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To cite this version:

J.M.L. Bernard. A global method for the scattering by a multimode plane with arbitrary primary sources and complete series using error functions. 2022. hal-03659812

HAL Id: hal-03659812
https://hal.science/hal-03659812
Preprint submitted on 5 May 2022

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A global method for the scattering by a multimode plane with arbitrary primary sources and complete series using error functions

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1) Introduction

In a recent paper [1], we considered the field scattered by an arbitrary impedance plane in electromagnetism. We here exploit this formalism to analyze the scattering by a structure composed of several homogeneous planar layers, with isotropy or uniaxial anisotropy, illuminated by arbitrary bounded sources above the multilayer, when it is grounded (multilayer on an impedance plane) or not (multilayer slab in free space), which generalizes our previous approach for a multilayer given in [2].

The field scattered of such structures is usually given by its plane wave expansion (Fourier representation) [3]-[6] with reflection coefficients that are meromorphic functions. Each one, when modeled as a rational function with a set of $N$ simple poles \( \{ -g_j \}_{j=1,\ldots,N} \), leads to a multimode boundary condition of order $N$ [2].

The Fourier expansion is well-adapted in far field or for plane wave illuminations, but is not suitable for an analysis at any distance or for complex incident waves. Even when double Fourier integrals are reduced to single Fourier-Bessel integrals, calculation remains lengthy and delicate because of functions in the integral that remains highly oscillating and, most often in literature [3]-[7], analytic expansions are not strictly convergent but asymptotic. Besides, an additional difficulty comes from that, in multimode case, we have to take account that the constants $g_j$ can have real parts of any sign, which signifies that passive but also active modes are present, even if the complete system is strictly passive.

In this frame, after expanding potentials into a combination of Fourier-Bessel integrals depending on each $g_j$, we are led to transform them and to derive an original integral representation, able to take account of active modes from the definition of a parameter $\epsilon$, which permits novel exact and asymptotic series with error functions. These series allow to exhibit guiding waves terms near and far from the sources, generalizing and refining [1].

Otherwise, our approach, as in [1], uses a new representation of potentials for the incident field, which possesses the originality to consider arbitrarily oriented electric and
magnetic primary currents sources. Thus, we have no more to solve separately the problem for vertical or horizontal dipolar source as commonly done in the literature for passive impedance planes [7]-[14], isotropic or uniaxial slabs [15]-[17], or multilayers [3]-[6],[18]-[22]. In practice, the analytic method so developed can be applied in whole generality to various problems, in particular for the determination of coupling between antennas above an imperfectly reflective plane, or for the calculus of Green's functions for planar lines printed on a multilayer.

This chapter is organized as follows. In section 2, we give a discussion on the representation of the field with potentials, on the boundary conditions, and on the positions of $g_j$ in the complex planes when metamaterials are present. Afterwards, we give a global expression of potentials attached to the fields radiated by arbitrary bounded sources in free space in section 3 and above the multilayer in section 4, that we develop and expand for arbitrarily oriented dipoles in section 5. In sections 6 and 7, we then detail a compact expression of the special function involved in the potentials attached to each mode, intimately depending of a parameter $\epsilon$ which is necessary to take correctly account of active modes. The definition of $\epsilon$ will be useful for the development of exact (section 6) and asymptotic (section 7) expansions with error functions for arbitrary cases, allowing in particular a general analysis of guided waves in section 8, including backward waves, near and far from the sources.

2) Potentials, reflection coefficients and multimode boundary conditions

2.1) Fields and Potentials

We consider the scattering by an imperfectly reflective plane when it is illuminated by the field radiated by a bounded source, composed of arbitrary electrical and magnetic currents $J$ and $M$ (see fig. 1). In the space of points $r$ with cartesian coordinates $(x, y, z)$, this plane is defined by $z = 0$. A harmonic time dependence $e^{i\omega t}$, from now on assumed, is suppressed throughout. The constants $\varepsilon_0$ and $\mu_0$ are respectively the permittivity and the permeability of the free space above the plane, and $k_0 = \omega(\mu_0\varepsilon_0)^{1/2}$ is its wavenumber. Each component of the scattered field is assumed to be regular in the domain $z > 0$, and $O(e^{-\gamma|z|})$ with $\gamma > 0$ as $|r| \to \infty$ when $|\arg(ik_0)| < \pi/2$ (note : no loss is a limit case).

The electric field $E$ and the magnetic field $H$ above the multilayer, following Harrington [23, p.131] (see also Jones [3, p.19]), can be written with two scalar potentials $\mathcal{E}$ and $\mathcal{H}$, following,
\[ E = -i k_0 \text{curl}(\mathcal{H} \hat{z}) + (\text{grad(\text{div}(.))} + k_0^2)(\mathcal{E} \hat{z}), \]
\[ Z_0 H = i k_0 \text{curl}(\mathcal{E} \hat{z}) + (\text{grad(\text{div}(.))} + k_0^2)(\mathcal{H} \hat{z}), \]

where \((\Delta + k_0^2)\mathcal{E} = 0\) and \((\Delta + k_0^2)\mathcal{H} = 0\) outside the sources, \(Z_0 = (\mu_0/\varepsilon_0)^{1/2}\).

Thereafter, we denote \((\mathcal{E}_i, \mathcal{H}_i)\) and \((\mathcal{E}_s, \mathcal{H}_s)\) the potentials corresponding respectively to the incident field (incoming wave) \((E^i, H^i)\) and the scattered field (outgoing wave) \((E^s, H^s)\), and we write (1) in the compact form,

\[
(E, Z_0 H) = (\mathcal{C}_1(\hat{z} \mathcal{E}, \hat{z} \mathcal{H}), \mathcal{C}_2(\hat{z} \mathcal{E}, \hat{z} \mathcal{H})) = \mathcal{L}(\hat{z} \mathcal{E}, \hat{z} \mathcal{H}),
\]

\[
\mathcal{L}_1(u, v) = ((\text{grad(\text{div}(.))} + k_0^2)(u) - i k_0 \text{curl}(v)),
\]

\[
\mathcal{L}_2(u, v) = (i k_0 \text{curl}(u) + (\text{grad(\text{div}(.))} + k_0^2)(v)).
\]

2.2) Boundary conditions on a multilayer backed by an impedance plane

Let us consider a multilayer plane composed of uniform isotropic (or z-axial anisotropic) layers. Following the appendix 1, when a plane wave \((E^i, H^i)\) is incident at angle \(\beta\) with the normal \(\hat{z}\), the scattered field is a reflected plane wave \((E^s, H^s)\) which satisfies \(E^s_{\hat{z}}|_{z=0^+} = R_{e,e} E^i_{\hat{z}}|_{z=0^-}\) (i.e. \(H^s_{\hat{z}}|_{z=0^+} = R_{e,e} H^i_{\hat{z}}|_{z=0^-}\) in TM polarization), and \(H^s_{\hat{z}}|_{z=0^+} = R_{h,h} H^i_{\hat{z}}|_{z=0^-}\) (i.e. \(E^s_{\hat{z}}|_{z=0^+} = R_{h,h} E^i_{\hat{z}}|_{z=0^-}\) in TE polarization) [4], [5], [21].

If the multilayer is backed by a constant impedance plane, the reflection coefficients \(R_{e,e}\) and \(R_{h,h}\) are meromorphic functions of the variable \(\cos \beta\), that we model as rational functions [2] with simple poles, following,

\[
R_{e,e}(\beta) = \prod_{j=1}^{N_e} \frac{\cos \beta - g_j^e}{\cos \beta + g_j^e}, \quad R_{h,h}(\beta) = \prod_{j=1}^{N_h} \frac{\cos \beta - g_j^h}{\cos \beta + g_j^h},
\]

for which we have the basic equalities (without superscripts \(e\) and \(h\)),

\[
\prod_{j=1}^{N} \frac{\cos \beta - g_j}{\cos \beta + g_j} = (\pm 1)^N + \sum_{j=1}^{N} a_j \left( \frac{-\cos \beta}{g_j} \right)^{\pm 1} \frac{1}{g_j},
\]

\[
\frac{a_j}{2g_j} = -\prod_{i \neq j}^N \frac{g_j + g_i}{g_j - g_i},
\]

where \(N \geq 1\), and we can let \(\prod_{i \neq j}^N \frac{g_j + g_i}{g_j - g_i} \equiv 1\) for \(N = 1\). The constants \((\pm 1)^N\) refer to limit values when \(|\cos \beta|^{\pm 1} \rightarrow \infty\).
The \( g_j^{e,h} \) are constants attached to complex modes, passive when \( \text{Re}(g_j^{e,h}) \geq 0 \) or active when \( \text{Re}(g_j^{e,h}) < 0 \), and ordered such that \( |g_j^{e,h}| \geq |g_j^{e,h}| \). We assume, from now on, that \( N_e \) and \( N_h \), and thus \( N \), are positive odd numbers, as in [26]-[28].

Considering plane waves representation of fields (appendix 1), we can then write a multimode boundary conditions at \( z = 0^+ \) [2],

\[
\prod_{j=1}^{N_e} \left( \frac{\partial}{\partial z} - i k_0 g_j^e \right) E_z^e(z)|_{0^+} = \prod_{j=1}^{N_e} \left( \frac{\partial}{\partial z} + i k_0 g_j^e \right) E_z^i( -z)|_{0^+},
\]

\[
\prod_{j=1}^{N_h} \left( \frac{\partial}{\partial z} - i k_0 g_j^h \right) H_z^e(z)|_{0^+} = \prod_{j=1}^{N_h} \left( \frac{\partial}{\partial z} + i k_0 g_j^h \right) H_z^i( -z)|_{0^+}. \tag{5}
\]

From the symmetry at normal incidence, the condition \( R_{h,h}(0) = - R_{e,e}(0) \) must apply, which leads us to write,

\[
\prod_{j=1}^{N_h} \frac{\pm 1 - g_j^h}{\pm 1 + g_j^h} = - \prod_{j=1}^{N_e} \frac{\pm 1 - g_j^h}{\pm 1 + g_j^h}, \tag{6}
\]

and implies that \( R_{h,h}(\pi) = - R_{e,e}(\pi) \). The condition (6) has crucial importance to avoid non physical behaviours of fields derived from potentials, as examined further in this paper. Besides, the reader will notice that (6) implies \( g_1^e = 1/g_1^h \) when \( N_{e,(h)} = 1 \), as well-known for monomode (impedance) boundaries conditions [1]. The numbers \( N_{e,(h)} \) correspond to truncated infinite products, where the less significant \( g_j^{e,(h)} \) have been neglected, while some \( g_j^{e,(h)} \) have to be modified so that \( R_{h,h}(0) = - R_{e,e}(0) \) remains.

Considering (1), we can use,

\[
E_z = \frac{\partial^2 \mathcal{E}}{\partial z^2} + k_0^2 \mathcal{E}, \quad Z_0 H_z = \frac{\partial^2 \mathcal{H}}{\partial z^2} + k_0^2 \mathcal{H}, \tag{7}
\]

in (5), and we are led to search scattered potentials \( \mathcal{E}_s \) and \( \mathcal{H}_s \), satisfying the Helmholtz equation as \( z > 0 \), regular and vanishing as \( z \to \infty \) when \( |\arg(i k_0)| < \pi/2 \), that verify,

\[
\prod_{i=1}^{N_e} \left( \frac{\partial}{\partial z} - i k_0 g_j^e \right) \mathcal{E}_s(z) = \prod_{j=1}^{N_e} \left( \frac{\partial}{\partial z} + i k_0 g_j^e \right) \mathcal{E}_i( -z),
\]

\[
\prod_{j=1}^{N_h} \left( \frac{\partial}{\partial z} - i k_0 g_j^h \right) \mathcal{H}_s(z) = \prod_{j=1}^{N_h} \left( \frac{\partial}{\partial z} + i k_0 g_j^h \right) \mathcal{H}_i( -z), \tag{8}
\]

where \( \mathcal{E}_i \) and \( \mathcal{H}_i \) potentials are attached to radiation of arbitrary primary sources.
2.3) Extended boundary conditions

More generally, we can consider an extended form when we want to include the case of a multilayer slab in free space, composed of isotropic [4]-[5] (or $z$-axial anisotropic [21]) layers. The reflection coefficients $R_{e,e}$ and $R_{h,h}$ remain meromorphic functions of $\cos \beta$, but we now model them in a more general form, following,

$$
R_{e,e}(\beta) = R_0^e \prod_{j=1}^{N_e} \frac{(\cos \beta - g_j^e)}{(\cos \beta + g_j^e)}, \quad R_{h,h}(\beta) = R_0^h \prod_{j=1}^{N_h} \frac{(\cos \beta - g_j^h)}{(\cos \beta + g_j^h)},
$$

with simple poles $-g_j^{(e,h)}$, $N_{e(h)}^{'} \leq N_{e(h)}$, constants $R_0^e$ and $R_0^h$, for which we notice the basic equalities (without superscripts $e$ and $h$),

$$
R_0^e = -\frac{\prod_{j=1}^{N_e}(\cos \beta - g_j^e)}{\prod_{j=1}^{N_e}(\cos \beta + g_j^e)}, \quad a_{0e} = \sum_{j=1}^{N_e} \frac{(-\cos \beta)^{l-\frac{1}{2}}}{\cos \beta + g_j^e} = \sum_{j=1}^{N_e} \frac{\tau_j}{g_j^e}(\cos \beta)^{\tau} + 1,
$$

where the constants $a_{0e}$ refer to limit values when $|\cos \beta|^{\tau} \to \infty$ with $\tau = +1$ or $-1$. We have $N \geq N'$, $N \geq 1$, $N' \geq 0$, and we can let $\prod_{i \neq j}(g_i - g_j) \equiv 1$ for $N = 1$ and $\prod_{i=1}^{N'}(g_i - g_j) \equiv 1$ for $N' = 0$.

As in previous section, we consider that $R_{h,h}(0) = -R_{e,e}(0)$, and assume that $R_{h,h}(\pi) = -R_{e,e}(\pi)$. This implies, after using (10) when $\cos \beta = \pm 1$,

$$
\sum_{j=1}^{N_e} \frac{\tau_e}{g_j^e}(g_j^e)^{\tau_e} \pm 1 + \sum_{j=1}^{N_h} \frac{\tau_h}{g_j^h}(g_j^h)^{\tau_h} \pm 1 = -(a_{0e} + a_{0h}),
$$

with $\tau_{e(h)} = +1$ or $-1$. The condition (11), as previously noticed for (6), has a crucial importance to avoid non physical behaviours of fields derived from potentials.
Considering (9) and plane waves representation of fields (appendix 1), we can write

$$\prod_{j=1}^{N_i} \left( -\frac{\partial}{ik_0\partial z} + g_j^e \right) E_z^s(z) |_{z=0} = R_0^e \prod_{j=1}^{N_i'} \left( -\frac{\partial}{ik_0\partial z} - g_j'^e \right) E_z^i(-z) |_{z=0},$$

$$\prod_{j=1}^{N_i} \left( -\frac{\partial}{ik_0\partial z} + g_j^h \right) H_z^s(z) |_{z=0} = R_0^h \prod_{j=1}^{N_i'} \left( -\frac{\partial}{ik_0\partial z} - g_j'^h \right) H_z^i(-z) |_{z=0},$$

which applies for any primary sources that illuminates the plane. Using (7), we can solve (12) with \((\mathcal{E}_s, \mathcal{H}_s)\) that satisfy as \(z > 0\),

$$\prod_{i=1}^{N_i} \left( -\frac{\partial}{ik_0\partial z} + g_j^e \right) \mathcal{E}_s(z) = R_0^e \prod_{j=1}^{N_i'} \left( -\frac{\partial}{ik_0\partial z} - g_j'^e \right) \mathcal{E}_i(-z),$$

$$\prod_{j=1}^{N_i} \left( -\frac{\partial}{ik_0\partial z} + g_j^h \right) \mathcal{H}_s(z) = R_0^h \prod_{j=1}^{N_i'} \left( -\frac{\partial}{ik_0\partial z} - g_j'^h \right) \mathcal{H}_i(-z).$$

(13)

figure 1) geometry: sources \((J, M)\) and observation point above the plane \(z = 0\)

Remark: For \(z\)-uniaxial chiral layers [17], non-diagonal terms in (81) don't vanish, and

$$\begin{pmatrix}
\mathcal{P}_{N_i}^e \left( \frac{\partial}{\partial z} \right) \mathcal{E}_s(z) \\
\mathcal{P}_{N_i}^h \left( \frac{\partial}{\partial z} \right) \mathcal{H}_s(z)
\end{pmatrix} =
\begin{pmatrix}
Q_{N'_i}^{e,e} \left( \frac{\partial}{\partial z} \right) & Q_{N'_i}^{e,h} \left( \frac{\partial}{\partial z} \right) \\
Q_{N'_i}^{h,e} \left( \frac{\partial}{\partial z} \right) & Q_{N'_i}^{h,h} \left( \frac{\partial}{\partial z} \right)
\end{pmatrix}
\begin{pmatrix}
\mathcal{E}_i(-z) \\
\mathcal{H}_i(-z)
\end{pmatrix},$$

(14)

applies with polynomials \(\mathcal{P}^{e,(h)}\) and \(Q^{e,(h)}\). By linearity, it can be solved by addition of terms similar to ones derived for the solution of equations (13).

Remark: Letting \(R = (\cos\beta - \mathcal{Z}(\cos\beta))/(\cos\beta + \mathcal{Z}(\cos\beta))\), \(\mathcal{Z}\) is a rational function which is even when (3) applies [4,5,21] (see also appendix A of [2]).

Remark: Considering (11), we can assume that, when \(g_j^{e,(h)} \to \pm 1\), it exists a \(g_m^{h,(e)}\) such that \(a_j^{e} (g_j^{e,(h)} \pm 1)^{-1} + a_m^{h} (g_m^{h,(e)} \pm 1)^{-1}\) is bounded.
2.4) Properties of $R_{e,e}$ and $R_{h,h}$, and consequences on the $g_j^{e,h}$

The multilayer is characterized by permittivity $\varepsilon(z)$ and permeability $\mu(z)$ as $z$ varies from $z = 0$ to $-h_L$, which corresponds to a set of complex $\varepsilon_j$ and $\mu_j$ specified in each layer $-h_j \leq z \leq -h_{j-1}$ with $1 \leq j \leq L$, $h_0 = 0$. Considering the hypotheses for (3) and (9), the last numbered layer shall be impenetrable or is bounded by the free space at $z = -h_L$. The plane wave spectrum of fields follows certain rules (see appendix 1), and the reflection coefficients, called $R$ to simplify, have some elementary properties:

(a) $R$ is an analytic function of the complex variables $k_0$, $\cos \beta$, $i \omega \varepsilon$, $i \omega \mu$, and of layers depths. In the domain of passivity $\Omega_{\varepsilon,\mu}$ with $\text{Re}(i \omega \varepsilon) > 0$ and $\text{Re}(i \omega \mu) > 0$, $R$ has no singularity when $\beta$ varies from 0 to $i \infty + \arg(i k_0)$ as $\text{Re}(i k_0 \sin \beta) = 0$, and at infinity. This regularity applies in the whole domain with $|\arg(i \omega)| \leq \pi/2$ from causality principle. Its highest modulus is obtained for real $\omega$ from maximum modulus principle, while, from (84), $|R| \leq 1$ when $k_0$ is real and $0 \leq \beta \leq \pi/2$.

(b) When $i \omega \varepsilon$ and $i \omega \mu$ are purely real positive (perfect lossy case), the multilayer is purely resistive, and, in these circumstances, $R$ is real when $k_0$ and $\beta$ are real.

(c) From (a) and (b), we can apply the Schwarz reflection principle (or edge-of-the-wedge theorem) [24, sect. 5] (see also [25]), and deduce that $R$ satisfies in $\Omega_{\varepsilon,\mu}$,

$$R((k_0)^*, (\cos \beta)^*, (i \omega \varepsilon)^*, (i \omega \mu)^*)$$

$$= (R(k_0, \cos \beta, i \omega \varepsilon, i \omega \mu))^*.$$  \hspace{1cm} (15)

(d) Using $(i \varepsilon)^* = i(-\varepsilon^*)$ and $(i \mu)^* = i(-\mu^*)$ in (15), we conclude that, for real $\omega$, if we change $\varepsilon$ and $\mu$ for their 'anticonjugate' $-\varepsilon^*$ and $-\mu^*$ (possible in practice with metamaterials), then every pole $\cos \beta = (-g_j^{e,h})$ of $R$ in (3) and (9) changes for its conjugate $(-g_j^{e,h})^*$.

Adding that the set \{ $g_j^{e,h}$ \} has elements with positive and negative real parts [26]-[28], we then have from (d),

(e) The set \{ $g_j^{e,h}$ \} $\cup$ \{ $(g_j^{e,h})^*$ \} leaves no quarter-plane of the complex plane empty, which explains why we have a so wide range of properties when multilayers include metamaterials.

Let us now consider the example of a lossless system where $\varepsilon$ and $\mu$ have purely positive (resp. negative) real values. As defined in (d), both cases are respectively anticonjugate. Pure imaginary $g_j^{e,h}$ are generally of finite number. From [19] (resp. [16]) the other complex $g_j^{e,h}$ with non null real part are of infinite number, and they are complex
numbers with negative (resp. positive) imaginary parts, associated with improper (resp. proper) modes. Consequently, the domains of $g_{j}^{(h)}$ for both cases are conjugated each other, which plainly illustrates and confirms the property (d).

Remark: Considering the appendix 1, the conditions (a)-(e) also apply for anisotropic multilayers, when $\varepsilon$ and $\mu$ are tensors.

Remark: $k_j = \omega(\mu_j\varepsilon_j)^{1/2}$ with $(\mu_j\varepsilon_j)^{1/2} = \sqrt{\mu_j}\sqrt{\varepsilon_j}$ can be $\neq \omega\sqrt{\mu_j\varepsilon_j}$, and thus, a usual term like $((\mu_j\varepsilon_j)^{1/2}/\varepsilon_j)\tan(\omega(\mu_j\varepsilon_j)^{1/2}d)$ can be $\neq \sqrt{\mu_j}\varepsilon_j\tan(\omega\sqrt{\mu_j\varepsilon_j}d)$.

3) Incident potentials ($\mathcal{E}_i, \mathcal{H}_i$) for arbitrary bounded primary sources

To solve problems with conditions (8) or (13), we need a correct explicit expression of ($\mathcal{E}_i, \mathcal{H}_i$). We begin by considering the incident field ($E^i, H^i$) at point $r$ of coordinates $(x, y, z)$, radiated by arbitrary electric and magnetic bounded sources $J$ and $M$ in free space [3],

$$E^i = \text{curl}(G*J) + \frac{i}{\omega\varepsilon_0} (\text{grad} \text{(div(.$))) + k_0^2)(G*J),$$

$$H^i = - \text{curl}(G*J) + \frac{i}{\omega\mu_0} (\text{grad} \text{(div(.$))) + k_0^2)(G*J),$$

(16)

where $G(r) = -\frac{e^{-ik_0|r|}}{4\pi|r|}$ with $|r| = \sqrt{x^2 + y^2 + z^2}$ verifies $(\Delta + k_0^2)G = \delta$, $*$ is the convolution product, and $J$ and $M$ are generalized functions [29]. Considering arbitrary $J$ and $M$ in the domain $\pm z > 0$, the potentials ($\mathcal{E}_i, \mathcal{H}_i$) in the representation of fields $E^i$ and $H^i$ with (1), which satisfy the Helmholtz equation as $\mp z > 0$ and vanish as $\mp z \to \infty$ when $|\text{arg}(ik_0)| < \pi/2$, can be written when $\mp z > 0$,

$$\mathcal{E}_i, \mathcal{H}_i = \frac{2}{8\pi k_0} \mathcal{L}(Z_0J, M) \star \mathcal{W},$$

(17)

where $(A, B)\star C \equiv (A\star C, B\star C)$, $\mathcal{L}$ is defined as in (2),

$$\mathcal{L}(Z_0J, M) = (\mathcal{L}_1(Z_0J, M), \mathcal{L}_2(Z_0J, M)),
\mathcal{L}_1(u,v) = ((\text{grad} \text{(div(.$))) + k_0^2)(u) - ik_0\text{curl}(v)),
\mathcal{L}_2(u,v) = (ik_0\text{curl}(u) + (\text{grad} \text{(div(.$))) + k_0^2)(v)),$$

and $\mathcal{W}$ has the remarkable compact form [1]-[2],

$$\mathcal{W}(r) = e^{ik_0|z|}E_1(ik_0(|r| + |z|))
+ e^{-ik_0|z|}E_1(ik_0(|r| - |z|) + 2\ln\rho),$$

(18)
with \( \rho = \sqrt{x^2 + y^2} \), \( E_1 \) being the exponential integral function \([30]\), which verifies, as \( \mp z > 0 \),

\[
\left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \frac{W(r)}{8\pi i k_0} = G(r), (\Delta + k_0^2)W(r) = 0. \tag{19}
\]

Remark : It is worth noticing that we have \( \Delta_{xy} \ln(\rho) = 2\pi \delta(x)\delta(y) \), which implies that \( \ln \rho \) in \( \mathcal{W} \) has no influence on the expression of the field except by its singularity at \( \rho = 0 \).

Remark : Dividing \( J \) and \( M \) into the sources above and below a plane \( z = z_1 \), \( J_{z_1}^\pm = U_s(\pm (z - z_1))J \) and \( M_{z_1}^\pm = U_s(\pm (z - z_1))M \), we can write for arbitrary \( z \),

\[
(E^i, Z_0 H^i)(x, y, z_1) = \sum_\pm (E^\pm, Z_0 H^\pm)(x, y, z_1 \mp 0^\mp), \tag{20}
\]

where \( (E^\pm, H^\pm) \) are the field radiated by the sources \( J_{z_1}^\pm \) and \( M_{z_1}^\pm \), following \( (E^\pm, Z_0 H^\pm) = \mathcal{L}(\bar{z}\mathcal{E}^\pm, \bar{z}\mathcal{H}^\pm) \) from \( (2) \) with \( (\mathcal{E}^\pm, \mathcal{H}^\pm) = \frac{3}{8\pi k_0} \mathcal{L}(Z_0 J_z^\pm, M_z^\pm)*\mathcal{W} \) from \( (17) \), and \( U_s \) is the unit step function \([30]\) with \( U_s(z) = (\text{sign}(z) + 1)/2 \).

4) Scattered potentials \( (\mathcal{E}_s, \mathcal{H}_s) \) for arbitrary primary sources

4.1) A global expression of \( (\mathcal{E}_s, \mathcal{H}_s) \) for a multimode plane

Considering the potentials \( (\mathcal{E}_i, \mathcal{H}_i) \) attached to the radiation of arbitrary sources from \( (17) \), we can now express the scalar potentials \( \mathcal{E}_s \) and \( \mathcal{H}_s \) which satisfy the multimode conditions \( (8) \) or \( (13) \), as \( z \geq 0 \). From the method developed in \([1]-[2] \), we have for conditions \( (8) \),

\[
\mathcal{E}_s(x, y, z) = \pm \mathcal{E}_i(x, y, -z) + \left( \frac{\bar{z}}{\omega \varepsilon_0} \text{grad(div}(J)) + k_0^2 J \right) \frac{8\pi k_0}{8\pi k_0} + \frac{\bar{z}}{k} \left( -ik_0 \text{curl}(M) \right) \frac{8\pi k_0}{8\pi k_0} \times \sum_{\alpha = -1, 1} \left( \prod_{j=1}^{N_e} \frac{\varepsilon'_e + g_j^e}{\varepsilon'_e - g_j^e} \right) \mathcal{N}_e \left( \sum_{j=1}^{N_e} \varepsilon'_e a_j^e \bar{K}_j^e \right) (x, y, -z), \tag{21}
\]

and
\[ H_s(x, y, z) = \mp H_i(x, y, -z) \]
\[ + \left( \frac{2}{\omega \varepsilon_0} - \frac{2}{8\pi k_0} + \frac{2}{k} \right) (\text{grad(div}(M)) + k_0^2 M) \]
\[ * \sum_{\varepsilon' = -1, 1} \left( \frac{N_0^j \varepsilon' + g_j^h}{\sum_{j=1}^{N_0^j} \varepsilon' a_j^h K_j^g} \right) \right) (x, y, -z), \quad (22) \]

while we have, for extended conditions (13),
\[ E_s(x, y, z) = a_0^e E_i(x, y, -z) \]
\[ + \left( \frac{2}{\omega \varepsilon_0} - \frac{2}{8\pi k_0} + \frac{2}{k} \right) (\text{grad(div}(M)) + k_0^2 M) \]
\[ * \sum_{\varepsilon' = -1, 1} \left( \frac{N_0^j \varepsilon' + g_j^h}{\sum_{j=1}^{N_0^j} \varepsilon' a_j^h K_j^g} \right) \right) (x, y, -z), \quad (23) \]

and
\[ H_s(x, y, z) = a_0^h H_i(x, y, -z) \]
\[ + \left( \frac{2}{\omega \varepsilon_0} - \frac{2}{8\pi k_0} + \frac{2}{k} \right) (\text{grad(div}(M)) + k_0^2 M) \]
\[ * \sum_{\varepsilon' = -1, 1} \left( \frac{N_0^j \varepsilon' + g_j^h}{\sum_{j=1}^{N_0^j} \varepsilon' a_j^h K_j^g} \right) \right) (x, y, -z), \quad (24) \]

where \( V_e, K_g \) are given by,
\[ V_e(x, y, -z) = e^{ik_0z} (E_1(ik_0(|r| + \varepsilon' z)) + (1 - \varepsilon') \ln \rho), \]
\[ K_g(x, y, -z) = e^{ik_0z} \mathcal{J}_g(r, -z), \quad (25) \]

with \( \rho = \sqrt{x^2 + y^2} \), as \( z \geq 0 \). The function \( \mathcal{J}_g(\rho, -z) \) is given by the integral,
\[ \mathcal{J}_g(\rho, -z) = \frac{e^{ik_0z}}{2} \int_{\mathcal{D}} \frac{H_0^{(2)}(k_0 \rho \sin \beta) e^{-ik_0z \cos \beta}}{\cos \beta + g} \sin \beta d\beta, \quad (26) \]

with \( \text{Re}(ik_0 \sin \beta) = 0 \) on \( \mathcal{D} \) from \( -i \infty - \arg(ik_0) \) to \( i \infty + \arg(ik_0) \), \( H_0^{(2)} \) being the Hankel function [30], which is a Fourier-Bessel integral also called a Sommerfeld-type integral [31]. Letting \( g = \sin \theta_1 \) with \( \text{Re}(\theta_1) \leq \pi/2 \), we notice the cut \( \text{Re}(ik_0 \cos \theta_1) = 0 \) as \( g \) varies, due to poles of \( (\cos \beta + g)^{-1} \) that can go through \( \mathcal{D} \) when \( \text{Re}(g) < 0 \). The function \( \mathcal{J}_g \) will be rewritten and developed in an efficient way in section 6.

The choice of sign, \( \pm \) in (21)-(22) and \( \tau_{\varepsilon' h} \equiv \pm \) in (23)-(24), is free (note: algebraic modifications can apply as detailed in appendix 2).
4.2) Reciprocity principle from our expressions

Let us consider the scattered fields \((E^s_m, H^s_m)\) for \(m = 1, 2\), excited by primary bounded sources \((J_m, M_m)\) above the plane and corresponding to the potentials \((\mathcal{E}_{ms}, \mathcal{H}_{ms})\). We have, using (2) and integration by parts in sense of generalized functions [29],

\[
\int_{R^3} (Z_0 J_1(r) . E_2^s(r) - M_1(r) . Z_0 H_2^s(r)) dV_r \\
= \int_{R^3} (L_1(Z_0 J_1, M_1) \mathcal{E}_{2s} \hat{\hat{\gamma}} - L_2(Z_0 J_1, M_1) \mathcal{H}_{2s} \hat{\hat{\gamma}}) dV_r,
\]

while, from (21)-(24), \((\mathcal{E}_{ms}, \mathcal{H}_{ms})\) expresses in the form,

\[
(\mathcal{E}_{ms}, \mathcal{H}_{ms})(r) = \frac{\hat{\hat{\gamma}}}{8\pi k_0^2} \cdot (L_1(Z_0 J_m, M_m), \mathcal{L}_2(Z_0 J_m, M_m) * \mathcal{R}_s)(x, y, -z),
\]

with \(\mathcal{R}_s(x - x', y - y', - z - z')\) unchanged as \(r \leftrightarrow r'\). We can then write,

\[
\int_{R^3} (Z_0 J_1(r) . E_2^s(r) - M_1(r) . Z_0 H_2^s(r)) dV_r = \\
\frac{1}{8\pi k_0^2} \int_{R^3} dV_r \int_{R^3} dV_r' \sum_{j=1,2} (-1)^{j-1} [(L_j(Z_0 J_1, M_1)(r) \hat{\hat{\gamma}}) \\
\times (L_j(Z_0 J_2, M_2)(r') \hat{\hat{\gamma}})] \mathcal{R}_s(x - x', y - y', - z - z') \\
= \int_{R^3} (Z_0 J_2(r') . E_1^s(r') - M_2(r') . Z_0 H_1^s(r')) dV_r',
\]

Thus, we can exchange indices 1 and 2 without changing the result, and reciprocity theorem [3, p.118] is satisfied.

5) When primary sources \(J\) and \(M\) are arbitrarily oriented dipoles

5.1) Tensorial expressions of fields

Considering (21)-(24), we now develop the expression of scattered field for the case with electric and magnetic point sources, \(J = J'_r \delta(r - r')\) and \(M = M'_r \delta(r - r')\), where \(\delta(r)\) is the Dirac generalized function [29], while the expressions of scattered fields for volume and surface sources are analyzed in appendix 3. Using (1) with (21)-(24), we have for the scattered field \((E^s, H^s)\),
\[ E^s(r) = -ik_0 \text{curl}(\mathcal{H}_s \hat{z}) + (\text{grad}(\text{div}.)) + k_0^2)(\mathcal{E}_s \hat{z}) \]
\[ = \frac{1}{8\pi k_0^2}([\mathcal{F}_{he}(r, r')].Z_0 J_r - [\mathcal{D}_{he}(r, r')].M_r), \]  
(30)

and

\[ Z_0 H^s(r) = ik_0 \text{curl}(\mathcal{E}_s \hat{z}) + (\text{grad}(\text{div}.)) + k_0^2)(\mathcal{H}_s \hat{z}) \]
\[ = \frac{1}{8\pi k_0^2}([\mathcal{D}_{eh}(r, r')].Z_0 J_r + [\mathcal{F}_{eh}(r, r')].M_r), \]  
(31)

where \([\mathcal{F}_{he(eh)}(r, r')]\) and \([\mathcal{D}_{eh(eh)}(r, r')]\) are dyadic tensors, with \(C.[\hat{a} \hat{b}] = (C. \hat{a}) \hat{b}\) and \([\hat{a} \hat{b}].C = \hat{a} (\hat{b}.C)\), \(r = x \hat{x} + y \hat{y} + z \hat{z}, r' = x' \hat{x}' + y' \hat{y}' + z' \hat{z}'\), \(\hat{x} \equiv \hat{x}', \hat{y} \equiv \hat{y}', \hat{z} \equiv \hat{z}'\).

Noting that we have,

\[ \varsigma(\partial_z f \ast g)(x, y, \varsigma z) = \partial_z(\langle f \ast g \rangle(x, y, \varsigma z)), \]
\[ (\partial_x f \ast g)(x, y, \varsigma z) = \partial_x(\langle f \ast g \rangle(x, y, \varsigma z)), \]
\[ (\partial_y f \ast g)(x, y, \varsigma z) = \partial_y(\langle f \ast g \rangle(x, y, \varsigma z)), \]
\[ \partial^i_x \partial^j_y \partial^k_z \equiv \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} \text{and} \partial_{xy} \equiv \partial_x \partial_y, \]
(32)

these tensors express following,

\[ [\mathcal{D}_{he(eh)}(r, r')] = \]
\[ = ik_0[(\hat{x} \partial_y - \hat{y} \partial_x)(\hat{x}' \partial_x + \hat{y}' \partial_y)\varsigma \partial_z \mathcal{S}_{he(eh)}(r, r') \]
\[ + (\hat{x} \partial_x + \hat{y} \partial_y)(\hat{y}' \partial_x - \hat{x}' \partial_y)\varsigma \partial_z \mathcal{S}_{he(eh)}(r, r') \]
\[ + (\hat{x} \partial_y - \hat{y} \partial_x)\varsigma'(\partial_z^2 + k_0^2)\mathcal{S}_{he(eh)}(r, r') \]
\[ + \varsigma(\hat{y}' \partial_x - \hat{x}' \partial_y)(\partial_z^2 + k_0^2)\mathcal{S}_{he(eh)}(r, r') \]
\[ = ik_0[\langle \hat{x} \partial_y - \hat{y} \partial_x \rangle \partial_{xy} + \hat{x} \hat{y} \partial^2_y \]
\[ - \hat{y} \partial_z \partial^2_x \partial_z (\varsigma \mathcal{S}_{he(eh)}(r, r') - \mathcal{S}_{he(eh)}(r, r')) \]
\[ + ik_0[(-\hat{x} \hat{y}' - \hat{y} \hat{x}') \partial_z (\partial_z^2 + k_0^2) \mathcal{S}_{he(eh)}(r, r') \]
\[ + (\hat{x} \partial_y - \hat{y} \partial_x)\varsigma'(\partial_z^2 + k_0^2)\mathcal{S}_{he(eh)}(r, r') \]
\[ + \varsigma(\hat{y}' \partial_x - \hat{x}' \partial_y)(\partial_z^2 + k_0^2)\mathcal{S}_{he(eh)}(r, r')], \]  
(33)

and
\[
\begin{align*}
[F_{he(eh)}(r, r')] &= \\
&= \left[ (\hat{x}\partial_x + \hat{y}\partial_y)(\hat{x}'\partial_x + \hat{y}'\partial_y) (\hat{\zeta}\partial_z^2 S_{e,(h)}(r, r') + k_0^2 S_{h,(e)}(r, r')) \right] \\
&\quad + [b_0^2 (\hat{x}\hat{\zeta} + \hat{y}\hat{\zeta}')(\partial_z^2 + k_0^2)](S_{h,(e)}(r, r')) \\
&\quad + (\hat{x}\partial_x + \hat{y}\partial_y)^2 \partial_z(\hat{\zeta}\partial_z^2 + k_0^2)S_{e,(h)}(r, r') \\
&\quad + \hat{z}(\hat{x}'\partial_x + \hat{y}'\partial_y)\hat{z}'\partial_z(\partial_z^2 + k_0^2)S_{e,(h)}(r, r') \\
&\quad + \hat{z}\hat{z}'(\partial_z^2 + k_0^2)(\partial_z^2 + k_0^2)S_{e,(h)}(r, r'), \\
\end{align*}
\]

where \( \zeta = -1 \). Using (86) of appendix 2, the \( S_{e,(h)}(r, r') \) can be written,

\[
S_e(r, r') = c^e S_i(r_{im} - r') + \sum_{j=1}^{N_e} \frac{\tau_e a_j^e}{g_j^e} ((g_j^e)^{\tau_e}) \\
\times \sum_{e'=-1,1} \frac{V_{e'} + e'K_{g_j^e}}{(g_j^e)^{\tau_e} - e'\tau_e})(r_{im} - r'),
\]

\[
S_h(r, r') = c^h S_i(r_{im} - r') + \sum_{j=1}^{N_h} \frac{\tau_h a_j^h}{g_j^h} ((g_j^h)^{\tau_h}) \\
\times \sum_{e'=-1,1} \frac{V_{e'} + e'K_{g_j^h}}{(g_j^h)^{\tau_h} - e'\tau_h})(r_{im} - r'),
\]

where \( r_{im} \equiv (x, y, -z) \) and \( r' \equiv (x', y', z') \), and we have \( c^e = -c^h = \pm 1 \) and \( \tau_e = -\tau_h = \pm 1 \) as (3)-(8) with (21)-(22) applies, while \( c^{e,(h)} = a_0^{e,(h)} \) with \( \tau_{e,(h)} = \pm 1 \) or \(-1\) for the extended case as (9)-(13) with (23)-(24) applies. The functions \( V_{e'}, K_g, \) and \( S_i \) satisfy the Helmholtz equation above the plane, and are defined, following,

\[
V_{e'}(x, y, -z) = e^{i\xi_0\rho}(E_1(i\xi_0\rho) + e'\ln\rho),
\]

\[
K_g(x, y, -z) = e^{ik_0\rho} J_g(\rho, -z), \quad S_i = \sum_{e'=-1,1} V_{e'},
\]

for \( z \geq 0, \rho = \sqrt{x^2 + y^2}, r^2 = \rho^2 + z^2 \), and they verify,
\[ \partial_z \mathcal{V}_e(x, y, -z) = -\epsilon \frac{e^{-ik_0|z|}}{|r|} + ik_0 \epsilon' \mathcal{V}_e(x, y, -z), \]
\[ \partial_z \mathcal{K}_g(x, y, -z) = \frac{e^{-ik_0|z|}}{|r|} + ik_0 g \mathcal{K}_g(x, y, -z), \]
\[ \partial_z ((\mathcal{V}_e + \epsilon' \mathcal{K}_g)(x, y, -z)) = ik_0 \epsilon' (\mathcal{V}_e + g \mathcal{K}_g)(x, y, -z), \]
\[ (\partial_z^2 + k_0^2) \mathcal{V}_e = ( -\epsilon' \partial_z - ik_0 ) \frac{e^{-ik_0|z|}}{|r|}, \]
\[ (\partial_z^2 + k_0^2) \mathcal{K}_g = (\partial_z + ik_0 g) \frac{e^{-ik_0|z|}}{|r|} + (k_0^2 - (k_0 g)^2) \mathcal{K}_g, \]
\[ \partial_z^2 (\mathcal{V}_e + \epsilon' \mathcal{K}_g) = -ik_0 \epsilon' (\epsilon' - g) \frac{e^{-ik_0|z|}}{|r|}, \]
\[ -k_0^2 (\mathcal{V}_e + g^2 \epsilon' \mathcal{K}_g). \]  

Remark: we notice that,
\[ \text{curl}_r ([\mathcal{D}_{he(eh)}(r, r')].C_{r'}) = ik(\mathcal{F}_{eh(eh)}(r, r')].C_{r'}), \]
\[ \text{curl}_r ([\mathcal{D}_{he(eh)}(r, r')].C_{r'}) = -ik(\mathcal{D}_{eh(eh)}(r, r')].C_{r'}), \]  

and
\[ D_r.\mathcal{F}_{he(eh)}(r, r').C_{r'} = C_{r'} \mathcal{F}_{he(eh)}(r', r).D_r, \]
\[ D_r.\mathcal{D}_{he(eh)}(r, r').C_{r'} = C_{r'} \mathcal{D}_{eh(eh)}(r', r).D_r, \]  

with \( C_{r'} = c_x \hat{x}' + c_y \hat{y}' + c_z \hat{z}', \) \( D_r = d_x \hat{x} + d_y \hat{y} + d_z \hat{z} \) being two constant vectors.

5.2) Reduction of sums over \( \epsilon' \) indices

We can simplify in (33)-(35) the sums of terms over \( \epsilon' \) indices, and then reduce the derivative order of the following terms,
\[ \mathcal{U}_a = (\partial_z^2 + k_0^2) \mathcal{S}_{e(h)}(r, r'), \]
\[ \mathcal{U}_b = \partial_z (\mathcal{S}_{h(e)}(r, r') + \mathcal{S}_{e(h)}(r, r')) \]
\[ \mathcal{U}_c = -\partial_z^2 \mathcal{S}_{e(h)}(r, r') + k_0^2 \mathcal{S}_{h(e)}(r, r'), \]  

for the field expressions (30)-(31).
For $U_a = (\partial^2 + k_0^2)S_{c(h)}(r, r')$ term, we use (37), and noting that,

$$
\sum_{e'=-1,1} \left( \partial^2 + k_0^2 \right) \left( \frac{\mathcal{V}_{e'} + e'\mathcal{K}_g}{g - e'} \right) = -2k_0^2\mathcal{K}_g,
$$

$$
\sum_{e'=-1,1} \left( \partial^2 + k_0^2 \right) \frac{1}{g} \left( \frac{\mathcal{V}_{e'} + e'\mathcal{K}_g}{g - e'} \right) = -2ik_0 \frac{e^{-ik_0|r|}}{|r|} + 2k_0^2g\mathcal{K}_g,
$$

$$
\sum_{e'=-1,1} \left( \partial^2 + k_0^2 \right) \mathcal{V}_{e'} = -2ik_0 \frac{e^{-ik_0|r|}}{|r|}.
$$

we have,

$$
U_a = -2k_0^2 \left( \left( 2e^{(h)} + \tau^{(h)} - \frac{1}{2} \sum_{j=1}^{N_{c(h)}} \frac{a_j^{(h)} g_j^{(h)}}{g_j^{(h)}} \right) + \frac{1}{2} \sum_{j=1}^{N_{c(h)}} a_j^{(h)} \mathcal{K}_g^{(h)} \right),
$$

with $\sum_{j=1}^{N_{c(h)}} a_j^{(h)} = -2$ (resp. $= a_{0-} - a_{0+}$) when (3) (resp. (9)) applies.

For $U_b = \partial_z (S_{h(e)}(r, r') + S_{c(h)}(r, r'))$ and $U_c = -\partial_z^2 S_{c(h)}(r, r') + k_0^2 S_{h(e)}(r, r')$, it will be done provided the condition (6) or (11) is used, which suppresses pure residual terms $\mathcal{V}_{e'}$ in them, in particular $\mathcal{V}_{-1}$ that diverges as $\rho \to \infty$. Considering (85) given in appendix, we note that $\mathcal{V}_{e'}$ factor vanishes following,

$$
\sum_{j=1}^{N_{c(h)}} \frac{\tau_{e(h)} a_j^{(h)}}{g_j^{(h)}} (g_j^{(h)}) \tau_{e(h)} + \sum_{j=1}^{N_{c(h)}} \frac{\tau_{h(e)} a_j^{(h)}}{g_j^{(h)}} (g_j^{(h)}) \tau_{h(e)}
$$

$$
\frac{1}{2} \left( g_j^{(h)} \mathcal{K}_g^{(h)} - (g_j^{(h)}) \mathcal{B} \right)
$$

Using (37) and $\frac{(g)^{e+1}}{g^{e+2}} = \pm \frac{g}{g-1}$, we then derive for $U_b$ that,

$$
U_b = 2ik_0 \sum_{v, e, h} \left[ \frac{e^{-ik_0|r|}}{ik_0|r|} \sum_{j=1}^{N_{c(h)}} a_j^v (1 - (g_j^v)^\tau \mathcal{B}') + \sum_{j=1}^{N_e} a_j^v \frac{g_j^v \mathcal{K}_g^{(h)} - (g_j^v)^\tau \mathcal{B}}{(g_j^v)^2 - 1} - c^v \left( \frac{e^{-ik_0|r|}}{2ik_0|r|} \mathcal{B}' + \frac{\mathcal{B}}{2} \right) \right].
$$

In this expression, $\mathcal{B}$ and $\mathcal{B}'$ are independent of $v$ and their choice is free. When a $g_j^v = g$ is closed to 1 and (11) applies (note: when $g_j^v \to 1$, it exists a $g_{m}^h$ and a constant $C$ such that $(g_j^v - 1)^{-1} \sim C(g_j^v - 1)^{-1}$), or when $N_{c,h} = 1$ and (6) applies (note: $g_i^h$ and $g_i^h$ are then related by $g_i^h = 1/g_i^h = g$), we can take $\mathcal{B}' = (g)^{-\tau}$ and $\mathcal{B} = (g)^{1-\tau} \mathcal{K}_g$. 


In a similar manner, concerning \( \mathcal{U}_e \), we can write,

\[
\mathcal{U}_e = \left[ - (\partial^2_x + k_0^2) \sum_{\epsilon'=-1,1} \left( \sum_{j=1}^{N_{d,(\epsilon)}} \frac{\tau_{e,(\epsilon)j}^{(\epsilon)}}{g_j} \left( \mathcal{V}_{\epsilon'} + \epsilon' \mathcal{K}_{e,(\epsilon)j} \right) \right) \right.
\]

\[
+ c^{(\epsilon)} \mathcal{V}_{\epsilon'} \right] + \left[ k_0^2 \sum_{\epsilon'=-1,1} \left( \sum_{j=1}^{N_{d,(\epsilon)}} \frac{\tau_{e,(\epsilon)j}^{(\epsilon)}}{g_j} \left( \mathcal{V}_{\epsilon'} + \epsilon' \mathcal{K}_{e,(\epsilon)j} \right) \right) \right.
\]

\[
+ c^{(\epsilon)} \mathcal{V}_{\epsilon'} \right]
\]

\[
= 2k_0^2 \left[ \left( (\tau_{e,(\epsilon)} - 1) \sum_{j=1}^{N_{d,(\epsilon)}} \frac{a^{(\epsilon)}_{j}^{(\epsilon)}}{g_j} + 4c^{(\epsilon)} \right) \frac{e^{-ik_0|z|}}{2k_0|z|} \right.
\]

\[
+ \sum_{j=1}^{N_{d,(\epsilon)}} a^{(\epsilon)}_{j}^{(\epsilon)} \mathcal{K}_{e,(\epsilon)j} + \left[ \sum_{j=1}^{N_{d,(\epsilon)}} \frac{a^{(\epsilon)}_{j}^{(\epsilon)}}{g_j} \frac{\mathcal{K}_{e,(\epsilon)j} - \left( g_j^{(\epsilon)} \right)^2 - 1}{\left( g_j^{(\epsilon)} \right)^2 - 1} - \frac{B}{2} \right] \left( c^{(\epsilon)} + c^{(\epsilon)} \right) \right]
\]

(45)

where we have used (41)-(43) and \( \left( g_j^{(\epsilon)} \right)^{\pm 1} = \pm \frac{g}{g^{(\epsilon)}} \). Let us note that the choice of \( B \) is free, while \( \sum_{j=1}^{N_{d,(\epsilon)}} a^{(\epsilon)}_{j}^{(\epsilon)} = -2 \) (resp. \( = a_{0-} - a_{0+} \)) as (3) (resp. (9)) applies.

Therefore, after summations over \( \epsilon' \) and use of analytical reductions (42), (44) and (45) in (33)-(34), the field expressions (30)-(31) are reduced to expansions with second order partial derivatives, covering any cases of dipole sources over arbitrary multimode planes.

Remark : \( \mathcal{S}_{e,(\epsilon)} \) remains definite for \( g^{e,(\epsilon)} = 1 \), because \( \mathcal{V}_{\epsilon'=1} + \mathcal{K}_g \to 0 \) when \( g \to 1 \), while it is singular for \( g^{e,(\epsilon)} = -1 \). Moreover, we have \( gK_g \to -\frac{e^{ik_0|z|}}{ik_0|z|} \) when \( g \to \infty \).

Remark : We can use (30)-(31) for the incident field \( (E^i, H^i) \) in place of \( (E^s, H^s) \), if we take \( \xi = 1 \), instead of \( -1 \), in (33)-(34) and \( \mathcal{S}_i(r - r') \) for \( \mathcal{S}_{e,(\epsilon)}(r, r') \).

6) Integral reduction of \( \mathcal{J}_g \) and exact remarkable expansion with error functions

6.1) A non-singular integral for \( \mathcal{J}_g \)
To avoid any restriction on the sign of the real part of \( g^{e^{ib}} \), we developed in [1] a novel expression of \( J_g \) for arbitrary \( g = \sin \theta_1 \) with \( |\text{Re}(\theta_1)| \leq \pi/2 \) as \( |\text{arg}(ik_0)| \leq \pi/2 \),

\[
J_g(\rho, -z) = i \int_b^{i\infty} e^{-\alpha \cos \theta} d\alpha, \tag{46}
\]

where the parameters \( a \) and \( b \), with \( |\text{Re}(b)| \leq \pi, \text{Re}(a) > 0 \), are defined following,

\[
e^{\pi ib} = \frac{ik_0R}{a}(1 \pm \sin \theta_1)(1 \pm \cos \varphi), \quad a = \epsilon ik_0R \sin \varphi \cos \theta_1, \quad b = i \ln(\epsilon \cot(\pi/4 - \theta_1/2)) - i \ln(\tan(\varphi/2)),
\]

\[
\partial b \quad \partial \varphi = \frac{-i}{\sin \varphi}, \quad \frac{\partial \varphi}{\partial \rho} = \frac{z}{R^2}, \quad \frac{\partial \varphi}{\partial z} = -\frac{\rho}{R^2};
\]

and \( z = R \cos \varphi, \rho = R \sin \varphi, R = \sqrt{\rho^2 + z^2}, \epsilon = \text{sign}(\text{Re}(ik_0R \sin \varphi \cos \theta_1)) \) (\( \text{Re}(a) = 0 \) being a limit case), \( 0 \leq \varphi < \pi/2 \). So defined, we can write,

\[
e^{\pi ib} = \epsilon \cot \left( \frac{\pi}{4} \pm \frac{\theta_1}{2} \right) \cot \left( \frac{\pi}{4} \pm \frac{\varphi}{2} \right),
\]

\[
\left| \tan^2 \left( \frac{b}{2} \right) \right| = \left| \frac{\arccos \epsilon - e\alpha}{\arccos \epsilon + e\alpha} \right| = \left| \frac{1 + \sin(\theta_1 - \varphi)}{1 + \sin(\theta_1 + \varphi)} \right|
\]

\[
e^{\frac{i\theta_1 e^{(\frac{b}{2} - \text{Re}\theta_1)} + e^{-i\varphi}}{e^{\text{Im}\theta_1} e^{(\frac{b}{2} - \text{Re}\theta_1)} + e^{i\varphi}}} \leq 1, \tag{48}
\]

and thus \( \frac{1-\epsilon}{4} \pi \leq |\text{Re}(b)| \leq \frac{3-\epsilon}{4} \pi \), \( \text{sign}(\text{Re}(b)) = -\epsilon \text{sign}(\text{Im}(\sin \theta_1)) \), while we have \( \text{sign}(\text{Im}(a)) = -\text{sign}(\text{arg}(ik_0)) \) when \( \epsilon = -1 \).

It is worth noticing that, as \( g = \sin \theta_1 \) varies in the complex plane, this expression has a correct cut as \( \epsilon \) changes of sign for \( \text{Re}g \leq 0 \), is singular for \( g = -1 \), and is regular elsewhere (note: for \( \text{Re}g > 0 \), the change of sign of \( \epsilon \) does not induce a cut as \( g \) varies).

After writing,

\[
J_g(\rho, -z) = i \left( \int_0^{i\infty} e^{-\alpha \cos \theta} d\alpha - \int_0^b e^{-\alpha \cos \theta} d\alpha \right)
\]

\[
= -\left( K_0(a) + M(a, b) \right), \tag{49}
\]

with \( K_0 \) a modified Bessel function [30], the aim of the calculus will be the reduction of \( M(a, b) = i \int_0^b e^{-\alpha \cos \theta} d\alpha \).

In this respects, we will consider it by regions with value of \( \epsilon \) defined so that,

\[
\left| \tan^2 \left( \frac{b}{2} \right) \right| \leq 1 \text{ with } \epsilon = +1 \text{ or } -1, \text{ as } |\text{Re}(b)| \leq \pi, \tag{50}
\]
as already noted in (48). Indeed, we can exhibit a remarkable link between values of $\mathcal{M}$ in each region, leading us to derive exact and asymptotic expansions everywhere.

Remark : Considering the definitions of $a$ and $b$ given in (47)-(48), we have,

\[
( - i\tan(b/2))^\epsilon = \frac{1 + \sin(\theta_1 - \varphi)}{\cos \varphi + \sin \theta_1} = \frac{\cos \varphi + \sin \theta_1}{1 + \sin(\theta_1 + \varphi)},
\]

\[
2\sin^2(b/2) = \arccos b - a = ik_0 R_l^{(1+\sin(\theta_1 - \varphi))} \text{ when } \epsilon = 1,
\]

\[
2\cos^2(b/2) = \arccos b + a = ik_0 R_l^{(1+\sin(\theta_1 + \varphi))} \text{ when } \epsilon = -1. \tag{51}
\]

Remark : $\mathcal{J}_g$ is also related to the incomplete cylindrical function [32] and to the leaky aquifer function [33]-[34].

6.2) Exact remarkable expansion of $\mathcal{M}(a, b)$ with error functions

6.2.1) The link between the cases $\epsilon = +1$ and $\epsilon = -1$

The function

\[
\mathcal{M}(a, b) = i \int_0^b e^{-a \cos \alpha} d\alpha, \tag{52}
\]

is an entire function (holomorphic at any finite points). It has particular properties in each subset defined in (50) with $\epsilon = \pm 1$, starting with a remarkable link between the cases $\epsilon = +1$ and $\epsilon = -1$.

Indeed, if $\epsilon = -1$ and thus $|\tan(b/2)| \geq 1$, we have,

\[
\mathcal{M}(a, b) = i \int_{b/2}^b e^{-a \cos \alpha} d\alpha + i \int_0^{b/2} e^{-a \cos \alpha} d\alpha = \mathcal{M}(-a, b - \nu \pi) + i \pi \nu I_0(a), \tag{53}
\]

where $\nu = \text{sign}(\text{Re}(b))$, $I_0$ is a modified Bessel function [30], with $|\text{Re}(b - \nu \pi)| \leq \pi$ as $|\text{Re}(b)| \leq \pi$. Considering that $b - \nu \pi$ satisfies,

\[
|\tan^2(b/2)| = 1/|\tan^2(b/2)| \leq 1, \tag{54}
\]

the cases $\epsilon = +1$ and $\epsilon = -1$ perfectly corresponds each other from (53).

Remark : we note that $i\pi I_0(a) = \text{sign}(\text{Im}(a))(K_0(-a) - K_0(a))$ for $\text{Im}(a) \neq 0$ [30].
6.2.2) Exact series for $\mathcal{M}$ with error functions as $|\text{Re}(b)| \leq \pi$

- In the case $\epsilon = 1$, i.e. as $|\tan^2(b/2)| \leq 1$, we use $\sin(\alpha/2) = \sin(b/2)(1 - t)^{1/2}$ in (52), and obtain, in first step,

$$
\mathcal{M}(a, b) = - \int_0^1 \frac{e^{-a \cos b} e^{2a (-i \sin(b/2))^2 t} (-i \sin(b/2))}{(1 - t)^{1/2} (1 + (-i \sin(b/2))^2 (1 - t))^{1/2}} dt. 
$$

(55)

If we only expand for small $\sin(b/2)$, we would obtain a series similar to the ones used by Fisher and Chang [13], but we want to go beyond and define an exact series for the case $\epsilon = 1$, i.e. for arbitrary finite $b$ with $|\tan^2(b/2)| \leq 1$ as $|\text{Re}(b)| \leq \pi$.

For that, noting that $\text{Re}(1/\cos^2(b/2)) = \text{Re}(1 + \tan^2(b/2)) \geq 0$, and thus $|\arg(\cos(b/2))| \leq \pi/4$, we can let

$$(1 + (-i \sin(b/2))^2 (1 - t))^{1/2} = \cos\left(\frac{b}{2}\right)(1 + \tan^2\left(\frac{b}{2}\right)t)^{1/2},$$

(56)

as $0 \leq t \leq 1$, and thus derive,

$$
\mathcal{M}(a, b) = e^{-a \cos b} \int_0^1 \frac{i \tan\left(\frac{b}{2}\right) e^{2a (-i \sin(b/2))^2 t}}{(1 - t)^{1/2} (1 + \tan^2(b/2)t)^{1/2}} dt,
$$

(57)

so that, using the binomial expansion of $(1 + \tan^2(b/2)t)^{-1/2}$, we have

$$
\mathcal{M}(a, b) = i \sum_{m \geq 0} c_m \left(\tan\left(\frac{b}{2}\right)\right)^{2m+1} e^{-a \cos b} \int_0^1 \frac{e^{2a (-i \sin(b/2))^2 t} t^m}{(1 - t)^{1/2}} dt
$$

$$
= i e^{-a \cos b} \sum_{m \geq 0} c_m \left(\tan\left(\frac{b}{2}\right)\right)^{2m+1} \frac{\partial^m \left(\frac{\sqrt{\pi} \text{erf}(s^{1/2})}{s^{1/2}}\right)}{\partial s^m} \bigg|_{s = -2 \sin^2\left(\frac{b}{2}\right)},
$$

(58)

for $\tan^2(b/2) \neq -1$, i.e. $|b| < \infty$. In these expressions, $c_m = (-1)^m (2m)!/((m!)^2 2^{2m})$ is the binomial coefficient for the function $(1 + t)^{-1/2}$, and erf is the error function [30].

We remark that the function $W_m(s) = \frac{e^{-\partial^m}}{\partial s^m} \left(\frac{e^{\text{erf}(s^{1/2})}}{s^{1/2}}\right)$ for large $m \gg |s|$ satisfies,

$$
W_m(s) = W_m(0) + O(s/m^{3/2}) = m^{-1/2} + O(m^{-3/2}),
$$

(59)

$\text{Re}(b)$, while $c_m W_m(0)$ are real negative numbers, which confirms, from Abel's test theorem, the convergence of (58) for finite $b$ with $|\tan^2(b/2)| \leq 1$. 
This convergence can be improved, by letting \( W_m(s) = (W_m(s) - W_m(0)) + W_m(0) \) and noticing that the sum with \( W_m(0) \) is \( e^{-a} \mathcal{M}(0, b) = ibe^{-a} \), which gives us,

\[
\mathcal{M}(a, b) = ie^{-a} \left( \sum_{m \geq 0} c_m \left( \tan \left( \frac{b}{2} \right) \right)^{2m} \right) \times \left( W_m(-2\sin^2 \left( \frac{b}{2} \right)) - W_m(0) \right) + b. \tag{60}
\]

- In the case \( \epsilon = -1 \), i.e. as \( |\tan^2 \left( \frac{b}{2} \right)| > 1 \), we simply exploit the equations (53)-(54) to reduce the problem of calculus of \( \mathcal{M}(a, b) \) to the one of \( \mathcal{M}(a, b - \nu \pi) \), and letting in equations (58)-(60), \( a \to -a \) and \( b \to b - \nu \pi \), with \( \nu = \text{sign}(\text{Re}(b)) \), we have,

\[
\mathcal{M}(a, b) = -ie^{-a\cos b} \left( \sum_{m \geq 0} c_m \left( \cot \left( \frac{b}{2} \right) \right)^{2m} \right) \times \left( e^{a} \left( \sum_{m \geq 0} c_m \left( \cot \left( \frac{b}{2} \right) \right)^{2m} \right) \right) \times \left( W_m(2\cos^2 \left( \frac{b}{2} \right)) - W_m(0) \right) + (b - \nu \pi) + i\pi \nu I_0(a). \tag{61}
\]

6.3) Exact expansion of \( \mathcal{J}_g(\rho, -z) \) for arbitrary parameters

We can now use \( \mathcal{M}(a, b) \) given in (58)-(61), and directly derive, from (49) and (51), an exact convergent expansion of \( \mathcal{J}_g(\rho, -z) \) with error function, as \( z = R\cos \varphi > 0 \) and \( \rho = R\sin \varphi > 0 \), for arbitrary \( g = \sin \theta_1 \), when \( |\text{arg}(ik_0)| \leq \pi/2 \) and \( |\text{Re}(\theta_1)| \leq \pi/2 \) (i.e. \( |\text{arg}(\cos \theta_1)| \leq \pi/2 \)), following,

\[
\mathcal{J}_g(\rho, -z) = f_g(\rho, -z) \quad \text{if} \quad \epsilon = 1,
\]

\[
\mathcal{J}_g(\rho, -z) = f_g(\rho, -z) \quad \text{if} \quad \epsilon = -1, \text{Re}(g) > 0,
\]

\[
\mathcal{J}_g(\rho, -z) = f_g(\rho, -z) - C_g(\rho) \quad \text{if} \quad \epsilon = -1, \text{Re}(g) < 0, \tag{62}
\]
where,

\[
f_g(\rho, -z) = -ie^{-ik_0R(1 + \sin \theta_1 \cos \varphi)} \sum_{m \geq 0} c_m \left( \frac{1 + \sin(\theta_1 - \varphi)}{\cos \varphi + \sin \theta_1} \right)^{2m+1} \times \frac{\partial^m}{\partial s^m} \left( \frac{\sqrt{\pi} e^s \text{erf}(s^{1/2})}{s^{1/2}} \right) \bigg|_{s = ik_0R(1 + \sin(\theta_1 - \varphi))} - K_0(ik_0 \rho \cos \theta_1)
\]

\[
= -ie^{-ik_0 \rho \cos \theta_1} \left( \sqrt{\pi} \sum_{m \geq 0} c_m \left( \frac{1 + \sin(\theta_1 - \varphi)}{\cos \varphi + \sin \theta_1} \right)^{2m+1} \times (W_m(ik_0R(1 + \sin(\theta_1 - \varphi))) - W_m(0)) + b_g(\varphi) \right),
\]

while,

\[
C_g(\rho) = K_0(-ik_0 \rho \cos \theta_1) - K_0(ik_0 \rho \cos \theta_1),
\]

\[
b_g(\varphi) = i \ln(\cot(\frac{\pi}{4} - \frac{\theta_1}{2}) \cot(\frac{\varphi}{2})) - ie^{ik_0 \rho \cos \theta_1} K_0(ik_0 \rho \cos \theta_1).
\]

Since \( \rho = R \sin \varphi \) and \( K_0(a) \sim -\ln(a) \) as \( a \to 0 \), \( b_g(\varphi) \) diverges as \( R \to 0 \) or \( \theta_1 \to -\pi/2 \), but is bounded elsewhere, as \( J_g(\rho, -z) \). In these expressions, \( c_m = (-1)^m (2m)! / ((m!)^2 2^{2m}) \) is the binomial coefficient for the function \( (1 + t)^{-1/2} \), and \text{erf} \ is the error function [30].

Remark : We can write \( C_g(\rho) = i \pi w I_0(ik_0 \rho \cos \theta_1) \) with \( w = \text{sign}(\arg(ik_0 \rho \cos \theta_1)) \) [30]. Since \( |\text{Re}(ik_0)| \leq \pi/2 \) and \( |\text{Re}(\theta_1)| \leq \pi/2 \), \( |\arg(ik_0) + \arg(\cos \theta_1)| > \pi/2 \) only if \( \arg(ik_0) \) and \( \arg(\cos \theta_1) \) have the same sign, and thus, by definition, we note that \( w = \text{sign}(\arg(ik_0)) \) when \( \epsilon = -1 \).

7) Asymptotic expansion of \( J_g \) with error functions

7.1) Asymptotics of \( M(a, b) \)

7.1.1) For large \( a \) and arbitrary \( b \) with \( |\text{Re}(b)| \leq \pi \)

We can let \( \frac{\text{erf}(u)}{u} = \frac{1 - \text{erfc}(\delta u)}{\delta u} \) in (58) and (61), with the goal of using the well-known asymptotic of complementary error function \text{erfc} [30].

Subsequently, when \( \epsilon = 1 \), we write,
\[ \mathcal{M}(a, b) = \lim_{M \to \infty} \left[ i e^{-a} \tan \left( \frac{b}{2} \right) \sum_{m=0}^{M} c_m (\tan \left( \frac{b}{2} \right))^m \left( e^{-s} \frac{\partial^m (\sqrt{s} e^s)}{\partial s^m} \right) \right] \]

\[ - i e^{-\text{acosh}} \sum_{m=0}^{M} c_m (\tan \left( \frac{b}{2} \right))^m \left( e^{-s} \frac{\partial^m (\sqrt{s} \text{erfc}(s^{1/2}))}{\partial s^m} \right) \bigg|_{s^{1/2} = -i \delta \sqrt{\sin(\frac{b}{2})}}. \]

(65)

At this step, we then notice that the term with \( e^{-a} \) factor has the same asymptotics as \( -\delta K_0(a) \). Indeed, using that

\[
\frac{\partial^m}{\partial s^m} \left( \frac{e^s}{s^{1/2}} \right) = \frac{e^s}{s^{1/2}} \sum_{p=0}^{m} \frac{(-1)^{p+1} m! (-1 \times 1 \times \ldots \times (2p - 1))}{2^p p! (m - p)!},
\]

(66)

with \( s^{1/2} = \delta z^{1/2} \), we remark that, noting \( z^{1/2} = -i \sqrt{2 a \sin \left( \frac{b}{2} \right)} \) and \( v = \tan^2 \left( \frac{b}{2} \right) \), we can write, as \( \epsilon = 1 \) and thus \( |v| < 1 \), for large \( a \) and \( M \),

\[
- i \sqrt{\pi} e^{-a} \tan \left( \frac{b}{2} \right) \sum_{m=0}^{M} c_m (\tan \left( \frac{b}{2} \right))^m \left( e^{-z} \frac{\partial^m}{\partial z^m} \left( \frac{e^z}{z^{1/2}} \right) \right)
\]

\[ = i \sqrt{\pi} e^{-a} \sum_{p=0}^{M} (\tan \left( \frac{b}{2} \right))^{2p+1} \frac{(-1)^p (-1 \times \ldots \times (2p - 1))}{2^p (2^p(2a)^{p+1}p!)} \left( -i \sqrt{2 a \sin \left( \frac{b}{2} \right)} \right)^{2p+1}
\]

\[ \times \left[ \frac{1}{p!} \sum_{m \geq p} c_m \frac{m!}{(m-p)!} (m-p)! \frac{1}{2^{m-p}} \right] \left[ \frac{\partial^p \left( \frac{1}{(1+u)} \right)}{\partial v^p} \right]
\]

\[ = \sqrt{\pi} e^{-a} \sum_{p \geq 0} \frac{(-1)^{p+1} (-1 \times \ldots \times (2p - 1))}{2^p (2a)^{p+1} p!} \left( -i \sqrt{2 a \sin \left( \frac{b}{2} \right)} \right)^{2p+1},
\]

(67)

which is equal to the known asymptotics of \( K_0(a) \) [30], and we then have,

\[ \mathcal{M}(a, b) \sim -\delta K_0(a) \]

\[ - i e^{-\text{acosh}} \tan \left( \frac{b}{2} \right) \sum_{m=0}^{M} c_m (\tan \left( \frac{b}{2} \right))^2 \]

\[ \times \left[ \frac{\partial^m}{\partial s^m} \left( \frac{\sqrt{s} e^s}{\sqrt{s^{1/2}} \text{erfc}(s^{1/2})} \right) \right] \bigg|_{s^{1/2} = -i \delta \sqrt{2 \sin(\frac{b}{2})}}, \]

(68)

where \( \delta = + \) or \( -1 \) is chosen such that \( \text{Re}(s^{1/2}) > 0 \), and \( \frac{\partial^m}{\partial s^m} \left( \frac{s^a}{\sqrt{s^{1/2}} \text{erfc}(s^{1/2})} \right) = O \left( \frac{1}{s^{m+1}} \right) \) for large \( s \), with \( \text{erfc}(-u) = 2 - \text{erfc}(u) \) [30].
In complements, when \( \epsilon = -1 \), we can use (53)-(54), so that, after letting \( a \to -a \) and \( b \to b - \nu \pi \) with \( \nu = \text{sign}(\text{Re}(b)) \) in (68) to express \( \mathcal{M}(a, b - \nu \pi) \), we derive that,

\[
\mathcal{M}(a, b) \sim - \delta' K_0(-a) + i \pi \nu I_0(a) + ie^{-a \cos b} \cot \left( \frac{b}{2} \right) \sum_{m=0}^{M} c_m \left( \cot \left( \frac{b}{2} \right) \right)^{2m} \times \frac{\partial^n}{\partial s^n} \left( \frac{\sqrt{\pi} e^s}{s^{1/2}} \text{erfc}(s^{1/2}) \right) \bigg|_{s^{1/2} = -i \delta' \sqrt{-2 \text{Re}(s^{1/2})}},
\]

where \( \delta' = + \) or \( -1 \) is chosen such that \( \text{Re}(s^{1/2}) > 0 \), with \( \text{erfc}(-u) = 2 - \text{erfc}(u) \). Besides, we note that, when \( \text{Re}(a) > 0 \),

\[
-i \sqrt{-2 \text{asinf}((b - \nu \pi)/2)} = - \tau \sqrt{2 \cos(b/2)},
\]

with \( \tau = - \nu \text{sign}(\text{Im}(a)) \), \( i \pi I_0(a) = \text{sign}(\text{Im}(a))(K_0(-a) - K_0(a)) \).

7.1.2) Refining the asymptotics when \( \text{Im}(b) \) is large but \( a \) is not

In previous section, we derived asymptotics with \( \text{erfc} \) functions from our exact series with \( \text{erf} \) functions, when \( a \) is large. We now proceed to consider an asymptotics in regions where \( \text{Im}(b) \) is large but \( a \) is not, when \( \text{Re}(a) > 0 \) and \( |\text{Re}(b)| \leq \pi \).

Letting \( s = -2 \text{asinf}^2 \left( \frac{b}{2} \right) \neq 0 \) and \( \text{Re}(s^{1/2}) = -i \delta \sqrt{2 \text{asinf}(b/2)} \) > 0 with \( \delta = +1 \) or \( -1 \), we can rearrange (55) and write,

\[
\mathcal{M}(a, b) = i \delta \int_0^{\chi b} e^{-a \cos \alpha} d\alpha = -i \delta \int_{\chi b}^{i \infty} e^{-a \cos \alpha} d\alpha - \delta K_0(a) = \int_{-\infty}^{0} \left( 1-t \right)^{\frac{1}{2}} \left( 1 + \left( -i \sin \left( \frac{b}{2} \right) \right)^{2} \left( 1 - t \right) \right)^{\frac{1}{2}} dt - \delta K_0(a) = \int_{1}^{\chi \infty} \frac{-i \sin \left( \frac{b}{2} \right) e^{-a \cos \left( 1 - t' \right)} e^{s t'}}{(1-t')^{\frac{1}{2}} \left( \cos \left( \frac{\chi}{2} \right) + \tan \left( \frac{b}{2} \right) (1-t')^{\frac{1}{2}} \right)} dt' - \delta K_0(a),
\]

with \( \chi = e^{-i \arg(s)} \), taking \( t' = 1 - t \) and \( st \leq 0 \) along the integration paths. The function \( \mathcal{M}(a, b) \) is holomorphic at any finite complex point, and as we let continuously vary \( a \) and \( b \) in (71), we must not let vanish the terms \( (1-t)^{\frac{1}{2}} \) and \( (1 + (-i \sin^{2} \left( \frac{b}{2} \right))^{2} (1-t)^{\frac{1}{2}} \), i.e. \( s - st \neq 0 \) and \( 2 \cos^{2} \left( \frac{b}{2} \right) - st \neq 0 \) as \( st \leq 0 \).

So defined, the expression (71) is valid as \( \text{Re}(a) > 0 \) and \( |\text{Re}(b)| \leq \pi \), when \( \delta \) is chosen \( +1 \) or \( -1 \) such that \( \text{Re}(s^{1/2}) = -i \delta \sqrt{2 \text{asinf}(b/2)} \) > 0,
We can then expand (71) for large $\sqrt{2a \sin \left(\frac{b}{2}\right)}$. For that, we consider a Taylor expansion of $(\cos^2 \left(\frac{b}{2}\right)(1 + \tan^2 \left(\frac{b}{2}\right)(1 - t'))^{-1/2}$ in vicinity of $t' = 1$ where exponential term is not negligible, then using $(t')^{1/2}$ as a new variable of integration, we derive, from complementary error function definition [30], an asymptotic series which, as expected, perfectly corresponds to (68).

On this occasion, we have then enlarged the domain of validity of (68) to the case where $b$ is large but $a$ is not, as $\text{Re}(a) > 0$ and $|\text{Re}(b)| \leq \pi$, provided (72) is verified.

7.2) Asymptotics of $J'$ for $|a| = |k_0 \rho \cos \theta_1| \gg 1$, or $|k_0 R (1 + \sin(\theta_1 \mp \varphi))| \gg 1$ as $\epsilon = \pm 1$

Considering the expressions given in (47)-(48) and (51), we now use previous asymptotic expansion (68)-(69) of $\mathcal{M}(a, b)$ when $a$ or $2a \sin^2 \left(\frac{b}{2}\right)$ (and thus, in particular if $k_0 R$ or $g$) is large. On this occasion, we refine the asymptotics given in [1] in a part of the domain with $\epsilon = -1$ and $\text{Re} \theta_1 < 0$, considering that (68) is also the asymptotics of (71) provided (72) is satisfied.

Noticing that (72) is verified on the left of $S'$ with $ik_0 R (1 + \sin(\theta_1 + \varphi)) \leq 0$ (figure 2), we can give the asymptotics of $J'$, from (68) when $\epsilon = 1$ and when $\epsilon = -1$ on the left of $S'$, and from (69) when $\epsilon = -1$ on the right of $S'$. 

\begin{equation}
2a \cos^2 \left(\frac{b}{2}\right) < 0. \tag{72}
\end{equation}
7.2.1) Asymptotics of $\mathcal{J}_g$ from (68) when $\epsilon = 1$ and when $\epsilon = -1$ on the left of $S'$

In the domain of validity of (68), we can write,

\[
\mathcal{J}_g(\rho, z) \sim - (1 - \delta_\epsilon) K_0(a) + i \sqrt{\pi} e^{-a \cos b} \tan \left( \frac{b}{2} \right) \\
\times \sum_{m \geq 0} c_m \left( \tan \left( \frac{b}{2} \right) \right)^{2m} \frac{\partial^m \left( e^{\text{erfc}(s^2)} \right)}{\partial s^m} \bigg|_{s^2 = -i \delta_\epsilon \sqrt{2 \sin(\frac{b}{2})}},
\]

(73)

when we have $\epsilon = 1$, but also, in complements, when $\epsilon = -1$ on the left of $S'$ (figure 2) where (72) is verified.

In this expression, $\delta_\epsilon$ denotes $((1 + \epsilon) \delta + (1 - \epsilon) \delta')/2$ where the parameter $\delta$ (resp. $\delta'$), chosen + or −1 so that Res$^{1/2} > 0$ as $\epsilon = 1$ (resp. $\epsilon = -1$), changes of sign when we cross the line $S$ with $ik_0 R(1 + \sin(\theta_1 - \varphi)) < 0$ (resp. the line $S'$ with $ik_0 R(1 + \sin(\theta_1 + \varphi)) < 0$) (figure 2), i.e. the line with Re$(\theta_1) = \varphi - \frac{\pi}{2} + G(\text{Im} \theta_1)$ (resp. with Re$(\theta_1) = - \varphi - \frac{\pi}{2} + G(\text{Im} \theta_1)$), where $G(x) = 2 \arctan(\tan(\frac{\arg(ik_0)}{2}) \tanh(\frac{x}{2}))$.

while (47)-(48) and (51) apply, and thus we have,

\[
- i \delta_\epsilon \sqrt{2 \sin(\frac{b}{2})} = \sqrt{ik_0 R(1 + \sin(\theta_1 - \epsilon \varphi))},
\]

\[
\tan(\frac{b}{2}) = i \frac{1 + \sin(\theta_1 - \epsilon \varphi)}{\cos \varphi + \sin \theta_1}, \quad a \cos b = ik_0 R(1 + \sin \theta_1 \cos \varphi),
\]

\[
a = \epsilon ik_0 R \sin \varphi \cos \theta_1 \text{ with } \epsilon = \text{sign}(\text{Re}(ik_0 R \sin \varphi \cos \theta_1)),
\]

(74)

where the term $\sqrt{(.)}$ has always positive real part [30] (in contrary of $(.)^{1/2}$ in sense of Riemann).
7.2.2) Asymptotics of $J_\rho$ from (69) when $\epsilon = -1$ on the right of $S'$.  

On the right of the line $S'$ (the line in figure 2 with $ik_0R(1 + \sin(\theta_1 + \varphi)) < 0$) when $\epsilon = -1$, we can use (69), and write,

$$J_\rho(\rho, -z) \sim - (1 + \tau)K_0(a) + \tau(\delta + 1)K_0(-a)$$

$$- i \sqrt{\pi e^{-\cot(b/2)}} \sum_{m \geq 0} c_m(\cot(b/2))^{2m+1} \frac{\partial^m(e^{s\varphi}/\sqrt{s+1})}{\partial s^m} \bigg|_{s=\delta \sqrt{2 \text{acos}(b/2)}},$$

where $\delta$ is chosen + or $-1$ so that Re($s^{1/2}$) > 0 and $\frac{\partial^m(e^{s\varphi}/\sqrt{s+1})}{\partial s^m}$ is $O\left(\frac{1}{s^{m+1}}\right)$ for $\text{Re}(s)$, with $\tau = -\nu \text{sign(Im}(a))$, $\nu = \text{sign(Re}(b)) = -\epsilon \text{sign(Im}(\sin\theta_1))$, and $\text{sign(Im}(a)) = -\text{sign(\text{arg}(ik_0))) as previously noticed for } \epsilon = -1$, while (47)-(48) and (51) apply, and thus we have,

$$- \delta \sqrt{2 \text{acos}(b/2)} = i k_0 R(1 + \sin(\theta_1 - \varphi)),$$

$$\cot(b/2) = - i \frac{1 + \sin(\theta_1 - \varphi)}{\cos\varphi + \sin\theta_1},$$

and $\tau = \text{sign(Im}(\sin\theta_1))\text{sign(\text{arg}(ik_0))) for } (75)$, with $\text{Re}(\sqrt{\cdot}) > 0$ [30]. By definition, the parameter $\delta$ changes of sign when we cross the line $S$ with $ik_0 R(1 + \sin(\theta_1 - \varphi)) < 0$ (figure 2).

7.2.2) Concerning $\delta, \delta'$ and $\delta_e$

Considering (73)-(75), we note that, in the region with $\epsilon = 1$ (figure 2),

$\delta_e = \delta = 1$ on the right side of $S$,

$\delta_e = \delta = -1$ on the left side of $S$,

$$\delta_e = \delta = 1$$

while, in the zone with $\epsilon = -1$ (figure 2),

$\delta = \tau$ on the right side of $S$,

$\delta = -\tau$ on the left side of $S$,

$\delta_e = \delta' = 1$ on the left side of $S'$,

with $\tau = \text{sign(Im}(\sin\theta_1))\text{sign(\text{arg}(ik_0)))$. From $\epsilon = \text{sign(Re}(ik_0\cos\theta_1))$, we have $\epsilon = 1$ between $D_+$ and $D_-$, and $\epsilon = -1$ elsewhere (figure 2).
8) Forward and backward guided waves

8.1) guided waves from the asymptotics (73)-(75)

Excited guided waves refer to \( K_0(\pm \epsilon i k_0 \rho \cos \theta_1) e^{i k_0 z \sin \theta_1} \) terms in the expression of \( J_0(\rho, -z) e^{i k_0 z \sin \theta_1} \) with \( g = \sin \theta_1 \), for each mode attached to \( g = g_j e^{i \varphi_j} \).

They behave like \( \sqrt{\frac{\pi}{\pm 2 \epsilon i k_0 \rho \cos \theta_1}} e^{i k_0 R \sin(\theta_1 \mp \varphi)} \) for large \( R \). Relatively to their asymptotic phase, these guided waves propagate like forward waves \( e^{-i k_0 (\rho \cos \theta_1 - z \sin \theta_1)} \) as \( \pm \epsilon = 1 \) and backward waves \( e^{i k_0 (\rho \cos \theta_1 + z \sin \theta_1)} \) as \( \pm \epsilon = -1 \). They are present in a angular domain (depending on \( \varphi \)) where \( e^{i k_0 R \sin(\theta_1 \mp \varphi)} \) remain bounded at infinity, as already noticed by Tamir and Oliner [35] for 2D case. We can then analyze the situation from (73)-(75) and figure 2.

8.1.1) In passive zone as Re(\( \theta_1 \)) > 0

In passive zone with Re(\( \theta_1 \)) > 0, we have only \( K_0(i k_0 \rho \cos \theta_1) e^{i k_0 z \sin \theta_1} \) term which is present in the region on the left of \( S \) where \( \epsilon = 1 \), and is clearly associated to the excitation of forward proper waves, vanishing as \( \rho \) or \( z \) increases (proper mode).

8.1.2) In active zone as Re(\( \theta_1 \)) < 0

In the active zone with Re(\( \theta_1 \)) < 0, the situation is more complex. Both \( K_0(\pm i k_0 \rho \cos \theta_1) e^{i k_0 z \sin \theta_1} \) terms can be present, and excited waves can be attached to improper or backward modes:

a) in the region with \( \epsilon = -1 \) and Re(\( \theta_1 \)) < 0, we have a mix of backward and of forward waves in a limited domain of excitation \( \Omega_g^\varphi \), while, in the domain limited by \( S \) and \( S' \) depending on \( \varphi \), the forward waves disappear and backward term \( K_0(-i k_0 \rho \cos \theta_1) e^{i k_0 z \sin \theta_1} \) remains alone. In these domains depending on \( \varphi \), forward and backward waves are bounded, even if forward waves grows at fixed \( z \) as \( \rho \) increases.

b) in the region with \( \epsilon = 1 \) and Re(\( \theta_1 \)) < 0, we have forward guided wave term \( K_0(i k_0 \rho \cos \theta_1) e^{i k_0 z \sin \theta_1} \) excited on the left of \( S \), which grows at fixed \( \rho \) as \( z \) increases, but remains bounded because of its limited region of excitation depending on \( \varphi \).

Remark: Let us note that, for meromorphic reflection coefficients, excited guided waves does not involve lateral waves (branch-cut waves) [5].
Remark: For positive $\text{Re}(\varepsilon)$ and $\text{Re}(\mu)$ with real $k_0$, quarter-plane with $\text{Re}(g_j^{\varepsilon(h)}) < 0$ and $\text{arg}(ik)\text{Im}(g_j^{\varepsilon(h)}) > 0$ is generally empty, which bans the presence of backward waves. This explains why backward waves is generally attached to the presence of metamaterials (see also (15) and properties (d)-(e)).

8.2) guided-like waves from the exact expansion (62)-(63)

In addition to asymptotics given in (73)-(75), we have developed in (62)-(63) the exact expansions of $J_j(\rho, -z)$ with error functions for any mode parameter $g = \sin\theta_1$. It allows to consider in general, without any complex approximation, its exact behaviour at any finite distance from the sources. When the term $ik_0 R(1 + \sin(\theta_1 - \varphi))$ is small, $K_0(\pm ik_0 \rho \cos\theta_1)$ terms are predominant in (62)-(63). This guided-like waves behaviour is important at small distance but also at large bounded distance for very small $g$ when $\varphi \approx \pi/2$ (closed to the plane). It has to be distinguished from actual guided waves behaviour, attached to $K_0(\pm ik_0 \rho \cos\theta_1)$ terms in asymptotics (73)-(75) for large unbounded $R$.

This behaviour confirms, in a simple exact way, the comments of J.R. Wait given in [36]-[37] that surface waves like excitation is possible in limited region around the source, even if they are attached to modes not excited at very large distance, or said more precisely, every mode attached to a pole $g = \sin\theta_1$ of the reflection coefficients gives a guided-like waves contribution in vicinity of the source provided $ik_0 R(1 + \sin(\theta_1 - \varphi))$ is small, but only some of them develop excited guided waves in asymptotics at large unbounded distance.

9) Conclusion

We have developed simple exact expressions of the field scattered by a multimode impedance passive plane for arbitrary bounded primary sources. The modes can be passive but also active, which is crucial to explain leaky and backward waves. These expressions apply when the multilayer, composed of metamaterials, is isotropic or with uniaxial anisotropy. The multilayer can be backed by impedance plane but also in free space, which generalizes our previous works on impedance [1] and on multimode plane [2]. An expansion of the special function involved in the expression is given from the use of error function, which has the particularities to be convergent and to lead us to simple complete asymptotics. The asymptotics is presented here in the most general form, and
we show in which circumstances guided complex waves, forward and backward ones, are excited.

Appendix 1: Complex spectral properties

1) Wave spectrum and reflection coefficients

We consider the plane wave spectrum \( (F_E, F_H) \) attached to the representation [3] of an electromagnetic field \( (E, H) \), in an half-space free of sources \( \pm z \geq 0 \), except possibly at positions of some sources on boundary \( z = 0 \),

\[
E(x, y, z) = \frac{i}{4\pi} \int_{R^2 \setminus \{0\}} F_E(p, q) \frac{e^{-2\pi i(px + qy + zs)}}{s} \, dp \, dq,
\]

\[
H(x, y, z) = \frac{i}{4\pi} \int_{R^2 \setminus \{0\}} F_H(p, q) \frac{e^{-2\pi i(px + qy + zs)}}{s} \, dp \, dq,
\]

\[
\hat{u} = \frac{2\pi}{k_0} (p, q, \pm s)^t, \quad Z_0 F_H = \hat{u} \wedge F_E, \quad \hat{u}.F_E = 0, \tag{79}
\]

with \( \text{Re}(i) = 0 \), where \( 2\pi \frac{p}{k_0} = k_0 \sin \beta \frac{\cos \phi}{\sin \phi} \) and \( 2\pi s = k_0 \cos \beta \), \( 0 < \phi \leq 2\pi \), \( \beta \) varies from 0 to \( i \infty + \text{arg}(i k_0) \) as \( \text{Re}(i k_0 \sin \beta) = 0 \). From regularity of electromagnetic field as \( \pm z \geq 0 \), except possibly at \( z = 0 \), \( F_{E,H}(p, q) \) is analytic on \( R^2 \), except possibly at infinity. We note that \( (\frac{2\pi}{k_0})^2 dp \, dq = \sin \beta \cos \beta d\beta d\phi \), \( \hat{u} \) is the unit vector of propagation of each plane wave composing the spectrum, with \( \hat{u}.\hat{u} = 1 \) (\( \neq \hat{u}\hat{u}^* \) when \( \hat{u} \) is complex) and \( Z_0 = (\mu_0/\varepsilon_0)^{1/2} \).

The representation (79) applies for the field scattered by plane \( z = 0 \) with \( (F_E^p, F_H^p) \) spectrum, on and above it, and the field incident on this plane with \( (F_E^i, F_H^i) \) spectrum, on and below it. Denoting \( \hat{u}_\perp = (\hat{z} \wedge \hat{u})/\sin \beta \) and \( \hat{u}_\parallel = (\hat{u} \wedge (\hat{z} \wedge \hat{u}))/\sin \beta \), \( F_{Ez} = \hat{z}.F_E \) and \( F_{Hz} = \hat{z}.F_H \), we note in general that,

\[
F_H = \frac{-\hat{u}_\perp F_{Ez}/Z_0 + \hat{u}_\parallel F_{Hz}}{\sin \beta},
\]

\[
F_E = \frac{\hat{u}_\perp Z_0 F_{Hz} + \hat{u}_\parallel F_{Ez}}{\sin \beta}, \tag{80}
\]

whether \( \hat{u} \) is complex or not. Considering a multilayer invariant by translation at fixed \( z \), each element of the incident spectrum, at \( \hat{u} = \frac{2\pi}{k_0} (p, q, -s)^t \), induces a plane wave at \( \hat{u} = \frac{2\pi}{k_0} (p, q, s)^t \) in the scattering spectrum. In this circumstance, we generally have,
where the reflection coefficients \( R_{e,e}, R_{h,h}, R_{e,h} \) and \( R_{h,e} \) depend on frequency and on multilayer parameters, while varying with \( p, q, \) and \( s \) in general, or more simply with \( s \) when layers have z-axial bi-anisotropy [17]. When meromorphic, these coefficients are generally with simple poles. For isotropic [3]-[6] or z-axial anisotropic layers [20]-[22], we have \( TE \) and \( TM \) polarizations decoupling at interfaces, and incident \( TE \) (resp. \( TM \)) waves gives reflected \( TE \) (resp. \( TM \)) ones, which leads to \( R_{e,h} = R_{h,e} = 0 \).

Remark : Because of the regularity of the scatterer at \( z = 0 \), the scattered fields cannot be more singular than primary sources at \( z = 0 \) if any, and thus, the reflection coefficients are bounded as \( |p| \) or \( |q| \to \infty \).

2) Complex Poynting vector and reflection coefficients

The expression of the energy \( W \) outgoing in the region without sources \( \pm z > z_0 \) is given by an integral of complex Poynting vector \( \frac{1}{2}(\vec{E} \wedge \vec{H}^*) \) [3] along the plane \( z = z_0 \), that can be transformed with (79) from Parseval equality, following,

\[
W = W' + iW'' = \frac{1}{2} \int_{z=0} \pm \hat{z}(\vec{E} \wedge \vec{H}^*) \, dxdy \\
= \frac{\pi k_0}{Z_0(4\pi)^2|k_0|^2} \int_{R^2} e^{-2\pi i(s-s^*)z_0} \left( \frac{(s^* + s)(F_E F_E^*)}{2ss^*} \right) dpdq,
\]

where we have used \( k_0 \hat{z} (F_E \wedge (u^* \wedge F_E^*)) = 2\pi (s^*(F_E F_E^*) - (s^* - s)F^*_{Ez} F_{Ez}) \) (note : if we change \( F_E \) for \( Z_0 F_{Hz} \) in the integral, we obtain \( W^* \)). The real and imaginary parts of integral term appears clearly, and for real \( k_0 \), we notice that,

\[
W' = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{(F_E F_E^*)}{2Z_0(4\pi)^2} \sin \beta d\beta d\phi, \\
W'' = \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-2ik_0 \cos \beta} \frac{(F_E F_E^* - 2|F_{Ez}|^2)}{2iZ_0(4\pi)^2} \sin \beta d\beta d\phi.
\]
(83), we have for real $k_0$ and $0 \leq \beta \leq \pi/2$,

$$|R_{e,e}|^2 + |R_{h,e}|^2 \leq 1,$$

$$|R_{h,h}|^2 + |R_{e,h}|^2 \leq 1. \quad (84)$$

**Appendix 2 : Some equalities derived from (4) and (10)**

When we let, in (4) and (10), $\cos \beta = -\epsilon'$ with $\epsilon' = +1$ or $-1$, we have respectively,

$$\prod_{j=1}^{N} \frac{\epsilon' + g_j}{\epsilon' - g_j} - (\pm 1)^N = \sum_{j=1}^{N} a_j \frac{(g_j)^{\pm 1}}{g_j (g_j)^{\pm 1} - \epsilon'},$$

$$\prod_{j=1}^{N'} (-\epsilon' - g'_j) R_0 N \frac{\epsilon_0 \pm a_{0\pm}}{\prod_{j=1}^{N} (-\epsilon' + g_j)} = \sum_{j=1}^{N} a_j \frac{(g_j)^{\pm 1}}{g_j (g_j)^{\pm 1} - \epsilon'}, \quad (85)$$

with $(\pm 1)^N = \pm 1$ since $N$ is odd. Considering that $\sum_{\epsilon'=-1,1} g_j\frac{1}{1-\epsilon'g_j} = \sum_{\epsilon'=-1,1} \frac{\epsilon'}{1-\epsilon'g_j} = \frac{2g_j}{1-g_j}$, we can then write, for a use in (21)-(22) and in (23)-(24),

$$\sum_{\epsilon'=-1,1} \left( \left( \frac{\prod_{j=1}^{N} \frac{\epsilon' + g_j}{\epsilon' - g_j} \mp 1}{g_j (g_j)^{\pm 1} - \epsilon'} \right) \right) = \sum_{j=1}^{N} \frac{a_j}{g_j} \left( g_j \sum_{\epsilon'=-1,1} \frac{\epsilon'K_{g_j}}{\epsilon'(g_j)^{\pm 1} - \epsilon'} \right),$$

$$\sum_{\epsilon'=-1,1} \left( \left( \frac{\prod_{j=1}^{N} (-\epsilon' - g'_j) R_0 \frac{\epsilon_0 \mp a_{0\pm}}{\prod_{j=1}^{N} (-\epsilon' + g_j)} \right) = \sum_{j=1}^{N} \frac{a_j}{g_j} \left( \epsilon' \sum_{\epsilon'=-1,1} \frac{\epsilon'K_{g_j}}{(g_j)^{\pm 1} - \epsilon'} \right), \quad (86)$$

**Appendix 3 : Other types of bounded sources**

Our expressions (21)-(24) can be used indifferently for dipole, volume, and surface primary sources (in the sense of generalized functions) illuminating a multilayer, and this appendix is devoted to particular simplifications or expansions in the two latter cases.

1) Bounded volume sources
Let consider the case of volume sources, $J = J_0 + J_0$ and $M = M_0 + M_0$ when $J_0$ and $M_0$ are analytic, where $1_\Omega(r') = \frac{1}{4\pi} \int_{\partial\Omega} \hat{n} \cdot \nabla(\frac{1}{|r-r'|})dS$ is the indicator function for $\Omega$, such that $1_\Omega(r') = 1$ in $\Omega$, $0$ in $R^3 \setminus \Omega$ ($\hat{n}$ is the unit normal, directed inside $\Omega$).

Since $v(x, y, z)$ and $k_g(x, y, z)$ satisfy the Helmholtz equation for $z > 0$, we can write $\text{curl(curl(.))}$, which is equal to $\text{grad(div(.))} - \Delta(.)$, in place of $\text{grad(div(.))} + k_0^2$ in our expressions of potentials above the plane.

Letting $A = 1_\Omega(r')A_0(r')$ and $S = \partial\Omega$, we notice the following identities for curl$(A)$ and curl(curl$(A)$) terms,

$$\text{curl}(A) = 1_\Omega(r')\text{curl}(A_0) + \hat{n} \wedge A_0 \delta_S,$$

$$\text{curl(curl}(A)) = 1_\Omega(r')\text{curl(curl}(A_0)) +$$

$$+ \hat{n} \wedge \text{curl}(A_0) \delta_S + \text{curl}(\hat{n} \wedge A_0 \delta_S),$$

and thus simplify our expression of potentials, until reducing all volume integrals to surface ones when $J_0$ and $M_0$ are constants. Since $J$ and $M$ are distributions, we have

$$\int_{R^3} C \cdot \partial_t \mathbf{f} = - \int_{R^3} \partial_t C \cdot \mathbf{f},$$

where $\partial_t$ denotes a derivative along a cartesian coordinates, and note that $\int_{R^3} \text{curl}(C) \cdot \mathbf{f} = \int_{R^3} \text{curl}(C) \cdot \text{curl}(\mathbf{f})$, $\int_{R^3} \text{curl(curl}(C) \cdot \mathbf{f} = \int_{R^3} \text{curl}(C) \cdot \text{curl}(\mathbf{f})$, while

$$\int_{R^3} \text{grad(div}(C) \cdot \mathbf{f} = - \int_{R^3} \text{div}(C) \cdot \text{div}(\mathbf{f}) = \int_{R^3} C \cdot \text{grad}(\text{div}(\mathbf{f})).$$

2) Bounded surface sources

The equivalent (or Huygens) surface sources, $M = - \hat{n} \wedge [E'] \delta_S$ and $J = \hat{n} \wedge [H'] \delta_S$, where $[E']$ and $[H']$ denote fields discontinuities in free space through $S$ along normal $\hat{n}$ are commonly used in Boundary Integral Equation (exact method) or in Physical Optics (approximation) [23], [38]. In this case, we can use

$$Z_0 \text{div}(J) = Z_0 \text{div}_S(\hat{n} \wedge [H']) \delta_S - Z_0(\hat{n} \wedge [H']).\hat{n} \delta_{\partial S},$$

$$\text{div}(M) = - \text{div}_S(\hat{n} \wedge [E']) \delta_S + (\hat{n} \wedge [E']).\hat{n} \delta_{\partial S},$$

where $\text{div}_S$ is the surface divergence [p.330, 38], $\hat{n}$ is the geodesic normal to $\partial S$ directed outside $S$, and $\delta_S$ (resp. $\delta_{\partial S}$) is the Dirac surface (resp. line) function, while $\text{div}_S(\hat{n} \wedge [H']) = - i\omega \varepsilon_0 \hat{n}[E']$ and $\text{div}_S(\hat{n} \wedge [E']) = i\omega \mu_0 \hat{n}[H']$ from Maxwell equations, which directly simplify our expression of potentials.

Appendix 4 : Some miscellaneous results on $J_g$
Let us notice that we have, for \( \text{Re} \ g > 0 \) and \( \text{arg}(i k_0) = \pi/2 \),
\[
e^{i k_0 g z} J_g(\rho, -z) = \int_{-i \infty}^{0} e^{-i k_0 g z_1} e^{-i k_0 R (z_1 - z)} \frac{dz_1}{R (z_1 - z)},
\]
with \( R (z - z) = \sqrt{\rho^2 + z^2} \), and that,
\[
(\Delta + k_0^2)(e^{i k_0 g z} J_g(\rho, -z)) = 4\pi e^{i k_0 g z} U_s( -z) \delta(x) \delta(y),
\]
\[J_g(\rho, -z) = \pm i \int_{\pm b}^{i \infty} e^{-ac \cos \alpha} d\alpha - (1 \mp 1) K_0(a),
\]
\[J_g(\rho, -z) = -J_{-g}(\rho, z) - 2K_0(a),
\]
where \( U_s(z) = (1 + \text{sign}(z))/2 \) is the unit step function, \( \delta \) is the Dirac function [29], \( a \)
and \( b \) are defined in (47). In other respects, for the calculus of our series with the error
functions, it is worth noticing [30], that
\[
W_m(s) = \frac{e^{-s}}{\partial s^m} \left( e^s \text{erf}(s^{1/2}) \right) = \int_0^1 \frac{e^{-st}(1 - t)^m}{\sqrt{\pi t^{1/2}}} dt
\]
\[= m! M \left( \frac{1}{2}, m + \frac{3}{2}, -s \right) \frac{\Gamma(m + \frac{3}{2})}{\Gamma(m + 3/2)},
\]
\[
\frac{\partial^m}{\partial s^m} \left( \frac{e^s \text{erfc}(s^{1/2})}{m! s^{1/2}} \right) = \frac{(-1)^m}{\sqrt{\pi}} U(m + 1, m + \frac{3}{2}, s)
\]
\[= \frac{(-1)^m}{\sqrt{\pi} s^{m+1}} (1 + O(s^{-1})),
\]
where \( M \) and \( U \) are confluent hypergeometric functions [30]. These special functions are
well-tabulated, and we remark in particular that, for large \( m \gg |s|, \)
\[M(1/2, m + 3/2, -s) = 1 + O(s/m) \text{ and } W_m(0) = \frac{\Gamma(m+1)}{\Gamma(m+3/2)} = m^{-1/2} + O(m^{-3/2}).
\]

**Remark :** Letting \( V_m^v = W_m(s) - v W_m(0) \), with \( v = 0 \) or 1, we derive, from [30],
\[2sV_{m+1}^v(s) + V_m^v(s)(1 - 2s + 2m) - 2m V_{m-1}^v(s)
\]
\[= v \frac{s\Gamma(m+1)}{\Gamma(m+3/2)},
\]

**Remark :** We can write [30], as \( \text{Re}(u) > 0 \) and \( M \geq 0 \),
\[
\text{erfc}(u) = \frac{e^{-u^2}}{\sqrt{\pi} u} \sum_{m=0}^{M-1} \frac{(-1)^m (2m)!}{2^m m!(2u^2)^m} + \mathcal{R}_M(u),
\]
\[
\mathcal{R}_M(u) = \frac{(-1)^M (2M)!}{2^M M! (2u^2)^M} \int_0^{\infty} \frac{e^{-(t+u^2)}/\sqrt{\pi}}{(t/u^2 + 1)^{M+2}} dt
\]
References:
[18] M.A. Marin, P.H. Pathak, 'An asymptotic closed-form representation for the
[19] G. Valerio, D.R. Jackson, A. Galli, 'Fundamental properties of surface waves in
[22] C.W. Qiu, Li Hu, X. Xu, Y. Feng, 'Spherical cloaking with homogeneous isotropic
1, pp.77-86, 1985.
[27] A.V. Osipov, 'Diffraction by a wedge with higher-order boundary conditions', Radio
[28] A.A. Tuzhilin, 'Diffraction of plane sound wave in an angular domain with absolutely
hard and slippery face bounded by thin elastic plates', Different Urav., 9, 10, pp. 1875-
[29] V.S. Vladimirov, 'Generalized functions in mathematical physics', MIR publishers,
Moscow, 1979.
[31] A. Sommerfeld, 'Uber die ausbreitung der wellen in der drahtlosen telegraphie',
[33] F.E. Harris, 'Incomplete Bessel, generalized incomplete gamma or leaky aquifer
[34] N. M. Temme, 'The leaky aquifer function revisited', Int. J. Quantum Chem., 109,
[36] J.R. Wait, 'Excitation of surface waves on conducting, stratified, dielectric clad,
[38] D.S. Jones, 'Methods in electromagnetism wave propagation', IEEE Press,