = O\left(\frac{1}{\sqrt{N \log N}}\right), \quad (6)$$

and

$$\forall 1 \leq i \leq d, \quad \mathbb{E} \sup_{N \geq 1} \sqrt{\frac{N \log^+(|X_N^{(i)}|)}{q_N^2}} < \infty, \quad (7)$$

where the symbol $X_N^{(i)}$ denotes the i^{st} component of the random vector X_N . Then we have the following asymptotic estimation

$$\mathbb{E} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^N \exp 2i\pi \langle \alpha, X_k \rangle - \mathbb{E} \exp 2i\pi \langle \alpha, X_k \rangle \right| < \infty. \quad (8)$$

Notice that the condition (7) is not empty since it contains the following example: $X_k = u_k + \theta_k$ where (u_k) is any strictly increasing sequence of positive integers ($d = 1$) and (θ_k) a sequence of independent and identically distributed, random variables satisfying

$$\exists \delta > 0 / \mathbb{E} |\theta_1|^\delta < \infty.$$

We can show that the condition is satisfied when $q_N = \sqrt{N \log(u_N)}$ [23].

Moreover if we suppose that for some $\delta \in]0, 1[$ there exists $\gamma \in]0, 1[$ such that $\mathbb{E} (\|X_N\|^\delta) = O(2^{N^\gamma})$ then we will prove in section 2.3 that the assumptions of theorem 1.5 hold with $q_N = N^{\gamma'}$ where $\gamma' \in]\frac{\gamma+1}{2}, 1[$.

Now, from the spectral Lemma, we have

$$\begin{aligned} & \left\| \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} - \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k} \right\|_{L^2(\mu)}^2 = \\ & \int_{[0,1]^d} \left| \frac{1}{N} \sum_{k=1}^N \exp 2i\pi \langle \alpha, X_k(\omega) \rangle - \frac{1}{N} \sum_{k=1}^N \mathbb{E} \exp 2i\pi \langle \alpha, X_k \rangle \right|^2 \mu_f(d\alpha), \end{aligned} \quad (9)$$

where μ_f is the spectral measure of the operator T at the point f . Theorem 1.5 gives us therefore a bound for (9) as follows

$$(9) \leq \|f\|_{L^2(\mu)}^2 \left(\frac{q_N}{N}\right)^2 \chi^2(\omega).$$

with

$$\chi(\omega) = \sup_{N \geq 1} \sup_{\alpha \in [0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^N \exp 2i\pi \langle \alpha, X_k(\omega) \rangle - \frac{1}{q_N} \sum_{k=1}^N \mathbb{E} \exp 2i\pi \langle \alpha, X_k \rangle \right|.$$

But χ is a positive random variable which is integrable because of Theorem 1.5. And it is obviously independent of the choice of the dynamical system.

Hence, as a corollary of Theorem 1.5, we obtain the following result which studies the asymptotic behavior of the ergodic averages introduced in (1), within the framework of multidimensional dynamical systems, that is to say that the transform T is no longer a \mathbb{Z} -action but a \mathbb{Z}^d -action, $d \geq 1$.

Corollary 1.1 *Let $\mathcal{N} \subset \mathbb{N}$. For all independent process $\{X_k, k \in \mathbb{N}\}$ with values in \mathbb{N}^d , and all normalization q_N satisfying the conditions (6) and (7) of Theorem 1.5, there exists a measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, there exists a finite positive constant $C = C(\omega)$ such that for all measurable dynamical system (Y, \mathcal{A}, μ, T) , T being a \mathbb{Z}^d -action, for all $f \in L^2(\mu)$, we have*

$$\forall N \in \mathcal{N}, \quad \left\| \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} - \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k} \right\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)} \frac{q_N}{N}.$$

This property is actually satisfied, more generally, for all probability space (Y, \mathcal{A}, μ) and all $L^2(\mu)$ -contraction T .

2 Gaussian Tools and proof of Theorem 1.5

2.1 technical lemmas

Let us present now the Gaussian tools proved by X.Fernique that we shall use in the sequel (see [23] for more details).

Proposition 2.1 *Let $\{G_k, k \in \mathbb{N}\}$ be a sequence of Gaussian vectors with values in a Banach space $(B, \|\cdot\|)$. Then*

$$\mathbb{E} \sup_{k \in \mathbb{N}} \|G_k\| \leq K_1 \left\{ \sup_{k \in \mathbb{N}} \mathbb{E} \|G_k\| + \mathbb{E} \sup_{k \in \mathbb{N}} |\lambda_k \sigma_k| \right\},$$

where $\{\lambda_k, k \geq 0\}$ is an iso-normal sequence, K_1 an absolute constant and for all $k \in \mathbb{N}$,

$$\sigma_k = \sup_{f \in B', \|f\| \leq 1} \| \langle G_k, f \rangle \|_{2, \Omega}.$$

Proposition 2.2 *Let g be a real-valued Gaussian random function which is stationary, separable and continuous in quadratic mean. Moreover, m denotes its associated spectral measure on \mathbb{R}^+ such that*

$$\mathbb{E}(|g(s) - g(t)|^2) = 2 \int (1 - \cos 2\pi u(s-t)) m(du).$$

Then

$$\mathbb{E} \sup_{\alpha \in [0,1]} g(\alpha) \leq K_2 \left\{ \left(\int \min(u^2, 1) m(du) \right)^{1/2} + \int (m(\cdot \exp x^2, \infty))^{1/2} dx \right\},$$

where K_2 is an absolute constant.

Proposition 2.3 [Slépián-Fernique] *Let T be a finite set with cardinal n , X and Y be two Gaussian vectors with values in \mathbb{R}^d , d_X and d_Y be the associated deviations (that is to say, for instance, $d_Y(s, t) = \sqrt{\mathbb{E} |Y(s) - Y(t)|^2}$, for $(s, t) \in T \times T$). Assume that*

$$\forall (s, t) \in T \times T, \quad d_Y(s, t) \leq d_X(s, t).$$

Then, we have also

$$\mathbb{E} \sup_{t \in T} Y(t) \leq \mathbb{E} \sup_{t \in T} X(t).$$

We give one more spectral estimation which associates the operator T at the point f , with its spectral measure on the torus of dimension d , that we identify, once for all, with $[0, 1]^d$.

First case: $d = 1$, the \mathbb{Z} -actions.

Proposition 2.4 *Let T be a contraction on a Hilbert space \mathcal{H} , and let $p(x)$ be a polynomial defined on the unit circle $D = \{x, x \in \mathbb{C}, |x| = 1\}$. Then for all f in \mathcal{H} , there exists a positive Borel measure, called μ_f , that is bounded over D and such that we have*

$$\|p(T)f\|_{\mathcal{H}}^2 \leq \int_D |p(x)|^2 \mu_f(dx).$$

This result is a consequence of a simple inductive argument on the degree of the polynomial and from the equality $(T^n f, f)_{\mathcal{H}} = \int_D x^n \mu_f(dx)$ is accessible at the Appendix in Riesz-Szokefalvi's book (see [19]).

Second case: $d \geq 2$, \mathbb{Z}^d -actions.

Now let $\mathbb{T}^d = [0, 1]^d$ denote the dual group of \mathbb{Z}^d and recall that if $\{T^k, k \in \mathbb{Z}^d\}$ is a group of unitary operators acting on $L^2(\mu)$, then it is easy to see that for each $f \in L^2(\mu)$, if we put for $k \in \mathbb{Z}^d$: $\gamma(k) = \int T^k f \cdot \bar{f} d\mu$, then γ is a positive definite function, hence one has by the Herglotz-Bochner-Weil Theorem the existence of a unique non-negative bounded measure μ_f on the Borel σ -algebra $B([0, 1]^d)$, such that

$$\forall k \in \mathbb{Z}^d, \quad \int T^k f \cdot \bar{f} d\mu = \int_{[0, 1]^d} \exp 2i\pi \langle k, \alpha \rangle \mu_f(d\alpha).$$

Proposition 2.5 *Let $n \in \mathbb{Z}^d$ and denote by $P(T) = \sum_{k \in \mathbb{Z}^d, 0 \leq k \leq n} a_k T^k$ where $\{a_k, k \in \mathbb{Z}^d\}$ is a sequence of complex numbers, then one has the spectral equality*

$$\forall f \in L^2(\mu), \quad \|P(T)(f)\|_{L^2(\mu)}^2 = \int_{[0, 1]^d} \left| \sum_{k \in \mathbb{Z}^d, 0 \leq k \leq n} a_k \exp 2i\pi \langle k, \alpha \rangle \right|^2 \mu_f(d\alpha),$$

and by $k \leq n$ for $k, n \in \mathbb{Z}^d$ we mean $k_j \leq n_j$ for each $1 \leq j \leq d$ (see [1]).

Here are two other technical lemmas that we shall need.

Lemma 2.1 [20] *Let $\{f_n, n \geq 1\}$ be a sequence of non-negative and measurable functions defined on a measured space (X, \mathcal{A}, μ) with $\mu(X) = 1$. Assume that for all fixed $\rho > 1$,*

$$\mu \left\{ x : \lim_{N \in \mathcal{N}_\rho \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(x) \text{ exists} \right\} = 1,$$

where \mathcal{N}_ρ is defined in (3). Then for all $\rho > 1$, the limit is the same, let us call it L , and we have

$$\mu \left\{ x : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(x) = L(x) \right\} = 1.$$

For a proof see [20] Lemma 1.5, page 17.

Lemma 2.2 [3][28] *Let $\{f_n, n \geq 1\}$ be a sequence of measurable functions defined on a measured space (X, \mathcal{A}, μ) with $\|f_n\|_{L^2(\mu)} = 1, \forall n \geq 1$. Let $\rho > 1$ and for all sequence of integer $(N_j)_{j \geq 1}$ with $N_{j+1} \geq 2N_j$, let*

$$M_j(x) := \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N f_k(x) - \frac{1}{N_j} \sum_{l=1}^{N_j} f_l(x) \right|,$$

where \mathcal{N}_ρ^j is defined in Definition 1.3. Assume that $\sum_{j=1}^J \|M_j\|_{L^2(\mu)} / J$ tends to 0, when N tends to ∞ , at a speed which does not depend on $(N_j)_{j \geq 1}$. Then

$$\mu \left\{ x : \lim_{N \in \mathcal{N}_\rho \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(x) \text{ exists} \right\} = 1.$$

2.2 Dilation theorem and oscillation functions

We shall also use the following tools. Let us start with a *dilation* theorem from Akcoglu-Sucheston ([15], p.192, Th.2.9).

Theorem 2.1 [Dilation theorem] *Let $T : L \rightarrow L$ a non-negative contraction defined on a L^p -space L ($1 \leq p < \infty$). Then there exists another L^p -space \hat{L} and a non-negative isometry $\hat{T} : \hat{L} \rightarrow \hat{L}$ such that, for all $n \in \mathbb{N}$, $DT^n = P\hat{T}^nD$, where $D : L \rightarrow \hat{L}$ is a non-negative isometric embedding from L to \hat{L} , and $P : \hat{L} \rightarrow \hat{L}$ is a non-negative contraction.*

Consider now a probability space (X, \mathcal{A}, μ) and a sequence $\{f_k, k \geq 1\}$ of elements of $L^2(\mu)$ that satisfy: $\forall k \geq 1, \|f_k\|_{L^2(\mu)} = 1$. For all $\rho > 1$, consider, at last, a family of partial indexes $\mathcal{N}_\rho = \{[\rho^k], k \geq 0\}$, and an increasing sequence of integers, let us say $\{N_j, j \geq 0\}$, that satisfies: $N_{j+1} \geq 2N_j$, for all $j \geq 2$.

We are interested in the evaluation of the oscillation function associated with the sequence (f_k) of elements in $L^2(\mu)$, along the sequence $\{N_j, j \geq 0\}$. This \mathbb{R} -valued function is defined as follows:

$$O_{L^2(\mu)}((f_k), (N_j), \mathcal{N}_\rho) := \sqrt{\sum_{j \geq 1} \left\| \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N f_k - \frac{1}{N_j} \sum_{k=1}^{N_j} f_k \right| \right\|_{L^2(\mu)}^2}, \quad (10)$$

where \mathcal{N}_ρ^j is given in Definition 1.3. For convenience, we shall also use, sometimes, the notation

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}_\rho) := \sqrt{\sum_{j \geq 1} \left\| \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N f \circ T_k - \frac{1}{N_j} \sum_{k=1}^{N_j} f \circ T_k \right| \right\|_{L^2(\mu)}^2},$$

or

$$O_{L^2(\mu)}(T, f, (N_j), \mathcal{N}_\rho) := \sqrt{\sum_{j \geq 1} \left\| \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N f \circ T^{u_k} - \frac{1}{N_j} \sum_{k=1}^{N_j} f \circ T^{u_k} \right| \right\|_{L^2(\mu)}^2},$$

where the choice of the integer-valued sequence (u_k) will be given by the context.

Proposition 2.6 *Assume there exists a positive constant $K < \infty$ which does not depends on the sequence $\{N_j, j \geq 1\}$, such that*

$$O_{L^2(\mu)}((f_k), (N_j), \mathcal{N}_\rho) \leq K.$$

Then we have:

non-negativity: *If we assume that, for all $k \geq 1, f_k \geq 0$, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(x) \text{ exists } \mu - pp. \quad (11)$$

boundedness: *If we assume that $\sup_{k \geq 1} \|f_k\|_\infty < \infty$, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(x) \text{ exists } \mu - pp. \quad (12)$$

Proof. We use, in both cases, Lemma 2.1. In the non-negative case, the conclusion comes down when it is combined with Lemma 2.2. In the second case, one replaces the argument of Lemma 2.2 by a standard reasoning that is clearly set in [28], p.719-04. \diamond

2.3 Proof of Theorem 1.5

The data of the problem are the followings: Let (Ω, \mathcal{B}, P) be a complete probability space on which is defined a random function $\{X_k, k \in \mathbb{N}\}$ with values in \mathbb{R}^d , $d \geq 1$. In order to set the results in the most general framework, assume that this function is m -dependent, that is to say, by definition,

Definition 2.1 *A random process $\{X_k, k \in \mathbb{N}\}$ is m -dependent if and only if, for any integer $k \geq 1$, X_{k+m} is independent of $\{X_k, X_{k-1}, \dots, X_0\}$.*

Examples: A sequence of independent random variables $\{X_k, k \in \mathbb{N}\}$ is by definition 1-dependent. On the other hand, a random walk $\{S_k = \sum_{j=0}^k X_j, k \in \mathbb{N}\}$ where $\{X_k, k \in \mathbb{N}\}$ is a sequence of independent and identically distributed random variables, is ∞ -dependent if we fit the preceding definition in consequence.

Concerning the applications in ergodic theory, of course we shall consider only the case when $m = 1$.

We are going to estimate the asymptotic behavior of the quantity

$$\sup_{\alpha \in [0,1]^d} \left| \sum_{k=1}^N \exp 2i\pi \langle \alpha, X_k \rangle - \mathbb{E} \exp 2i\pi \langle \alpha, X_k \rangle \right|, \quad (13)$$

uniformly in α when N tends to infinity. That is the essential point of this work that will permit to separate the conditions to impose, on one hand, on the random process, and on the other hand, on the dynamical system that is likely to model a concrete situation, as for instance, the position of an energetic particle in the space.

-Step 1: This step consists in transferring the problem to a problem of regularity of Gaussian random functions' trajectories.

Let us take an independent copy of $X = (X_k)_{k \geq 0}$ that we denote $X' = (X'_k)_{k \geq 0}$. Usual convexity properties shows that, to prove (8), we just need to prove

$$\mathbb{E}_{X, X'} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0,1]^d} \left| \frac{1}{qN} \sum_{k=1}^N \exp 2i\pi \langle \alpha, X_k \rangle - \exp 2i\pi \langle \alpha, X'_k \rangle \right| < \infty.$$

Symetrization of the problem: Let us consider the following family of random functions with continuous trajectories,

$$\{f_k(\alpha) = \exp 2i\pi \langle \alpha, X_k \rangle - \exp 2i\pi \langle \alpha, X'_k \rangle, \alpha \in [0,1]^d, k \geq 1\}.$$

The hypothesis of m -dependence on the process $\{X_k, k \geq 1\}$ suggests to make a partition of \mathbb{N} , the set of positive integers, as follows:

$$\mathbb{N} = I_1 \cup I_2 \cup \dots \cup I_m,$$

where, for all integers $l \leq m$, $I_l = \{l + rm, r \in \mathbb{N}\}$. Our goal is to construct a finite family (with m elements) of symmetric random functions associated with m arithmetic progressions.

For all integers $l \geq m$, we denote the random function

$$f^l = \{f_k, k \in I_l\}.$$

Note that, by construction, every f^l forms a symmetric family of random functions, that is to say that their law is not modified by change of sign. More precisely, let us denote $\{\epsilon_k, k \geq 1\}$ a sequence of Rademacher, independent, random variables (which takes the values $+1$ and -1 with probability $1/2$) that we assume to be independent of the random variables X and X' .

Hence, we have

$$\sum_{k=1}^N f_k(\alpha) = \sum_{l=1}^m \sum_{k \in I^l, k \leq N} f_k(\alpha),$$

and so

$$\mathbb{E}_{X, X'} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0,1]^d} \left| \frac{1}{qN} \sum_{k=1}^N f_k(\alpha) \right| \leq \sum_{l=1}^m \mathbb{E}_{X, X'} \sup_{N \geq 1} \sup_{\alpha \in [0,1]^d} \left| \frac{1}{qN} \sum_{k \in I^l, k \leq N} f_k(\alpha) \right|.$$

Let us fix now an integer $l \leq m$ and study

$$\mathbb{E}_{X, X'} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0, 1]^d} \left| \frac{1}{q_N} \sum_{k \in I^l, k \leq N} f_k(\alpha) \right|. \quad (14)$$

We know that $\{f_k, k \in I^l\}$ and $\{\epsilon_k f_k, k \in I^l\}$ have the same law. Thus (14) can be written with a larger space of integration,

$$\begin{aligned} & \mathbb{E}_{X, X', \epsilon} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0, 1]^d} \left| \frac{1}{q_N} \sum_{k \in I^l, k \leq N} \epsilon_k f_k(\alpha) \right| \leq \\ & 2 \mathbb{E}_{X, \epsilon} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0, 1]^d} \left| \frac{1}{q_N} \sum_{k \in I^l, k \leq N} \epsilon_k \exp 2i\pi \langle \alpha, X_k \rangle \right|. \end{aligned} \quad (15)$$

We shall use a tool which is recurrent in the study of Gaussian processes: the principle of contraction. This tool is based on a very simple idea which allows us to replace the change of sign in (15) by a sequence of centered and reduced, normal, random variables. For more details, see [23] page 169.

Hence, to show that the right term of (15) is finite, we just need to prove

$$\mathbb{E}_{X, g, g'} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0, 1]^d} \left| \frac{1}{q_N} \sum_{k \in I^l, k \leq N} g_k \cos 2\pi \langle \alpha, X_k \rangle + g'_k \sin 2\pi \langle \alpha, X_k \rangle \right| < \infty, \quad (16)$$

where $\{g_k, k \geq 1\}$ and $\{g'_k, k \geq 1\}$ are two sequences of independent random variables of law $\mathcal{N}(0, 1)$ which are independent of X, ϵ and ϵ' .

The problem is then reduced to the study of the regularity of Gaussian random functions' trajectories if we fix the variable of integration X . This concludes the first step of the proof.

-*Step 2: Use of Gaussian tools.*

Let us start by denoting, conditionally to X , the random Gaussian functions

$$G_N(\alpha) = \frac{1}{q_N} \sum_{k \in I^l, k \leq N} g_k \cos 2\pi \langle \alpha, X_k \rangle + g'_k \sin 2\pi \langle \alpha, X_k \rangle.$$

Because of Proposition 2.6, we can bound (16) from above by

$$\mathbb{E} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0, 1]^d} |G_N(\alpha)| \leq K_1 \mathbb{E}_X \left(\sup_{N \geq 1} \mathbb{E}_{g, g'} \sup_{\alpha \in [0, 1]^d} |G_N(\alpha)| + \mathbb{E}_\lambda \sup_{N \geq 1} |\lambda_N \sigma_N| \right),$$

where $\{\lambda_N, N \geq 1\}$ is an isonormal sequence, that is to say, a sequence of independent and identically distributed, random variables with centered and reduced, Gaussian law and

$$\sigma_N \leq \sup_{\alpha \in [0, 1]^d} \|G_N(\alpha)\|_{2, g, g'}.$$

This estimation of σ_N permits to obtain easily

$$\sigma_N \leq \frac{1}{q_N} \sqrt{\text{card}\{k \in I^l, k \leq N\}}.$$

or

$$\sigma_N \leq \frac{\sqrt{N}}{q_N}.$$

But the hypothesis (6) tells us that $\sigma_N = O(1/\sqrt{\log N})$.

Let g be a centered Gaussian random variable such that $\mathbb{E}|g|^2 = \sigma^2$. Hence for all real a , we have

$$P(g \geq a) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^\infty e^{-t^2/2\sigma^2} dt \leq K \exp\left(\frac{-a^2}{2\sigma^2}\right).$$

For $a \geq 2$, if we denote by $(T_N)_{N \geq 1}$ a sequence of independent and centered Gaussian random variables such that

$$\sqrt{\mathbb{E}|T_N|^2} \leq \frac{1}{\sqrt{1 + \log N}},$$

then we obtain

$$\begin{aligned} P(\exists N \geq 1, |T_N| \geq a) &\leq 2 \sum_{N \geq 1} \exp\left(\frac{-a^2}{2}(1 + \log N)\right) \leq 2 \exp\left(\frac{-a^2}{2}\right) \sum_{N \geq 1} N^{-a^2/2} \\ &\leq K \exp\left(\frac{-a^2}{2}\right). \end{aligned} \quad (17)$$

Finally, we have

$$\mathbb{E} \sup_{N \geq 1} |T_N| < \infty.$$

Hence, if we denote $T_N = \lambda_N \sigma_N$, we obtain

$$\mathbb{E}_\lambda \sup_{N \in \mathcal{N}} |\lambda_N \sigma_N| < \infty,$$

and finally,

$$\mathbb{E}_X \mathbb{E}_\lambda \sup_{N \in \mathcal{N}} |\lambda_N \sigma_N| < \infty.$$

Hence to show (16), we just need to prove

$$\mathbb{E}_X \sup_{N \in \mathcal{N}} \mathbb{E}_{g, g'} \sup_{\alpha \in [0, 1]^d} |G_N(\alpha)| < \infty. \quad (18)$$

In the sequel, since X is fixed, we shall fix also the integer N . We are going to compare the random function $G(\alpha) = G_N(\alpha)$ with every one of its 1-dimensional margins in order to use Proposition 4.1. The random function G represents a stationary Gaussian random function which is continuous in probability and thus, possess a modification with continuous trajectory that will be also denoted by G . On the space $[0, 1]^d$, G induces an Hilbert distance defined by

$$\forall (s, t) \in [0, 1]^d \times [0, 1]^d, \quad d_G(s, t) = \sqrt{\mathbb{E} |G(s) - G(t)|^2}.$$

For a positive integer $j \leq d$, let us consider the family of stationary, Gaussian, random functions which are continuous in probability

$$\forall \alpha_j \in [0, 1], \quad G^j(\alpha_j) = \frac{1}{q_N} \sum_{k \in I^j, k \leq N} g_{k,j} \cos 2\pi \alpha_j X_k^{(j)} + g'_{k,j} \sin 2\pi \alpha_j X_k^{(j)},$$

where $\{g_{k,j}, k \geq 1\}$ and $\{g'_{k,j}, k \geq 1\}$ are families of sequences of centered and reduced, Gaussian, random variables which are independent for $j \leq d$ and $l \leq m$.

For all t_j and s_j in $[0, 1]$, denote

$$d_{G^j}(s_j, t_j) = \sqrt{\mathbb{E} |G^j(s_j) - G^j(t_j)|^2},$$

the Hilbert distance on $[0, 1]$, induced by G^j . In this context, remark that we have

$$\forall t = (t_1, \dots, t_d) \in [0, 1]^d, \forall s = (s_1, \dots, s_d) \in [0, 1]^d, \quad d_G(s, t) \leq \sqrt{d} \sqrt{\sum_{j=1}^d d_{G^j}^2(s_j, t_j)}.$$

Using Slépian-Fernique's Theorem (see Proposition 2.3), we deduce from the preceding estimation

$$\mathbb{E}_{g, g'} \sup_{\alpha \in [0, 1]^d} G_N(\alpha) \leq \sqrt{d} \sum_{j=1}^d \mathbb{E}_{g, g'} \sup_{\alpha_j \in [0, 1]} G_N^j(\alpha_j).$$

For all $j \leq d$, we associate the random function G^j with its spectral measure on \mathbb{R}^+ defined by

$$\forall 1 \leq j \leq d, \quad m^{(j)} = \frac{1}{q_N^2} \sum_{k \in I^j, k \leq N} \delta_{|X_k^{(j)}|},$$

where δ_u is the Dirac measure at the point u .

Because of Proposition 2.2, we obtain that (18) will be achieved as soon as the two families of following conditions will be satisfied for all $1 \leq j \leq d$,

$$\mathbb{E} \sup_{N \in \mathcal{N}} \left[\frac{1}{q_N} \right] \sqrt{\sum_{k \in I^j, k \leq N} \min[(X_k^{(j)})^2, 1]} < \infty. \quad (19)$$

$$\mathbb{E} \sup_{N \in \mathcal{N}} \left[\frac{1}{q_N} \right] \int_0^\infty \sqrt{\sum_{k \in I^j, k \leq N} I_{\{|X_k^{(j)}| > e^{x^2}\}}} dx < \infty. \quad (20)$$

Note that (19) can be reduced to $\sup_{N \in \mathcal{N}} (\sqrt{N}/q_N) < \infty$. But this is an immediate consequence of the hypothesis (6). Finally, the condition (20) can be easily reduced to the condition (7) by using the non-increasing condition on (\sqrt{N}/q_N) . This ends the proof of Theorem 1.5. \diamond

Now, we shall prove if we suppose that for some $\delta \in]0, 1[$ there exists $\gamma \in]0, 1[$ such that $\mathbb{E} (\|X_N\|^\delta) = O(2^{N^\gamma})$ the assumptions of theorem 1.5 are holds with $q_N = N^{\gamma'}$ where $\gamma' \in]\frac{\gamma+1}{2}, 1[$.

Conditions of non-increasing and (6) are obvious. We have to prove Condition (7).

Multiplying (7) by $\sqrt{\delta}$ and changing q_N in $N^{\gamma'}$, we have to bound

$$\forall 1 \leq i \leq d, \quad \mathbb{E} \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N^{(i)}|^\delta)}{N^{2\gamma'-1}}},$$

from above. It is actually smaller than

$$\left(\mathbb{E} \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N^{(i)}|^\delta / 2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}} + \sup_{N \geq 1} \sqrt{\frac{\log(2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}} \right),$$

where the second term is bounded from above by 1 and the first term is bounded by

$$\sum_{N \geq 1} \sqrt{\frac{\mathbb{E} \log^+ (|X_N^{(i)}|^\delta / 2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}} \leq \sum_{N \geq 1} \sqrt{\frac{\mathbb{E} (|X_N^{(i)}|^\delta)}{2^{N^{2\gamma'-1}} N^{2\gamma'-1}}}.$$

Using the hypothesis made on X_N , one proves that it is smaller than

$$\sum_{N \geq 1} \sqrt{\frac{2^{N^\gamma}}{2^{N^{2\gamma'-1}} N^{2\gamma'-1}}} < \infty.$$

2.4 Another result

At last, we shall also need the following proposition relating to some random trigonometric polynomials and that we prove with the help of the Gaussian techniques clearly set now.

Proposition 2.7 *Let (Ω, \mathcal{B}, P) be a probability space on which is defined a sequence of centered, independent, identically distributed and reals valued random variables $(\theta_k, k \geq 1)$ which satisfies $\mathbb{E}|\theta_k|^2 < \infty$. Let $(u_k, k \geq 1)$ be an increasing sequence of reals such that there exists $c_1 > 0$ and $c_2 > 0$ with $u_N \geq c_1 N^{c_2}$.*

Then, we have

$$\mathbb{E} \sup_{N \geq 1} \sup_{\alpha \in [0, 1]} \left| \frac{1}{\sqrt{N \log u_N}} \sum_{k=1}^N \theta_k \exp(2i\pi\alpha u_k) \right| < \infty.$$

For the proof of this result, the reader is referred to [27].

3 First approach: an application of the transference method

3.1 A transference theorem

The aim of the following theorem is to explore a transference principle from a model with one deterministic orbit (\mathbb{Z}^d , the shift S) to any random dynamical system.

Definition 3.1 For all real $\rho > 1$, define a partial index

$$\mathcal{N}_\rho = \{[\rho^k], k \in \mathbb{N}\},$$

and for all sequence of integers $(N_j)_{j \geq 1}$ with $N_{j+1} \geq 2N_j$,

$$\mathcal{N}_\rho^j = \{[\rho^k], k \in \mathbb{N}, N_j \leq [\rho^k] < N_{j+1}\}.$$

For all $v \in l^2(\mathbb{Z}^d)$, define also

$$M^\rho v(n) = \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(n + X_k) \right|, \quad (21)$$

and

$$M_j^\rho v(n) = \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(n + X_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} \mathbb{E} v(n + X_k) \right|. \quad (22)$$

Now, we give a sufficient deterministic condition based on the model with one orbit in order to obtain almost everywhere convergence's properties for random ergodic averages.

Theorem 3.1 Let be a \mathbb{N}^d -valued random process $\{X_k, k \in \mathbb{N}\}$ which satisfies

$$\mathbb{E} \|X_k\|^\delta = O(2^{k^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$, and assume there exist a constant C and a number $A(J)$ independent of $(N_j)_{j \geq 1}$ such that, for all $v \in l^2(\mathbb{Z}^d)$,

$$\|M^\rho v\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)}, \quad (23)$$

$$\sum_{j=1}^J \|M_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}, \quad (24)$$

and $A(J)/J$ tends to 0 when $J \rightarrow \infty$ at a speed which depends only on ρ . Then there exists an absolute, measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, for all measurable dynamical system (Y, \mathcal{A}, μ, T) , T being a \mathbb{Z}^d -action, for all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

This result gives us deterministic criteria for a particular dynamical system \mathbb{Z}^d with the shift, which lead to general properties of pointwise convergence of random ergodic averages within a universal framework.

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.1 Let $\rho > 1$ and let $\{X_k, k \in \mathbb{N}\}$ be a \mathbb{N}^d -valued random process such that

$$\forall v \in l^2(\mathbb{Z}^d), \quad \|M^\rho v\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)},$$

where $M^\rho v$ is defined by (21). Then, for all dynamical system (Y, \mathcal{A}, μ, T) , we have

$$\forall f \in L^2(\mu), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E}(f \circ T^{X_k}) \right| \right\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)}.$$

Lemma 3.2 *Let $\rho > 1$ and let $(N_j)_{j \geq 1}$ be a sequence of integers such that $N_{j+1} \geq 2N_j$ and let M_j^ρ be defined by (22). Assume there exist, for all fixed J , a number $A(J)$ independent of v and $(N_j)_{j \geq 1}$ such that*

$$\sum_{j=1}^J \|M_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

Then, for all dynamical system (Y, \mathcal{A}, μ, T) , we have, $\forall f \in L^2(\mu)$,

$$\sum_{j=1}^J \left\| \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E}(f \circ T^{X_k}) - \frac{1}{N_j} \sum_{k=1}^{N_j} \mathbb{E}(f \circ T^{X_k}) \right| \right\|_{L^2(\mu)} \leq A(J) \|f\|_{L^2(\mu)}.$$

Lemma 3.1 is a generalization of transference principles based on a model with one orbit of Calderón [4] (the novelty here is the expectation \mathbb{E}). To prove it, fix $m \in \mathbb{N}$ and take an integer $J > m$ and consider

$$\varphi_x(n) = \begin{cases} T^n \circ f(x) & \text{if } n \in \{0, \dots, J\}^d \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.2 is an application of results [13] because the generalization to \mathbb{Z}^d is obvious.

Using Corollary 1.1 together with Lemma 3.1, we obtain

Corollary 3.1 *Let be a \mathbb{N}^d -valued independent random process $\{X_k, k \in \mathbb{N}\}$ which satisfies*

$$\mathbb{E}(\|X_N\|^\delta) = O(2^{N^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$. Let $\rho > 1$ and assume there exists a constant C independent of $v \in l^2(\mathbb{Z}^d)$ such that

$$\|M^\rho v\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)},$$

then there exists an absolute measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, there exists a finite, positive constant $C = C(\omega)$ such that for all measurable dynamical system (Y, \mathcal{A}, μ, T) , T being a \mathbb{Z}^d -action, for all $f \in L^2(\mu)$, we have

$$\left\| \sup_{N \geq 1} \left| \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} \right| \right\|_{L^2(\mu)} \leq C(\omega) \|f\|_{L^2(\mu)}.$$

Proof. First of all, let $\gamma' \in ((\gamma+1)/2, 1)$ and notice that $q_N := N^{\gamma'}$ satisfies the conditions of non-increasing, (6), (7) and $\sum_{N \in \mathcal{N}_\rho} q_N/N < \infty$.

Letting

$$A_N := \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)},$$

we have next by using Lemma 3.1,

$$\| \sup_{N \in \mathcal{N}_\rho} |A_N| \|_{L^2(\mu)} \leq \| \sup_{N \in \mathcal{N}_\rho} |A_N - \mathbb{E}A_N| \|_{L^2(\mu)} + C \|f\|_{L^2(\mu)}$$

The non-negativity of T permits to change the supremum over \mathbb{N} into the supremum over \mathcal{N}_ρ (for $\rho^p \leq N \leq \rho^{p+1}$, we have $|A_N| \leq \rho |A_{\rho^{p+1}}|$).

Now, by using Theorem 1.5 and spectral Lemma for \mathbb{Z}^d -actions

$$\| \sup_{N \in \mathcal{N}_\rho} |A_N - \mathbb{E}A_N| \|_{L^2(\mu)} \leq \chi(\omega) \|f\|_{L^2(\mu)} \sum_{N \in \mathcal{N}_\rho} q_N/N < \infty,$$

where χ is a positive integrable random variable, independent of the dynamical system. \diamond

Corollary 3.1 means that the class of convergence of the random ergodic averages above, for the almost everywhere convergence, is closed in $L^2(\mu)$.

The problem of pointwise convergence is thus linked to the possibility of making an approximation: find a class which is dense in $L^2(\mu)$ and for which we have almost-everywhere convergence. We have actually the following corollary of Corollary 1.1, Lemma 3.2 and spectral Lemma.

Corollary 3.2 *Let the process $\{X_k, k \in \mathbb{N}\}$ satisfies*

$$\mathbb{E}(\|X_N\|^\delta) = O(2^{N^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$. Let $(N_j)_{j \geq 1}$ be a sequence of integers such that $N_{j+1} \geq 2N_j$ and let M_j^ρ be defined by (22). Assume there exists, for all fixed J , a number $A(J)$ independent of v and $(N_j)_{j \geq 1}$ such that

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|M_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)},$$

where $A(J)/J$ tends to 0 when $J \rightarrow \infty$ at a speed which depends only on $\rho > 1$, then we obtain:

There exists an absolute, measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, for all measurable dynamical system (Y, \mathcal{A}, μ, T) , T being a \mathbb{Z}^d -action, for all $f \in L^\infty(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

Proof. Using the same notation as above, Lemmas 3.2 and 2.2 tells us that $\mathbb{E}A_N$ converges when $N \in \mathcal{N}_\rho$ and $N \rightarrow \infty$. But, from Corollary 1.1,

$$\left\| \sum_{N \in \mathcal{N}_\rho} |A_N - \mathbb{E}A_N| \right\|_{L^2(\mu)} \leq \sum_{N \in \mathcal{N}_\rho} \|A_N - \mathbb{E}A_N\|_{L^2(\mu)} < \infty.$$

Therefore

$$\sum_{N \in \mathcal{N}_\rho} |A_N - \mathbb{E}A_N| < \infty, \quad \mu - a.e.$$

which means that $A_N - \mathbb{E}A_N$ tends to 0, and A_N converges μ -almost everywhere, when $N \in \mathcal{N}_\rho$. And, since T is non-negative, we know, from Lemma 2.1, that the convergence still occurs when N describes \mathbb{N} .

◇

Corollary 3.1 together with Corollary 3.2 lead obviously to Theorem 3.1.

3.2 Application to perturbed MOI-good sequences (proof of Theorem 1.1)

Our purpose, here, is to study the behavior of MOI-good sequences when they are perturbed by a multiplicative, independent, identically distributed, random variable. First of all introduce the following notation: for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ and $y = (y_1, \dots, y_d) \in \mathbb{Z}^d$ we note $x.y$ the element of \mathbb{Z}^d defined by $x.y = (x_1 y_1, \dots, x_d y_d)$.

Now, we have

Lemma 3.3 *Let $(u_n)_{n \in \mathbb{N}}$ be $\mathbb{Z}N^d$ -valued sequence which satisfies (2), then for all $m = (m_i)_{i=1}^d \in \mathbb{Z}^d$, we have*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + m.u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)},$$

with the same constant C .

To prove Lemma 3.3, we remark that it is just a special case of the transference principle: if there is an inequality on \mathbb{Z}^d , then the same inequality holds for any \mathbb{Z}^d -action, hence, in particular, for $Tx = x + m$.

◇

Corollary 3.3 *Let $\{\theta_k, k \in \mathbb{N}\}$ be a family of independent, identically distributed, \mathbb{Z}^d -valued random variables and $(u_k)_{k \in \mathbb{N}}$ a \mathbb{Z}^d -valued sequence which satisfies (2), then we have*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(\cdot + \theta_k.u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)}.$$

For $j \in \mathbb{Z}^d$, denote $P_j = P(\{\omega \in \Omega : \theta_0(\omega) = j\})$. Since the family $\{\theta_N, N \in \mathbb{N}\}$ is identically distributed, we have also $P_j = P(\{\omega \in \Omega : \theta_N(\omega) = j\})$. Hence

$$\begin{aligned} \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(n + \theta_k \cdot u_k) \right| &= \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \sum_{j \in \mathbb{Z}^d} P_j v(n + j \cdot u_k) \right| \\ &\leq \sum_{j \in \mathbb{Z}^d} P_j \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(n + j \cdot u_k) \right|. \end{aligned}$$

Using the triangle inequality, we can bound the l^2 -norm from above as follow

$$\left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(\cdot + \theta_k \cdot u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq \sum_{j \in \mathbb{Z}^d} P_j \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + j \cdot u_k) \right| \right\|_{l^2(\mathbb{Z}^d)}.$$

And using Lemma 3.3, we obtain

$$\left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} v(\cdot + \theta_k \cdot u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq \sum_{j \in \mathbb{Z}^d} P_j C \|v\|_{l^2(\mathbb{Z}^d)} = C \|v\|_{l^2(\mathbb{Z}^d)}.$$

◇

We have found therefore a process $(X_k = \theta_k \cdot u_k)$ which satisfies (23). To apply Theorem 3.1, we are going to show that it satisfies also (24).

Recall that \tilde{M}_j^ρ is defined in (4), and for $m = (m_i)_{i=1}^d \in \mathbb{Z}^d$, define

$$\forall v \in l^2(\mathbb{Z}^d), \quad \tilde{M}_j^{\rho, m} v(n) = \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N v(n + m \cdot u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(n + m \cdot u_k) \right|.$$

Lemma 3.4 *Let $(u_k)_{k \in \mathbb{N}}$ be an \mathbb{Z}^d -valued sequence such that there exists, for all fixed J , a number $A(J)$ independent of v and $(N_j)_{j \geq 1}$ which satisfies*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|\tilde{M}_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

Then, we have, for all $m \in \mathbb{Z}^d$,

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|\tilde{M}_j^{\rho, m} v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

The proof of this lemma is very close to Lemma 3.3's. In fact the same remark applies: the results [13] apply to this situation as well, hence the inequality is true for any \mathbb{Z}^d -actions, hence for $Tx = x + m$, with the same constant.

An immediate consequence of this is the next corollary

Corollary 3.4 *Let $(u_k)_{k \in \mathbb{N}}$ be a \mathbb{Z}^d -valued sequence such that there exists, for all fixed J , a number $A(J)$ independent of v and $(N_j)_{j \geq 1}$ which satisfies*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|\tilde{M}_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

Let $\{\theta_k, k \in \mathbb{N}\}$ be a family of independent, identically distributed, \mathbb{Z}^d -valued random variables without condition of moment. Let $M_j^\rho v$ be defined by (22) with $\{X_N = \theta_N \cdot u_N, N \geq 1\}$. Then, we have

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|M_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

Let us denote once again $P_m = P(\{\omega \in \Omega : \theta_0(\omega) = m\})$. Then we have

$$\begin{aligned} \tilde{M}_j^\rho v(n) &= \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N \sum_{m \in \mathbb{Z}^d} P_m v(n + m \cdot u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{m \in \mathbb{Z}^d} P_m v(n + m \cdot u_k) \right| \\ &\leq \sum_{m \in \mathbb{Z}^d} P_m \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^N v(n + m \cdot u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(n + m \cdot u_k) \right| \\ &= \sum_{m \in \mathbb{Z}^d} P_m M_j^{\rho, m} v(n) \end{aligned}$$

and, using lemma 3.4,

$$\sum_{j=1}^J \|\tilde{M}_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq \sum_{m \in \mathbb{Z}^d} P_m A(J) \|v\|_{l^2(\mathbb{Z}^d)} = A(J) \|v\|_{l^2(\mathbb{Z}^d)}.$$

Finally, we have found easy conditions on the process $\{X_N = \theta_N \cdot u_N, N \geq 1\}$ to satisfy the hypotheses of Theorem 3.1. We just need the extra condition

$$\forall 1 \leq i \leq d, \quad \mathbb{E} |\theta_N^{(i)} u_N^{(i)}|^\delta = O(2^{N^\gamma}),$$

which is satisfied as soon as

$$\forall 1 \leq i \leq d, \quad \mathbb{E} |\theta_0^{(i)}|^\delta < \infty \text{ and } u_N^{(i)} = O(2^{N^{\gamma'}}),$$

for some $\gamma' < \gamma$. So we obtain the following theorem :

Theorem 3.2 *Let $\{\theta_N, N \in \mathbb{N}\}$ be a family of independent, identically distributed, \mathbb{N}^d valued random variables such that there exists $\delta > 0$ with $\mathbb{E} \|\theta_1\|^\delta < \infty$. Let $(u_n)_{n \in \mathbb{N}}$ be a MOI-good sequence on \mathbb{N}^d such that $\|u_n\| = O(2^{n^\gamma})$ (with $\gamma < 1$). Then there exists an universal measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$:*

- 1) *The sequence $\{\theta_k(\omega) \cdot u_k, k \geq 1\}$ is MOI-good.*
- 2) *For all measurable dynamical system (Y, \mathcal{A}, μ, T) , T being a \mathbb{Z}^d -action,*

$$\forall f \in L^2(\mu), \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{\theta_k(\omega) \cdot u_k} \text{ exists } \mu\text{-almost everywhere.}$$

Remark: conclusion (2) is an obvious consequence of conclusion (1).

In order to obtain the first conclusion (MOI-good), using corollaries 3.3 and 3.4 we have to prove that there exists an universal measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ there exists a random variable $C = C(\omega) < \infty$, so that

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + \theta_k \cdot u_k) - \mathbb{E} v(\cdot + \theta_k \cdot u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C(\omega) \|v\|_{l^2(\mathbb{Z}^d)}.$$

and

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^J \|M_j^\rho(\omega) v - \mathbb{E} M_j^\rho v\|_{l^2(\mathbb{Z}^d)} \leq C(\omega) A(J) \|v\|_{l^2(\mathbb{Z}^d)},$$

where

$$M_j^\rho(\omega) = \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + \theta_k(\omega) \cdot u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(\cdot + \theta_k(\omega) \cdot u_k) \right|.$$

We prove the first estimation. By standard troncation arguments and harmonic analysis (see [3]), we can write

$$\left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + \theta_k \cdot u_k) - \mathbb{E} v(\cdot + \theta_k \cdot u_k) \right| \right\|_{l^2(\mathbb{Z}^d)}$$

$$\leq \sum_{N \in \mathcal{N}_\rho} \left[\int_{[0,1]^d} \left| \frac{1}{N} \sum_{k=1}^N \exp 2i\pi \langle \alpha, \theta_k \cdot u_k \rangle - \mathbb{E} \exp 2i\pi \langle \alpha, \theta_k \cdot u_k \rangle \right|^2 |\hat{v}(\alpha)|^2 d\alpha \right]^{\frac{1}{2}} := A$$

where $\hat{v}(\alpha) = \sum_{k \in \mathbb{Z}^d} v(k) \exp 2i\pi \langle \alpha, k \rangle$.

Now, by virtue of Theorem 1.5, there exists χ a positive integrable random variable, independent of v , ρ and α such that

$$A \leq \chi \left(\sum_{N \in \mathcal{N}_\rho} \frac{q_N}{N} \right) \|v\|_{l^2(\mathbb{Z}^d)}.$$

This prove the first point of the MOI-good condition.

Computations of previous type will do the job for the second point of the MOI-good condition.

Remark: Using the same arguments, we can find a result of [23] again: the convergence of the ergodic average associated with $X_N = u_N + \theta_N$, where $\mathbb{E} \left| \prod_{i=1}^d \theta_0^{(i)} \right|^\delta < \infty$ for some $\delta > 0$. More generally, we have, the same way, the convergence of the ergodic mean associated with $X_N = \theta_N u_N + \tilde{\theta}_N$, where $(\theta_N)_{N \in \mathbb{N}}$ and $(\tilde{\theta}_N)_{N \in \mathbb{N}}$ are two i.i.d. random processes (see Theorem 1.1).

We finish this section by given the end of the proof of Theorem 1.1 in the case where we do not know if $(u_n)_{n \in \mathbb{N}}$ is MOI-good sequence. We shall suppose only that sequence (u_n) of integers is L^2 -good and that its maximal function is type (2, 2), i.e.

$$\forall v \in l^2(\mathbb{Z}^d), \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^N v(\cdot + u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C \|v\|_{l^2(\mathbb{Z}^d)}.$$

Indeed, the purpose of the averages $\frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{X_k(\omega)} f$ is to serve as a deterministic comparison to the random averages $\frac{1}{N} \sum_{k=1}^N T^{X_k(\omega)} f$.

Let $\{\theta_n, n \geq 1\}$ be an integer valued, i.i.d sequence of random variables (so no moment assumption). Suppose the sequence $(u_n)_{n \geq 1}$ of integers is good in $L^2(\mu)$ and that its maximal function is type (2,2). Then the averages $\frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_k u_k} f(x)$ converge for x μ -a.e. It will be clear that the proof clearly applies to the perturbed averages $\frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{u_n + \theta_n} f(x)$ or even to $\frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_n u_n + \theta'_n} f(x)$.

This result is stated only for \mathbb{Z} for simplicity.

We can assume that f is nonnegative. Introduce $p_q = P(\theta_1 = q)$, and choose a large Q so that $\sum_{|q| > Q} p_q < \epsilon$ for a given $\epsilon > 0$.

Now we write

$$\begin{aligned} \frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_k u_k} f(x) &= \sum_{|q| \leq Q} p_q \frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{q u_k} f(x) \\ &+ \sum_{|q| > Q} p_q \frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{q u_k} f(x). \end{aligned}$$

For each fixed q , the averages $\frac{1}{N} \sum_{k=1}^N T^{q u_k} f(x)$ converge μ -a.e. since $(u_n)_{n \geq 1}$ is a L^2 -good sequence, hence it is good for the transformation T^q . Now we denote the pointwise oscillation by Δ

$$\Delta(f)(x) = \limsup_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_k u_k} f(x) - \liminf_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_k u_k} f(x)$$

Then Δ can be estimated from above by

$$\Delta(f)(x) \leq 2 \cdot \sum_{|q|>Q} p_q \sup_{N \geq 1} \left[\frac{1}{N} \sum_{k=1}^N T^{qu_k} f(x) \right].$$

It follows that the $L^2(\mu)$ norm of the oscillation by using the triangular inequality is less than

$$\|\Delta(f)\|_{L^2(\mu)} \leq 2 \cdot \sum_{|q|>Q} p_q \left\| \frac{1}{N} \sum_{k=1}^N T^{qu_k} f(x) \right\|_{L^2(\mu)}.$$

Now the maximal function for the averages along u_k is assumed to be of type (2,2), which means, in particular, that there exists an absolute constant $C > 0$, such that we have

$$\left\| \sup_{N \geq 1} \left[\frac{1}{N} \sum_{k=1}^N T^{qu_k} f(x) \right] \right\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)}.$$

Since we chose Q so that $\sum_{|q|>Q} p_q < \epsilon$, we get

$$\|\Delta(f)\|_{L^2(\mu)} \leq 2C\epsilon.$$

Since ϵ is arbitrary, the $L^2(\mu)$ norm of the oscillation must be 0, which means $\frac{1}{N} \sum_{k=1}^N \mathbb{E} T^{\theta_k u_k} f(x)$ converges x μ -a.e. ◇

4 Second application: study of the oscillation function

4.1 Proof of Theorem 1.2

We start with properties of oscillation functions associated with some probability system and non-negative contractions (see for examples [12]). Consider therefore a contraction T on $L^2(\mu)$, a function f in $L^2(\mu)$ and $\{u_k, k \geq 1\}$ a sequence of positive integers.

Proposition 4.1 *In the case when T is a non-negative contraction over $L^2(\mu)$ and $u_k = k$, we have, for all $f \in L^2(\mu)$,*

$$O_{L^2(\mu)}(T, f, (N_j), \mathcal{N}_\rho) \leq K \|f\|_{L^2(\mu)},$$

(see notations 10) where K is an absolute constant.

To prove Proposition 4.1, first we give a theorem obtained by R. Jones, L. Ostrovskii and J. Rosenblatt:

Theorem 4.1 [14] *Let T a non-negative isometry on $L^2(\mu)$, then, for all increasing sequence (N_j) of positive integers, there exists an absolute constant $K < \infty$ such that*

$$\sum_{j=1}^{\infty} \left\| \sup_{N_j < N \leq N_{j+1}} |A_N(f) - A_{N_j}(f)| \right\|_{L^2(\mu)}^2 \leq K \|f\|_{L^2(\mu)}^2, \quad (25)$$

where $A_N(f) = (1/N) \sum_{k=1}^N T^k(f)$.

Moreover, the equation (25) remains true when the means A_N are generated by an arbitrary contraction on $L^2(\mu)$, under the extra condition $\sup_{j \geq 1} N_{j+1}/N_j < \infty$. The constant K in (25) depends therefore on this quantity.

The main difficulty consists, of course, in proving the preceding theorem. With the help of this theorem, the dilation theorem, where the argument of non-negativity is crucial, permits to prove Proposition 4.1.

We shall note that Lifshits and Weber have obtained another proof of the previous theorem by a spectral regularization principle ([29], Corollary 6.4.3, p. 110) in the case when T is a non-negative isometry on an arbitrary L^2 -space.

Indeed, let $\mathcal{H} = L^2(\mu)$. By hypothesis, T is a non-negative contraction. In virtue of the dilation theorem, there exists another L^2 -space that we denote by $\hat{\mathcal{H}}$, and a non-negative isometry \hat{T} such that, $DT^n = P\hat{T}^nD$, where $D : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ is a non-negative isometric embedding from $\hat{\mathcal{H}}$ to \mathcal{H} and $P : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$ a non-negative projection. The non-negativity of the mentioned operators and the fact that a non-negative isometry on a L^2 -space transforms functions with disjointed supports into functions with disjointed supports (see [15], p.186) leads to

$$O_{\mathcal{H}}(T, f, (N_j), \mathcal{N}_\rho) \leq 2O_{\hat{\mathcal{H}}}(\hat{T}, Df, (N_j), \mathcal{N}_\rho).$$

Then finally, R. Jones, L. Ostrovskii and J. Rosenblatt's result states that

$$\forall f \in \mathcal{H}, \quad O_{\hat{\mathcal{H}}}(\hat{T}, Df, (N_j), \mathcal{N}_\rho) \leq K\|Df\|_{\hat{\mathcal{H}}} = K\|f\|_{\mathcal{H}}.$$

Hence the conclusion. ◇

Remarks:

1. Proposition 4.1 gives us directly the theorem of ergodic domination by Akcoglu (Maximal inequality) of an ergodic mean over a L^2 -space for non-negative contractions ([15], Th.2.5, p.189).
2. Proposition 2.6 gives us directly Akcoglu's ergodic theorem for non-negative contractions defined on a L^2 -space ([15], Th.2.6, p.190).

We shall use Proposition 4.1 in order to bound the oscillation function of ergodic averages along some random sequences.

Proposition 4.2 *Let (Ω, \mathcal{B}, P) be a complete probability space on which is defined a sequence of integer-valued independent random variables $\{X_k, k \in \mathbb{N}\}$. Moreover, assume that the law of the X_k 's is generated by the convolution of a given law, that is to say, there exists an integrable random variable, say Y , such that*

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(S_k)},$$

where (S_k) is a random walk generated by a sequence of non-centered i.i.d. random variables which have a moment of order two, and are defined on another space $(\Omega', \mathcal{B}', P')$ independent of (Ω, \mathcal{B}, P) .

Then there exists a measurable set $\Omega_0 \subset \Omega \times \Omega'$, with $P \times P'(\Omega_0) = 1$, such that, for all fixed $(\omega, \omega') \in \Omega_0$, we have:

For all measurable dynamical system (X, \mathcal{A}, μ, T) , and for all $f \in L^2(\mu)$, the oscillation function $O_{L^2(\mu)}(T, (f), (N_j), \mathcal{N}_\rho)$ associated with the sequence of ergodic averages

$$\left\{ \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k}, N \geq 1 \right\},$$

satisfies

$$O_{L^2(\mu)}(T, (f), (N_j), \mathcal{N}_\rho) \leq K(\omega, \omega') \|f\|_{L^2(\mu)},$$

where $K(\omega, \omega') < \infty$, $P \times P'$ -a.e.

To prove this result, one of the tools we shall use consists in the following proposition that comes from a collective work of M.T. Lacey, K. Petersen, D. Rudolph and M. Wierdl [18].

Proposition 4.3 *Given a 1-dimensional, non-centered random walk which has a moment of order two, we know that there exist a measurable set $\Omega_0 \subset \Omega$ such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, there exists $C_\omega < \infty$ such that,*

$$\forall N \geq 2, \quad \sup_{\alpha \in [-1/2, 1/2]} \left| \frac{1}{N} \sum_{k=1}^N \exp 2i\pi\alpha S_k(\omega) - \mathbb{E} \frac{1}{N} \sum_{k=1}^N \exp 2i\pi\alpha S_k \right| \leq C_\omega \frac{\sqrt{\log N}}{N^{1/6}}.$$

For an exhaustive proof of the latter estimation as $O\left(\frac{\sqrt{\log N}}{N^{1/6}}\right)$ we can see the paper of N. Guillotin ([11]).

Let us prove Theorem 1.2.

Proof. First of all, recall that, for $\rho > 1$ and a given increasing sequence of positive integers (N_j) , we denote by

$$\mathcal{N}_\rho^j := \{[\rho^k], k \in \mathbb{N}, N_j \leq [\rho^k] < N_{j+1}\}.$$

Denote also by

$$A_N = \frac{1}{N} \sum_{k=1}^N T^{X_k}(f).$$

We suggest a proof in three steps.

-*Step 1:* Fix $\omega' \in \Omega'$. Hence, we assume that a realization of the law of the process (X_k) is given with the help of the sequence of positive number $\{S_k(\omega'), N \geq 1\}$. From the application of the strong law of large numbers to the sequence (S_k) (that is to say, $\sup_{k \geq 1} S_k/k < \infty$, P' -a.e.), we can omit to suppose that, for all ω' ,

$$\mathbb{E}_\omega |X_k| = O(k).$$

The arguments used here are, on one hand, the hypothesis of integrability of the law Y and, on the other hand, the independence between the spaces on which are defined Y and the random walk (S_k) .

We need to introduce the following quantity: $\mathbb{E}_\omega A_N$. It gives us the following immediate bound,

$$\begin{aligned} \left\| \sup_{N \in \mathcal{N}_\rho^j} |A_N - A_{N_j}| \right\|_{L^2(\mu)}^2 &\leq 3 \sum_{N \in \mathcal{N}_\rho^j} \|A_N - \mathbb{E}_\omega A_N\|_{L^2(\mu)}^2 + 3 \|A_{N_j} - \mathbb{E}_\omega A_{N_j}\|_{L^2(\mu)}^2 \\ &\quad + 3 \left\| \sup_{N \in \mathcal{N}_\rho^j} |\mathbb{E}_\omega A_N - \mathbb{E}_\omega A_{N_j}| \right\|_{L^2(\mu)}^2. \end{aligned}$$

From Corollary 1.1 and Theorem 1.5 with $q_N = \sqrt{N \log N}$, we have therefore

$$\sum_{N \in \mathcal{N}_\rho^j} \|A_N - \mathbb{E}_\omega A_N\|_{L^2(\mu)}^2 \leq K(\omega, \omega')^2 \sum_{N \in \mathcal{N}_\rho^j} \frac{\log N}{N},$$

and

$$\|A_{N_j} - \mathbb{E}_\omega A_{N_j}\|_{L^2(\mu)}^2 \leq K(\omega, \omega')^2 \frac{\log N_j}{N_j},$$

where $K(\omega, \omega') < \infty$ for almost every ω and ω' .

Then, using the arguments detailed in the proof of Proposition 5.1, we show that

$$\left[\sum_{N \in \mathcal{N}_\rho^j} \frac{\log N}{N} + \frac{\log N_j}{N_j} \right]$$

is the general term of a convergent serie of index j , and this happen independently of the lacunary sequence (N_j) , for all geometric partial index \mathcal{N}_ρ .

It gives us a bound for the oscillation function we are interested in,

$$O_{L^2(\mu)}((T^{X_k}), f, (N_j), \mathcal{N}_\rho) \leq \|f\|_{L^2(\mu)} K(\omega, \omega') + 3 O_{L^2(\mu)}((\mathbb{E}_\omega T^{X_k}), (f), (N_j), \mathcal{N}_\rho).$$

-*Step 2:* The “transferred” problem consists therefore in finding an estimation for

$$O_{L^2(\mu)}((\mathbb{E}_\omega T^{X_k}), f, (N_j), \mathcal{N}_\rho).$$

Denote by \tilde{T} the non-negative contraction on $L^2(\mu)$ defined by $\tilde{T}(\cdot) = \mathbb{E}T^Y(\cdot)$. Then the sequence of non-negative operators on $L^2(\mu)$ defined by $\mathbb{E}T^{X_k}(\cdot)$ can be rewritten, without difficulty, in

$$\forall k \geq 1, \quad \mathbb{E}_\omega T^{X_k}(\cdot) = \tilde{T}^{S_k(\omega')}(\cdot).$$

Hence, in order to bound $O_{L^2(\mu)}((\tilde{T}^{S_k(\omega')}), f, (N_j), \mathcal{N}_\rho)$, we need to introduce the quantity $\mathbb{E}_{\omega'} \tilde{T}^{S_k}$. The technique used in the first step of the proof, together with, this time, Proposition 4.3 and the spectral lemma applied to the contraction \tilde{T} , gives us

$$O_{L^2(\mu)}((\tilde{T}), (f), (N_j), \mathcal{N}_\rho) \leq \|f\|_{L^2(\mu)} K(\omega, \omega') + 3 O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), (f), (N_j), \mathcal{N}_\rho),$$

where $K(\omega, \omega') < \infty$ for almost every ω and ω' .

In that case it will appear the serie

$$\left[\sum_{N \in \mathcal{N}_\rho^j} \frac{\log N}{N^{1/3}} + \frac{\log N_j}{N_j^{1/3}} \right].$$

-*Step 3:* The problem “transferred” a second time consists therefore in finding an estimation for

$$O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), f, (N_j), \mathcal{N}_\rho).$$

Denote by $\tilde{\tilde{T}}$ the non-negative contraction on $L^2(\mu)$ defined by $\tilde{\tilde{T}}(\cdot) = \mathbb{E} \tilde{T}^{S_1}(\cdot)$ where S_1 is the i.i.d increment of the random walk.

Hence, the sequence of contractions $\mathbb{E}_{\omega'} \tilde{T}^{S_k}(\cdot)$ on $L^2(\mu)$ can be written as

$$\forall k \geq 1, \quad \tilde{\tilde{T}}^k(\cdot) = \mathbb{E}_{\omega'} \tilde{T}^{S_k}(\cdot).$$

Using Proposition 4.1, we show that

$$O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), (f), (N_j), \mathcal{N}) \leq K \|f\|_{L^2(\mu)},$$

where K is an absolute constant. This concludes the proof of Proposition 4.2. \diamond

This means, in particular, that, according to Proposition 2.6, $P \times P'$ -almost everywhere, the sequence (X_k) is L^2 -good.

We have, in fact, the expression of the limit which is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} = \mathbb{E}(f|\mathcal{F}), \quad \mu\text{-a.e.}$$

where \mathcal{F} is the σ -field generated by the functions in $L^2(\mu)$ which are invariant under the action of T . To show this result, we just need to observe that the function which are invariant under T are the same as the ones under $\tilde{\tilde{T}}$. We conclude using Hopf’s ergodic theorem.

4.2 Sketch of the proof of Theorem 1.3

We start as in the first step of the proof of Theorem 1.2. Then we show that one can extend, without difficulty, the maximal and oscillation inequalities for non-negative contractions. We leave the details to the reader.

Remark 1: In the case when, for instance, u_k is a polynomial or the sequence of ordered primes numbers, we know that the oscillation inequalities are satisfied in the framework of non-negative isometries (see [3]).

Remark 2: In the case when $u_k = k$ and $\mathcal{L}(Y)$ is a Poisson law of parameter 1, a direct calculation gives an estimation of the oscillation function of the ergodic average above, by application of result of [12] (Theorem 4.1), without any argument of dilation.

Indeed, for all $N \in \mathbb{N}$, let X_N be a Poisson of parameter N . We can easily check that such a process $\{X_N : N \in \mathbb{N}\}$ satisfies the condition (7) with $q_N = \sqrt{N \log N}$. Corollary 1.1 together with a reasoning

analogous to the one made in the proof of Corollary 3.2, show therefore that, for all $\omega \in \Omega_0$, (Y, \mathcal{A}, μ, T) and $f \in L^2(\mu)$, the a.e. convergence of $(1/N) \sum_{k=1}^N f \circ T^{X_k}$ is equivalent to the one of $(1/N) \sum_{k=1}^N \mathbb{E} f \circ T^{X_k}$.

We shall focus now on the convergence of that last average. The method is nearly the same. Since the ergodic average $(1/N) \sum_{k=1}^N f \circ T^k$ is known to converge a.e. for all $f \in L^2(\mu)$, we just need to show that there exists a sequence a_N such that, for all $\rho > 1$, $\sum_{N \in \mathcal{N}_\rho} a_N < \infty$ and

$$\left\| \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k} - \frac{1}{N} \sum_{k=1}^N f \circ T^k \right\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)} a_N.$$

From the Spectral Lemma, we know that the left side of this last inequality is equivalent to

$$\left(\int_{[-1/2, 1/2]} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} e^{2\pi i \alpha X_k} - \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2} \leq I_1 + I_2 + I_3,$$

where μ_f still denotes the spectral measure of the operator at the point f , and

$$I_1 := \left(\int_{|\alpha| < \epsilon} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} e^{2\pi i \alpha X_k} - \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

$$I_2 := \left(\int_{|\alpha| \in (\epsilon, 1/2)} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} e^{2\pi i \alpha X_k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

$$I_3 := \left(\int_{|\alpha| \in (\epsilon, 1/2)} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

for some arbitrary $\epsilon \in]0, 1/2[$. To bound I_1 from above, we use the well-known property of Poisson laws,

$$\mathbb{E} e^{2\pi i \alpha X_k} = e^{k(e^{2\pi i \alpha} - 1)}.$$

We have therefore to bound

$$\begin{aligned} \left| \frac{1}{N} \sum_{k=1}^N \left(e^{k(e^{2\pi i \alpha} - 1)} - e^{2\pi i \alpha k} \right) \right| &\leq \frac{1}{N} \sum_{k=1}^N \left| e^{k(e^{2\pi i \alpha} - 1)} - e^{2\pi i \alpha k} \right| \\ &\leq \frac{1}{N} \sum_{k=1}^N \left| e^{k O(\alpha^2)} - 1 \right| \\ &\leq \frac{1}{N} \sum_{k=1}^N k O(\alpha^2) \\ &\leq N O(\alpha^2) \\ &\leq N O(\epsilon^2). \end{aligned}$$

The last inequality comes from the fact that the integration is over $[-\epsilon, \epsilon]$. We obtain finally

$$I_1 \leq C \|f\|_{L^2(\mu)} N \epsilon^2.$$

To bound I_2 and I_3 from above, we use, respectively

$$\left| \frac{1}{N} \sum_{k=1}^N e^{k(e^{2\pi i \alpha} - 1)} \right| \leq \frac{C}{N |\alpha|},$$

and

$$\left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \alpha k} \right| \leq \frac{C}{N |\alpha|}.$$

We obtain easily

$$I_2 + I_3 \leq \frac{C}{N\epsilon} \|f\|_{L^2(\mu)},$$

since the integration is done, this time, over $\{\alpha \in [-1/2, 1/2] : |\alpha| > \epsilon\}$.

Finally, putting $\epsilon = N^{-2/3}$, we have the desired bound,

$$\left(\int_{[-1/2, 1/2]} \left| \frac{1}{N} \sum_{k=1}^N \mathbb{E} e^{i2\pi\alpha X_k} - \frac{1}{N} \sum_{k=1}^N e^{i2\pi\alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2} \leq C \|f\|_{L^2(\mu)} N^{-1/3}.$$

Observe that we have also proved that the limit of $(1/N) \sum_{k=1}^N f \circ T^{X_k}$ is the same as $(1/N) \sum_{k=1}^N f \circ T^k$ which is known to be $\mathbb{E}(f | \mathcal{F})$ where \mathcal{F} denotes the σ -field of invariants under T .

5 Third application: weighted ergodic averages (proof of Theorem 1.4)

We end this work by a study of the following problem: given a probability space (Ω, \mathcal{B}, P) on which is defined a sequence of centered, independent and identically distributed random variable $(X_k, k \geq 1)$ that satisfies a condition on the moments (see below), given an increasing sequence of integers $(u_k, k \geq 1)$, given a measurable dynamical system (X, \mathcal{A}, μ, T) , for $f \in L^2(\mu)$, what can we say about the pointwise convergence of the sequence of modulated ergodic averages

$$\left\{ \frac{1}{N} \sum_{k=1}^N X_k f \circ T^{u_k}, N \geq 1 \right\}?$$

Let us specify that

1. we do not assume that the sequence of random weight is bounded,
2. the sequence $(u_k, k \geq 1)$ is not supposed to be L^2 -good for the pointwise ergodic theorem. Thus, we shall be able, in no way, to use any maximal inequality which is known to be true, for instance, when the sequence $(u_k, k \geq 1)$ is L^2 -good.

In a previous work with Weber[27], one of the authors shows that when the sequence $(u_k, k \geq 1)$ is L^2 -good and when the sequence of weights is non-negative, for all $f \in L^2(\mu)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N X_k f \circ T^{u_k} \text{ exists } \mu\text{-almost everywhere,}$$

universally on the space where the sequence (X_k) is defined.

More recently, Assani[2] proved a result of the same kind in the case when $u_k = k$, the weights are centered and the spaces are of L^p -style, $p > 1$.

Given a sequence of centered weights, say (X_k) , writing it as a difference of two non-negative sequences, $X_k = X_k \cdot I_{X_k \geq 0} - (-X_k \cdot I_{X_k < 0})$, and applying [27] to both non-negative weights, we obtain immediately Assani's result for $p = 2$. It is therefore not difficult to observe that we have also the result for all $p > 1$.

To prove Theorem 1.4, we start by determining the oscillation function associated with the sequence of operators, $T_k := X_k T^{u_k}$. For that, recall that

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}) = \sqrt{\sum_{j \geq 1} \left\| \sup_{N \in \mathcal{N}_j} \left| \frac{1}{N} \sum_{k=1}^N T_k(f) - \frac{1}{N_j} \sum_{k=1}^{N_j} T_k(f) \right| \right\|_{L^2(\mu)}^2},$$

where \mathcal{N} is a partial index, (N_j) is a sequence of integers such that $N_{j+1} \geq 2N_j$, and $f \in L^2(\mu)$.

Proposition 5.1 *Let (Ω, \mathcal{B}, P) be a probability space on which is defined a sequence of centered independent and identically distributed random variables $\{X_k, k \in \mathbb{N}\}$ that satisfies $\mathbb{E}|X_1|^2 < \infty$. Let $(u_k, k \geq 1)$ be an strictly increasing sequence of integer such that there exists $\beta \in (0, 1)$ with $u_k = O(2^{k^\beta})$. Then there exists a \mathcal{B} -measurable set Ω_0 , with $P(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, there exists a positive constant $K(\omega) < \infty$ such that, for all measurable dynamical system (X, \mathcal{A}, μ, T) , and all $f \in L^2(\mu)$,*

$$O_{L^2(\mu)}((T_k), (f), (N_j), \mathcal{N}) \leq K(\omega) \|f\|_{L^2(\mu)}.$$

Moreover, the constant K is independent of the sequence (N_j) , and we can choose \mathcal{N} among the following indexes: \mathcal{N}_ρ for $\rho > 1$ (geometrical index, see Section 1), $\mathcal{N}^\gamma = \{k^\gamma, k \geq 1\}$ where $\gamma > 1/(1 - \beta)$ is an integer (β is given in the theorem).

A consequence of Proposition 5.1, is that we have the theorem with only partial indexes, and, a priori, we cannot obtain immediately the total index by arguments of non-negativity and boundedness (see Proposition 2.6). Actually, the lemma below permits to conclude. Indeed, it exhibits a connection between the sequence of random variables (X_k) and the index $\mathcal{N}^\gamma = \{k^\gamma, k \geq 1\}$ where $\gamma \geq 3$ is an integer.

Lemma 5.1 *Under the hypotheses of Proposition 5.1, for all integer $\gamma \geq 3$, we have*

$$\mathbb{E} \sup_{N \geq 1} \frac{1}{N^{\gamma-1}} \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} |X_k| < \infty.$$

Remark: The sequence does not need to be centered.

Let us prove, first, Proposition 5.1.

Proof. We suggest a proof relying on Proposition 2.7 and the spectral lemma. Let us first introduce a notation: for all integer N and all $f \in L^2(\mu)$, denote

$$A_N := \frac{1}{N} \sum_{k=1}^N T_k(f).$$

Hence, we have

$$\left\| \sup_{N \in \mathcal{N}^j} |A_N - A_{N_j}| \right\|_{L^2(\mu)}^2 \leq 2 \sum_{N \in \mathcal{N}^j} \|A_N\|_{L^2(\mu)}^2 + 2 \|A_{N_j}\|_{L^2(\mu)}^2,$$

where $\mathcal{N}^j := \{N \in \mathcal{N}, N_j \leq N < N_{j+1}\}$. Then, using the spectral lemma and Proposition 2.7, we obtain

$$\left\| \sup_{N \in \mathcal{N}^j} |A_N - A_{N_j}| \right\|_{L^2(\mu)}^2 \leq 2 C(\omega) \|f\|_{L^2(\mu)}^2 \left[\sum_{N \in \mathcal{N}^j} \frac{\log u_N}{N} + \frac{\log u_{N_j}}{N_j} \right].$$

From the hypothesis of increase made on (u_k) , and the fact that $N_j \geq 2^j$, we have the estimation

$$\left\| \sup_{N \in \mathcal{N}^j} |A_N - A_{N_j}| \right\|_{L^2(\mu)}^2 \leq 2 C(\omega) \|f\|_{L^2(\mu)}^2 \left[\tilde{C} \sum_{N \in \mathcal{N}^j} \frac{1}{N^{1-\beta}} + \frac{1}{2^{j(1-\beta)}} \right],$$

where \tilde{C} is a constant which depends only on the sequence (u_k) .

- In the case when $\mathcal{N} = \mathcal{N}_\rho = \{\lceil \rho^k \rceil, k \geq 1\}$ with $\rho > 1$, it is clear that we obtain

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}_\rho) \leq K(\rho, (u_k), \omega) \|f\|_{L^2(\mu)},$$

with $K(\rho, (u_k), \omega) < \infty$ for all $\rho > 1$.

- In the case when $\mathcal{N} = \mathcal{N}^\gamma = \{k^\gamma, k \geq 1\}$ with $\gamma > 1/(1 - \beta)$ an integer, it is clear that we obtain

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}^\gamma) \leq K(\gamma, (u_k), \omega) \|f\|_{L^2(\mu)},$$

with $K(\gamma, (u_k), \omega) < \infty$ for all $\gamma > 1/(1 - \beta)$. Hence the proof of Proposition 5.1.

◇

Next, we give a proof of Lemma 5.1 which relies on an argument of Gaussian randomization.
Proof. Introducing the quantity $\mathbb{E}|X_k|$, we have the following bound

$$\mathbb{E} \sup_{N \geq 1} \frac{1}{N^{\gamma-1}} \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} |X_k| \leq \mathbb{E} \sup_{N \geq 1} \frac{1}{N^{\gamma-1}} \left| \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} |X_k| - \mathbb{E}|X_k| \right| + C(\gamma) \mathbb{E}|X_1|,$$

where $C(\gamma)$ is a constant which depends on γ .

From symetrization and contraction principles, it remains us to prove that

$$\mathbb{E}_{X,g} \sup_{N \geq 1} \frac{1}{N^{\gamma-1}} \left| \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} g_k |X_k| \right| < \infty,$$

where $\{g_k, k \geq 1\}$ is an isonormal sequence.

Fix the variable X , and integrate over g , the Gaussian random variable indexed on positive integers, $G_N := \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} g_k |X_k|$. For all fixed X and all N , denote

$$\Gamma_N = \frac{1}{\sqrt{\sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} |X_k|^2}} G_N.$$

Note that, for all N , $\mathcal{L}(\Gamma_N) = \mathcal{N}(0, 1)$. Therefore,

$$\mathbb{E}_{X,g} \sup_{N \geq 1} \frac{1}{N^{\gamma-1}} |G_N| \leq \mathbb{E}_X 2^\gamma \sup_{M \geq 1} \sqrt{\frac{1}{M} \sum_{k=1}^M |X_k|^2} \left[\mathbb{E}_g \sup_{N \geq 1} \left| \frac{1}{N^{\gamma/2-1}} \Gamma_N \right| \right].$$

As $\gamma/2 - 1 > 0$ ($\gamma \geq 3$ by hypothesis), we know that $\mathbb{E}_g \sup_{N \geq 1} |\Gamma_N / N^{\gamma/2-1}| < \infty$ (see 17). And the condition on the moments imposed to the sequence (X_k) leads to the conclusion that

$$\mathbb{E}_X 2^\gamma \sup_{M \geq 1} \sqrt{\frac{1}{M} \sum_{k=1}^M |X_k|^2} < \infty.$$

◇

Finally, we prove Theorem 1.4

Proof. From Proposition 5.1, we know that, for all dynamical system and all $f \in L^2(\mu)$, on the indexes $\mathcal{N}_\gamma = \{k^\gamma, k \geq 1\}$ with γ an integer large enough (which depends on the increase of the sequence (u_k)), we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^\gamma} \sum_{k=1}^{N^\gamma} X_k(\omega) f(T^{u_k} x) = 0, \mu\text{-a.e.}$$

Lemma 5.1 permits us to show the existence of a positive random variable C which is integrable and such that, for all $N \geq 1$, we have

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^{\gamma-1}} |X_k(\omega)| \leq \frac{C(\omega)}{N}.$$

For all positive integer M , there exists an integer N such that

$$N^\gamma \leq M < (N+1)^\gamma.$$

We have the following inequality

$$\left| \frac{1}{M} \sum_{k=1}^M X_k(\omega) f(T^{u_k} x) \right| \leq$$

$$\frac{N^\gamma}{M} \left| \frac{1}{N^\gamma} \sum_{k=1}^{N^\gamma} X_k(\omega) f(T^{u_k} x) \right| + \frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma-1} |X_k(\omega)| |f(T^{u_k} x)|.$$

The first term converges to 0 when N tends to ∞ . Let us focus on the second term. Without loss of generality, we can assume that f is non-negative. Then, we decompose f as follows, $f = f_1 + f_2 = f \mathbb{1}_{f \leq \sqrt{N}} + f \mathbb{1}_{f > \sqrt{N}}$.

For the first term, we have,

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma-1} |X_k(\omega)| |f_1(T^{u_k} x)| \leq C(\omega) \frac{\sqrt{N}}{N}$$

which tends to 0 when N tends to ∞ because of the Lemma 5.1.

For the second term, by integrating with the measure μ which is invariant under the action of the operator T , we have again, from the lemma,

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma-1} |X_k(\omega)| \int |f_2(T^{u_k} x)| \mu(dx) \leq C(\omega) \int \frac{f \mathbb{1}_{f > \sqrt{N}}}{N} d\mu. \quad (26)$$

We finish by remarking that

$$\sum_{N \geq 1} \int \frac{f \mathbb{1}_{f > \sqrt{N}}}{N} d\mu < \infty.$$

Indeed, applying Schwarz inequality to the sum and the integral, we obtain

$$\sum_{N \geq 1} \int \frac{f \mathbb{1}_{f \leq \sqrt{N}}}{N} d\mu \leq \sqrt{\int \sum_{N \geq 1} \frac{f^2}{N^2} d\mu} \sqrt{\sum_{N \geq 1} \mu(f^2 > N)} < \infty,$$

because f is in $L^2(\mu)$.

Remark finally that the same calculations would lead to the same kind of result in L^p . \diamond

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