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DRIFT ESTIMATION UNDER STRONG MIXING

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Abstract: It is shown that a kernel-type estimate for the drift function is pointwise as well as uniformly strongly consistent, under strong mixing condition.

Keywords: drift, strong mixing, kernel estimate.

1. Introduction

The diffusion model has been widely used in the literature to describe the behavior of a dynamical system disturbed by white noise.

It can be defined as the solution to the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t; \quad t \geq 0,$$

where $(W_t; t \in \mathbb{R}^+)$ is a standard Brownian motion, and b and σ two continuous and unknown functions. We assume (X_t) to be a stationary process with density f and we are interested in estimation of $b(x)$ for each $x \in E$, where E is the nonempty set $\{x \in \mathbb{R}/f(x) > 0\}$.

The above problem has been considered by Brown and Hewitt [3] in a parametric setting, in which b is assumed to be a linear combination of known functions Φ_1, \dots, Φ_k with unknown coefficients a_1, \dots, a_k to be estimated. Whereas Banon and N'Guyen [1] considered the nonparametric estimation of b by an indirect method using kernel estimates of f and its derivatives. Recently Pham [9] gave a kernel estimate of the drift function from the regression function $E(X_{t+\Delta}|X_t = \cdot)$ for which he establishes convergence in quadratic mean.

In the framework, Genon-Catalot *et al.* [5] estimated the variance function using the wavelets methods, Kutoyants and Pilibossian [8] considered

the parametric estimation of the parameter of the Ornstein-Uhlenbeck process.

Let Δ be positive and fixed and $n \in \mathbb{N}$; the Markov observation $(X_{i\Delta}, 0 \leq i \leq n)$ permits to write:

$$X_{i\Delta+\Delta} - X_{i\Delta} = b_{\Delta}(X_{i\Delta}) + \sigma_{\Delta}(X_{i\Delta})\epsilon_{i\Delta+\Delta}$$

where

$$b_{\Delta}(X_t) = E(X_{t+\Delta} - X_t | X_t)$$

$$\sigma_{\Delta}^2(X_t) = V(X_{t+\Delta} | X_t)$$

are supposed to exist and define discrete versions of b and σ^2 , (ϵ_t) being a stationary Gaussian process such that:

$$E(\epsilon_{t+\Delta} | X_s; s \leq t) = 0$$

$$E(\epsilon_{t+\Delta}^2 | X_s; s \leq t) = 1$$

A natural estimator of b_{Δ} is:

$$b_{\Delta,n}(x) = \frac{\sum_{i=0}^{n-1} K\left(\frac{X_{i\Delta} - x}{h_n}\right) (X_{i\Delta+\Delta} - X_{i\Delta})}{\sum_{i=0}^{n-1} K\left(\frac{X_{i\Delta} - x}{h_n}\right)} \quad \forall x \in E;$$

where (h_n) is a positive sequence of real numbers such that $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ when $n \rightarrow \infty$, and K a Parzen-Rosenblatt kernel type (see D. Bosq, J.P. Lecoutre [2]) that is a bounded function satisfying:

$\int_{\mathbb{R}} K(t) dt = 1$ and $\lim_{|t| \rightarrow \infty} |t|K(t) = 0$; moreover the kernel K will be assumed to be positive and with bounded variation.

We establish the almost sure convergence of $b_{\Delta,n}$ to b_{Δ} under strong mixing hypothesis and using the fact that:

$$b(x) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} E(X_{t+\Delta} - X_t | X_t = x)$$

we give an estimate $\frac{b_{\Delta,n}}{\Delta}$ of b by letting $\Delta \rightarrow 0$, such that $n\Delta \rightarrow \infty$ with n .

2. Main results

We introduce the following assumptions:

H1. The process $(X_{i\Delta})$, $i \in \mathbb{N}$, is strictly stationary and strong mixing in the sense that:

$\alpha_n = \sup\{|P(A \cap B) - P(A)P(B)|; A \in \mathcal{M}_0^t, B \in \mathcal{M}_{t+n}^\infty\} \rightarrow 0, n \rightarrow \infty$;
 where \mathcal{M}_0^t (resp \mathcal{M}_{t+n}^∞) denotes the σ -field generated by $(X_s; s \leq t)$ [resp $(X_s; s \geq t+n)$].

H2. The $X_{i\Delta}$ have a continuous and bounded density f in E .

H3. The functions b and σ and their discretized versions b_Δ and σ_Δ are Lipschitz and bounded in E , where the Lipschitz condition is defined by: $|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq c|x - y| \forall (x, y) \in \mathbb{R} \times \mathbb{R}$, where c is a given constant. Moreover σ is assumed to be strictly positive.

H4. The density f is twice differentiable in E and its seconde derivative is bounded on a specified compact C included in E .

Remarks

A) If we assume that the initial condition X_0 is independent of $(W_i; t \in \mathbb{R}^+)$ with density f , then a condition such as:

$$\forall x \in \mathbb{R}, |b(x)| + \sigma(x) \leq a\sqrt{1+x^2}$$

where a is a strictly positive and given constant, implies that the process (X_t) is stationary (E.Wong [10]).

B) Assumptions H1 to H3 are satisfied in the case of an Ornstein-Uhlenbeck process if X_0 follows a centered normal law.

We give the following results:

Theorem 1.

Under assumptions H1 to H3, if (h_n) is a sequence satisfying with m_n :

$$\lim_{n \rightarrow \infty} \frac{n^{1-\xi} h_n}{m_n \ln n} = \infty \text{ for some } \xi \in]0, 1[$$

where m_n is an unbounded and nondecreasing sequence with $1 \leq m_n \leq n/2$, and such that there exists a bounded constant A satisfying:

$$\frac{n}{m_n} \left(\frac{1}{m_n} \sum_{j=1}^{m_n} \alpha_j + e\sqrt{\epsilon} \alpha_{m_n}^{2m_n/3n} \right) < A;$$

then for all x in E we have:

$$b_{\Delta,n}(x) \xrightarrow{a.s.} b_{\Delta}(x), \quad n \rightarrow \infty.$$

Theorem 2.

Under assumptions of theorem 1 and if the kernel K is Lipschitz, we have:

$$\sup_{x \in C} |b_{\Delta,n}(x) - b_{\Delta}(x)| \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

Theorem 3.

Under assumptions of theorem 2 and H4, if the kernel K is even with $\int z^2 K(z) dz < \infty$; then:

$$\Theta_n^{-1} \sup_{x \in C} |b_{\Delta,n}(x) - b_{\Delta}(x)| = O(1) \text{ a.s.}, \quad n \rightarrow \infty,$$

if we choose the sequence h_n such that:

$$\Theta_n = \frac{m_n \ln n}{n^{1-\xi} h_n} \rightarrow 0, \quad \Theta_n m_n \rightarrow \infty, \quad n \rightarrow \infty$$

for some $\xi \in]0, 1[$

and if there exists a finite positive constant D such that:

$$\forall n \in \mathbb{N}, \quad n \geq 3 \quad h_n^2 \Theta_n^{-1} \leq D.$$

Corollary

Under assumptions of theorem 1, if we choose h_n and Δ such that:

$$\Delta \rightarrow 0, \frac{h_n}{\Delta} = o(1), \frac{n^{1-\xi}\Delta h_n}{m_n \ln n} \rightarrow \infty, \frac{n^\xi}{\Delta m_n} \rightarrow 0,$$

for some $\xi \in]0, 1[$, then for all x in E we have:

$$\frac{b_{\Delta,n}(x)}{\Delta} \xrightarrow{a.s.} b(x), n \rightarrow \infty.$$

Remark

For example, if $\alpha_n \leq C \exp(-cn^2)$, $C > 0$, $c > 0$:
 we can choose $m_n = m_0 \sqrt{n}$ and $\Delta = n^{-\tau}$ with $0 < \tau < 1$ then, the condition
 $\frac{n^{1-\xi}\Delta h_n}{m_n \ln n} \rightarrow \infty$ becomes $\frac{n^{1/2-(\xi+\tau)h_n}}{\ln n} \rightarrow \infty$ for some $\xi + \tau < 1/2$ and Θ_n
 becomes $\frac{\ln n}{n^{1/2-\xi}h_n} \rightarrow 0$, $n \rightarrow \infty$.

3. Preliminary results

We put:

$$b_{\Delta,n}(x) - b_{\Delta}(x) = f_n^{-1}(x)[A_1(x) + A_2(x) + b_{\Delta}(x)A_3(x)]$$

with

$$A_1(x) = \frac{1}{nh_n} \sum_{i=0}^{n-1} K\left(\frac{X_{i\Delta} - x}{h_n}\right) b_{\Delta}(X_{i\Delta}) - b_{\Delta}(x)f(x)$$

$$A_2(x) = \frac{1}{nh_n} \sum_{i=0}^{n-1} K\left(\frac{X_{i\Delta} - x}{h_n}\right) \sigma_{\Delta}(X_{i\Delta})\epsilon_{i\Delta+\Delta}$$

$$A_3(x) = f(x) - f_n(x)$$

$$f_n(x) = \frac{1}{nh_n} \sum_{i=0}^{n-1} K\left(\frac{X_{i\Delta} - x}{h_n}\right)$$

Lemma 1. (Carbon [4])

Let Z_i be a real valued strong mixing process, such that:

$$\forall n \in \mathbb{N}, \forall i \in \mathbb{N}, 1 \leq i \leq n, EZ_i = 0, |Z_i| \leq d_n, EZ_i^2 \leq D_n$$

then: $\forall \epsilon > 0, \forall n \in \mathbb{N}, n \geq 3$

$$P\left\{\left|\sum_{i=1}^n Z_i\right| > \epsilon\right\} \leq 2\exp\left\{-\gamma\epsilon + 4\gamma^2 ne\left[D_n + 8d_n^2 \sum_{j=1}^k \alpha_j\right] + 2e\sqrt{\epsilon}\alpha_k^{2k/3n} \frac{n}{k}\right\}$$

where k is an integer less or equal than $n/2$, γ satisfying $0 < \gamma \leq 1/4ked_n$.

Lemma 2.

If the sequence (h_n) is submitted to the conditions of theorem 1, then for all x in E :

$$f_n(x) \xrightarrow{a.s.} f(x), n \rightarrow \infty.$$

Proof:

By lemma 1 we have:

$$f_n(x) - Ef_n(x) \xrightarrow{a.s.} 0, n \rightarrow \infty.$$

And by Bochner's lemma we have:

$$Ef_n(x) - f(x) \rightarrow 0, n \rightarrow \infty.$$

Lemma 3.

Under hypothesis of theorem 1 we have:

$$A_1(x) \xrightarrow{a.s.} 0, n \rightarrow \infty.$$

Proof:

We write:

$$A_1(x) = [\overline{A_1}(x) - E\overline{A_1}(x)] + [E\overline{A_1}(x) - b_\Delta(x)f(x)]$$

where

$$\overline{A}_1(x) = \frac{1}{nh_n} \sum_{i=0}^{n-1} K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}).$$

We have $E\overline{A}_1(x) - b_{\Delta}(x)f(x) \rightarrow 0$, $n \rightarrow \infty$ (see Pham [9]).

It remains to show that:

$$\overline{A}_1(x) - E\overline{A}_1(x) \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

We put

$$\overline{A}_1(x) - E\overline{A}_1(x) = \sum_{i=0}^{n-1} Z_i,$$

with

$$(3.1) \quad Z_i = \frac{1}{nh_n} \left\{ K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}) - E \left[K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}) \right] \right\}$$

By construction we have: $EZ_i = 0$. If \overline{K} , Γ and ρ_1 are upperbounds of K , f and b_{Δ} respectively, we have:

$$|Z_i| \leq \frac{2\overline{K}\rho_1}{nh_n} = d_n$$

$$EZ_i^2 \leq \frac{2\overline{K}\Gamma\rho_1^2}{n^2h_n} = D_n$$

Then we apply lemma 1 with $k = m_n$ and $\gamma = \frac{1}{4m_n d_n e}$

$$P \left\{ \left| \sum_{i=0}^{n-1} Z_i \right| > \epsilon \right\} \leq 2 \exp \left\{ -\frac{nh_n}{8e\overline{K}\rho_1 m_n} \left(\epsilon - \frac{\Gamma\rho_1}{m_n} \right) + \frac{2n}{em_n} \left(\frac{1}{m_n} \sum_{j=1}^{m_n} \alpha_j + e\sqrt{e}\alpha_{m_n}^{2m_n/3n} \right) \right\}$$

For sufficiently large n , we have:

$$\frac{\Gamma\rho_1}{m_n} < \frac{\epsilon}{2};$$

therefore, assumptions of theorem 1 imply that there exist two positive constants C_1 and C_2 such that:

$$(3.2) \quad P \left\{ \left| \sum_{i=0}^{n-1} Z_i \right| > \epsilon \right\} \leq C_1 \exp \left(-C_2 \epsilon \frac{nh_n}{m_n} \right)$$

and the choice of h_n shows that:

$$\forall \epsilon > 0, \quad \sum_n P \left\{ \left| \sum_{i=0}^{n-1} Z_i \right| > \epsilon \right\} < \infty.$$

Lemma 4.

Under hypothesis of theorem 1, if the sequence m_n is satisfying:

$$\frac{n^\xi}{m_n} \rightarrow 0, \quad n \rightarrow \infty;$$

for some $\xi \in]0, 1[$ then, we have:

$$A_2(x) \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

Proof:

The study of $A_2(x)$ cannot be made directly because of the possible large values for the variables $\epsilon_{i\Delta+\Delta}$.

We use a truncation technique which consists in decomposing $A_2(x)$ in $A_2^+(x)$ and $A_2^-(x)$.

Where:

$$A_2^+(x) = \frac{1}{nh_n} \sum_{i=0}^{n-1} K \left(\frac{X_{i\Delta} - x}{h_n} \right) \sigma_\Delta(X_{i\Delta}) \epsilon_{i\Delta+\Delta} I_{\{|\epsilon_{i\Delta+\Delta}| \geq M_n\}}$$

and $A_2^-(x) = A_2(x) - A_2^+(x)$, M_n being a nondecreasing sequence satisfying $M_n = n^\xi$ for some $\xi \in]0, 1[$.

We have

$$|A_2^+(x)| \leq \frac{\bar{K} \rho_2}{nh_n} \sum_{i=0}^{n-1} |\epsilon_{i\Delta+\Delta}| I_{\{|\epsilon_{i\Delta+\Delta}| \geq M_n\}}$$

where ρ_2 is an upper bound of σ_Δ .

Leading by Schwartz inequality to:

$$E|A_2^+(x)| \leq \frac{\bar{K}\rho_2}{nh_n} \sum_{i=0}^{n-1} (E\epsilon_{i\Delta+\Delta}^2)^{1/2} (P\{|\epsilon_{i\Delta+\Delta}| \geq M_n\})^{1/2}$$

Using Markov inequality we get for any sequence (e_n)

$$P\{|A_2^+(x) - EA_2^+(x)| \geq e_n\} \leq \frac{M}{e_n h_n M_n^{\beta/2}}$$

M being a positive constant and β such that $\beta > 4/\xi - 2$.

It suffices to choose now $e_n = e_0(n^{1-\xi}h_n)^{-1}$ for a certain $e_0 > 0$ to get:

$$(4.1) \quad P\{n^{1-\xi}h_n|A_2^+(x) - EA_2^+(x)| \geq e_0\} \leq Ln^{1-\xi(1+\beta/2)}$$

where L is a positive constant.

The choice of β makes that the upperbound is the general term of a convergent series, hence:

$$A_2^+(x) - EA_2^+(x) \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

To prove that $A_2^-(x) - EA_2^-(x) \xrightarrow{a.s.} 0$ we put:

$$\Psi_i = \frac{1}{nh_n} \left\{ K_i(x)\sigma_\Delta(X_{i\Delta})\epsilon_{i\Delta+\Delta} I_{\{|\epsilon_{i\Delta+\Delta}| < M_n\}} - E \left[K_i(x)\sigma_\Delta(X_{i\Delta})\epsilon_{i\Delta+\Delta} I_{\{|\epsilon_{i\Delta+\Delta}| < M_n\}} \right] \right\}$$

where $K_i(x) = K\left(\frac{X_{i\Delta} - x}{h_n}\right)$.

By the lemma 1 we get:

$$(4.2) \quad \forall \epsilon > 0 \quad P \left\{ \left| \sum_{i=0}^{n-1} \Psi_i \right| > \epsilon \right\} \leq C_3 \exp \left(-C_4 \epsilon \frac{n^{1-\xi}h_n}{m_n} \right)$$

where C_3 and C_4 are two positive constants; and the choice of h_n shows that:

$$\sum_n P \left\{ \left| \sum_{i=0}^{n-1} \Psi_i \right| > \epsilon \right\} < \infty.$$

Now, $A_2(x) = A_2(x) - EA_2(x)$ permits to conclude.

Lemma 5.

Under hypothesis of theorem 2, we have:

$$\sup_{x \in C} |A_2(x)| \xrightarrow{a.s.} 0, n \rightarrow \infty.$$

Proof:

By the proof of lemma 4, we get $\sup_{x \in C} |A_2^+(x) - EA_2^+(x)| \xrightarrow{a.s.} 0, n \rightarrow \infty.$

It remains to show that: $\sup_{x \in C} |A_2^-(x) - EA_2^-(x)| \xrightarrow{a.s.} 0, n \rightarrow \infty.$

Let $\{B_j, j = 1, 2, \dots, l_n\}$ be a cover of C by l_n spheres with center v_j and radius less than h_n^η with $l_n \leq \bar{l} h_n^{-\eta}$, where η is a fixed number such that for $\gamma_1 > 0, \eta > 1 + \frac{1}{\gamma_1}$ and \bar{l} a given positive constant.

If x is a fixed point in C , there exists a point v_j such that $x \in B_j$ and we write:

$$\Psi_i(x) = \Psi_i(v_j) + \tilde{\Psi}_i(x),$$

where $\tilde{\Psi}_i(x) = \Psi_i(x) - \Psi_i(v_j).$

The kernel K being Lipschitz of order γ_1 , there exists a finite positive constant λ such that:

$$\sup_{x \in C} \left| \sum_{i=0}^{n-1} \tilde{\Psi}_i(x) \right| \leq \lambda h_n^{\gamma_1(\eta-1)-1} n^{-1} \sum_{i=0}^{n-1} [|\epsilon_{i\Delta+\Delta}| + E|\epsilon_{i\Delta+\Delta}|]$$

By the law of large numbers we have:

$$n^{-1} \sum_{i=0}^{n-1} |\epsilon_{i\Delta+\Delta}| \xrightarrow{a.s.} E|\epsilon_{i\Delta+\Delta}|$$

then, we get:

$$\sup_{x \in C} \left| \sum_{i=0}^{n-1} \tilde{\Psi}_i(x) \right| \leq \Lambda h_n^{\gamma_1(\eta-1)-1} \text{ a.s.}$$

where Λ is a positive constant.

It remains to show that:

$$\max_{1 \leq j \leq l_n} \left| \sum_{i=0}^{n-1} \Psi_i(v_j) \right| \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

For any $\epsilon > 0$, we have:

$$P \left\{ \max_{1 \leq j \leq l_n} \left| \sum_{i=0}^{n-1} \Psi_i(v_j) \right| > \epsilon \right\} \leq l_n \sup_{x \in \mathcal{C}} P \left\{ \left| \sum_{i=0}^{n-1} \Psi_i(x) \right| > \epsilon \right\}$$

Inequality (4.2) shows that there exists a positive constant C_5 such that

$$P \left\{ \max_{1 \leq j \leq l_n} \left| \sum_{i=0}^{n-1} \Psi_i(v_j) \right| > \epsilon \right\} \leq C_5 h_n^{-\eta} \exp \left(-C_4 \epsilon \frac{n^{1-\xi} h_n}{m_n} \right)$$

the right-hand side of the last inequality could be written as follows:

$$\frac{C_5}{(n h_n)^\eta} n^{\eta - C_4 \mu_n \epsilon};$$

where $\mu_n = \frac{n^{1-\xi} h_n}{m_n \ln n}$; it is the general term of a convergent series, hence the result.

Lemma 6.

Under hypothesis of lemma 2, there exists $\delta > 0$ such that:

$$\sum_n P(\inf_{x \in \mathcal{C}} f_n(x) \leq \delta) < \infty.$$

Proof:

It suffices to write:

$$f_n(x) = f(x) - (f(x) - f_n(x))$$

$$\text{then, } \inf_{x \in \mathcal{C}} f_n(x) \geq \inf_{x \in \mathcal{C}} f(x) - \sup_{x \in \mathcal{C}} |f(x) - f_n(x)|$$

4. Proofs of theorems.

4.1 Proof of theorem 1.

We obtain:

$$|b_{\Delta,n}(x) - b_{\Delta}(x)| \leq \frac{|A_1(x)| + |A_2(x)| + |b_{\Delta}(x)||A_3(x)|}{f(x) - |f(x) - f_n(x)|}$$

Lemmas 2, 3 and 4 permit to conclude that:

$$b_{\Delta,n}(x) \xrightarrow{a.s.} b_{\Delta}(x), \quad n \rightarrow \infty.$$

4.2 Proof of theorem 2.

We obtain:

$$\sup_{x \in C} |b_{\Delta,n}(x) - b_{\Delta}(x)| \leq \frac{\sup_{x \in C} |A_1(x)| + \sup_{x \in C} |A_2(x)| + \sup_{x \in C} |b_{\Delta}(x)||A_3(x)|}{\inf_{x \in C} f_n(x)}$$

We find a proof of $\sup_{x \in C} |f_n(x) - f(x)| \xrightarrow{a.s.} 0$, $n \rightarrow \infty$ in Györfi *and al.* [6], and since the function b_{Δ} is bounded on E we conclude that:

$$\sup_{x \in C} |b_{\Delta}(x)||A_3(x)| \xrightarrow{a.s.} 0, \quad n \rightarrow \infty.$$

Now we pick up again the decomposition of $A_1(x)$ in the proof of lemma 3 and we write:

$$\sup_{x \in C} |A_1(x)| \leq \sup_{x \in C} |\overline{A}_1(x) - E\overline{A}_1(x)| + \sup_{x \in C} |E\overline{A}_1(x) - b_{\Delta}(x)f(x)| = T_1 + T_2$$

with

$$T_1 = \sup_{x \in C} \left| \frac{1}{nh_n} \sum_{i=0}^{n-1} \left\{ K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}) - E \left[K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}) \right] \right\} \right|$$

and

$$T_2 = \sup_{x \in C} \left| \frac{1}{nh_n} \sum_{i=0}^{n-1} E \left[K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_{\Delta}(X_{i\Delta}) \right] - b_{\Delta}(x)f(x) \right|$$

By Bochner's lemma and the fact that b_Δ is Lipschitz we get $T_2 \rightarrow 0$ $n \rightarrow \infty$

Now, using the technique of covering C by a finite number of spheres as in the proof of lemma 5 gives:

$$\forall \epsilon > 0, P \left\{ \max_{j=1, \dots, n} \left| \sum_{i=0}^{n-1} Z_i(v_j) \right| > \epsilon \right\} \leq C_6 h_n^{-\eta} \exp \left(-C_7 \epsilon \frac{nh_n}{m_n} \right)$$

where C_6 and C_7 are two positive constants and Z_i the variables defined in (3.1). Moreover, if we write:

$$\tilde{Z}_i(x) = Z_i(x) - Z_i(v_j)$$

we get:

$$\sup_{x \in C} \left| \sum_{i=0}^{n-1} \tilde{Z}_i(x) \right| \leq h_n^{\gamma_1(n-1)-1}$$

which leads to:

$$T_1 \xrightarrow{a.s.} 0, n \rightarrow \infty.$$

Lastly, the lemmas 5 and 6 permit to finish the proof of theorem 2.

4.3 Proof of theorem 3.

We proceed as in the proof of theorem 2 with $\epsilon_0 \Theta_n$ instead of ϵ , for a certain $\epsilon_0 > 0$, and we establish that:

$$\Theta_n^{-1} \sup_{x \in C} |\overline{A}_1(x) - E\overline{A}_1(x)| = O(1) \text{ a.s., } n \rightarrow \infty.$$

and proceeding as in the proof of lemma 5 with $\epsilon_0 \Theta_n$ instead of ϵ for a certain $\epsilon_0 > 0$ we establish that:

$$\Theta_n^{-1} \sup_{x \in C} |A_2^-(x) - EA_2^-(x)| = O(1) \text{ a.s., } n \rightarrow \infty.$$

Furthermore, we have:

$$T_2 \leq \sup_{x \in C} \frac{1}{h_n} \int K \left(\frac{u-x}{h_n} \right) |b_\Delta(u) - b_\Delta(x)| f(u) du + \sup_{x \in C} \frac{|b_\Delta(x)|}{h_n} \int K \left(\frac{u-x}{h_n} \right) |f(u) - f(x)| du$$

and if we put $z = \frac{u-x}{h_n}$ and use the fact that b_Δ is Lipschitz of order 1 and ratio $r < \infty$ we get:

$$T_2 \leq r h_n \sup_{x \in C} \int |z| K(z) f(z h_n + x) dz + \rho_1 \sup_{x \in C} \int K(z) |f(z h_n + x) - f(x)| dz$$

Now, the hypothesis of theorem 3 and a Taylor expansion in the second term of the right-hand side of the inequality above shows that there exists a finite constant $D > 0$ such that: $T_2 \leq D h_n^2$.

Hence, $\Theta_n^{-1} \sup_{x \in C} |A_1(x)| = O(1)$ a.s., $n \rightarrow \infty$.

Moreover, by the proof of lemma 4 we have:

$$P\{\sup_{x \in C} |A_2^+(x) - EA_2^+(x)| \geq e_n\} \leq \frac{M n^{-\xi\beta/2}}{e_n h_n}$$

If we choose $e_n = e_0 m_n^{-1} \Theta_n$ for a certain $e_0 > 0$ we get:

$$P\{m_n \Theta_n^{-1} \sup_{x \in C} |A_2^+(x) - EA_2^+(x)| \geq e_0\} \leq \bar{L} \frac{1}{n^{\xi(1+\beta/2)-1} \ln n}$$

where \bar{L} is a positive constant, and then we conclude with Borel-Cantelli lemma.

Lastly, we find in Györfi, Härdle, Sarda and Vieu [6] a proof of:

$$\Theta_n^{-1} \sup_{x \in C} |f_n(x) - f(x)| = O(1) \text{ a.s., } n \rightarrow \infty.$$

4.4 Proof of corollary.

We obtain:

$$\begin{aligned} \frac{b_{\Delta,n}(x)}{\Delta} - b(x) &= \frac{b_{\Delta,n}(x) - b_\Delta(x)}{\Delta} + \left(\frac{b_\Delta(x)}{\Delta} - b(x) \right) \\ \frac{1}{\Delta} |b_{\Delta,n}(x) - b_\Delta(x)| &\leq \frac{1}{\Delta} \times \frac{|A_1(x)| + |A_2(x)| + |b_\Delta(x)| |A_3(x)|}{f(x) - |f(x) - f_n(x)|} \end{aligned}$$

First we have:

$$\frac{|b_\Delta(x)|}{\Delta} |A_3(x)| \leq \left| \frac{b_\Delta(x)}{\Delta} - b(x) \right| |A_3(x)| + |b(x)| |A_3(x)|$$

lemma 2 and the fact that $\frac{b_\Delta(x)}{\Delta} \rightarrow b(x)$, $\Delta \rightarrow 0$ give:

$$\frac{|b_\Delta(x)|}{\Delta} |A_3(x)| \xrightarrow{a.s.} 0, \Delta \rightarrow 0, n \rightarrow \infty.$$

Now proceeding as in the proof of lemma 3 we get:

$$\frac{1}{\Delta} A_1(x) = R_1 + R_2$$

with

$$R_2 = \frac{1}{n\Delta h_n} \sum_{i=0}^{n-1} E \left[K \left(\frac{X_{i\Delta} - x}{h_n} \right) [b_\Delta(X_{i\Delta}) - b_\Delta(x)] \right] + \frac{b_\Delta(x)}{\Delta} (E f_n(x) - f(x))$$

with

$$\frac{b_\Delta(x)}{\Delta} (E f_n(x) - f(x)) = \left(\frac{b_\Delta(x)}{\Delta} - b(x) \right) (E f_n(x) - f(x)) + b(x) (E f_n(x) - f(x))$$

using Bochner's lemma, the fact that $b(x)$ is bounded and the fact that b_Δ is Lipschitz we easily show that $R_2 \rightarrow 0$, $\Delta \rightarrow 0$, $n \rightarrow \infty$.

Moreover;

$$R_1 = \frac{1}{n\Delta h_n} \left\{ K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_\Delta(X_{i\Delta}) - E \left[K \left(\frac{X_{i\Delta} - x}{h_n} \right) b_\Delta(X_{i\Delta}) \right] \right\}$$

then, using lemma 1 we get $R_1 \xrightarrow{a.s.} 0$, $\Delta \rightarrow 0$, $n \rightarrow \infty$.

Now,

$$\frac{1}{\Delta} A_2(x) = \frac{1}{\Delta} A_2^+(x) + \frac{1}{\Delta} A_2^-(x)$$

Proceeding as in the proof of lemma 4 we get:

$$P \left\{ \frac{1}{\Delta} |A_2^+(x) - EA_2^+(x)| \geq e_n \right\} \leq \frac{M}{\Delta e_n h_n n^{\xi\beta/2}}$$

if we choose $e_n = e_0(n^{1-\xi}\Delta h_n)^{-1}$ for a certain $e_0 > 0$, we get:

$$P \{ n^{1-\xi} h_n |A_2^+(x) - EA_2^+(x)| \geq e_0 \} \leq M e_0^{-1} n^{1-\xi(1+\beta/2)}$$

then we conclude as in the proof of lemma 4.

Lastly if we apply lemma 1 to $\frac{1}{\Delta}[A_2^-(x) - EA_2^-(x)]$ we get:

$$P \left\{ \left| \sum_{i=0}^{n-1} \Psi_i \right| > \epsilon \right\} \leq 2 \exp \left\{ -\frac{n^{1-\xi} \Delta h_n}{m_n 8 \bar{K} e \rho_2} \left(\epsilon - \frac{n^\xi \Gamma \rho_2}{\Delta m_n} \right) + \frac{2n}{em_n} \left(\frac{1}{m_n} \sum_{j=1}^{m_n} \alpha_j + e \sqrt{\epsilon} \alpha_{m_n}^{2m_n/3n} \right) \right\}$$

For sufficient large n , we have:

$$\frac{n^\xi \Gamma \rho_2}{\Delta m_n} < \frac{\epsilon}{2};$$

therefore assumptions of corollary imply that there exist two positive constants C_8 and C_9 such that:

$$P \left\{ \left| \frac{1}{\Delta} [A_2^-(x) - EA_2^-(x)] \right| > \epsilon \right\} \leq C_8 \exp \left(-C_9 \epsilon \frac{n^{1-\xi} \Delta h_n}{m_n} \right)$$

and the choice of h_n shows that:

$$\sum_n P \left\{ \left| \frac{1}{\Delta} [A_2^-(x) - EA_2^-(x)] \right| > \epsilon \right\} < \infty.$$

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