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Mohammed El Azhari

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A note on n -Jordan homomorphisms

M. El Azhari

Abstract. By using a variation of a theorem on n -Jordan homomorphisms due to Herstein, we deduce the following G. An's result: Let A and B be two rings where A has a unit and $\text{char}(B) > n$. If every Jordan homomorphism from A into B is a homomorphism (anti-homomorphism), then every n -Jordan homomorphism from A into B is an n -homomorphism (anti- n -homomorphism).

Keywords: Jordan homomorphism, homomorphism, n -Jordan homomorphism, n -homomorphism.

1. Preliminaries

Let A, B be two rings and $n \geq 2$ an integer. An additive map $h : A \rightarrow B$ is called an n -Jordan homomorphism if $h(x^n) = h(x)^n$ for all $x \in A$. Also, an additive map $h : A \rightarrow B$ is called an n -homomorphism or an anti- n -homomorphism if $h(\prod_{i=1}^n x_i) = \prod_{i=1}^n h(x_i)$ or $h(\prod_{i=1}^n x_i) = \prod_{i=0}^{n-1} h(x_{n-i})$, respectively, for all $x_1, \dots, x_n \in A$.

In the usual sense, a 2-Jordan homomorphism is a Jordan homomorphism, a 2-homomorphism is a homomorphism and an anti-2-homomorphism is an anti-homomorphism. It is obvious that n -homomorphisms are n -Jordan homomorphisms. Conversely, under certain conditions, n -Jordan homomorphisms are n -homomorphisms.

We say that a ring A is of characteristic greater than n ($\text{char}(B) > n$) if $n!x = 0$ implies $x = 0$ for all $x \in A$.

2. Results

Lemma 2.1 [4, Lemma 1]. Let A, B be two rings, $n \geq 2$ be an integer and $f : A^n \rightarrow B$ be a multi-additive map such that $f(x, x, \dots, x) = 0$ for all x in A . Then $\sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = 0$ for all $x_1, \dots, x_n \in A$, where S_n is the set of all permutations on $\{1, \dots, n\}$.

By using Lemma 2.1, we have the following lemma:

Lemma 2.2. Let A, B be two rings, $n \geq 2$ be an integer and $h : A \rightarrow B$ be an n -Jordan homomorphism. Then $\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$ for all $x_1, \dots, x_n \in A$, where S_n is the set of all permutations on $\{1, \dots, n\}$.

Proof. Consider the map $f : A^n \rightarrow B$,

$f(x_1, x_2, \dots, x_n) = h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)$, f is clearly multi-additive and $f(x, x, \dots, x) = h(x^n) - h(x)^n = 0$ for all $x \in A$. By Lemma 2.1, $\sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$ for all $x_1, \dots, x_n \in A$.

It was shown in [3] that if $n \geq 2$, A, B are commutative rings, $\text{char}(B) > n$ and $h : A \rightarrow B$ is an n -Jordan homomorphism, then h is an n -homomorphism. The same was also proved for algebras in [2] and [5]. Here we obtain this result as a consequence of Lemma 2.2.

Theorem 2.3. Let A, B be two commutative rings and $\text{char}(B) > n$. Then every n -Jordan homomorphism from A into B is an n -homomorphism.

Proof. Let $h : A \rightarrow B$ be an n -Jordan homomorphism. By Lemma 2.2 and since A, B are commutative,

$\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = n!(h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)) = 0$ for all $x_1, \dots, x_n \in A$, hence h is an n -homomorphism since $\text{char}(B) > n$.

Now we give the following variation of a theorem on n -Jordan homomorphisms due to Herstein [4, Theorem K].

Theorem 2.4. Let h be an n -Jordan homomorphism from a ring A into a ring B with $\text{char}(B) > n$. Suppose further that A has a unit e , then $h = h(e)\tau$ where $h(e)$ is in the centralizer of $h(A)$ and τ is a Jordan homomorphism.

Proof. Since h is an n -Jordan homomorphism, $h(e) = h(e^n) = h(e)^n$. By Lemma 2.2 and putting $x_1 = x, x_2 = x_3 = \dots = x_n = e$, $n!h(x) = (n-1)!(h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \dots + h(x)h(e)^{n-1})$ and so $nh(x) = h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \dots + h(x)h(e)^{n-1}$ (1) since $\text{char}(B) > n$.

By multiplying on the right by $f(e)$ both sides of the equality (1), $nh(x)h(e) = h(e)^{n-1}h(x)h(e) + h(e)^{n-2}h(x)h(e)^2 + \dots + h(x)h(e)$ (2)

Also, by multiplying on the left by $f(e)$ both sides of the equality (1), $nh(e)h(x) = h(e)h(x) + h(e)^{n-1}h(x)h(e) + \dots + h(e)h(x)h(e)^{n-1}$ (3)

By (2) and (3), $(n-1)h(x)h(e) = (n-1)h(e)h(x)$ and consequently $h(x)h(e) = h(e)h(x)$ (4) since $\text{char}(B) > n$. Then $h(e)$ is in the centralizer of $h(A)$. By (1) and (4), $nh(x) = nh(e)^{n-1}h(x)$ and so $h(x) = h(e)^{n-1}h(x)$ (5)

since $\text{char}(B) > n$. By Lemma 2.2, (4) and putting $x_1 = x_2 = x, x_3 = \dots = x_n = e$, $n!(h(x^2) - h(e)^{n-2}h(x)^2) = 0$ and hence $h(x^2) = h(e)^{n-2}h(x)^2$ (6) since $\text{char}(B) > n$. Consider the map $\tau : A \rightarrow B$, $\tau(x) = h(e)^{n-2}h(x)$, τ is clearly additive. By (5), $h(x) = h(e)^{n-1}h(x) = h(e)h(e)^{n-2}h(x) = h(e)\tau(x)$ for all $x \in A$. By (6), $\tau(x^2) = h(e)^{n-2}h(x^2) = h(e)^{2(n-2)}h(x)^2 = (h(e)^{(n-2)}h(x))^2 = \tau(x)^2$ for all $x \in A$, thus τ is a Jordan homomorphism.

As a consequence, we obtain the following result of G. An [1].

Corollary 2.5 [1, Theorem 2.4]. Let A and B be two rings where A has a unit e and $\text{char}(B) > n$. If every Jordan homomorphism from A into B is a homomorphism (anti-homomorphism), then every n -Jordan homomorphism from A into B is an n -homomorphism (anti- n -homomorphism).

Proof. Let $h : A \rightarrow B$ be an n -Jordan homomorphism. By Theorem 2.4, $h = h(e)\tau$ where $h(e)$ is in the centralizer of $h(A)$ and τ is a Jordan homomorphism. If τ is a homomorphism,

$$\begin{aligned} h(x_1x_2 \cdots x_n) &= h(e)\tau(x_1x_2 \cdots x_n) \\ &= h(e)\tau(x_1)\tau(x_2) \cdots \tau(x_n) \\ &= h(e)^n\tau(x_1)\tau(x_2) \cdots \tau(x_n) \text{ since } h(e) = h(e^n) = h(e)^n \\ &= h(e)\tau(x_1)h(e)\tau(x_2) \cdots h(e)\tau(x_n) \text{ since } h(e) \text{ commutes with each } \tau(x) \\ &= h(x_1)h(x_2) \cdots h(x_n) \text{ for all } x_1, \dots, x_n \in A, \end{aligned}$$

hence h is an n -homomorphism. Similarly, if τ is an anti-homomorphism, then h is an anti- n -homomorphism.

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Ecole Normale Supérieure
Avenue Oued Akreuch
Takaddoum, BP 5118, Rabat
Morocco

E-mail: mohammed.elazhari@yahoo.fr