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Abstract

Let $(X, \mathcal{A}, \mu)$ be a probability space, let T be a contraction on $L^p(\mu)$ and let f in $L^p(\mu)$, $(p > 1)$. We provide suitable conditions over sequences (w_k), (u_k) and (A_k) in such a way that the weighted ergodic limit $\lim_{N\to\infty} \frac{1}{A_N} \sum_{k=0}^{N-1} w_k T^{u_k}(f) = 0$ μ-a.e.. As a consequence of our main theorems, we also deal with the so-called one-sided weighted ergodic Hilbert transforms.

Keywords weighted ergodic averages, contractions of Hilbert space, one-sided weighted ergodic Hilbert transformation

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1 Introduction

Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \to X$ be a transformation preserving μ, i.e. $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{A}$. In what follows, we write $(T^k f)(x) = f(T^k x)$ for any measurable function $f : X \to \mathbb{R}$, $x \in X$ and $k \geq 1$. The study of averages over iterations of T started in 1930s with the classical Birkhoff's Ergodic Theorem [12]. Wiener and Wintner [25] generalized this result by showing that, for any $f \in L^1(\mu)$, $\mu-$almost everywhere (μ-a.e.), for any $\theta \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2i\pi k\theta} T^k(f) \text{ exists.}$$

Lesigne [13] went on to prove the same property when the term $e^{2i\pi k\theta}$ is replaced by any weight of the form $w_k = e^{2i\pi P(k)}$, where P is a polynomial. In this way, many results on the convergence of averages of the form

$$\frac{1}{N} \sum_{k=0}^{N-1} w_k T^{u_k}(f)$$

were established for various weights (w_k) and for various sequences of integers (u_k). Bellow and Losert in [1] proved that, if we have $(\Omega, \mathcal{B}, \nu, S)$ a dynamical system where S has countable Lebesgue spectrum, then for any $g \in L^\infty(\Omega)$ there exists a set Ω_g of probability one such that for any $\omega \in \Omega_g$: for any ergodic dynamical system $(X, \mathcal{A}, \mu, T)$, for any $f \in L^\infty(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} g(S^k\omega)T^k(f) \quad \text{exists} \quad \mu - a.e..$$

As a famous extension of the work of Bellow and Losert, we have the Bourgain's return times theorem in [20], which states that if we take $(\Omega, \mathcal{B}, \nu, S)$ an ergodic dynamical system the previous result stays true, see also Assani [2] and Demeter, Lacey, Tao and Thiele [9]. For the Von Mangoldt weight in relation with the prime numbers see Eisner and Lin [29]. For more weighted ergodic theorems, refer to Berend, Lin, Rosenblatt and Tempelman [11], Lin, Olsen and Tempelman [22]. For a different type of weighted ergodic theorems with arithmetic weights we refer to Cuny and Weber [8] and Buczolich [32].

In this paper, we consider the question for the rate of convergence. More precisely, we deal with the following problem : let (w_k) be a sequence of complex numbers such that $\sum_{k=0}^{N-1} |w_k| \sim CN$ where C is a positive constant and let (u_k) and (A_k) be sequences of integers and real numbers, respectively. We provide suitable conditions over these sequences to ensure that the weighted ergodic average

$$\frac{1}{A_N} \sum_{k=0}^{N-1} w_k T^{u_k}(f), \tag{1}$$

converges to 0 μ-a.e., where T is a contraction on $L^p(\mu)$ and $f \in L^p(\mu)$, $(1 < p < \infty)$. Depending on p, many papers [8, 18, 21, 23, 26, 27, 28] are devoted to the convergence (i.e. a.e. or norm convergence) to 0 of the averages considered in (1). Another important matter is to investigate the almost everywhere convergence of the so-called one-sided weighted ergodic Hilbert transforms, namely

$$\sum_{k\geq 1} \frac{w_k}{A_k} T^{u_k}(f).$$

For results on such subject, we can refer to [3, 5, 6, 10, 16, 24]. In what follows, we denote by C a generic positive constant which may differ from line to line. One possible condition that we shall make on the sequences (w_k) and (u_k) is as follows : there exists constants $0 < \alpha \leq 1$ and $\beta \in \mathbb{R}$ such that $\left|\left|\sum_{k=0}^{N-1} w_k T^{u_k}(f)\right|\right|_{p,\mu} \leq CN^\alpha \log^\beta N$. Two useful tools are the Riesz-Thorin interpolation theorem (see p.95 in [4]) and the spectral inequality, that is presented later in Section 2, which transfers the problem to the study of uniform estimates of the form $\sup_{\theta\in\mathbb{R}} \left|\sum_{k=0}^{N-1} w_k e^{2i\pi\theta u_k}\right| \leq CN^\alpha \log^\beta N$.

We now state our main result.

Theorem 1. *Let (w_k) be a bounded sequence of complex numbers and let $(u_n) \subset \mathbb{N}$ be a sequence of integers. Let $1 < p < \infty$, $0 < \alpha < 1$, $\beta \geq 0$, T be a contraction on $L^p(\mu)$ and $f \in L^p(\mu)$.*

1. *If* $\left|\left|\sum_{k=0}^{N-1} w_k T^{u_k}(f)\right|\right|_{p,\mu} \leq CN^{\alpha}\log^{\beta} N$. *Then for* $H > \beta + \frac{1}{p}$, *we have*

$$\lim_{N\to\infty} \frac{1}{N^{\frac{(p-1)\alpha+1}{p}}\log^{H} N}\sum_{k=0}^{N-1} w_k T^{u_k}(f) = 0 \quad a.e. \quad \& \quad \sup_{N>1}\left|\frac{\sum_{k=0}^{N-1} w_k T^{u_k}(f)}{N^{\frac{(p-1)\alpha+1}{p}}\log^{H} N}\right| \in L^p(\mu).$$

2. *If* $\left|\left|\sum_{k=M}^{N-1} w_k T^{u_k}(f)\right|\right|_{p,\mu} \leq C(N-M)^{\alpha}\log^{\beta} N$. *Then for* $H > \beta + \frac{1}{p}$ *when* $p\alpha > 1$ *and for* $H > \beta + \frac{3}{p}$ *when* $p\alpha = 1$, *we have*

$$\lim_{N\to\infty} \frac{1}{N^{\alpha}\log^{H} N}\sum_{k=0}^{N-1} w_k T^{u_k}(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N>1}\left|\frac{\sum_{k=0}^{N-1} w_k T^{u_k}(f)}{N^{\alpha}\log^{H} N}\right| \in L^p(\mu).$$

This theorem provides optimal results for this type of weighted ergodic averages, which improves a recent result derived by Fan [18] that has studied weighted ergodic averages under Condition $\sup_{\theta\in\mathbb{R}}\left|\sum_{k=M}^{N-1} w_k e^{2i\pi\theta u_k}\right| \leq C(N-M)^{\alpha}$ with $\frac{1}{2} \leq \alpha < 1$, which implies that $\left|\left|\sum_{k=M}^{N-1} w_k T^{u_k}(f)\right|\right|_{2,\mu} \leq C(N-M)^{\alpha}$ by the spectral Lemma. More precisely, for any dynamical system $(X, \mathcal{A}, \mu, T)$ and any $f \in L^2(\mu)$, he obtained the almost everywhere convergence to 0 with the normalization $N^{\alpha}\log^2 N \log^{1+\delta}\log N$, $\delta > 0$. Moreover, Theorem 1 can be applied to enhance ideas of Theorem 4.2 in [23]. A weaker version of Theorem 1 was obtained in [27].

As a consequence of Theorem 1, for a Dunford-Schwartz operator on $L^1(\mu)$, i.e., a contraction of each $L^p(\mu)$, $1 \leq p \leq \infty$, we obtain the following.

Corollary 1. *Assume that* $\sup_{\theta\in\mathbb{R}}\left|\sum_{k=0}^{N-1} w_k e^{2i\pi\theta u_k}\right| \leq CN^{\alpha}\log^{\beta} N$. *Then for any Dunford-Schwartz operator* T *on* $L^1(\mu)$, *and any* $f \in L^p(\mu)$, *any* $H > 2\beta\left(1-\frac{1}{p}\right)+\frac{1}{p}$ *where* $1 < p \leq 2$, *any* $H > \frac{2\beta}{p}+\frac{1}{p}$ *where* $2 < p < \infty$, *we have*

$$\lim_{N\to\infty} \frac{1}{N^{\Gamma}\log^{H} N}\sum_{k=0}^{N-1} w_k T^{u_k}(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N>1}\left|\frac{1}{N^{\Gamma}\log^{H} N}\sum_{k=0}^{N-1} w_k T^{u_k}(f)\right| \in L^p(\mu),$$

where $\Gamma = \begin{cases} \frac{2(p-1)^2\alpha+p+(p-1)(2-p)}{p^2} & \text{if} \quad 1 < p \leq 2 \\ \frac{2(p-1)\alpha+p+(p-1)(p-2)}{p^2} & \text{if} \quad 2 < p < \infty \end{cases}$.

Notice that, in the particular case where $p = 2$, the fact that T is a contraction in $L^2(\mu)$ is sufficient to obtain the Corollary 1.

Concerning the almost everywhere convergence of the so-called one-sided weighted ergodic Hilbert transforms, we have the following result.

Corollary 2. *Under the same assumption as in the first statement of Theorem 1, for* $H > \frac{1}{p}+\beta$, *the series*

$$\sum_{k>1}\frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}}\log^{H} k}T^{u_k}(f) \text{ exists } \mu - a.e. \quad \& \quad \sup_{N>1}\left|\sum_{k=2}^{N}\frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}}\log^{H} k}T^{u_k}(f)\right| \in L^p(\mu).$$

Motivated by the classical Davenport estimate [17] : for any $\beta > 0$, there exists a constant C_β such that

$$\sup_{\theta\in\mathbb{R}}\left|\sum_{k=1}^{N}\mu(k)e^{2i\pi k\theta}\right| \leq C_\beta\frac{N}{\log^\beta N},$$

where $(\mu(k))$ is a Möbius sequence. Several studies on the convergence of the averages considered in (1), have been conducted based on the above estimation. Recall that a Möbius sequence is defined as $\mu(1) = 1$, $\mu(k) = (-1)^j$ if k is a product of j distinct primes, and $\mu(k) = 0$ otherwise. Let $(X, \mathcal{A}, \mu, \tau)$ be a measurable dynamical system. It is mentioned in [26] that for any $f \in L^2(\mu)$, $\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mu(k)f\circ\tau^k = 0$, $\mu-a.e.$. Abdalaoui *et al* [21] proved that, the previous result remains true for any $f \in L^1(\mu)$. In [8] Cuny and Weber obtained a rate of convergence. More precisely, for any $f \in L^p(\mu)$, $p > 1$ and for any $H > 0$,

$$\lim_{N\to\infty}\frac{\log^H N}{N}\sum_{k=1}^{N}\mu(k)f\circ\tau^k = 0 \quad \mu-a.e. \quad \text{and} \quad \sup_{N\geq 1}\left|\frac{\log^H N\sum_{k=1}^{N}\mu(k)f\circ\tau^k}{N}\right| \in L^p(\mu).$$

Eisner [28] showed that the previous result of Abdalaoui *et al* [21] remains true if we replace $T^k(f)$ by $T^{P(k)}(f)$, where P is a polynomial. Recently, Fan [18] extended these results to any Bourgain's sequence (see Section 2 in [18] for a more precise definition), and for any bounded sequence (w_k) satisfying : $\sup_{\theta\in\mathbb{R}}\left|\sum_{k=0}^{N-1}w_k e^{2i\pi\theta u_k}\right| \leq C\frac{N}{\log^\beta N}$, with $\beta > \frac{1}{2}$. In the following results, we deal with condition of Davenport type. Our results deal with a more general problem since we consider a general sequence of integers (u_k), and not necessarily Bourgain's sequences.

Theorem 2. *Let $1 < p < \infty$, let T be a contraction on $L^p(\mu)$ and let $f \in L^p(\mu)$, such that,*

$$\left|\left|\sum_{k=0}^{N-1}w_k T^{u_k}(f)\right|\right|_{p,\mu} \leq C\frac{N}{\log^\beta N}, \tag{2}$$

for some $\beta > \frac{1}{p-1}$. Then, for $0 < H < \frac{\beta(p-1)-1}{p}$, we have

$$\lim_{N\to\infty}\frac{\log^H N}{N}\sum_{k=0}^{N-1}w_k T^{u_k}(f) = 0 \quad \mu-a.e. \quad \& \quad \sup_{N\geq 1}\left|\frac{\log^H N}{N}\sum_{k=0}^{N-1}w_k T^{u_k}(f)\right| \in L^p(\mu).$$

If we consider that, for any $\beta > 0$, there exists a constant C_β, such that (2) is satisfied (as in the Davenport estimate), then the result of Theorem 2 remains true for any $H > 0$.

Corollary 3. *Suppose that*

$$\sup_{\theta\in\mathbb{R}}\left|\sum_{k=0}^{N-1}w_k e^{2i\pi\theta u_k}\right| \leq C\frac{N}{\log^\beta N},$$

for some $\beta > 0$. Then for any Dunford-Schwartz operator T on $L^1(\mu)$ and for any $f \in L^p(\mu)$, if

$$\begin{cases} \beta > \frac{1}{2}\left(\frac{Hp^2}{(p-1)^2} + \frac{p}{(p-1)^2}\right) & \text{where} \quad 1 < p \leq 2 \\ \beta > \frac{1}{2}\left(\frac{Hp^2}{p-1} + \frac{p}{p-1}\right) & \text{where} \quad 2 < p < \infty \end{cases},$$

we have

$$\lim_{N\to\infty} \frac{\log^H N}{N} \sum_{k=0}^{N-1} w_k T^{u_k}(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N\geq 1} \left| \frac{\log^H N}{N} \sum_{k=0}^{N-1} w_k T^{u_k}(f) \right| \in L^p(\mu).$$

Notice that, if $p = 2$, T is a contraction that is sufficient to obtain Corollary 3. When p is large enough and $H = 0$, the result remains true for $\beta > \frac{1}{2}$.

Corollary 4. *Assume that* (2) *holds with $\beta > \frac{pH}{p-1} + \frac{1}{p-1}$. Then we have*

$$\sum_{k\geq 1} \frac{w_k \log^H k}{k} T^{u_k}(f) \quad \text{exists } \mu - a.e. \quad \& \quad \sup_{N\geq 1} \left| \sum_{k=1}^{N} \frac{w_k \log^H k}{k} T^{u_k}(f) \right| \in L^p(\mu).$$

The rest of this paper is organized as follows. In Section 2, we recall some known results, which will be used in the proofs of our main theorems, including spectral theorem, Móricz's lemma and van der Corput's theorem. In Section 3, we prove our theorems and their corollaries. One of the main analytical tools used in proofs is Móricz's lemma. In the last section, we provide several examples illustrating our main results. More precisely, we construct sequences (w_k) and (u_k) that are difficult to deal with previously, which satisfy the conditions of our theorems. The applications in this context concretely prove the effectiveness of the obtained theorems.

2 Preliminaries

In this section, we recall some known results. The first one, referred to as the spectral lemma, reduces the problem of evaluating norms to Fourier analysis questions. Let T be a contraction in a Hilbert space $\mathcal{H}$, that is, a linear operator such that $||T(f)|| \leq ||f||$ for each $f \in \mathcal{H}$, and put

$$P_n(f) = \langle T^n(f), f\rangle \quad \text{for} \quad n \geq 0 \quad \text{and} \quad P_n(f) = \overline{P_{-n}(f)} \quad \text{for} \quad n < 0.$$

The sequence $(P_n(f))_{n\in\mathbb{Z}}$ is non-negative definite. By Herglotz's theorem, there exists a finite positive measure μ_f on $\mathcal{B}(\mathbb{R}/\mathbb{Z})$ (called the spectral measure of f) such that for all $n \geq 0$, we have

$$\langle T^n(f), f\rangle = \int_{\mathbb{R}/\mathbb{Z}} \exp(2i\pi nt)\mu_f(\mathrm{d}t).$$

The spectral lemma can be stated as follows.

Lemma 1. *(Spectral lemma). Let T be a contraction in a Hilbert space $\mathcal{H}$ and let $f \in \mathcal{H}$. Let $P(x) = \sum_{k=0}^{N} a_k x^k$ be a complex polynomial of degree $N \geq 0$. Then we have*

$$||P(T)f||^2 \leq \int_{\mathbb{R}/\mathbb{Z}} |P(\exp(2i\pi t)|^2 \mu_f(\mathrm{d}t).$$

The second result is provided by Móricz. A direct consequence of Theorems 1 and 3 in [14] is the following lemma.

Lemma 2. *(Móricz's lemma). Let $1 < p < \infty$. Let (N_k) be an increasing sequence of integers and let $(X_n)_{n\geq 1}$ be a sequence of random variables on a probability space $(X, \mathcal{A}, \mathbb{P})$. Assume that there exist $\lambda \geq 1$ and a non-negative function $g : \mathbb{N} \times \mathbb{N} \longmapsto \mathbb{R}^+$, such that for any $j \in \mathbb{N}^*$, $m, n \in [\![N_j, N_{j+1}]\!]$ with $m < n$*

1. $\mathbb{E}(|\sum_{k=m+1}^{n} X_k|^2) \leq C g^\lambda(m,n)$,
2. $g(m,j) + g(j,n) \leq g(m,n)$ *for all* $j \in [\![m,n]\!]$.

Then, if $\lambda > 1$

$$\mathbb{E}\left(\sup_{j=m+1}^{n} \left|\sum_{k=m+1}^{j} X_k\right|^p\right) \leq C(\lambda) g^\lambda(m,n) \qquad \text{for any } m,n \in [\![N_j, N_{j+1}]\!] \text{ with } m < n,$$

where $C(\lambda)$ is a constant only depending on λ.
If $\lambda = 1$,

$$\mathbb{E}\left(\sup_{j=m+1}^{n} \left|\sum_{k=m+1}^{j} X_k\right|^p\right) \leq C \log^p\left(2(n-m)\right) g(m,n) \quad \text{for any } m,n \in [\![N_j, N_{j+1}]\!] \text{ with } m < n,$$

where C is an absolute constant.

The following lemmas are due to van der Corput (see Theorems 3 and 4 in [30]) and will be used in our examples.

Lemma 3. *Let a and b be two integers, with $a < b$, and let f be a function on $[a,b]$. Assume that the second derivative is such that $-f''(x) \geq \rho$ for any $x \in [a,b]$, and for some $\rho > 0$. Then*

$$\left|\sum_{k=a}^{b} e^{2i\pi f(k)}\right| \leq \left(|f'(b) - f'(a)| + 2\right)\left(\frac{4}{\sqrt{\rho}} + 3\right).$$

Lemma 4. *Let $n \geq 2$ be an integer and put $K = 2^n$. Suppose that $a \leq b \leq a + N$ and that $f : [a,b] \to \mathbb{R}$ has continuous nth derivative that satisfies the inequality $0 < \lambda \leq |f^{(n)}(x)| \leq h\lambda$ for any $x \in [a,b]$. Then*

$$\left|\sum_{k=a}^{b} e^{2i\pi f(k)}\right| \leq ChN\left(\lambda^{\frac{1}{K-2}} + N^{\frac{-2}{K}} + (N^n\lambda)^{\frac{-2}{K}}\right).$$

3 Proofs

Let f be a measurable function and denote by $S_N(f)$ the weighted sums associated with f, i.e.

$$S_N(f) = \sum_{k=0}^{N-1} w_k T^{u_k}(f).$$

Let $(A_N)_{N>0}$ be a sequence converging to infinity as N goes to infinity. In this section, we prove Theorems 1, 2 and their corollaries. Our proofs use Móricz's lemma in particular and operate in the following way. Let (N_k) be an increasing sequence of integers. Let N be a sufficiently large integer and let $j \in \mathbb{N}^*$ be such that $N_j < N \leq N_{j+1}$ and $N_{j+1} < CN_j$. The main idea is to prove that for any $f \in L^p(\mu)$,

$$\lim_{j\to\infty} \left| \frac{S_{N_j}(f)}{A_{N_j}} \right| = 0 \quad \mu - a.e. \tag{3}$$

and that

$$\lim_{j\to\infty} \sup_{N=N_j+1}^{N_{j+1}} \left| \frac{S_N(f)}{A_N} - \frac{S_{N_j}(f)}{A_{N_j}} \right| = 0 \quad \mu - a.e.. \tag{4}$$

Proving Equations (3) and (4) is sufficient since for any $j \in \mathbb{N}$, we have

$$\begin{aligned} \left| \frac{S_N(f)}{A_N} \right| &\leq \left| \frac{S_N(f)}{A_N} - \frac{S_{N_j}(f)}{A_{N_j}} \right| + \left| \frac{S_{N_j}(f)}{A_{N_j}} \right| \\ &\leq \sup_{N=N_j+1}^{N_{j+1}} \left| \frac{S_N(f)}{A_N} - \frac{S_{N_j}(f)}{A_{N_j}} \right| + \left| \frac{S_{N_j}(f)}{A_{N_j}} \right|. \end{aligned} \tag{5}$$

To deal with (4), we introduce for any $k \geq 1$ the following random variable

$$X_k = \frac{S_{k-1}(f)}{A_{k-1}} - \frac{S_k(f)}{A_k}.$$

Notice that for any $n, m \in [\![N_j, N_{j+1}]\!]$ with $m < n$,

$$\mathbb{E}\left| \sum_{k=m+1}^{n} X_k \right|^p = \left|\left| \frac{S_m(f)}{A_m} - \frac{S_n(f)}{A_n} \right|\right|_{p,\mu}^p,$$

The main idea is to bound $\left|\left| \frac{S_m(f)}{A_m} - \frac{S_n(f)}{A_n} \right|\right|_{p,\mu}^p$ by $Cg^\lambda(m,n)$ for $\lambda > 1$ and for some sub-additive function g. Indeed, if we do it, then according to Móricz's lemma we have

$$\mathbb{E}\left(\sup_{N=N_j+1}^{N_{j+1}} \left| \frac{S_N(f)}{A_N} - \frac{S_{N_j}(f)}{A_{N_j}} \right|^p \right) \leq Cg^\lambda(N_j, N_{j+1}).$$

By taking the sum over j, and by applying Beppo Levi theorem, this proves (4). Before starting the proofs, we notice that, trivially, for $2 \leq m < n$ and for $A_N = N^\alpha \log^\beta N$, where $0 \leq \alpha \leq 1$ and $\beta \in \mathbb{R}$, we have

$$\left(\frac{1}{A_m} - \frac{1}{A_n}\right) \leq C \frac{(n-m)^\alpha}{n^\alpha \times m^\alpha \log^\beta m}. \tag{6}$$

Proof of Theorem 1. Let $N_j = [j^{\frac{1}{1-\alpha}}]$. Notice that, for any $m, n \in [\![N_j, N_{j+1}]\!]$ with $m < n$, we have $m < n \leq 4^{\frac{1}{1-\alpha}} m$. First, we deal with (3) for $A_N = N^{\frac{(p-1)\alpha+1}{p}} \log^H N$. It follows from the assumption of Theorem 1, that

$$\left|\left| \frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}} \log^H N_j} \right|\right|_{p,\mu}^p \leq \frac{1}{N_j^{1-\alpha} \log^{p(H-\beta)} N_j}.$$

Since $H > \frac{1}{p} + \beta$, this implies

$$\sum_{j \geq 1} \left|\left| \frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}} \log^H N_j} \right|\right|_{p,\mu}^p < \infty.$$

Consequently

$$\lim_{j \to \infty} \frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}} \log^H N_j} = 0 \quad \mu - a.e..$$

Now, we have to prove that the limit of the oscillation as defined in (4) is equal to 0 μ-a.e.. For any $f \in L^p(\mu)$, we write

$$\left|\left| \frac{S_m(f)}{A_m} - \frac{S_n(f)}{A_n} \right|\right|_{p,\mu}^p \leq C \left(\frac{1}{A_m} - \frac{1}{A_n}\right)^p ||S_m(f)||_{p,\mu}^p + C \frac{1}{A_n^p} ||S_n(f) - S_m(f)||_{p,\mu}^p .$$

Since (w_k) is a bounded sequence of complex numbers and $||T^{u_k}(f)||_p \leq ||f||_p$ this gives that $||S_n(f) - S_m(f)||_{p,\mu}^p \leq C||f||_p^p (n-m)^p$. According to (6) and to the assumption of theorem, we have

$$\begin{aligned}
\left|\left| \frac{S_m(f)}{A_m} - \frac{S_n(f)}{A_n} \right|\right|_{p,\mu}^p &\leq C \frac{(n-m)^{(p-1)\alpha+1} m^{p\alpha} \log^{p\beta} m}{m^{(p-1)\alpha+1} n^{(p-1)\alpha+1} \log^{pH} m} + C \frac{(n-m)^p}{n^{(p-1)\alpha+1} \log^{pH} n} \\
&\leq C \frac{(n-m)^{(p-1)\alpha+1}}{m^{2(p-1)\alpha+2-p\alpha} \log^{p(H-\beta)} m} + C \frac{(n-m)^p}{n^{(p-1)\alpha+1} \log^{pH} n} \\
&\leq C \frac{(n-m)^p}{m^{(p-1)\alpha+1} \log^{p(H-\beta)} m},
\end{aligned}$$

where the last inequality comes from the fact that $(p-1)\alpha+1 < p$ and $(p-1)\alpha+1 \leq 2(p-1)\alpha+2-p\alpha$. Therefore, for any m,n belonging in $[\![N_j, N_{j+1}]\!]$ with $m<n$, we have

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq C\left(\frac{(n-m)}{m^{\frac{(p-1)\alpha+1}{p}}\log^{H-\beta} m}\right)^p.$$

Applying Móricz's lemma with $\lambda = p$ and $g(m,n) = \frac{n-m}{m^{\frac{(p-1)\alpha+1}{p}}\log^{H-\beta} m}$, we obtain

$$\mathbb{E}\left(\sup_{N=N_j+1}^{N_{j+1}}\left|\frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}}\log^H N_j} - \frac{S_N(f)}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N}\right|^p\right) \leq C g^p(N_j, N_{j+1}).$$

Since, $N_{j+1}-N_j \leq C j^{\frac{1}{1-\alpha}-1}$. Therefore

$$g^p(N_j, N_{j+1}) \leq \frac{C}{j\log^{p(H-\beta)} j}.$$

By assumption we have $H > \frac{1}{p}+\beta$, then the last term is a term of a convergent series . By Beppo Levi's theorem, we have

$$\lim_{j\to\infty}\sup_{N=N_j+1}^{N_{j+1}}\left|\frac{S_N(f)}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N} - \frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}}\log^H N_j}\right| = 0 \quad \mu-a.e..$$

Now, we prove the strong maximal inequality. To do it, we use (5), and we take the supremum over j. This gives

$$\sup_{N>1}\left|\frac{1}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N}\sum_{k=0}^{N-1} w_k T^{u_k}(f)\right|^p \leq$$
$$C\sup_{j\geq 1}\sup_{N=N_j+1}^{N_{j+1}}\left|\frac{S_N(f)}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N} - \frac{S_{N_j}(f)}{N_j^{\frac{(p-1)\alpha+1}{p}}\log^H N_j}\right|^p + C\sup_{j\geq 1}\left|\frac{S_{N_j}(f)}{N_j^{\frac{(p-1)+\alpha+1}{p}}\log^H N_j}\right|^p.$$

Integrating over μ, we get

$$\left\|\sup_{N>1}\left|\frac{1}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N}\sum_{k=0}^{N-1} w_k T^{u_k}(f)\right|\right\|_{p,\mu}^p \leq C\sum_{j\geq 1} g^p(N_j, N_{j+1})$$
$$+C\sum_{j\geq 1}\frac{1}{N_j^{1-\alpha}\log^{p(H-\beta)} N_j}.$$

This concludes the proof of the first statement of Theorem 1. The proof of the second statement can be accomplished in a similar way as the first. □

Proof of Corollary 1. Let $1 < p \leq 2$ and let $0 < t \leq 1$, such that $\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$.

For any $f \in L^1(\mu)$, we have

$$||S_n(f)||_{1,\mu} \leq n||f||_{1,\mu}$$

and by the spectral Lemma, we have, for any $f \in L^2(\mu)$

$$||S_n(f)||_{2,\mu} \leq Cn^{\alpha} \log^{\beta} n||f||_{2,\mu}.$$

Now using the Riesz-Thorin interpolation theorem [4] between $L^1(\mu)$ and $L^2(\mu)$, we obtain

$$||S_n(f)||_{p,\mu} \leq Cn^{\frac{(2-p)+2\alpha(p-1)}{p}} \log^{2\beta\frac{p-1}{p}} n||f||_{p,\mu}.$$

It follows from Theorem 1 that if $H > 2\beta\frac{p-1}{p} + \frac{1}{p}$, we have the desired result.

Now, let $2 < p < \infty$ and take $0 < t < 1$ such that $\frac{1}{p} = \frac{1-t}{2}$. Notice that for any $f \in L^{\infty}(\mu)$, we have

$$||S_n(f)||_{\infty} \leq n||f||_{\infty}.$$

Hence, performing interpolation between $L^2(\mu)$ and $L^{\infty}(\mu)$, we easily obtain

$$||S_n(f)||_{p,\mu} \leq Cn^{\frac{(p-2)+2\alpha}{p}} \log^{\frac{2\beta}{p}} n||f||_{p,\mu}.$$

According to Theorem 1, the desired result holds if $H > \frac{2\beta}{p} + \frac{1}{p}$. □

Proof of Corollary 2. By Abel's summation formula, we have

$$\begin{aligned}
&\sum_{k=2}^{N-1} \frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}} \log^H k} T^{u_k}(f) \\
&= \sum_{k=2}^{N-2} \left(\frac{1}{k^{\frac{(p-1)\alpha+1}{p}} \log^H k} - \frac{1}{(k+1)^{\frac{(p-1)\alpha+1}{p}} \log^H (k+1)} \right) \sum_{j=2}^{k} w_j T^{u_j}(f) \\
&\qquad + \frac{1}{N^{\frac{(p-1)\alpha+1}{p}} \log^H N} \sum_{k=2}^{N-1} w_k T^{u_k}(f).
\end{aligned}$$

According to Theorem 1, the last term of the right-hand side converges to 0 μ-a.e.. Moreover, according to Lemma 1 and the fact that

$$\left| \frac{1}{k^{\frac{(p-1)\alpha+1}{p}} \log^H k} - \frac{1}{(k+1)^{\frac{(p-1)\alpha+1}{p}} \log^H (k+1)} \right| \leq C \frac{1}{k^{\frac{(p-1)\alpha+1}{p}+1} \log^H k},$$

we obtain

$$\left|\left|\sum_{k>1}\left(\frac{1}{k^{\frac{(p-1)\alpha+1}{p}}\log^H k}-\frac{1}{(k+1)^{\frac{(p-1)\alpha+1}{p}}\log^H(k+1)}\right)\left|\sum_{j=2}^{k}w_jT^{u_j}(f)\right|\right|\right|_{p,\mu}$$
$$\leq C\sum_{k>1}\frac{1}{k^{\frac{(p-1)\alpha+1}{p}+1}\log^H k}\left|\left|\sum_{j=1}^{k}w_jT^{u_j}(f)\right|\right|_{p,\mu}$$
$$\leq C||f||_{p,\mu}\sum_{k>1}\frac{1}{k^{\frac{1-\alpha}{p}+1}\log^{H-\beta}k}.$$

The last term is finite because $\alpha<1$. According to Beppo Levi's theorem, this proves that $\sum_{k>1}\frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}}\log^H k}T^{u_k}(f)$ exists μ-a.e.. Moreover,

$$\left|\left|\sup_{N>1}\left|\sum_{k=2}^{N}\frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}}\log^H k}T^{u_k}(f)\right|\right|\right|_{p,\mu}$$
$$\leq\left|\left|\sum_{k>1}\left(\frac{1}{k^{\frac{(p-1)\alpha+1}{p}}\log^H k}-\frac{1}{(k+1)^{\frac{(p-1)\alpha+1}{p}}\log^H(k+1)}\right)\left|\sum_{j=2}^{k}w_jT^{u_j}(f)\right|\right|\right|_{p,\mu}$$
$$+\left|\left|\sup_{N>1}\left|\frac{1}{N^{\frac{(p-1)\alpha+1}{p}}\log^H N}\sum_{k=2}^{N-1}w_kT^{u_k}(f)\right|\right|\right|_{p,\mu}.$$

Therefore

$$\left|\left|\sup_{N>1}\left|\sum_{k=2}^{N}\frac{w_k}{k^{\frac{(p-1)\alpha+1}{p}}\log^H k}T^{u_k}(f)\right|\right|\right|_{p,\mu}\leq C||f||_{p,\mu}.$$

□

Proof of Theorem 2. Here $A_N=\frac{N}{\log^H N}$. Choosing $N_j=2^j$ and following the same lines as in the proof of Theorem 1, we obtain for $\beta>\frac{1}{p}+H$

$$\sum_{j\geq1}\left|\left|\frac{\log^H N_jS_{N_j}(f)}{N_j}\right|\right|_{p,\mu}^p\leq C\sum_{j\geq1}\frac{1}{j^{p(\beta-H)}}<\infty,$$

which showed (3). It remains to prove (4), to do it, we write, for any m,n in $[\![N_j,N_{j+1}]\!]$, with $m<n$ and for any $f\in L^p(\mu)$,

$$\left|\left|\frac{\log^H mS_m(f)}{m}-\frac{\log^H nS_n(f)}{n}\right|\right|_{p,\mu}^p$$
$$\leq C\,||S_m(f)||_{p,\mu}^p\left|\frac{\log^H m}{m}-\frac{\log^H n}{n}\right|^p+C\frac{\log^{pH}n}{n^p}\,||S_n(f)-S_m(f)||_{p,\mu}^p\,.$$

We now fix $a > \frac{p}{p-1} > 1$. Using the assumption of theorem, we obtain

$$||S_m(f)||_{p,\mu}^p \left| \frac{\log^H m}{m} - \frac{\log^H n}{n} \right|^p \leq C \frac{(n-m)^p}{m^p \log^{p(\beta-H)} m} \leq C \frac{(n-m)^{\frac{p(a-1)}{a}} m^{\frac{p}{a}}}{m^p \log^{p(\beta-H)} m},$$

and

$$\begin{aligned} ||S_n(f) - S_m(f)||_{p,\mu}^p &\leq ||S_n(f) - S_m(f)||_{p,\mu}^{\frac{p}{a}} ||S_n(f) - S_m(f)||_{p,\mu}^{\frac{p(a-1)}{a}} \\ &\leq \frac{n^{\frac{p}{a}}}{\log^{\frac{p\beta}{a}} m} (n-m)^{\frac{p(a-1)}{a}}. \end{aligned}$$

Hence

$$\left|\left| \frac{\log^H m S_m(f)}{m} - \frac{\log^H n S_n(f)}{n} \right|\right|_{p,\mu}^p \leq C \frac{(n-m)^{\frac{p(a-1)}{a}}}{m^{\frac{p(a-1)}{a}} \log^{p(\frac{\beta}{a}-H)} m}.$$

By Móricz's lemma applied to $\lambda = \frac{p(a-1)}{a}$ and $g(m,n) = \frac{n-m}{m \log^{\frac{\beta - aH}{a-1}} m}$, we obtain

$$\sum_{j \geq 1} \mathbb{E} \left(\sup_{N=N_j+1}^{N_{j+1}} \left| \frac{\log^H N_j S_{N_j}(f)}{N_j} - \frac{\log^H N S_N(f)}{N} \right|^p \right) \leq C \sum_{j \geq 1} \frac{1}{\log^{p(\frac{\beta}{a}-H)} N_j},$$

which is finite when $N_j = 2^j$ and $\beta > \frac{1}{p-1} + \frac{pH}{p-1}$. The Theorem now follows from the Beppo Levi's theorem. □

Proof of Corollary 3. We proceed as in the proof of Corollary 1. Notice that, $||S_n(f)||_{p,\mu} \leq n||f||_{p,\mu}$ for any $f \in L^p(\mu)$ and by the spectral Lemma, we have for any $f \in L^2(\mu)$

$$||S_n(f)||_{2,\mu} \leq C \frac{n}{\log^\beta n} ||f||_{2,\mu}.$$

Hence, performing interpolation between $L^1(\mu)$ and $L^2(\mu)$ on the on hand and between $L^2(\mu)$ and $L^\infty(\mu)$ on the other hand and using Theorem 2, we can easily deduce the proof. □

Proof of Corollary 4. The proof follows from Abel's summation formula and Theorem 2. □

4 Appications

In this section, we present several examples illustrating our main results.

Example 1. *Let $1 < p < \infty$. Let $d \in \mathbb{N}^*$ and $\delta > d$. In what follows, we let $||\delta|| = \min(\{\delta\}, 1 - \{\delta\})$, where $\{\delta\}$ denotes the fractional part of δ. As a first example, under*

suitable conditions over K which will be given later, we obtain that, for any Dunford-Schwartz operator T, any $f \in L^p(\mu)$ and any $H > \frac{1}{p}$, if $1 < p \leq 2$,

$$\sum_{k>1} \frac{e^{2i\pi k^\delta}}{k^{1-||\delta||\left(\frac{p-1}{p}\right)^2 \frac{4}{3(K-2)}} \log^H k} T^{k^d}(f) \tag{7}$$

exists $\mu - a.e.$. If $2 < p < \infty$,

$$\sum_{k>1} \frac{e^{2i\pi k^\delta}}{k^{1-||\delta||\frac{p-1}{p^2}\frac{4}{3(K-2)}} \log^H k} T^{k^d}(f) \tag{8}$$

exists $\mu - a.e.$. Before proving this, we notice that Krause, Lacey and Wierdl [5] in their study of the convergence of oscillatory ergodic Hilbert transforms showed that, for any $f \in L^r(\mu)$, $1 \leq r < \infty$, the limit

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{e^{2i\pi h(k)}}{k} T^k(f)$$

exists μ-a.e., where $h(k)$ is a Hardy field function. Recall that (see Section 3 in [5]) a function of the form $k^\delta = e^{\delta \log k}$ is a Hardy field function. In that case, our result is more precise than [5].

Now, we prove that (7) and (8) exist $\mu - a.e.$. According to the proof of Corollary 1 and to Corollary 2 it is sufficient to prove that

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{N} e^{2i\pi(k^\delta + \theta k^d)} \right| \leq C N^{1-||\delta||\frac{2}{3(K-2)}}, \tag{9}$$

where $K = 2^n$ with $n - 1 < \delta < n$.

In what follows, we consider the function $g(x) = \theta x^d + x^\delta$. For any $x \in [N^{1-\epsilon}, N]$, where ϵ is a positive constant which will be defined later, we have

$$g^{(n)}(x) = \delta(\delta - 1)\dots(\delta - (n-1))x^{\delta - n}.$$

Consequently,

$$\frac{C(n,\delta)}{N^{n-\delta}} \leq f^{(n)}(x) \leq \frac{C(n,\delta)}{N^{(1-\epsilon)(n-\delta)}},$$

where $C(n,\delta) = \delta(\delta-1)\dots(\delta-(n-1))$. According to Lemma 4 with $\lambda = \frac{C(n,\delta)}{N^{n-\delta}}$ and $h = N^{\epsilon(n-\delta)}$, we get

$$\left| \sum_{k=\lfloor N^{1-\epsilon} \rfloor}^{N} e^{2i\pi(k^\delta + \theta k^d)} \right| \leq CN \left(\frac{1}{N^{\frac{n-\delta}{K-2} - \epsilon(n-\delta)}} + \frac{1}{N^{\frac{2}{K} - \epsilon(n-\delta)}} + \frac{1}{N^{\frac{2\delta}{K} - \epsilon(n-\delta)}} \right).$$

Now, choosing $\epsilon = \frac{1}{3(K-2)}$ *and using the fact that* $\delta > 1$*, we have*

$$\left|\sum_{k=\lfloor N^{1-\epsilon}\rfloor}^{N} e^{2i\pi(k^\delta+\theta k^d)}\right| \leq CN\left(\frac{1}{N^{||\delta||\left(\frac{1}{K-2}-\epsilon\right)}} + \frac{1}{N^{\frac{2}{K}-\epsilon(n-\delta)}}\right)$$
$$\leq CN\left(\frac{1}{N^{||\delta||\frac{2}{3(K-2)}}} + \frac{1}{N^{\frac{6-n+\delta}{3(K-2)}}}\right).$$

Since $||\delta|| \leq \frac{1}{2}$ *and* $\delta > n-1$*, we obtain*

$$\left|\sum_{k=\lfloor N^{1-\epsilon}\rfloor}^{N} e^{2i\pi(k^\delta+\theta k^d)}\right| \leq CN\left(\frac{1}{N^{||\delta||\frac{2}{3(K-2)}}}\right).$$

It follows that

$$\left|\sum_{k=1}^{N} e^{2i\pi(k^\delta+\theta k^d)}\right| \leq \left|\sum_{k=1}^{\lfloor N^{1-\epsilon}\rfloor-1} e^{2i\pi(k^\delta+\theta k^d)}\right| + \left|\sum_{k=\lfloor N^{1-\epsilon}\rfloor}^{N} e^{2i\pi(k^\delta+\theta k^d)}\right|$$
$$\leq C\left(N^{1-\frac{1}{3(K-2)}} + N^{1-||\delta||\frac{2}{3(K-2)}}\right)$$
$$\leq CN^{1-||\delta||\frac{2}{3(K-2)}}.$$

This proves (9)*.*

Notice that (9) *and Corollary 1 also imply that, for any Dunford-Schwartz operator* T*, any* $f \in L^p(\mu)$ *and any* $H > \frac{1}{p}$*,*

$$\lim_{N\to\infty} \frac{1}{N^\Gamma \log^H N} \sum_{k=0}^{N-1} e^{2i\pi k^\delta} T^{k^d}(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N>1}\left|\frac{\sum_{k=0}^{N-1} e^{2i\pi k^\delta} T^{k^d}(f)}{N^\Gamma \log^H N}\right| \in L^p(\mu),$$

where $\Gamma = \begin{cases} 1 - ||\delta||\left(\frac{p-1}{p}\right)^2 \frac{4}{3(K-2)} & \text{if} \quad 1 < p \leq 2 \\ 1 - ||\delta||\frac{p-1}{p^2}\frac{4}{3(K-2)} & \text{if} \quad 2 < p < \infty \end{cases}$.

Example 2. *In this example, we take* $u_k = k$ *and* $w_k = e^{2i\pi k^\delta}$ *for some* $1 > \delta > 0$*. First we prove the following inequality*

$$\sup_{\theta\in\mathbb{R}}\left|\sum_{k=1}^{N} e^{2i\pi(k^\delta+\theta k)}\right| \leq CN^{1-\frac{\delta}{2}}. \tag{10}$$

To do it, the main idea is to apply the van der Corput theorem. Let $a = \sqrt{N}$*,* $b = N$ *and* $f(x) = \theta x + x^\delta$*. In particular, we have*

$$f'(x) = \theta + \delta x^{\delta-1} \quad \text{and} \quad f''(x) = \delta(\delta-1)x^{\delta-2}.$$

Moreover for any $x \in [\sqrt{N}, N]$, we have

$$-f''(x) \geq CN^{\delta-2} := \rho.$$

According to Lemma 3, this gives

$$\begin{aligned}\left|\sum_{k=1}^{N} e^{2i\pi(k^\delta+\theta k)}\right| &\leq \left|\sum_{k=1}^{\lfloor\sqrt{N}\rfloor} e^{2i\pi(k^\delta+\theta k)}\right| + \left|\sum_{k=\lfloor\sqrt{N}\rfloor}^{N} e^{2i\pi(k^\delta+\theta k)}\right| \\ &\leq \sqrt{N} + \left(|\delta N^{\delta-1} - \delta N^{\frac{\delta-1}{2}}| + 2\right)\left(\frac{C}{N^{\frac{\delta-2}{2}}} + 3\right) \\ &\leq CN^{1-\frac{\delta}{2}}.\end{aligned}$$

This proves Equation (10)*. Now, according to Corollary 1, for any Dunford-Schwartz operator T on $L^1(\mu)$, any $f \in L^p(\mu)$ and for $H > \frac{1}{p}$, we have*

$$\lim_{N\to\infty} \frac{1}{N^\Gamma \log^H N} \sum_{k=1}^{N} e^{2i\pi k^\delta} T^k(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N>1} \left|\frac{\sum_{k=1}^{N} e^{2i\pi k^\delta} T^k(f)}{N^\Gamma \log^H N}\right| \in L^p(\mu),$$

where $\Gamma = \begin{cases} 1 - \left(\frac{p-1}{p}\right)^2 \delta & \text{if} \quad 1 < p \leq 2 \\ 1 - \frac{p-1}{p^2}\delta & \text{if} \quad 2 < p < \infty \end{cases}$.

Moreover, by Corollary 2, the series

$$\sum_{k>1} \frac{e^{2i\pi k^\delta}}{k^\Gamma \log^H k} T^k(f) \text{ exists } \mu - a.e. \quad \& \quad \sup_{N>1} \left|\sum_{k=2}^{N} \frac{e^{2i\pi k^\delta}}{k^\Gamma \log^H k} T^k(f)\right| \in L^p(\mu).$$

Notice that our results also improve [5].

As a remark, if we take $w_k = e^{2i\pi \log^\delta k}$ for some $\delta > 2$. Similarly to the proof of (10)*, a simple calculation yields*

$$\sup_{\theta\in\mathbb{R}} \left|\sum_{k=1}^{N} e^{2i\pi(\log^\delta k+\theta k)}\right| \leq C\frac{N}{\log^{\frac{\delta-1}{2}} N}.$$

Then we can apply Corollaries 3 and 4.

Example 3. *Here we deal with an example which was considered in [15, 23]. Let (X_k) be a sequence of i.i.d random variables defined on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and let $W_k = e^{2i\pi X_k}$ such that $\mathbb{E}(W_k) = 0$. Let (u_k) be an increasing sequence of integers such that $u_k = O\left(e^{\frac{k}{\log^{2\beta} k}}\right)$ for some $1 < \beta < \frac{3}{2}$.*

According to Theorem 1.1 in [15] in the one-dimensional setting, there exists some random variable $C(\omega)$ which is almost surely (a.s.) finite and independent of M, N and θ such that,

for almost every $\omega \in \Omega$,

$$\sup_{\theta\in\mathbb{R}}\left|\sum_{k=M+1}^{N} W_k(\omega)e^{2i\pi\theta u_k}\right| \leq C(\omega)\sqrt{\sum_{k=M+1}^{N}|W_k|^2}\times\sqrt{\log u_N}$$
$$\leq C(\omega)\sqrt{N-M}\times\frac{\sqrt{N}}{\log^{\beta}N}.$$

Theorem 2 applied to $p=2$ *ensures that there exists a measurable set* $\Omega^*\subset\Omega$ *of full measure, such that for any* $\omega\in\Omega^*$, *any contraction* T *on* $L^2(\mu)$ *and any* $f\in L^2(\mu)$, *if* $0<H<\frac{\beta-1}{2}$, *then*

$$\lim_{N\to\infty}\frac{\log^H N}{N}\sum_{k=0}^{N-1}W_k(\omega)T^{u_k}(f)=0 \quad \mu-a.e. \text{ and } \sup_{N\geq 1}\left|\frac{\log^H N}{N}\sum_{k=0}^{N-1}W_k(\omega)T^{u_k}(f)\right|\in L^2(\mu). \tag{11}$$

Moreover, by Corollary 4, we also obtain that

$$\sum_{k\geq 1}\frac{W_k(\omega)\log^H k}{k}T^{u_k}(f) \quad exists \quad \mu-a.e..$$

A weak version of (11) *could be derived from a theorem due to Weber. Indeed, according to Theorem 4.2 in [23] with* $\phi(x)=\frac{x}{\log^{2\beta}x}$, *for any* $\tau>\frac{3}{2}$, *there exists a measurable set* $\Omega^*\subset\Omega$ *of full measure, such that for any* $\omega\in\Omega^*$, *any contraction* T *on* $L^2(\mu)$ *and any* $f\in L^2(\mu)$,

$$\lim_{N\to\infty}\frac{1}{N\log^{\tau-\beta}N}\sum_{k=0}^{N-1}W_k(\omega)T^{u_k}(f)=0 \quad \mu-a.e.,$$

and

$$\left|\left|\sup_{N>1}\left|\frac{1}{N\log^{\tau-\beta}N}\sum_{k=0}^{N-1}W_k(\omega)T^{u_k}(f)\right|\right|\right|_{2,\mu}\leq C(\omega)\,||f||_{2,\mu}\,.$$

However, (11) *is more precise.*

Example 4. *Let* $(p_k)_{k\geq 1}$ *be a squence of real numbers such that*

$$0\leq p_k\leq 1, \quad \sum_{k=1}^{\infty}p_k=\infty, \quad p_k\downarrow 0, \quad N^{\alpha}=O\left(\sum_{k=1}^{N}p_k\right) \quad \& \quad \sup_{k\geq 1}\frac{k(p_k-p_{k+1})}{p_k}=O(1). \tag{12}$$

where $\alpha>\frac{1}{2}$. *Let* (X_k) *be a sequence of independent Bernoulli random variables on some probability space* $(\Omega,\mathcal{B},\mathbb{P})$ *such that* $\mathbb{P}(X_k=1)=p_k=1-\mathbb{P}(X_k=0)$. *In this example, we*

consider the random set of integers $\mathcal{N}(\omega) := \{k \geq 1 : X_k(\omega) = 1\}$. *In particular, we have* $\mathcal{N}(\omega) = \{u_k(\omega), k \geq 1\}$, *where*

$$u_1(\omega) = \inf\{i \geq 1 : X_i(\omega) = 1\} \quad \textit{and} \quad u_{k+1}(\omega) = \inf\{i > u_k(\omega) : X_i(\omega) = 1\}.$$

In what follows we denote by

$$\Pi(N) = \#\mathcal{N} \cap [1, N] = \sum_{k=1}^{N} X_k.$$

The above random variable satisfies the following law of large numbers: a.s. $\lim_{N\to\infty} \frac{\Pi(N)}{\sum_{k=1}^{N} p_k} = 1$ *(see Proposition 6.2. in [19]). Our purpose in this example is to give an estimate of Davenport type. More precisely, for the Möbius sequence, we prove the following estimate.*

Proposition 1. *Almost surely*

$$\sum_{k=1}^{N} |\mu(u_k)| \sim CN \tag{13}$$

and, there exists some random variable $C'(\omega)$ *which is almost surely finite and independent of* N *and* θ *such that, for almost every* $\omega \in \Omega$,

$$\forall\, \beta > 0, \exists\, C_\beta : \sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{N} \mu(u_k) e^{2i\pi u_k \theta} \right| \leq C'(\omega) C_\beta \frac{N}{\log^\beta N}. \tag{14}$$

Proof. To deal with (13), we write

$$\frac{\sum_{k=1}^{N} |\mu(u_k)|}{N} = \frac{\sum_{k=1}^{u_N} X_k |\mu(k)|}{N} = \frac{\sum_{k=1}^{u_N} (X_k - p_k)|\mu(k)|}{N} + \frac{\sum_{k=1}^{u_N} p_k |\mu(k)|}{N}. \tag{15}$$

Now, let $s_N = \sum_{k=1}^{N} p_k$ with $s_0 = p_1$ and consider the martingale

$$M_N = \sum_{k=1}^{N} \frac{(X_k - p_k)|\mu(k)|}{s_k}.$$

By computation we easily have

$$\mathbb{E}M_N^2 \leq \sum_{k=1}^{N} \frac{p_k(1-p_k)}{s_k^2} \leq \sum_{k=1}^{N} \frac{p_k}{s_k s_{k-1}} = \frac{1}{p_1} + \sum_{k=2}^{N} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k} \right) \leq \frac{2}{p_1}.$$

It follows from the Doob's martingale convergence theorem, that the martingale M_N converges almost surely. Then, using the Kronecker's lemma, we infer that, almost surely

$$\lim_{N\to\infty} \frac{1}{s_N} \sum_{k=1}^{N} (X_k - p_k)|\mu(k)| = 0.$$

Since $s_{u_N} \sim \Pi(u_N) = N$, we conclude that, almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{u_N} (X_k - p_k)|\mu(k)| = 0.$$

According to (15), we have, almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\mu(u_k)| = \lim_{N\to\infty} \frac{1}{s_{u_N}} \sum_{k=1}^{u_N} p_k|\mu(k)|$$

It is well-known (see e.g. p.6 of [8]) that $\sum_{k=1}^{N} |\mu(k)| \sim \frac{6}{\pi^2} N$. Hence $\sum_{k=1}^{N} p_k|\mu(k)| \sim \frac{6}{\pi^2} s_N$. Therefore $\sum_{k=1}^{u_N} p_k|\mu(k)| \sim \frac{6}{\pi^2} N$. This concludes the proof of (13).

To deal with (14), we write

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{N} \mu(u_k) e^{2i\pi u_k \theta} \right| \leq \sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{u_N} (X_k - p_k)\mu(k) e^{2i\pi k\theta} \right| + \sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{u_N} p_k \mu(k) e^{2i\pi k\theta} \right|.$$

It follows from Theorem 1.1 in [15] in the one-dimensional setting, that there exists some random variable $C'(\omega)$ which is almost surely finite and independent of N and θ such that, for almost every $\omega \in \Omega$,

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{u_N} (X_k - p_k)\mu(k) e^{2i\pi k\theta} \right| \leq C'(\omega) \sqrt{u_N \log(u_N)}.$$

By induction, we have $u_N \leq C N^{\frac{1}{\alpha}}$. Hence $\sqrt{u_N \log(u_N)} \leq C N^{\frac{1}{2\alpha}} \log^{\frac{1}{2}} N$. Furthermore, for any $\beta > 0$, there exists $C_1(\beta) > 0$ such that $N^{\frac{1}{2\alpha}} \log^{\beta+\frac{1}{2}} N \leq C_1(\beta) N$, consequently

$$\sqrt{u_N \log(u_N)} \leq C_1(\beta) \frac{N}{\log^{\beta} N}.$$

It remains to prove that, for any $\beta > 0$, there exists $C_2(\beta) > 0$ such that

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{k=1}^{u_N} p_k \mu(k) e^{2i\pi k\theta} \right| \leq C_2(\beta) \frac{N}{\log^{\beta} N}. \tag{16}$$

To do it, we use the Abel's summation

$$\frac{\log^{\beta} N}{N} \sum_{k=1}^{u_N} p_k \mu(k) e^{2i\pi k\theta} = \frac{\log^{\beta} N}{N} \sum_{k=1}^{u_N - 1} (p_k - p_{k+1}) \sum_{j=1}^{k} \mu(j) e^{2i\pi j\theta} + \frac{p_{u_N} \log^{\beta} N}{N} \sum_{k=1}^{u_N} \mu(k) e^{2i\pi k\theta}.$$

This together with the Davenport estimate, implies that, there exists $C_3(\beta) > 0$ such that

$$\frac{\log^{\beta} N}{N} \left| \sum_{k=1}^{u_N} p_k \mu(k) e^{2i\pi k\theta} \right| \leq C_3(2\beta) \frac{\log^{\beta} N}{N} \sum_{k=2}^{u_N - 1} (p_k - p_{k+1}) \frac{k}{\log^{2\beta} k} + C_3(\beta) \frac{p_{u_N} u_N \log^{\beta} N}{N \log^{\beta}(u_N)}.$$

Since $p_N \downarrow 0$ as $N \to \infty$ and $\sum_{k=1}^{u_N} p_k \sim \Pi(u_N) = N$, we have $p_{u_N} u_N \leq \sum_{k=1}^{u_N} p_k \sim N$. Hence $\frac{p_{u_N} u_N \log^\beta N}{N \log^\beta(u_N)} \leq C$.

By assumption, we have $\sup_{k\geq 1} \left(\frac{(p_k - p_{k+1})k}{p_k} \right) \leq C$. Then

$$\frac{\log^\beta N}{\Pi(N)} \sum_{k=2}^{N} \frac{(p_k - p_{k+1})k}{\log^{2\beta} k} \leq C \frac{\log^\beta N}{\Pi(N)} \sum_{k=2}^{N} \frac{p_k}{\log^{2\beta} k}.$$

Moreover, by Abel's summation and the fact that $\frac{1}{\log^{2\beta}(k)} - \frac{1}{\log^{2\beta}(k+1)} \leq C \frac{1}{k \log^{2\beta+1} k}$, we obtain

$$\frac{\log^\beta N}{\Pi(N)} \sum_{k=2}^{N} \frac{p_k}{\log^{2\beta} k} \leq \frac{\log^\beta N}{\Pi(N)} \sum_{k=2}^{N-1} \frac{1}{k \log^{2\beta+1} k} \sum_{j=1}^{k} p_j + \frac{\sum_{k=2}^{N} p_k}{\Pi(N) \log^\beta N}.$$

Since $\frac{1}{k \log^{\beta+1} k}$ is a term of convergent series and $\frac{\sum_{k=1}^{N} p_k}{\log^\beta N} \uparrow \infty$ as $N \to \infty$, we have, by the Kronecker's lemma and using the fact that $\Pi(N) \sim \sum_{k=1}^{N} p_k$,

$$\frac{\log^\beta N}{\Pi(N)} \sum_{k=2}^{N} \frac{p_k}{\log^{2\beta} k} \leq C.$$

Which implies that

$$\frac{\log^\beta N}{N} \sum_{k=2}^{u_N} \frac{p_k}{\log^{2\beta} k} \leq C.$$

This proves the estimate in (16) and finishes the proof of (14). □

Now, applying Corollary 3 in this context, we obtain that, there exists a measurable set $\Omega^* \subset \Omega$ of full measure, such that for any $\omega \in \Omega^*$, any Dunford-Schwartz operator T on $L^1(\mu)$ and for any $f \in L^p(\mu)$, if $H > 0$, we have

$$\lim_{N\to\infty} \frac{\log^H N}{N} \sum_{k=0}^{N-1} \mu(u_k) T^{u_k}(f) = 0 \quad \mu - a.e. \quad \& \quad \sup_{N\geq 1} \left| \frac{\log^H N}{N} \sum_{k=0}^{N-1} \mu(u_k) T^{u_k}(f) \right| \in L^p(\mu).$$

Example 5. *Let $1 < p < \infty$. Let (X_k) be a sequence of independent Bernoulli random variables on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that $\mathbb{P}(X_k = 1) = \frac{1}{\log k} = 1 - \mathbb{P}(X_k = 0)$. In this example, we consider the random set of integers $\mathcal{N}(\omega) := \{k \geq 2 : X_k(\omega) = 1\}$. This set is referred to as the Cramer's random model of primes. In particular, we have $\mathcal{N}(\omega) = \{u_k(\omega), k \geq 1\}$, where*

$$u_1(\omega) = \inf\{i \geq 2 : X_i(\omega) = 1\} \quad \text{and} \quad u_{k+1}(\omega) = \inf\{i > u_k(\omega) : X_i(\omega) = 1\}.$$

In what follows we denote by

$$\Pi(N) = \#\mathcal{N} \cap [2, N] = \sum_{k=2}^{N} X_k.$$

The above random variable satisfies the following law of large numbers: $\lim_{N\to\infty} \frac{\Pi(N)\log N}{N} = 1$ *(see Proposition 6.2. in [19]). We deal with below the almost everywhere convergence of the following average*

$$\frac{1}{N^\beta}\sum_{k=1}^{N} T^{u_k}(f) = \frac{1}{\Pi(u_N)^\beta}\sum_{k=2}^{u_N} X_k T^k(f) \qquad \left(\frac{p-1}{p} < \beta \leq 1\right).$$

To do it, it is sufficient to deal with the almost everywhere convergence of

$$\frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N} X_k T^k(f).$$

First we write

$$\frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N} X_k T^k(f) = \frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N}\left(X_k - \frac{1}{\log k}\right)T^k(f) + \frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N}\frac{T^k(f)}{\log k}.$$

According to Theorem 1.1 in [15] in the one-dimensional setting, there exists some random variable $C(\omega)$ *which is* $\mathbb{P}-a.s.$ *finite and independent of* M, N *and* θ *such that, almost surely*

$$\begin{aligned}\sup_{\theta\in\mathbb{R}}\left|\sum_{k=M}^{N}\left(X_k - \frac{1}{\log k}\right)e^{2i\pi\theta k}\right| &\leq C(\omega)\sqrt{\sum_{k=M}^{N}\left|\left(X_k - \frac{1}{\log k}\right)\right|^2}\times\sqrt{\log N}\\ &\leq C(\omega)\sqrt{N-M}\times\sqrt{\log N}.\end{aligned}$$

According to the proof of Corollary 1 when $p \geq 2$ *and applying the second statement of Theorem 1, for* $H > 2$ *when* $p = 2$ *and for* $H > \frac{1}{P} + \frac{1}{2}$ *when* $p > 2$*, we get almost surely*

$$\lim_{N\to\infty}\frac{1}{N^{\frac{p-1}{p}}\log^H N}\sum_{k=2}^{N}\left(X_k - \frac{1}{\log k}\right)T^k(f) = 0 \quad \mu - a.e..$$

Since $\sup_{N>1}\frac{N^{\frac{p-1}{p}}\log^H N}{\Pi(N)^\beta}$ *is bounded, we have almost surely*

$$\lim_{N\to\infty}\frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N}\left(X_k - \frac{1}{\log k}\right)T^k(f) = 0 \quad \mu - a.e..$$

Now, by Abel's summation

$$\begin{aligned}\left|\frac{1}{\Pi(N)^\beta}\sum_{k=2}^{N}\frac{T^k(f)}{\log k}\right| &\leq \frac{1}{\Pi(N)^\beta}\left(\sum_{k=2}^{N-1}\frac{\log(k+1)-\log k}{\log(k+1)\log k}\left|\sum_{j=2}^{k}T^j(f)\right| + \frac{1}{\log N}\left|\sum_{k=2}^{N}T^k(f)\right|\right)\\ &\leq \frac{\log^\beta N}{N^\beta}\sum_{k=2}^{N-1}\frac{1}{k^{1-\beta}\log^2 k}\left|\frac{1}{k^\beta}\sum_{j=2}^{k}T^j(f)\right| + \frac{1}{\log^{1-\beta}N}\left|\frac{1}{N^\beta}\sum_{k=2}^{N}T^k(f)\right|.\end{aligned} \tag{17}$$

In what follows, we only deal with the case where the function f is such that

$$\lim_{N\to\infty}\frac{1}{N^\beta}\sum_{k=1}^{N}T^k(f)=0\quad \mu-a.e.. \tag{18}$$

Examples of function f satisfying the above equation are given in [7, 31, 23].

According to the above assumption, the second term of (17) *converges to* 0 *μ-a.e.. To deal with the first term of* (17)*, recall that*

$$\sum_{k\geq 1}b_k=+\infty\quad and\quad \lim_{k\to\infty}a_k=0\implies\lim_{N\to\infty}\frac{\sum_{k=1}^{N}b_ka_k}{\sum_{k=1}^{N}b_k}=0$$

for any sequences (a_k), (b_k). Applying this to $a_k=\frac{1}{k^\beta}\sum_{j=2}^{k}T^j(f)$ and $b_k=\frac{1}{k^{1-\beta}\log^2 k}$, and using the fact that $\frac{\log^\beta N}{N^\beta}\sum_{k=2}^{N}\frac{1}{k^{1-\beta}\log^2 k}$ is bounded, it follows that

$$\lim_{N\to\infty}\frac{\log^\beta N}{N^\beta}\sum_{k=2}^{N-1}\frac{1}{k^{1-\beta}\log^2 k}\left|\frac{1}{k^\beta}\sum_{j=2}^{k}T^j(f)\right|=0\quad \mu-a.e..$$

Consequently, almost surely

$$\lim_{N\to\infty}\frac{1}{N^\beta}\sum_{k=1}^{N}T^{u_k}(f)=0\quad \mu-a.e..$$

As an open question, is it true that (18) *implies that*

$$\lim_{N\to\infty}\frac{1}{N^\beta}\sum_{k=1}^{N}T^{p_k}(f)=0\quad \mu-a.e.\qquad \left(\frac{p-1}{p}<\beta\leq 1\right),$$

where $\mathcal{P}=\{p_k\}_{k\geq 1}$ denotes the set of prime numbers?

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