UNITLESS FROBENIUS QUANTALES *

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Abstract. It is often stated that Frobenius quantales are necessarily unital. By taking negation as a primitive operation, we can define Frobenius quantales that may not have a unit. We develop the elementary theory of these structures and show, in particular, how to define nuclei whose quotients are Frobenius quantales. This yields a phase semantics and a representation theorem via phase quantales.

Important examples of these structures arise from Raney's notion of tight Galois connection: tight endomaps of a complete lattice always form a Girard quantale which is unital if and only if the lattice is completely distributive. We give a characterisation and an enumeration of tight endomaps of the diamond lattices M_n and exemplify the Frobenius structure on these maps. By means of phase semantics, we exhibit analogous examples built up from trace class operators on an infinite dimensional Hilbert space.

Finally, we argue that units cannot be properly added to Frobenius quantales: every possible extention to a unital quantale fails to preserve negations.

Keywords. Quantale, Frobenius quantale, Girard quantale, unit, dualizing element, Serre duality, tight map, trace class operator, nuclear map.

Introduction

It is often stated, see for example [19, 18, 9], that a Frobenius quantale has a unit. Indeed, as far as these quantales are defined via a dualizing element (the linear falsity), this element necessarily is the unit of the dual multiplication. Then, by duality, the standard multiplication of the quantale has a unit. Negations are here taken as defined operators, by means of false and the implications.

It is possible, however, to consider negations as primitive operators and axiomatize them so to be coherent with the implications. We follow here this approach, thus exploring an axiomatization which might be considered folklore: while the axiomatization is explicitly considered in [10, §3.3], it is also closely related to the notions of *Girard couple* [8] and of *Serre duality* [20]. Models of the given axioms are quantales coming with a notion of negation, yet they might lack a unit. When they have a unit, these structures coincide with the standard Frobenius and Girard quantales. We call these structures *unitless Frobenius quantales.* It is the goal of this paper to explore in depth such an axiomatization. We get to the conclusion that unitless Frobenius and Girard quantales are structures of interest and worth further research.

In support of this conclusion we present several examples of these structures and characterize when they have units. Also, we show that the standard theory of

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quantic nuclei and phase quantales can be lifted to unitless Girard quantales and even to unitless Frobenius quantales. Slightly surprisingly, the new axiomatization immediately clarifies how to generalise to Frobenius quantales the usual double negation nucleus. As far as we are ware of, this nucleus is always described in the literature as arising from a cyclic element, yielding therefore a Girard quantale and, overall, a representation theorem for Girard quantales. We prove therefore that unitless Frobenius quantales can be represented as phase quantales, i.e. as quotients of free quantales over a semigroup by a double negation nucleus. As a consequence, many of our examples (and in principle all of them) arise as phase quantales.

We present and study the example that prompted us to develop this research. It is known [13, 8, 9, 23, 22] that the quantale of sup-preserving endomaps of L has the structure of a Frobenius quantale if and only if L is a completely distributive lattice. If L is completely distributive, then every sup-preserving endomap of L is tight, in the sense of [17], or nuclear, in the sense of [11]. If L is not completely distributive, then tight maps still form a Girard quantale, yet a unitless one as we define. A statement finer than the one above then asserts that the Girard quantale of tight endomaps of a complete lattice L is unital if and only if L is completely distributive. We illustrate the theory so far developed on this example by showing that the Girard quantale of tight endomaps can also be constructed via a double negation nucleus. We further illustrate this example by characterizing tight endomaps of M_n (the modular lattice of height 3 with n atoms), by enumerating these maps, by characterising the operations on tight maps arising from the quantale structure.

In this setting, a natural question is whether it is possible to embed unitless Frobenius quantales into unital ones. This question has always a trivial positive answer. Yet, given the choice of negations as primitive operations, a finer question is whether it is possible to find such an embedding that also preserves the negations. A simple but surprising argument shows that this is never possible, unless the given quantale has already a unit. Studying further this phenomenon, we emphasize that, contrary to the unital case, this is due to the fact that positive elements are not closed under infima.

Let us mention other motivations that prompted us to develop this research. The literature on quantales often emphasizes that these structures might not have units. We could not identify a work whose main topic are units in quantales, yet the role of units has been discussed from slightly different perspectives. For example, completeness and complexity results for the Lambek calculus—which we consider as a fragment of non-commutative linear logic whose connections to quantale theory are well-known [26]—have shown to importantly differ depending on the presence of a unit in the calculus [2, 14].

The need of considering unitless semigroups analogous to Frobenius quantales has shown up when devising categorical frameworks for quantum computation [1]. While the Frobenius algebras considered in that work are not quantales, the Frobenius quantales that we construct in Section 3 are built up from the same Frobenius algebras of trace class operators considered in [1]; the strict analogy between the two situations can be formalised using the language of autonomous categories.

Finally, coming to the past research of one of the authors, the constructions carried out in [25] rely on Girard quantales; however, to perform them, it is unclear whether the units play any role, units even appear to be problematic. Fundamental results in that paper depend on considering an infinite family of Girard quantale embeddings that preserve all the structure except for the units.

Contents

1. Elementary notions

Let us recall a few notions that we need in the course of the paper. We address the reader to standard monographies, e.g. [7, 19, 13, 10, 9], for an in-depth presentation of these notions.

Definition 1. *A* complete lattice *L is a poset such that suprema $\bigvee_{i\in I} x_i$ exists, for each family $\{ x_i \in L \mid i \in I \}$. A function $f : L_0 \longrightarrow L_1$ between two posets is* sup-preserving *if $f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i)$, for each family $\{ x_i \in L_0 \mid i \in I \}$ and whenever the supremum $\bigvee_{i\in I} x_i$ exists.*

If L is a complete lattice, then the poset L^{op} (with the opposite ordering, $x \leq_{op} y$ if and only if $y \leq x$) is also a complete lattice, where the suprema in L^{op} are infima in L. If $g : L_0 \longrightarrow L_1$ is sup-preserving, then there exists a unique sup-preserving map $g : L_1^{\mathrm{op}} \longrightarrow L_0^{\mathrm{op}}$ satisfying $f(x) \leq y$ if and only if $g(y) \leq_{op} x$—that is, $x \leq g(y)$—for each $x, y \in L$. We use the notation $\rho(f)$ for this map and say that $\rho(f)$ is the *right adjoint* of f. In a similar way, if $g = \rho(f)$, then we let $f = \ell(g)$ and say that f is the *left adjoint* of g. We also write $f \dashv g$ if $g = \rho(f)$ and $f = \ell(g)$.

Complete lattices and sup-preserving maps form a category, denoted by SLatt. The operation $(-)^{\mathrm{op}} : \mathsf{SLatt} \longrightarrow \mathsf{SLatt}$, that we can extend to sup-preserving maps by setting $f^{\mathrm{op}} := \rho(f)$, is a contravariant functor from SLatt to itself. More than that, it is actually a category isomorphism between SLatt and its opposite category $\mathsf{SLatt}^{\mathrm{op}}$. A map $g : L_0 \longrightarrow L_1$ is *inf-preserving* if it can be considered as a map $L_0^{\mathrm{op}} \longrightarrow L_1^{\mathrm{op}}$ in SLatt.

A *Galois connection* on complete lattices L_0, L_1 is a pair of maps $f : L_0 \longrightarrow L_1$ and $g : L_1 \longrightarrow L_0$ such that, for each $x \in L_0$ and $y \in L_1$, we have $y \leq f(x)$ if and only if $x \leq g(y)$. It is a consequence of this definition that we can see f as a sup-preserving map from L_0 to L_1^{op} and that $g = \rho(f) : L_1 = (L_1^{\mathrm{op}})^{\mathrm{op}} \longrightarrow L_0^{\mathrm{op}}$. Every Galois connection is uniquely determined in this way. We say that $f : L \longrightarrow L$ is *self-adjoint* if the pair f, f is a Galois connection.

A *closure operator* on a complete lattice L is a map $j : L \longrightarrow L$ which is isotone (i.e. order preserving), increasing ($x \leq j(x)$, for each $x \in L$), and idempotent ($j(j(x)) = j(x)$, for each $x \in L$). If $j : L \longrightarrow L$ is a closure operator, then

$L_j := \{ x \in L \mid j(x) = x \}$ is, with the ordering induced from L, a complete lattice, where suprema (in L_j, denoted by $\bigvee_j$) are computed as follows:

$$\bigvee_j \{ x_i \in L_j \mid i \in I \} = j(\bigvee \{ x_i \in L_j \mid i \in I \}) \,.$$

We shall say that $x \in L$ is *j-closed* if $x \in L_j$. The map $j : L \longrightarrow L_j$ is then a surjective sup-preserving map. If $f : L_0 \longrightarrow L_1$ is sup-preserving, then $\rho(f) \circ f$ is a closure operator in L_0, and every closure operator can be obtained in this way.

Definition 2. *A* quantale *is a pair $(Q, *)$ where Q is a complete lattice and $*$ is a semigroup operation that distributes with arbitrary suprema in each place:*

$$(\bigvee_{i \in I} x_i) * (\bigvee_{j \in J} y_j) = \bigvee_{i \in I, j \in J} x_i * y_j \,, \tag{1}$$

for each pair of families $\{ x_i \mid i \in I \}$ and $\{ y_j \mid j \in J \}$. If the semigroup operation $*$ has a unit, then we say that the quantale is unital. *If $(Q_i, *_i)$, $i = 0, 1$, are quantales, then a sup-preserving map $f : Q_0 \longrightarrow Q_1$ is a* quantale homomorphism *if $f(x *_0 y) = f(x) *_1 f(y)$, for each $x, y \in Q_0$.*

The category SLatt is actually a $*$-autonomous category, see e.g. [3, 11]. As such, it comes with a tensor product and a (unital) quantale is exactly a (monoid) semigroup object in the monoidal category SLatt. For a quantale $(Q, *)$ and fixed $x, y \in Q$, equation (1) exhibits the maps $x*(-) : Q \longrightarrow Q$ and $(-)*y : Q \longrightarrow Q$ as being sup-preserving. Thus thes two maps have right adjoints $x\backslash(-) : Q^{\mathrm{op}} \longrightarrow Q^{\mathrm{op}}$ and $(-)/y : Q^{\mathrm{op}} \longrightarrow Q^{\mathrm{op}}$. We have therefore

$$x * y \leq z \;\text{ iff }\; y \leq x\backslash z \;\text{ iff }\; x \leq z/y \,,$$

for each $x, y, z \in Q$. These operations, that we call the *implications*, have also the following explicit expressions:

$$x\backslash z = \bigvee \{ y \mid x * y \leq z \} \,, \qquad z/y = \bigvee \{ x \mid x * y \leq z \} \,.$$

Notice that if $f : Q_0 \longrightarrow Q_1$ is a quantale homomorphism, then $f(x\backslash z) \leq f(x)\backslash f(z)$ but in general this inequality might be strict (and a similar remark applies to $f(z/y)$).

2. Frobenius and Girard quantales

The usual definition of Frobenius and Girard quantales requires a dualizing possibliy cyclic element. It goes as follows:

Definition 3 (Definition A)**.** *Let $(Q, *)$ be a quantale and let $0 \in Q$. The element 0 is* dualizing *if, for every x in Q, we have*

$$0/(x\backslash 0) = (0/x)\backslash 0 = x \,.$$

The element 0 is cyclic *if, for every x in Q, we have*

$$x\backslash 0 = 0/x \,.$$

A Frobenius quantale *is a tuple $(Q, *, 0)$ where $(Q, *)$ is a quantale and $0 \in Q$ is dualizing. If moreover 0 is cyclic then $(Q, *, 0)$ is a* Girard quantale.

It is well known that a Frobenius quantale, if so defined, is unital, see [18, Corollary to Proposition 1] and [9, Proposition 2.6.3]. Indeed, $0\backslash 0 = 0/0$ turns out to be the unit of the quantale. We propose next a slightly different definition of Frobenius and Girard quantales, which does not rely on dualizing elements. The definition is more general in that, as we shall see, it allows to consider unitless quantales.

Definition 4 (Definition B)**.** *A* Frobenius quantale *is a tuple* $(Q, *, {}^{\perp}(-), (-)^{\perp})$ *where* $(Q, *)$ *is a quantale and* ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ *are inverse antitone maps satisfying*

$$x\backslash {}^{\perp}y = x^{\perp}/y\,, \qquad \textit{for every } x, y \in Q. \tag{2}$$

The map $(-)^{\perp}$ *is called the* right negation *while the map* ${}^{\perp}(-)$ *the* left negation*. A* Girard quantale *is a Frobenius quantale for which right and left negations coincide.*

We informally call a structure as defined in Definition 4 a *unitless Frobenius* (or *Girard*) *quantale*. This naming, however, is slight imprecise, as these quantales might well have a unit.

Let us make straight the relation between Definition 3 and Definition 4.

Lemma 5 (cf. [10, Lemma 3.18])**.** *The Frobenius quantale structures defined in Definition 3 and those defined in Definition 4 that moreover are unital, are in bijection.*

Proof. Let $(Q, *, 0)$ be a Frobenius quantale as defined in Definition 3. By letting $x^{\perp} := x\backslash 0$ and ${}^{\perp}x := 0/x$ one obtains a Frobenius quantale as defined in Definition 4, in view of the well known identity $x\backslash(z/y) = (x\backslash z)/y$.

In the opposite direction, let $(Q, *, {}^{\perp}(-), (-)^{\perp})$ be a Frobenius quantale as defined in Definition 4 which, moreover, is unital. Observe that ${}^{\perp}1 = 1\backslash {}^{\perp}1 = 1^{\perp}/1 = 1^{\perp}$. Thus, let $0 := {}^{\perp}1 = 1^{\perp}$ and observe that

$${}^{\perp}x = 1\backslash {}^{\perp}x = 1^{\perp}/x = 0/x$$

and, similarly, $x^{\perp} = x\backslash 0$. It is then immediate that 0 is a dualizing element. □

The identity (2) is considered in [10], where it is called *contraposition*. Let us consider a few consequences of (2). This identity is (under the assumption that negations are inverse to each other) clearly equivalent to the following ones:

$$x\backslash y = x^{\perp}/y^{\perp}\,, \qquad x/y = {}^{\perp}x\backslash {}^{\perp}y\,, \qquad {}^{\perp}x\backslash y = x/y^{\perp}\,. \tag{3}$$

Invertible maps satisfying the above kind of conditions were called Serre dualities in [20]. The last of the three identities above was considered in [8].

Definition 6. *We say that a pair of antitone maps* ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ *is* Serre *if they satisfy* (2)*. If they are inverse to each other, then we say that they are form a* Serre duality*.*

The following Lemma immediately follows from the previous definition.

Lemma 7. *A pair of antitone maps is Serre if and only if the equivalence below holds (for each* $x, y, z \in Q$*):*

$$x * z \leq {}^{\perp}y \;\textit{ iff }\; z * y \leq x^{\perp}\,. \tag{4}$$

Notice that the equivalence in (6) amounts to what we called elsewhere (see e.g. [24]) the *shift relations*:

$$x * y \leq z \ \text{ iff } \ {}^{\perp}z * x \leq {}^{\perp}y \ \text{ iff } \ y * z^{\perp} \leq x^{\perp}\,,$$

which also appear in [10, §3.2.2] under the naming of (IGP).

If Q is unital, then a Serre pair ${}^{\perp}(-), (-)^{\perp}$ necessarily form a Galois connection. If Q is not unital, then a Serre pair need not to be a Galois connection. For example, if $x * y = 0$ for each $x, y \in Q$, then the constant map $x^{\perp} = 0$, $x \in Q$, is Serre with itself, but it is not self-adjoint if Q has at least two elements. We shall restrict ourself, notably in the next section, to consider Serre pairs forming a Galois connection. Of course, if ${}^{\perp}(-), (-)^{\perp}$ are inverse to each other, then they are also adjoint.

Given a quantale $(Q, *)$ and pair of antitone maps ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ inverse to each other, we can define two distinct dual operators:

$$x \oplus_{\perp} y := {}^{\perp}(y^{\perp} * x^{\perp})\,, \qquad\qquad x \,{}_{\perp}\!\oplus y := ({}^{\perp}y * {}^{\perp}x)^{\perp}\,.$$

Proposition 8 (cf. [10, Lemma 3.17])**.** *If ${}^{\perp}(-), (-)^{\perp}$ is a Serre duality, then the two dual multiplications coincide and they are determined by the implication and the negations as follows*

$$x \oplus_{\perp} y = {}^{\perp}x \backslash y = x / y^{\perp} = x \,{}_{\perp}\!\oplus y\,.$$

Proof. The chain of equivalences

$$z \leq ({}^{\perp}y * {}^{\perp}x)^{\perp} \ \text{ iff } \ {}^{\perp}y * {}^{\perp}x \leq {}^{\perp}z \ \text{ iff } \ {}^{\perp}x * z \leq y \ \text{ iff } \ z \leq {}^{\perp}x \backslash y$$

yields the identity $x \,{}_{\perp}\!\oplus y = {}^{\perp}x \backslash y$. In a similar way, we have

$$z \leq {}^{\perp}(y^{\perp} * x^{\perp}) \ \text{ iff } \ y^{\perp} * x^{\perp} \leq z^{\perp} \ \text{ iff } \ z * y^{\perp} \leq x \ \text{ iff } \ z \leq x / y^{\perp}$$

yielding $x \oplus_{\perp} y = x / y^{\perp}$. The equality $x \oplus_{\perp} y = x \,{}_{\perp}\!\oplus y$ then follows from (3). □

Our next aim is that of providing simple examples of unitless Frobenius quantales that indeed lack units.

Example 9: Chu construction. We give a first example of unitless Girard quantale by observing that the usual Chu construction (or Twist product, see e.g. [12, 3, 18]), does not require a quantale to be unital and yields a Girard quantale as defined in Definition 4. For a quantale $(Q, *)$, the quantale $C(Q) = (Q \times Q^{\mathrm{op}}, \star)$ is defined by

$$(x_1, x_2) \star (y_1, y_2) := (x_1 * y_1, y_1 \backslash x_2 \wedge y_2 / x_1)\,.$$

One can directly check the associativity of $\star$ and that it distributes in both arguments over joins of $Q \times Q^{\mathrm{op}}$. Recall that

$$(x_1, x_2) \backslash (z_1, z_2) = (x_1 \backslash z_1 \wedge x_2 / z_2, z_2 * x_1)\,,$$
$$(z_1, z_2) / (y_1, y_2) = (z_1 / y_1 \wedge z_2 \backslash y_2, y_1 * z_2)\,,$$

since the conditions $(x_1, x_2) \star (y_1, y_2) \leq (z_1, z_2)$, $(y_1, y_2) \leq (x_1, x_2) \backslash (z_1, z_2)$, and $(x_1, x_2) \leq (z_1, z_2)/(y_1, y_2)$, are all equivalent to the conjuction of the three conditions:

$$x_1 * y_1 \leq z_1\,, \quad y_1 * z_2 \leq x_2\,, \quad z_2 * x_1 \leq y_2\,.$$

The duality being given by

$$(x_1, x_2)^{\perp} := (x_2, x_1)\,,$$

so the computations

$$(x_1, x_2)\backslash(y_1, y_2)^{\perp} = (x_1, x_2)\backslash(y_2, y_1) = (x_1\backslash y_2 \wedge x_2/y_1, y_1 * x_1)$$
$$(x_1, x_2)^{\perp}/(y_1, y_2) = (x_2, x_1)/(y_1, y_2) = (x_2/y_1 \wedge x_1\backslash y_2, y_1 * x_1)$$

exhibit $(-)^{\perp}$ as Serre self-dual.

Proposition 10. *The quantale $C(Q)$ is unital if and only if Q is unital.*

Indeed, the first projection is a surjective semigroup homomorphism and so, if $C(Q)$ has a unit (u_1, u_2), then u_1 is a unit of Q. On the other hand, it is well known see that if u_1 is a unit of Q, then $(u_1, \top)$ is a unit of $C(Q)$. $\lozenge$

Example 11: Couples of quantales. In [8] the authors define a couple of quantales as a pair of quantales C, Q that are related by a sup-preserving map $\phi : C \longrightarrow Q$; moreover, C is asked to be a Q-bimodule and the folllowing equations

$$\phi(c_1) \cdot c_2 = c_1 \cdot \phi(c_2) = c_1 * c_2 \tag{5}$$

are to be satisfied. If Q is a quantale, then $C = Q^{\mathrm{op}}$ is a canonical Q-bimodule, with, for $x, y \in Q$ and $q \in Q^{\mathrm{op}}$,

$$y \cdot q = q/y\,, \qquad q \cdot x = x\backslash q\,.$$

Thinking of $\phi : Q^{\mathrm{op}} \longrightarrow Q$ as a sort of negation, the first equation in (5) yields

$$x/\phi(y) = \phi(x)\backslash y\,,$$

that is, the last equation in (3). Notice that, by taking the above equality as definition of a binary operator on Q^{op}, then this operator is necessarily associative. If, moreover, ϕ is an involution, then Q becomes a unitless Girard quantale. Viceversa, unitless Girard quantale structures on Q give rise to couples of quantales $\phi : C \longrightarrow Q$ with $C = Q^{\mathrm{op}}$ and ϕ an antitone involution. Let us notice, however, that *Girard* couples of quantales, in contrast to couples of quantales, turn out to have some unit [8, Proposition 8]. $\lozenge$

3. Nuclei and phase quantales

In this section we show how to generalise the elementary theory of nuclei from unital Girard quantales to unitless Girard quantales and also unitless Frobenius quantales.

Let us recall that a *quantic nucleus* (or simply a *nuclues*) on a quantale $(Q, *)$ is a closure operator j on Q such that, for all $x, y \in Q$,

$$j(x) * j(y) \leq j(x * y)\,.$$

It is easily seen that a closure operator j is a nucleus if and only if the two conditions below hold:

$$x * j(y) \leq j(x * y)\,, \qquad j(x) * y \leq j(x * y), \qquad\qquad \text{for all } x, y \in Q\,.$$

Given a nucleus j on $(Q, *)$, let Q_j be the set of fixed points of j. From the elementary theory of closure operators, Q_j is a complete lattice and $j : Q \longrightarrow Q_j$

is a surjective sup-preserving map. Q_j has a canonical structure of a quantale $(Q_j, *_j)$ as well, where the multiplication is given by

$$x *_j y := j(x * y)\,.$$

Moreover, $j : (Q, *) \longrightarrow (Q_j, *_j)$ is a surjective quantale morphism.

Definition 12. *For a quantale $(Q, *)$, a* Serre Galois connection *is a Galois connection $l, r : Q \longrightarrow Q$ such that $l \circ r = r \circ l$ and, for all $x, y, z \in Q$,*

$$x * y \leq r(z) \ \textit{ iff } \ z * x \leq l(y)\,. \tag{6}$$

That is, (l, r) is Serre if $l \circ r = r \circ l$ and the following identity holds:

$$r(z)/y = z\backslash l(y)\,.$$

We shall refer to (6) as the shift relation. Recall that if (l, r) is Serre and moreover (l, r) are inverse to each other, then we say that (l, r) is a Serre duality.

We say that $r : Q \longrightarrow Q$ is a Serre map if (r, r) is a Serre Galois connection.

Proposition 13. *If (l, r) is a Serre Galois connection, then $j := l \circ r = r \circ l$ is a nucleus and the restriction of (l, r) to Q_j yields a Frobenius quantale structure on $(Q_j, *_j)$.*

Proof. We firstly prove that $x*j(y) \leq j(x*y)$, that is, $x*rl(y) \leq rl(x*y)$. Using the shift relation, this inclusion is equivalent to $l(x*y)*x \leq lrl(y)$. Since $lrl(y) = l(y)$, the latter inclusion is $l(x * y) * x \leq l(y)$ and equivalent, by the shift relation, to $x * y \leq rl(x * y)$, where the last inclusion holds since $r \circ l$ is a closure operator. The inclusion $j(x) * y \leq j(x * y)$ is proved similarly, using now $j = l \circ r$.

Next, the restrictions of (l, r) to Q_j have Q_j as codomain: if $x \in Q_j$, then $j(r(x)) = r(l(r(x))) = r(x)$, so $r(x) \in Q_j$. Similarly, $l(x) \in Q_j$. These restrictions are also inverse to each other: for $x \in Q_j$, $l(r(x)) = r(l(x)) = j(x) = x$. Let us argue that these restrictions are Serre: for $x, y, z \in Q_j$, $x *_j y = j(x * y) \leq r(z)$ iff $x * y \leq r(z)$ (since $r(z) \in Q_j$) iff $z * x \leq l(y)$ iff $z *_j x = j(z * x) \leq l(y)$, so the shift relation holds in Q_j. □

Proposition 14. *Let j be a nucleus on $(Q, *)$ and suppose that $l, r : Q_j \longrightarrow Q_j$ is a Serre duality on $(Q_j, *_j)$. Then $(l \circ j, r \circ j)$ is a Serre Galois connection on $(Q, *)$ and the Frobenius quantale structure $(Q_j, *_j, l, r)$ is induced from $(l \circ j, r \circ j)$ as described in Proposition 13.*

Proof. Let j, l, r be as stated. For $x, y, z \in Q$, we have $x \leq r(j(y))$ iff $j(x) \leq r(j(y))$ iff $j(y) \leq l(j(x))$ iff $y \leq l(j(x))$. Moreover

$$\begin{aligned} x * y \leq r(j(z)) \ &\text{ iff } \ j(x * y) = j(x) *_j j(y) \leq r(j(z)) \\ &\text{ iff } \ j(z * x) = j(z) *_j j(x) \leq l(j(y)) \ \text{ iff } \ z * x \leq l(j(y))\,. \end{aligned}$$

Notice now that $j \circ r = r$ and $j \circ l = l$, since by assumption the codomain of r and l is Q_j. Then

$$(l \circ j) \circ (r \circ j) = l \circ r \circ j = j$$

and, similarly, $(r \circ j) \circ (l \circ j) = j$. Finally, the restriction of $r \circ j$ (resp., $l \circ j$) to Q_j is r (resp., l), for example, for $x \in Q_j$, we have $(r \circ j)(x) = r(j(x)) = r(x)$. □

We add next some remarks.

Definition 15. *If* 0 *is an element of a quantale* $(Q, *)$*, then we say that* 0 *is* weakly cyclic *if*

$$0/(x\backslash 0) = (0/x)\backslash 0\,, \quad \textit{for all } x \in Q.$$

We say that a Serre Galois connection (l, r) *on* $(Q, *)$ *is representable by* 0 *if*

$$r(x) = x\backslash 0\,, \qquad l(x) = 0/x\,, \qquad \textit{for all } x \in Q.$$

Observe that if a Serre Galois connection (l, r) is representable by $0 \in Q$, then 0 is a weakly cyclic element.

Lemma 16. *If* $(Q, *)$ *is unital, then every Serre Galois connection is representable.*

Proof. If 1 is the unit of $(Q, *)$, then

$$r(y) = r(y)/1 = y\backslash l(1) \qquad \text{and} \qquad l(y) = 1\backslash l(y) = r(1)/y\,.$$

Moreover

$$r(1) = r(1)/1 = 1\backslash l(1) = l(1)\,,$$

and therefore (l, r) is representable by $0 = r(1) = l(1) \in Q$. □

Lemma 17. *Let* 0 *be a weakly cyclic element of a quantale* $(Q, *)$ *and set* $r(x) := x\backslash 0$ *and* $l(x) := 0/x$*. Then* (l, r) *is a representable Serre Galois connection. If* 0 *is* j*-closed (with* $j = r \circ l = l \circ r$*), then* $(Q_j, *_j)$ *is unital. If* $(Q, *)$ *is unital, then* 0 *is* j*-closed.*

Proof. The first statement is obvious. If 0 is j-closed, then the relations $r(x) = x\backslash 0$ and $l(x) = 0/x$ hold within Q_j, whence 0 is a dualizing element of Q_j which therefore is unital. Finally, if 1 is the unit of $(Q, *)$, then $0 = r(1) = rlr(1) = l(1) = lrl(1)$, showing that 0 is j-closed. □

Lemma 18. *For* (l, r) *a Galois connection on* Q*, the condition* $l \circ r = r \circ l$ *holds if and only if the images of* l *and* r *in* Q *coincide.*

Proof. For a map f let $Img(f)$ denote its image. Notice that $Img(r) = Q_{r \circ l}$, since $r(x) = r(l(r(x)))$ and similarly $Img(l) = Q_{l \circ r}$. Therefore, if $r \circ l = l \circ r$, then $Img(r) = Img(l)$. Conversely, if $Img(r) = Img(l)$, then $Q_{r \circ l} = Q_{l \circ r}$ and since a closure operator is determined by the set of its fixed points, then $l \circ r = r \circ l$. □

Example 19: Trivial quantales. As we do not require units, each complete lattice Q can be endowed with the trivial quantale structure, defined by:

$$x * y := \bot\,, \quad \text{for each } x, y \in Q.$$

The trivial quantale structure clearly is not unital, unless Q is a singleton. Notice that, for the trivial structure,

$$x\backslash y = y/x = \top\,, \quad \text{for each } x, y \in Q.$$

If Q is autodual, say with $l, r : Q \longrightarrow Q$ inverse to each other, then $(Q, *, l, r)$ is a unitless Frobenius quantale. ◊

The trivial structure is a source of counter-examples to quick conjectures. Consider $(\mathbf{2}, *)$, the two-element Boolean algebra equipped with the trivial quantale structure. Then we have $x * y \leq r(z)$ if and only if $y * z \leq l(x)$ no matter how we choose (l, r). Thus, we might chose (l, r) not antitone and yet satisfying the shift relation (take $r = l$ be the identity of $\mathbf{2}$), or choose (l, r) antitone but not a Galois connection (take $r = l$ be the constant function taking $\perp$ as its unique value).

Also, consider the diamond lattice with 3 atoms (see figure 2) and endow it with the trivial quantale structure. Take for r a duality that cyclically permutes the 3 atoms and let l be its inverse, so to have $l \neq r$. Then (l, r) is a Serre duality on the commutative quantale over M_3. On the other hand, every representable Serre duality on a commutative quantale is necessarily self-adjoint (i.e. we have $r = l$).

It might be asked whether (l, r) representable on Q implies that (l, r) is representable on Q_j. This is not the case. On the chain $0 < 1 < 2$, consider the following commutative quantale structure:

$*$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

By commutativity, 0 is weakly cyclic and, letting $r(x) = x\backslash 0$, we have $r(0) = r(1) = 2$ and $r(2) = 1$. Therefore $Q_j = \{1, 2\}$ and the $*_j$ is the trivial quantale structure on Q_j. As the implication of the trivial structure is constant, it cannot yield the (unique) duality of $\{1, 2\}$. Considering Lemma 17, notice here that 0 is not j-closed.

It is not difficult to reproduce similar counter-examples with the multiplication being non-trivial (i.e. $x * y \neq \perp$, for some $x, y \in Q$).

Phase quantales. We explore next the construction described in Propositons 13 and 14 when $(Q, *)$ is the free quantale $(P(S), \bullet)$ over a semigroup $(S, \cdot)$. Recall (see e.g. [9]) that

$$\begin{aligned} X \bullet Y &:= \{ x \cdot y \mid x \in X,\, y \in Y \}, \\ X\backslash Y &= \{ s \in S \mid s \cdot x \in Y, \text{ for all } x \in X \}, \\ Y/X &= \{ s \in S \mid x \cdot s \in Y, \text{ for all } x \in X \}. \end{aligned}$$

It is well-known (see e.g. [15]) that Galois connections (l, r) on $P(S)$ bijectively correspond to binary relations R via the correspondence

$$xRy \text{ iff } x \in l(\{y\}) \text{ iff } y \in r(\{x\})$$

so

$$r(Z) = \{ u \in S \mid zRu, \text{ for all } z \in Z \}, \quad l(Y) = \{ u \in S \mid uRy, \text{ for all } y \in Y \}.$$

We aim ro characterize Serre Galois connections via properties of their corresponding relation.

Proposition 20. *A Galois connection (l, r) on $(P(S), \bullet)$ satisfies the shift relation (6) if and only if its corresponding binary relation R satisfies*

$$x \cdot yRz \textit{ iff } xRy \cdot z, \quad \textit{for all } x, y, z \in S. \tag{7}$$

Proof. If (l, r) satisfie (6), then the following is a chain of equivalent statements (where X, Y, Z are subsets of S):

$$\begin{aligned}\forall x \in X, y \in Y, z \in Z,\ zRx \cdot y \ &\text{iff}\ \ X \bullet Y \subseteq r(Z)\\ &\text{iff}\ \ Z \bullet X \subseteq l(Y)\\ &\text{iff}\ \ \forall x \in X, y \in Y, z \in Z,\ z \cdot xRy\,.\end{aligned}$$

Considering singletons we obtain that R satisfies

$$\forall x, y, z \in S\ \ zRx \cdot y\ \text{ iff }\ z \cdot xRy\,,$$

and clearly the latter condition suffices for (6) to hold. □

We call *associative* a relation satisfying (7). To complete the characterisation of Serre Galois connections, we need to characterise the condition $l \circ r = r \circ l$ in terms of the corresponding relation R. Notice first that the condition trivially holds if R is symmetric, in which case $r = l$.

By Lemma 18, $r \circ l = l \circ r$ if and only if $Im(r) = Im(l)$, that is

$$\forall X \exists Y\ \text{ s.t. }\ r(X) = l(Y)\,, \quad \text{and} \quad \forall Y \exists X\ \text{ s.t. }\ r(X) = l(Y)\,.$$

Clearly, these two conditions hold if and only if they hold with X (in the fist case, or Y in the second) restricted to singleton, yielding

$$\forall x \exists Y_x \forall z\ (xRz\ \text{ iff }\ zRy,\ \forall y \in Y_x)\,, \tag{8}$$

$$\forall y \exists X_y \forall z\ (zRy\ \text{ iff }\ xRz,\ \forall x \in X_y)\,. \tag{9}$$

We call *weakly-symmetric* a relation satisfying (8) and (9).

We exemplify our previous observations, in particular the use of Proposition 13, in different ways.

Example 21. Our first example illustrates how to construct ad-hoc unitless Frobenius quantales with almost no effort.

Consider $(\Sigma^+, \cdot)$, the free semigroup over the alphabet Σ. Let R be the binary relation total on letters and empty otherwise: wRu iff $w, u \in \Sigma$. Due to the absence of units, condition (7) is trivially satisfied. Notice also that R is symmetric. The closed subsets of Σ^+ are $\emptyset$, Σ, and Σ^+ and the table for the multiplication is as follows:

	$\emptyset$	Σ	Σ^+
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Σ	$\emptyset$	Σ^+	Σ^+
Σ^+	$\emptyset$	Σ^+	Σ^+

This quantale does not have units but it has a Serre duality, the unique duality of the chain $\emptyset \subseteq \Sigma \subseteq \Sigma^+$. ◊

Example 22: Unital Frobenius quantales from pregroups. Here we illustrate how the same tools can be used to build Frobenius quantales that are not Girard quantales. Recall that a pregroup, see e.g. [5], is an ordered monoid $(M, 1, \cdot, \leq)$ coming with functions $l, r : M \longrightarrow M$ satisfying

$$x \cdot x^r \leq 1\,, \qquad x^l \cdot x \leq 1\,, \qquad 1 \leq x^r \cdot x\,, \qquad 1 \leq x \cdot x^l\,,$$

for all $x \in M$. That is, a pregroup is a posetal rigid category. Consider the relation R defined by xRy iff $x \cdot y \leq 1$. Clearly, R is associative. Observe now that $x \cdot z \leq 1$

if and only if $z \cdot x^{rr} \leq 1$. That is, the condition in (8) is satisfied by letting $Y_x = \{ x^{rr} \}$. Similarly, (9) is satisfied by letting $X_y = \{ y^{ll} \}$. Since M has a unit, both $P(M)$ and $P(M)_j$ are unital. $\Diamond$

Example 23: Unitless Girard quantales from C*-algebras. Let us consider an algebra A coming with an associative symmetric pairing (i.e. a bilinear form) $\langle -, - \rangle$ into the base field $\mathbb{K}$. We do not assume that A has a unit. We recall that a pairing is said to be associative if it satisfies $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$, for each $x, y, z \in A$ (in which A is called a Frobenius algebra).

The binary relation R, defined by xRy if and only if $\langle x, y \rangle = 0$, is then an associative relation on the semigroup reduct $(A, \cdot)$ of A, so we can consider the powerset quantale $(P(A), \bullet)$ and the Serre Galois connection (l, r) the relation R gives rise to. Since R is symmetric, we denote $l(X) = r(X) = X^{\perp}$, as usual from standard algebra, and so $j(X) = X^{\perp\perp}$. Clearly, if A has a unit, then $P(A)$ and its quotient $P(A)_j$ have a unit as well.

We argue next that the converse holds, if we can transform the pairing into a sort of inner product (as for example with C*-algebras). Namely, suppose that A comes with an involution $(-)^* : A \longrightarrow A$ such that, for each $f \in A$,

$$\langle f, f^* \rangle = 0 \text{ implies } f = 0\,. \tag{10}$$

In particular, assuming (10), we have $A^{\perp} = \{ 0 \}$. Under these assumptions, the following statement holds:

Proposition 24. *If $P(A)_j$ has a unit, then A has a unit.*

Proof. Obviously, every set of the form $X^{\perp}$ is a subspace of A. Let us say that a subpace is closed if it is of the form $j(Y)$ or, equivalently, $X^{\perp}$ (for some X or some Y).

We firstly observe that every one dimensional subspace of A is closed. That is, we have $j(\{ f \}) = \mathbb{K}f$, for each $f \in A \setminus \{ 0 \}$. Indeed, for $g \in A$ and $k = \frac{\langle g, f^* \rangle}{\langle f, f^* \rangle}$, we can write $g = (g - kf) + kf$ with $g - kf \in \{ f^* \}^{\perp}$. Because of (10), we also have $f \notin \{ f^* \}^{\perp}$. That is, we have and $\mathbb{K}f \vee \{ f^* \}^{\perp} = A$ and $\mathbb{K}f \wedge \{ f^* \}^{\perp} = \{ 0 \}$ in the lattice of subspaces of A. The relation $\mathbb{K}f \vee \{ f^* \}^{\perp} = A$ a fortiori implies the relation $j(\mathbb{K}f) \vee_j \{ f^* \}^{\perp} = A$, taken in the lattice of closed subpaces of A. Using the duality, we derive $\{ 0 \} = A^{\perp} = (j(\mathbb{K}f) \vee_j \{ f^* \}^{\perp})^{\perp} = j(\mathbb{K}f)^{\perp} \wedge_j \{ f^* \}^{\perp\perp} = \{ f \}^{\perp} \wedge \{ f^* \}^{\perp\perp}$, where we have used that both meets $\wedge$ (the meet in the lattice of subspaces of A) and $\wedge_j$ (the meet in the lattice of closed subspace of A) are computed as intersection. Then we also have $\{ 0 \} = \{ f^* \}^{\perp} \wedge \{ f^{**} \}^{\perp\perp} = \{ f^* \}^{\perp} \wedge \{ f \}^{\perp\perp}$. It follows that both $\mathbb{K}f$ and $j(\{ f \}) = \{ f \}^{\perp\perp}$ are complements of $\{ f^* \}^{\perp}$ in the lattice of subspaces of A. Since $\mathbb{K}f \subseteq j(\{ f \})$, we derive $\mathbb{K}f = j(\{ f \})$ using modularity of this lattice.

Let now U be the unit of $P(A)_j$. Then $U \bullet \{ f \} = j(U) \bullet \{ f \} \subseteq j(U \bullet \{ f \}) \subseteq j(U \bullet j(\{ f \})) = U \bullet_j j(\{ f \}) = j(\{ f \}) = \mathbb{K}f$ and, similarly, $\{ f \} \bullet U \subseteq \mathbb{K}f$. Thus, for each $u \in U$ and $f \in A$, $u \cdot f = kf$ and $f \cdot u = k'f$ for some $k, k' \in \mathbb{K}$. In particular, for $u, v \in U$, then $ku = u \bullet v = k'v$, showing that U has dimension 1 (the unit U cannot be $\{ 0 \}$ unless $A = \{ 0 \}$). We claim now that, given $u \in U$, there exists $k \in \mathbb{K}$ such that, for all $f \in A$, $u \cdot f = kf$. Indeed, let $f, g \in A \setminus \{ 0 \}$ and write $u \cdot f = k_f f$ and $u \cdot g = k_g g$. If $f = kg$, then $k_f f = u \cdot f = u \cdot kg = k(u \cdot g) = kk_g g = k_g kg = k_g f$, thus $k_f = k_g$. Suppose now that f, g are linearly independent.

Then, for some $k \in \mathbb{K}$, $kf + kg = k(f+g) = u \cdot (f+g) = u \cdot f + u \cdot g = k_f f + k_g g$ which yields the relation $(k - k_f)f = (k_g - k)g$ and $k_f = k = k_g$. We have proved the claim from which it readily follows that $u' := \frac{1}{k}u$ is a left unit of A. Similarly, we can find a right unit $\tilde{u}$ and, as usual, $\tilde{u} = u'$. Thus A is unital. $\square$

The C*-algebra of $n \times n$ matrices over the complex numbers $\mathbb{C}$ comes with the pairing $\langle A, B\rangle = tr(A \cdot B)$, where A^* is the conjugate transpose of A. This algebra is unital and gives rise to a well known Girard quantale, see e.g [9, §2.6.15]. We can adapt this construction to consider classes of linear operators on an infinite dimensional Hilbert space H. A continuous linear mapping $f : H \longrightarrow H$ is trace class, see e.g. [6, §18], if

$$\sum_{e \in \mathcal{E}} \langle |f|e, e\rangle_H < \infty\,,$$

where $\langle -, -\rangle_H$ above is the inner product of the Hilbert space H, $\mathcal{E}$ is an orthonormal basis, and $|f|$ is the unique positive operator such that $f^* \circ f = |f|^2$. For such an operator, we can define

$$tr(f) := \sum_{e \in \mathcal{E}} \langle f(e), e\rangle_H\,, \qquad \langle f, g\rangle := tr(f \cdot g)\,, \tag{11}$$

yielding an associative symmetric pairing. Now, f is trace class if and only if its adjoint f^* is trace class. With respect to the involution given by the adjoint, this pairing satisfies (10). The algebra of trace class operators does not have a unit. First of all, it is easily seen that the identity is not trace class. Indeed, this algebra is a well-known proper (when H is infinite dimensional) ideal of the algebra of bounded linear operators on H. Most importantly, this algebra cannot have a unit. In fact, for each $h \in H$ there exists a trace class operator c_h and $p_h \in H$ such that $c_h(p_h) = h$. For example, if $\mathcal{E}$ is a basis for H, we can let $e_0 \in \mathcal{E}$, and define $c_h(e_0) = h$, and $c_h(e) = 0$, for $e \in \mathcal{E} \setminus \{\, e_0 \,\}$. Then a unit u is forcedly the identity:

$$u(h) = u(c_h(p_h)) = (u \circ c_h)(p_h) = c_h(p_h) = h\,, \qquad \text{for all } h \in H.$$

We collect these observations into a formal statement.

Theorem 25. *The collection of closed subspaces of the algebra of trace class operators is a unitless Girard quantale.*

Similar considerations can be developed for Hilbert-Schmidt operators, see [6]. Indeed, the composition of two such operators is trace class, showing that the formula for the pairing in (11) also applies to these operators. $\Diamond$

Our next goal is to give a representation theorem, analogous to the representation theorem for Girard quantales, see e.g. [18, Theorem 2]. The representation theorem is here extended to Frobenius and unitless quantales.

Proposition 26. *Given a Frobenius quantale $(Q, *, {}^{\perp}(-), (-)^{\perp})$, define xRy iff $x \leq {}^{\perp}y$. Then R is an associative weakly symmetric relation yielding a Serre Galois connection (l, r) on $(P(Q), \bullet)$, whose quotient $(P(Q)_j, \bullet_j, l, r)$ is isomorphic to $(Q, *, {}^{\perp}(-), (-)^{\perp})$.*

Proof. Let us verify that R is associative:

$$zRx * y \;\text{ iff }\; z \leq {}^{\perp}(x * y) \;\text{ iff }\; x * y \leq z^{\perp} \;\text{ iff }\; z * x \leq {}^{\perp}y \;\text{ iff }\; z * xRy\,.$$

Let (l, r) be the Galois connection corresponding to R. Let us argue that $r \circ l = l \circ r$ by showing that the images of l and r coincide, cf. Lemma 18. By definition, a subset $X \subseteq Q$ in the image of l is of the form $\{ y \mid y \leq {}^{\perp}x,$ for all $x \in X \}$. As the condition $y \leq {}^{\perp}x$, for all $x \in X$, is equivalent to $y \leq \bigwedge_{x \in X} {}^{\perp}x = {}^{\perp}(\bigvee X)$, we deduce that such set equals $\downarrow {}^{\perp}(\bigvee X)$. Since every element $y \in Q$ is of the form ${}^{\perp}z$ for some $z \in Q$, we deduce that the image of l is the set of principal downsets of Q. Similalry, a subset $X \subseteq Q$ in the image of r is of the form $\{ y \mid y \leq x^{\perp},$ for all $x \in X \} = \downarrow (\bigvee X)^{\perp}$ and the image of r is again the set of principal downsets of Q.

Finally, $j(X) = \downarrow \bigvee X$, since

$$j(X) = l(r(X)) = \downarrow ({}^{\perp}\bigvee(\downarrow (\bigvee X)^{\perp})) = \downarrow ({}^{\perp}(\bigvee X)^{\perp}) = \downarrow (\bigvee X).$$

It is then easily seen that the mapping $\downarrow : Q \longrightarrow P(Q)_j$ is inverted by $\bigvee : P(Q)_j \longrightarrow Q$. These maps are quantale homomorphisms, indeed we have, for all x and y in Q,

$$\downarrow x \bullet_j \downarrow y = \downarrow (\bigvee(\downarrow x \bullet \downarrow y)) = \{z \in Q \mid z \leq x * y\} = \downarrow (x * y). \qquad \square$$

4. The Girard quantale of tight maps

We present in this section the main example of a unitless Girard quantale, the one that prompted this research.

We denote by L^L the lattice of all function from L to L, by $[L, L]_{\vee}$ the lattice of all join-preserving endomaps of L, and by $[L, L]_{\wedge}$ the lattice of all meet-preserving endomaps of L. For a function $f : L \longrightarrow L$, its Raney's transforms $f^{\vee}$ and $f^{\wedge}$ are defined by

$$f^{\vee}(x) := \bigvee_{x \not\leq t} f(t), \qquad\qquad f^{\wedge}(x) := \bigwedge_{t \not\leq x} f(t).$$

Let us remark that, in the definition above, we do not require that f has any property, such as being monotone.

Definition 27. *An endomap $f : L \longrightarrow L$ is* tight *if $f = f^{\wedge\vee}$. We write $[L, L]^{t}_{\vee}$ for the set of tight endomaps of L. We say that $f : L \longrightarrow L$ is* cotight *if $f = f^{\vee\wedge}$ and write $[L, L]^{t}_{\wedge}$ for the set of cotight maps from L to L.*

Most of the properties of the Raney's transforms appear already in [11], we add proofs of lemmas. We aim at demonstrating the following theorem.

Theorem 28. *For any complete lattice L, the tuple $([L, L]^{t}_{\vee}, \circ, (-)^{*})$ is a Girard quantale (as defined in Definition 4), where $\circ$ is the function composition and the duality $(-)^{*}$ defined by $f^{*} := \rho(f)^{\vee}$.*

In order to prove the theorem, let us recall some elementary properties of Raney's transforms. In the following, L shall be an arbitrary but fixed complete lattice.

Lemma 29. *For any function $f : L \longrightarrow L$, $f^{\vee}$ has a right-adjoint, and, in particular $f^{\vee}$ is sup-preserving and monotone. If f is inf-preserving, then the following relation holds:*

$$\rho(f^{\vee}) = (\ell(f))^{\wedge}. \tag{12}$$

For $f : L \longrightarrow L$, $f^{\wedge}$ is inf-preserving and, when f is sup-preserving, his left-adjoint satisfies

$$\ell(f^{\wedge}) = (\rho(f))^{\vee}. \tag{13}$$

Proof. For each $x, y \in L$, we have

$$
\begin{aligned}
f^{\vee}(x) = \bigvee_{x \not\leq t} f(t) \leq y \ \text{ iff } \ &\text{for all } t\,,\ x \not\leq t \ \text{ implies } \ f(t) \leq y\,,\\
\text{iff } \ &\text{for all } t\,,\ f(t) \not\leq y \ \text{ implies } \ x \leq t\,,\\
\text{iff } \ &x \leq \bigwedge_{f(t) \not\leq y} t\,.
\end{aligned}
$$

Therefore, if we define $g(y) := \bigwedge_{f(t) \not\leq y} t$, then g is right-adjoint to $f^{\vee}$. Let us argue that equality (12) holds if f is inf-preserving. We shall show that, for all x and y in L, we have

$$f^{\vee}(x) \leq y \ \text{ iff } \ x \leq (\ell(f))^{\wedge}(y)\,.$$

On the one hand we have

$$\bigvee_{x \not\leq t} f(t) \leq y \ \text{ iff } \ \ \text{for all } t \in L \text{ s.t. } x \not\leq t,\ f(t) \leq y\,. \tag{a}$$

On the other hand we have

$$x \leq \bigwedge_{t \not\leq y} \ell(f)(y) \ \text{ iff } \ \ \text{for all } t \in L \text{ s.t. } t \not\leq y,\ x \leq \ell(f)(t). \tag{b}$$

For the implication (a) $\Rightarrow$ (b), let $t \in L$ be such that $t \not\leq y$. Suppose that $x \not\leq \ell(f)(t)$. Then, by (a), $f(\ell(f)(t)) \leq y$ and, by the adjunction, $t \leq f(\ell(f)(t))$, whence we deduce $t \leq y$, a contradiction. Therefore $x \leq \ell(f)(t)$. The implication (b) $\Rightarrow$ (a) is proved in a similar way.

The second part of the statement follows by duality. $\square$

Recall that the set L^L is pointwise ordered, i.e. we have $f \leq g$ iff $f(t) \leq g(t)$, for all $t \in L$.

Proposition 30. *The operation* $(-)^{\wedge} : L^L \longrightarrow L^L$ *is right-adjoint to* $(-)^{\vee} : L^L \longrightarrow L^L$.

Proof. Let f and g be two endomaps of L, we have

$$
\begin{aligned}
f^{\vee} \leq g \ \text{ iff } \ \ &\text{for all } x \in L,\ f^{\vee}(x) = \bigvee_{x \not\leq t} f(t) \leq g(x)\\
\text{iff } \ \ &\text{for all } x, t \in L,\ x \not\leq t \ \text{ implies } \ f(t) \leq g(x)\\
\text{iff } \ \ &\text{for all } t \in L,\ f(t) \leq \bigwedge_{x \not\leq t} g(x) = g^{\wedge}(t) \ \text{ iff } \ f \leq g^{\wedge}\,.
\end{aligned}
$$

$\square$

Let us briefly analyse the typing of the Raney's transforms. Lemma 29 exhibits the Raney's transforms as overloaded operators: for example, the codomain of $(-)^{\vee}$ can be taken as any of the complete lattices among L^L, $[L, L]_{\vee}$, $[L, L]_{\vee}^{t}$, as suggested in the left diagram in Figure 1.

To further analyse the situation, recall that a map $f : L \longrightarrow L$ is *cotight* if $f = f^{\vee\wedge}$ and that we use use $[L, L]_{\wedge}^{t}$ for the set of cotight maps from L to L. The adjunction $(-)^{\vee} \dashv (-)^{\wedge}$ of Lemma 30 restricts from L^L as suggested in the diagram on the left of Figure 1. This is further illustrated in the diagram on the right of the figure where the vertical inclusions on the left are inf-preserving and the vertical inclusions on the right are sup-preserving. The two Raney's transforms