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## A note on $n$ -Jordan homomorphisms

M. El Azhari

**Abstract.** By using a variation of a theorem on  $n$ -Jordan homomorphisms due to Herstein, we deduce the following G. An's result: Let  $A$  and  $B$  be two rings where  $A$  has a unit and  $\text{char}(B) > n$ . If every Jordan homomorphism from  $A$  into  $B$  is a homomorphism (anti-homomorphism), then every  $n$ -Jordan homomorphism from  $A$  into  $B$  is an  $n$ -homomorphism (anti- $n$ -homomorphism).

**Keywords:** Jordan homomorphism, homomorphism,  $n$ -Jordan homomorphism,  $n$ -homomorphism.

### 1. Preliminaries

Let  $A, B$  be two rings and  $n \geq 2$  an integer. An additive map  $h : A \rightarrow B$  is called an  $n$ -Jordan homomorphism if  $h(x^n) = h(x)^n$  for all  $x \in A$ . Also, an additive map  $h : A \rightarrow B$  is called an  $n$ -homomorphism or an anti- $n$ -homomorphism if  $h(\prod_{i=1}^n x_i) = \prod_{i=1}^n h(x_i)$  or  $h(\prod_{i=1}^n x_i) = \prod_{i=0}^{n-1} h(x_{n-i})$ , respectively, for all  $x_1, \dots, x_n \in A$ .

In the usual sense, a 2-Jordan homomorphism is a Jordan homomorphism, a 2-homomorphism is a homomorphism and an anti-2-homomorphism is an anti-homomorphism. It is obvious that  $n$ -homomorphisms are  $n$ -Jordan homomorphisms. Conversely, under certain conditions,  $n$ -Jordan homomorphisms are  $n$ -homomorphisms.

We say that a ring  $A$  is of characteristic greater than  $n$  ( $\text{char}(B) > n$ ) if  $n!x = 0$  implies  $x = 0$  for all  $x \in A$ .

### 2. Results

**Lemma 2.1** [4, Lemma 1]. Let  $A, B$  be two rings,  $n \geq 2$  be an integer and  $f : A^n \rightarrow B$  be a multi-additive map such that  $f(x, x, \dots, x) = 0$  for all  $x$  in  $A$ . Then  $\sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = 0$  for all  $x_1, \dots, x_n \in A$ , where  $S_n$  is the set of all permutations on  $\{1, \dots, n\}$ .

By using Lemma 2.1, we have the following lemma:

**Lemma 2.2.** Let  $A, B$  be two rings,  $n \geq 2$  be an integer and  $h : A \rightarrow B$  be an  $n$ -Jordan homomorphism. Then  $\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$  for all  $x_1, \dots, x_n \in A$ , where  $S_n$  is the set of all permutations on  $\{1, \dots, n\}$ .

**Proof.** Consider the map  $f : A^n \rightarrow B$ ,

$f(x_1, x_2, \dots, x_n) = h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)$ ,  $f$  is clearly multi-additive and  $f(x, x, \dots, x) = h(x^n) - h(x)^n = 0$  for all  $x \in A$ . By Lemma 2.1,  $\sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = 0$  for all  $x_1, \dots, x_n \in A$ .

It was shown in [3] that if  $n \geq 2$ ,  $A, B$  are commutative rings,  $\text{char}(B) > n$  and  $h : A \rightarrow B$  is an  $n$ -Jordan homomorphism, then  $h$  is an  $n$ -homomorphism. The same was also proved for algebras in [2] and [5]. Here we obtain this result as a consequence of Lemma 2.2.

**Theorem 2.3.** Let  $A, B$  be two commutative rings and  $\text{char}(B) > n$ . Then every  $n$ -Jordan homomorphism from  $A$  into  $B$  is an  $n$ -homomorphism.

**Proof.** Let  $h : A \rightarrow B$  be an  $n$ -Jordan homomorphism. By Lemma 2.2 and since  $A, B$  are commutative,

$\sum_{\sigma \in S_n} (h(\prod_{i=1}^n x_{\sigma(i)}) - \prod_{i=1}^n h(x_{\sigma(i)})) = n!(h(\prod_{i=1}^n x_i) - \prod_{i=1}^n h(x_i)) = 0$  for all  $x_1, \dots, x_n \in A$ , hence  $h$  is an  $n$ -homomorphism since  $\text{char}(B) > n$ .

Now we give the following variation of a theorem on  $n$ -Jordan homomorphisms due to Herstein [4, Theorem K].

**Theorem 2.4.** Let  $h$  be an  $n$ -Jordan homomorphism from a ring  $A$  into a ring  $B$  with  $\text{char}(B) > n$ . Suppose further that  $A$  has a unit  $e$ , then  $h = h(e)\tau$  where  $h(e)$  is in the centralizer of  $h(A)$  and  $\tau$  is a Jordan homomorphism.

**Proof.** Since  $h$  is an  $n$ -Jordan homomorphism,  $h(e) = h(e^n) = h(e)^n$ . By Lemma 2.2 and putting  $x_1 = x, x_2 = x_3 = \dots = x_n = e$ ,  $n!h(x) = (n-1)!(h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \dots + h(x)h(e)^{n-1})$  and so  $nh(x) = h(e)^{n-1}h(x) + h(e)^{n-2}h(x)h(e) + \dots + h(x)h(e)^{n-1}$  (1) since  $\text{char}(B) > n$ .

By multiplying on the right by  $f(e)$  both sides of the equality (1),  $nh(x)h(e) = h(e)^{n-1}h(x)h(e) + h(e)^{n-2}h(x)h(e)^2 + \dots + h(x)h(e)$  (2)

Also, by multiplying on the left by  $f(e)$  both sides of the equality (1),  $nh(e)h(x) = h(e)h(x) + h(e)^{n-1}h(x)h(e) + \dots + h(e)h(x)h(e)^{n-1}$  (3)

By (2) and (3),  $(n-1)h(x)h(e) = (n-1)h(e)h(x)$  and consequently  $h(x)h(e) = h(e)h(x)$  (4) since  $\text{char}(B) > n$ . Then  $h(e)$  is in the centralizer of  $h(A)$ . By (1) and (4),  $nh(x) = nh(e)^{n-1}h(x)$  and so  $h(x) = h(e)^{n-1}h(x)$  (5)

since  $\text{char}(B) > n$ . By Lemma 2.2, (4) and putting  $x_1 = x_2 = x, x_3 = \dots = x_n = e$ ,  $n!(h(x^2) - h(e)^{n-2}h(x)^2) = 0$  and hence  $h(x^2) = h(e)^{n-2}h(x)^2$  (6) since  $\text{char}(B) > n$ . Consider the map  $\tau : A \rightarrow B$ ,  $\tau(x) = h(e)^{n-2}h(x)$ ,  $\tau$  is clearly additive. By (5),  $h(x) = h(e)^{n-1}h(x) = h(e)h(e)^{n-2}h(x) = h(e)\tau(x)$  for all  $x \in A$ . By (6),  $\tau(x^2) = h(e)^{n-2}h(x^2) = h(e)^{2(n-2)}h(x)^2 = (h(e)^{(n-2)}h(x))^2 = \tau(x)^2$  for all  $x \in A$ , thus  $\tau$  is a Jordan homomorphism.

As a consequence, we obtain the following result of G. An [1].

**Corollary 2.5** [1, Theorem 2.4]. Let  $A$  and  $B$  be two rings where  $A$  has a unit  $e$  and  $\text{char}(B) > n$ . If every Jordan homomorphism from  $A$  into  $B$  is a homomorphism (anti-homomorphism), then every  $n$ -Jordan homomorphism from  $A$  into  $B$  is an  $n$ -homomorphism (anti- $n$ -homomorphism).

**Proof.** Let  $h : A \rightarrow B$  be an  $n$ -Jordan homomorphism. By Theorem 2.4,  $h = h(e)\tau$  where  $h(e)$  is in the centralizer of  $h(A)$  and  $\tau$  is a Jordan homomorphism. If  $\tau$  is a homomorphism,

$$\begin{aligned} h(x_1x_2 \cdots x_n) &= h(e)\tau(x_1x_2 \cdots x_n) \\ &= h(e)\tau(x_1)\tau(x_2) \cdots \tau(x_n) \\ &= h(e)^n\tau(x_1)\tau(x_2) \cdots \tau(x_n) \text{ since } h(e) = h(e^n) = h(e)^n \\ &= h(e)\tau(x_1)h(e)\tau(x_2) \cdots h(e)\tau(x_n) \text{ since } h(e) \text{ commutes with each } \tau(x) \\ &= h(x_1)h(x_2) \cdots h(x_n) \text{ for all } x_1, \dots, x_n \in A, \end{aligned}$$

hence  $h$  is an  $n$ -homomorphism. Similarly, if  $\tau$  is an anti-homomorphism, then  $h$  is an anti- $n$ -homomorphism.

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