

Appendices to "Financial Markets and the Real Economy: A Statistical Field Perspective on Capital Allocation and Accumulation"

Pierre Gosselin

Aileen Lotz

Marc Wambst

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Appendix 1 From large number of agents to field formalism

This appendix summarizes the most useful steps of the method developed in Gosselin, Lotz and Wambst (2017, 2020, 2021), to switch from the probabilistic description of the model to the field theoretic formalism and summarizes the translation of a generalization of (16) involving different time variables. By convention and unless otherwise mentioned, the symbol \int refers to all the variables involved.

A1.1 Probabilistic formalism

The probabilistic formalism for a system with N identical economic agents in interaction is based on the minimization functions described in the text. Classically, the dynamics derives through the optimization problem of these functions. The probabilistic formalism relies on the contrary on the fact, that, due to uncertainties, shocks... agents do not optimize fully these functions. Moreover, given the large number of agents, there may be some discrepancy between agents minimization functions, and this fact may be translated in an uncertainty of behavior around some average minimization, or objective function.

We thus assume that each agent chooses for his action a path randomly distributed around the optimal path. The agent's behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight.

In general, this weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget constraints and interactions among agents. These interactions may for instance arise through constraints, since income flows depend on other agents demand. The probabilistic description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given time span. They have the form of Euclidean path integrals.

In the context of the present paper, we have seen that the minimization functions for the system considered in this work have the form:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l\dots} f(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)^2 + \sum_i \left(\sum_{j,k,l\dots} g(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right) \right) \quad (109)$$

This minimization of this function will yield a dynamic equation for N agents in interaction described by a set of dynamic variables $\mathbf{A}_i(t)$ during a given timespan T .

The probabilistic description is straightforwardly obtained from (109). The probability associated to a configuration $(\mathbf{A}_i(t))_{i=1,\dots,N}$ $_{0 \leq t \leq T}$ is directly given by:

$$\mathcal{N} \exp \left(-\frac{1}{\sigma^2} \int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l\dots} f(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)^2 + \sum_i \left(\sum_{j,k,l\dots} g(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right) \right) \right) \quad (110)$$

where \mathcal{N} is a normalization factor and σ^2 is a variance whose magnitude describes the amplitude of deviations around the optimal path.

As in the paper, the system is in general modelled by several equations, and thus, several minimization function. The overall system is thus described by several functions, and the minimization function of the

whole system is described by the set of functions:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l,\dots} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)^2 + \sum_i \left(\sum_{j,k,l,\dots} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right) \right) \quad (111)$$

where α runs over the set equations describing the system's dynamics. The associated weight is then:

$$\mathcal{N} \exp \left(- \left(\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l,\dots} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)^2 + \sum_{i,\alpha} \left(\sum_{j,k,l,\dots} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right) \right) \right) \quad (112)$$

The appearance of the sum of minimization functions in the probabilistic weight (112) translates the hypothesis that the deviations with respect to the optimization of the functions (111) are assumed to be independent.

For a large number of agents, the system described by (112) involves a large number of variables $K_i(t)$, $P_i(t)$ and $X_i(t)$ that are difficult to handle. To overcome this difficulty, we consider the space H of complex functions defined on the space of a single agent's actions. The space H describes the collective behavior of the system. Each function Ψ of H encodes a particular state of the system. We then associate to each function Ψ of H a statistical weight, i.e. a probability describing the state encoded in Ψ . This probability is written $\exp(-S(\Psi))$, where $S(\Psi)$ is a functional, i.e. the function of the function Ψ . The form of $S(\Psi)$ is derived directly from the form of (112) as detailed in the text. As seen from (112), this translation can in fact be directly obtained from the sum of "classical" minimization functions weighted by the factors $\frac{1}{\sigma_\alpha^2}$:

$$\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l,\dots} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)^2 + \sum_{i,\alpha} \left(\sum_{j,k,l,\dots} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t) \dots) \right)$$

This is this shortcut we used in the text.

A1.2 Interactions between agents at different times

A straightforward generalization of (16) involve agents interactions at different times. The terms considered have the form:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l,\dots} \int f(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l) \dots, \mathbf{t}) d\mathbf{t} \right)^2 + \sum_i \sum_{j,k,l,\dots} \int g(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l) \dots, \mathbf{t}) d\mathbf{t} \quad (113)$$

where \mathbf{t} stands for $(t_i, t_j, t_k, t_l \dots)$ and $d\mathbf{t}$ stands for $dt_i dt_j dt_k dt_l \dots$

The translation is straightforward. We introduce a time variable θ on the field side and the fields write $|\Psi(\mathbf{A}, \theta)|^2$ and $|\hat{\Psi}(\hat{\mathbf{A}}, \hat{\theta})|^2$. The second term in (113) becomes:

$$\begin{aligned} & \sum_i \sum_j \sum_{j,k,\dots} \int g(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l) \dots, \mathbf{t}) d\mathbf{t} \\ \rightarrow & \int g(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}', \dots, \theta, \hat{\theta}) |\Psi(\mathbf{A}, \theta)|^2 |\Psi(\mathbf{A}', \theta')|^2 |\Psi(\mathbf{A}'', \theta'')|^2 d\mathbf{A} d\mathbf{A}' d\mathbf{A}'' \\ & \times |\hat{\Psi}(\hat{\mathbf{A}}, \hat{\theta})|^2 |\hat{\Psi}(\hat{\mathbf{A}}', \hat{\theta}')|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' d\theta d\hat{\theta} \end{aligned} \quad (114)$$

where $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are the multivariables $(\theta, \theta', \theta'' \dots)$ and $(\hat{\theta}, \hat{\theta}' \dots)$ respectively and $\mathbf{d}\boldsymbol{\theta}\mathbf{d}\hat{\boldsymbol{\theta}}$ stands for $d\theta d\theta' d\theta'' \dots$ and $d\hat{\theta} d\hat{\theta}' \dots$

Similarly, the first term in (113) translates as:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l,\dots} \int f(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l) \dots, \mathbf{t}) d\mathbf{t} \right)^2 \quad (115)$$

$$\rightarrow \int \Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma_{\mathbf{A}^{(\alpha)}}^2}{2} \nabla_{\mathbf{A}^{(\alpha)}} + \Lambda(\mathbf{A}, \theta) \right) \right) \Psi(\mathbf{A}, \theta) d\mathbf{A} d\theta \quad (116)$$

by:

$$\begin{aligned} \Lambda(\mathbf{A}, \theta) &= \int f^{(\alpha)}(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}' \dots, \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}) |\Psi(\mathbf{A}', \theta')|^2 |\Psi(\mathbf{A}'', \theta'')|^2 d\mathbf{A}' d\mathbf{A}'' \\ &\quad \times |\hat{\Psi}(\hat{\mathbf{A}}, \theta)|^2 |\hat{\Psi}(\hat{\mathbf{A}}', \theta'')|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' d\bar{\boldsymbol{\theta}} d\hat{\boldsymbol{\theta}} \end{aligned} \quad (117)$$

with $\mathbf{d}\bar{\boldsymbol{\theta}} = d\theta' d\theta''$.

Ultimately, as in the text, additional terms (25):

$$\begin{aligned} &\Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_\theta \left(\frac{\sigma_\theta^2}{2} \nabla_\theta - 1 \right) \right) \Psi(\mathbf{A}, \theta) \\ &+ \hat{\Psi}^\dagger(\hat{\mathbf{A}}, \theta) \left(-\nabla_\theta \left(\frac{\sigma_\theta^2}{2} \nabla_\theta - 1 \right) \right) \hat{\Psi}(\hat{\mathbf{A}}, \theta) + \alpha |\Psi(\mathbf{A})|^2 + \alpha |\hat{\Psi}(\hat{\mathbf{A}})|^2 \end{aligned} \quad (118)$$

are included to the action functional to take into account for the time variable.

A1.3 Translation of the minimization functions

Real economy

Translation of the minimization function: Physical capital allocation Let us start by translating in terms of fields the expression (26):

$$\sum_i \left(\left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 + \tau \sum_j \delta(X_i - X_j) \right) \quad (119)$$

To do so, we first consider the last term $\tau \sum_i \sum_j \delta(X_i - X_j)$. This term contains no derivative. The form of the translation is given by formula (17). Since the expression contains two indices, both of them are summed.

The first step of the translation is to replace X_i and X_j by two variables X et X' , and substitute:

$$\tau \delta(X_i - X_j) \rightarrow \tau \delta(X - X')$$

The sum over i and the sum over j are then replaced directly by the integrals $\int |\Psi(K, X)|^2 d(K, X)$, $\int |\Psi(K', X')|^2 d(K', X')$, which leads to the following translation:

$$\begin{aligned} \tau \sum_i \sum_j \delta(X_i - X_j) &\rightarrow \int |\Psi(K, X)|^2 d(K, X) \int |\Psi(K', X')|^2 d(K', X') \tau \delta(X - X') \\ &= \int \tau |\Psi(K, X)|^2 |\Psi(K', X)|^2 d(K, X) dK' \end{aligned} \quad (120)$$

To translate the first term in formula (119):

$$\sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \quad (121)$$

We use the translation (23) of a type-(22) expression. The gradient term appearing in equation (23) is ∇_X . We thus obtain the translation:

$$\begin{aligned} & \sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \\ \rightarrow & \int \Psi^\dagger(K, X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X + \Lambda(X, K) \right) \right) \Psi(K, X) dK dX \end{aligned} \quad (122)$$

Note that the variance σ_X^2 reflects the probabilistic nature of the model hidden behind the field formalism. This σ_X^2 represents the characteristic level of uncertainty of the sectors space dynamics. It is a parameter of the model. The term $\Lambda(X, K)$ is the translation of the term $-\nabla_X R(K_i, X_i) H(K_i)$ in the parenthesis of (121). This term is a function of one sole index "i". In that case, the term Λ is simply obtained by replacing (K_i, X_i) by (K, X) . We use the translation (21) of (19)-type term, so that Λ writes:

$$\Lambda(X, K) = -\nabla_X R(K, X) H(K)$$

and the translation of expression (121) is:

$$\begin{aligned} & \sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \\ \rightarrow & \int \Psi^\dagger(K, X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) \right) \Psi(K, X) dK dX \end{aligned} \quad (123)$$

Using equations (120) and (123), the translation of (119) is thus:

$$\begin{aligned} S_1 = & - \int \Psi^\dagger(K, X) \nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) \Psi(K, X) dK dX \\ & + \tau \int |\Psi(K', X)|^2 |\Psi(K, X)|^2 dK' dK dX \end{aligned} \quad (124)$$

Translation of the minimization function: Physical capital We can now turn to the translation of the second equation (27):

$$\sum_i \left(\frac{d}{dt} K_i + \frac{1}{\varepsilon} \left(K_i(t) - \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j(t))}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j(t))} \hat{K}_j(t) \right) \right)^2 \quad (125)$$

To detail the computations, we have kept the expanded formula (7) for $F_2(R(K_i(t), X_i(t)), \hat{X}_j(t))$. Once again, we use the translation (21) of (19)-type term, and start by building the field functional associated to the term inside the square:

$$K_i(t) - \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j(t))}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j(t))} \hat{K}_j(t)$$

We replace:

$$\begin{aligned} (K_i(t), X_i(t)) & \rightarrow (K, X) \\ (K_l(t), X_l(t)) & \rightarrow (K', X') \\ (\hat{K}_j(t), \hat{X}_j(t)) & \rightarrow (\hat{K}, \hat{X}) \end{aligned}$$

and:

$$K_i(t) - \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j(t))}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j(t))} \hat{K}_j(t) \rightarrow K - \sum_j \frac{F_2(R(K, X)) G(X - \hat{X})}{\sum_l F_2(R(K', X')) G(X' - \hat{X})} \hat{K} \quad (126)$$

The sum over l is then replaced by an integral $\int |\Psi(K', X')|^2 d(K', X')$:

$$\begin{aligned} & K_i(t) - \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j)}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j)} \hat{K}_j(t) \\ \rightarrow & K - \sum_j \frac{F_2(R(K, X)) G(X - \hat{X})}{\int |\Psi(K', X')|^2 d(K', X') F_2(R(K', X')) G(X' - \hat{X}_j)} \hat{K} \end{aligned} \quad (127)$$

Recall that investors' variables are denoted with an upper script $\hat{\cdot}$.

Finally, the sum over j and the second field are replaced by $\int |\hat{\Psi}(\hat{K}, \hat{X})|^2 d(\hat{K}, \hat{X})$. After introducing the characteristic factor $\frac{1}{\varepsilon}$ of the capital accumulation time scale (see (11)), the translation becomes:

$$\begin{aligned} & \frac{1}{\varepsilon} \left(K_i(t) - \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j)}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j)} \hat{K}_j(t) \right) \\ \rightarrow & \frac{1}{\varepsilon} \left(K - \int |\hat{\Psi}(\hat{K}, \hat{X})|^2 d(\hat{K}, \hat{X}) \frac{F_2(R(K, X)) G(X - \hat{X}) \hat{K}}{\int |\Psi(K', X')|^2 d(K', X') F_2(R(K', X')) G(X' - \hat{X})} \right) \\ \equiv & \Lambda(K, X) \end{aligned} \quad (128)$$

Using the translation (23) of (22)-type term, we are led to the translation of (125). Since the square (125) includes a derivative $\frac{d}{dt} K_i$, the expression starts with a gradient with respect to K , and we have:

$$\begin{aligned} & \sum_i \left(\frac{d}{dt} K_i + \frac{1}{\varepsilon} \left(K_i - \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j)}{\sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j)} \hat{K}_j(t) \right) \right)^2 \\ \rightarrow & \int \Psi^\dagger(K, X) \left(-\nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + \Lambda(K, X) \right) \right) \Psi(K, X) dK dX \end{aligned} \quad (129)$$

where, here again, the variance σ_K^2 reflects the probabilistic nature of the model that is hidden behind the field formalism. Recall that it represents the characteristic level of uncertainty in the dynamics of capital.

Inserting result (128) in equation (129), the translation of (??) becomes:

$$S_2 = - \int \Psi^\dagger(K, X) \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + \frac{1}{\varepsilon} \left(K - \int \hat{F}_2(R(K, X), \hat{X}) \hat{K} |\hat{\Psi}(\hat{K}, \hat{X})|^2 d\hat{K} d\hat{X} \right) \right) \Psi(K, X) \quad (130)$$

with:

$$\hat{F}_2(R(K, X), \hat{X}) = \frac{F_2(R(K, X)) G(X - \hat{X})}{\int F_2(R(K, X)) G(X - \hat{X}) |\Psi(K, X)|^2}$$

as quoted in the text.

Financial markets

The functions to be translated are those of the financial capital dynamics (28) and of the financial capital allocation (29). Both expressions include a time derivative and are thus of type (18). As for the real economy, the application of the translation rules is straightforward.

Translation of the minimization function: Financial capital dynamics We consider the function (28):

$$\sum_j \left(\frac{d}{dt} \hat{K}_j - \frac{1}{\varepsilon} \left(\sum_i \left(r_i + F_1 \left(\frac{R(K_i, X_i)}{\sum_l \delta(X_l - X_i) R(K_l, X_l)}, \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \frac{F_2(R(K_i, X_i)) G(X_i - \hat{X}_j)}{\sum_l F_2(R(K_l, X_l)) G(X_l - \hat{X}_j)} \hat{K}_j \right) \right)^2 \quad (131)$$

which translates, using the general translation formula of expression (22) in (23), into:

$$\int \hat{\Psi}^\dagger(\hat{K}, \hat{X}) \left(-\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} + \Lambda(\hat{K}, \hat{X}) \right) \right) \hat{\Psi}(\hat{K}, \hat{X}) d\hat{K} d\hat{X}$$

The function $\Lambda(\hat{K}, \hat{X})$ is obtained, as before, by translating the term following the derivative in the function (131):

$$\frac{1}{\varepsilon} \sum_i \left(r_i + F_1 \left(\frac{R(K_i, X_i)}{\sum_l \delta(X_l - X_i) R(K_l, X_l)}, \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \frac{F_2(R(K_i, X_i)) G(X_i - \hat{X}_j)}{\sum_l F_2(R(K_l, X_l)) G(X_l - \hat{X}_j)} \hat{K}_j \rightarrow \Lambda(\hat{K}, \hat{X}) \quad (132)$$

First, we use the price dynamics equation (2) at the zero-th order in fluctuations to translate the capital dynamics $\frac{\dot{K}_i(t)}{K_i(t)}$:

$$\begin{aligned} \frac{\dot{K}_i(t)}{K_i(t)} &= \sum_j \frac{F_2(R(K_i(t), X_i(t))) G(X_i(t) - \hat{X}_j)}{K_i \sum_l F_2(R(K_l(t), X_l(t))) G(X_l(t) - \hat{X}_j)} \hat{K}_j(t) - K_i(t) \\ &\rightarrow \Gamma(K, X) \end{aligned}$$

where:

$$\begin{aligned} \Gamma(K, X) &= \frac{\int \frac{F_2(R(K, X)) G(X - \hat{X})}{\int F_2(R(K, X)) G(X - \hat{X}) \|\Psi(K, X)\|^2} \hat{K} \left\| \hat{\Psi}(\hat{K}, \hat{X}) \right\|^2 d(\hat{K}, \hat{X}) - K}{K} \\ &= \int \frac{F_2(R(K, X)) G(X - \hat{X})}{K \int F_2(R(K, X)) G(X - \hat{X}) \|\Psi(K, X)\|^2} \hat{K} \left\| \hat{\Psi}(\hat{K}, \hat{X}) \right\|^2 d(\hat{K}, \hat{X}) - 1 \end{aligned} \quad (133)$$

Then, using the translation (21) of (19), we translate expression (132) by replacing:

$$\begin{aligned} (K_i, X_i) &\rightarrow (K, X) \\ (K_l, X_l) &\rightarrow (K', X') \\ (\hat{K}_j, \hat{X}_j) &\rightarrow (\hat{K}, \hat{X}) \end{aligned}$$

We also replace the sums by integrals times the appropriate square of field, which yields:

$$\begin{aligned} \Lambda(\hat{K}, \hat{X}) &= -\frac{\hat{K}}{\varepsilon} \int \left(r(K, X) - \gamma \frac{\int K' \|\Psi(K', X)\|^2}{K} + F_1 \left(\frac{R(K, X)}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')}, \Gamma(K, X) \right) \right) \\ &\quad \times \frac{F_2(R(K, X)) G(X - \hat{X})}{\int F_2(R(K', X')) G(X' - \hat{X}) \|\Psi(K', X')\|^2 d(K', X')} \|\Psi(K, X)\|^2 d(K, X) \end{aligned}$$

Ultimately, the translation of (28) is:

$$\begin{aligned}
S_3 = & - \int \hat{\Psi}^\dagger(\hat{K}, \hat{X}) \nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \frac{\hat{K}}{\varepsilon} \int \left(r(K, X) - \gamma \frac{\int K' \|\Psi(K', X)\|^2}{K} \right. \right. \\
& \left. \left. + F_1 \left(\frac{R(K, X)}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')}, \Gamma(K, X) \right) \right) \right) \\
& \times \frac{F_2(R(K, X)) G(X - \hat{X})}{\int F_2(R(K', X')) G(X' - \hat{X}) \|\Psi(K', X')\|^2 d(K', X')} \|\Psi(K, X)\|^2 d(K, X) \Big) \hat{\Psi}(\hat{K}, \hat{X})
\end{aligned}$$

Using expressions (32) and (34) yields the expression of the text.

Translation of the minimization function: Financial capital allocation The translation of the function for financial capital allocation (29) follows the previous pattern. We obtain:

$$\begin{aligned}
S_4 = & - \int \hat{\Psi}^\dagger(\hat{K}, \hat{X}) \nabla_{\hat{X}} \left(\sigma_{\hat{X}}^2 \nabla_{\hat{X}} - \int \left(\nabla_{\hat{X}} F_0(R(K, \hat{X})) + \nu \nabla_{\hat{X}} F_1 \left(\frac{R(K, \hat{X})}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')} \right) \right) \right. \\
& \left. \times \frac{\|\Psi(K, \hat{X})\|^2 dK}{\int \|\Psi(K', \hat{X})\|^2 dK'} \right) \hat{\Psi}(\hat{K}, \hat{X})
\end{aligned}$$

and (34) yields the formula quoted in the text.

Appendix 2 expression of $\Psi(K, X)$ as function of financial variables

A2.1 Finding $\Psi(K, X)$: principle

In this paragraph, we give the principle of resolution for $\Psi(K, X)$ for an arbitrary function H . The full resolution for some particular cases is given below. Given a particular state $\hat{\Psi}$, we aim at minimizing the action functional $S_1 + S_2 + S_3 + S_4$. However, given our assumptions, the action functional $S_3 + S_4$ depends on $\Psi(K, X)$, through average quantities, and moreover, we have assumed that physical capital dynamics depends on financial accumulation. Consequently, we can neglect, in first approximation, the impact of $\Psi(K, X)$ on $S_3 + S_4$ and consider rather the minimization of $S_1 + S_2$ which is given by:

$$\begin{aligned}
S_1 + S_2 = & - \int \Psi^\dagger(K, X) \left(\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) - \tau \left(\int |\Psi(K', X)|^2 dK' \right) \right. \\
& \left. + \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u(K, X, \Psi, \hat{\Psi}) \right) \right) \Psi(K, X) dK dX
\end{aligned} \quad (134)$$

with:

$$u(K, X, \Psi, \hat{\Psi}) = \frac{1}{\varepsilon} \left(K - \int \frac{F_2(R(K, X)) G(X - \hat{X})}{\int F_2(R(K, X)) G(X - \hat{X}) \|\Psi(K, X)\|^2} \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d\hat{K} d\hat{X} \right) \quad (135)$$

and:

$$\Gamma(K, X) = \int \frac{F_2(R(K, X)) G(X - \hat{X})}{K \int F_2(R(K, X)) G(X - \hat{X}) \|\Psi(K, X)\|^2} \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d(\hat{K}, \hat{X}) - 1$$

This is done in two steps. First, we find $\Psi(X)$, the background field for X when K determined by X . We then find the corrections to the particular cases considered and compute $\Psi(K, X)$.

A2.1.1 Particular case: K determined by X

A simplification arises, assuming K adapting to X . We assume that in first approximation K is a function of X , written K_X :

$$K = K_X = \int \frac{F_2(R(K_X, X)) G(X - \hat{X})}{\int F_2(R(K'_X, X')) G(X' - \hat{X}) \|\Psi(X')\|^2 dX'} \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d(\hat{K}, \hat{X}) \quad (136)$$

This means that for any sector X , the capital of all agents in this sector are equal. At the individual level, this corresponds to set $\frac{d}{dt} K_i(t) = 0$. The level of capital adapts faster than the motion in sector space and reaches quickly its equilibrium value. Incidentally, (136) implies that $\Gamma(K, X) = 0$. Actually, using (136):

$$\begin{aligned} \Gamma(K, X) &= \int \frac{F_2(R(K, X)) G(X - \hat{X})}{K \int F_2(R(K, X)) G(X - \hat{X}) \|\Psi(K, X)\|^2} \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d(\hat{K}, \hat{X}) - 1 \\ &= \int \frac{F_2(R(K, X)) G(X - \hat{X}) \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d\hat{K}}{\int F_2(R(K_X, X)) G(X - \hat{X}) \hat{K} \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 d(\hat{K}, \hat{X})} - 1 \\ &= 0 \end{aligned}$$

A2.1.1.1 Justification of approximation (136) Approximation (136) justifies in the following way. When F_2 is slowly varying with K , we perform the following change of variable in (134):

$$\begin{aligned} \Psi &\rightarrow \Psi \exp\left(-\frac{\int u(K, X, \Psi, \hat{\Psi}) dK}{\sigma_K^2}\right) \simeq \Psi \exp\left(-\frac{1}{2\sigma_K^2} \varepsilon u^2(K, X, \Psi, \hat{\Psi})\right) \\ \Psi^\dagger &\rightarrow \Psi^\dagger \exp\left(\frac{1}{\sigma_K^2} \int u(K, X, \Psi, \hat{\Psi}) dK\right) \simeq \Psi^\dagger \exp\left(-\frac{1}{2\sigma_K^2} \varepsilon u^2(K, X, \Psi, \hat{\Psi})\right) \end{aligned}$$

and this replaces S_2 in (134) by:

$$-\int \Psi^\dagger(K, X) \left(\frac{\sigma_K^2}{2} \nabla_K^2 - \frac{u^2}{2\sigma_K^2} (K, X, \Psi, \hat{\Psi}) + \frac{1}{2} \nabla_K u(K, X, \Psi, \hat{\Psi}) \right) \Psi(K, X) dK dX \quad (137)$$

The change of variable modifies S_1 in (134). Actually, the derivative ∇_X acts on $\exp\left(-\frac{1}{2\sigma_K^2} u^2(K, X, \Psi, \hat{\Psi})\right)$ and the term:

$$-\int \Psi^\dagger(K, X) \nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) \Psi(K, X) dK dX$$

becomes:

$$\begin{aligned} &-\int \Psi^\dagger(X) \nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) \Psi(X) dK dX \\ &+ \varepsilon \int \Psi^\dagger(K, X) \left(\frac{\sigma_X^2}{2\sigma_K^2} u \nabla_X u \right) \nabla_X \Psi(K, X) dK dX + \varepsilon \int \Psi^\dagger(K, X) \left(\frac{\sigma_X^2}{2\sigma_K^2} ((\nabla_X u)^2 + u \nabla_X^2 u) \right) \Psi(K, X) dK dX \\ &-\int \Psi^\dagger(K, X) \left(\varepsilon \frac{u \nabla_X u}{\sigma_K^2} \nabla_X R(K, X) H(K) + \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4} (u \nabla_X u)^2 \right) \Psi(K, X) dK dX \end{aligned} \quad (138)$$

Using that u is of order $\frac{1}{\varepsilon}$ (see(135)), the minimum of $S_1 + S_2$ is obtained when the potential:

$$\begin{aligned} &\int \Psi^\dagger(K, X) \left(\frac{u^2}{2\sigma_K^2} - \frac{1}{2} \nabla_K u \right) \Psi(K, X) dK dX \\ &+ \varepsilon \int \Psi^\dagger(K, X) \left(\frac{\sigma_X^2}{2\sigma_K^2} ((\nabla_X u)^2 + u \nabla_X^2 u) \right) \Psi(K, X) dK dX \\ &-\int \Psi^\dagger(K, X) \left(\varepsilon \frac{u \nabla_X u}{\sigma_K^2} \nabla_X R(K, X) H(K) + \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4} (u \nabla_X u)^2 \right) \Psi(K, X) dK dX \end{aligned} \quad (139)$$

is nul. The dominant term in (139) for $\varepsilon \ll 1$ is:

$$\int \Psi^\dagger(K, X) \left(\frac{u^2}{2\sigma_K^2} - \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4} (u \nabla_X u)^2 \right) \Psi(K, X) dK dX \quad (140)$$

For $\sigma_X^2 \ll \sigma_K^2$ it implies that the minimum for $S_1 + S_2$ is obtained for:

$$u(K, X, \Psi, \hat{\Psi}) \simeq 0$$

with solution (136).

A2.1.1.2 Rewriting the action $S_1 + S_2$ With our choice $G(X - \hat{X}) = \delta(X - \hat{X})$ we find:

$$K_X = \frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K}}{\left\| \Psi(X) \right\|^2} \quad (141)$$

and $\Psi(K, X)$ becomes a function $\Psi(X)$:

$$\Psi(K, X) \rightarrow \Psi(X)$$

To find the action for $\Psi(X)$ we evaluate (139) using $u(K_X, X, \Psi, \hat{\Psi}) = 0$, and compute the first term in (140) for $\Psi(X) = \Psi(K_X, X) \delta(u)$ by replacing:

$$\delta(u) \rightarrow \frac{\exp(-\varepsilon u^2)}{\sqrt{2\pi\varepsilon}}$$

We obtain:

$$\begin{aligned} - \int \Psi^\dagger(K, X) \left(\frac{\sigma_K^2}{2} \nabla_K^2 \right) \Psi(K, X) dK dX &= \frac{\sigma_K^2}{2} \int |\Psi(X)|^2 dX \int \frac{\exp(-\varepsilon u^2)}{\sqrt{2\pi\varepsilon}} \nabla_K^2 \frac{\exp(-\varepsilon u^2)}{\sqrt{2\pi\varepsilon}} dK \\ &\simeq \frac{\sigma_K^2}{2\varepsilon} \int |\Psi(X)|^2 dX \end{aligned}$$

and the action S_1 restricted to the variable X is given by:

$$\begin{aligned} S_1 &= \int \Psi^\dagger(X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - (\nabla_X R(X) H(K_X)) \right) + \tau |\Psi(X)|^2 \right) \Psi(X) \\ &\quad + \int \Psi^\dagger(K, X) \left(\frac{\sigma_X^2}{4\sigma_K^2} \left(\nabla_X u(K_X, X, \Psi, \hat{\Psi}) \right)^2 \right) \Psi(K, X) dK dX \\ &\quad + \int \left(\frac{\sigma_K^2}{2\varepsilon} - \frac{1}{2} \nabla_K u(K_X, X, \Psi, \hat{\Psi}) \right) |\Psi(X)|^2 dX \end{aligned}$$

In our order of approximation $\nabla_K u(K_X, X, \Psi, \hat{\Psi}) \simeq \varepsilon$. Ultimately, for $\sigma_X^2 \ll \sigma_K^2$, action S_1 reduces to:

$$S_1 = \int \Psi^\dagger(X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - (\nabla_X R(X) H(K_X)) \right) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X) \quad (142)$$

and we look for $\Psi(X)$ minimizing (142).

A2.1.1.3 Minimization of (142) To minimize (142), we assume for the sake of simplicity, that for $i \neq j$:

$$|\nabla_{X_i} \nabla_{X_j} R(X)| < |\nabla_{X_i}^2 R(X)|$$

which is the case for example if $R(X)$ is a function with separated variables : $R(X) = \sum R_i(X_i)$. This can be also realized if locally, one chooses the variables X_i to diagonalize $\nabla_{X_i} \nabla_{X_j} R(X)$ at some points in the sector space.

We then perform the change of variables:

$$\exp \left(\int^X \frac{\nabla_X R(X)}{\sigma_X^2 \|\nabla_X R(X)\|} H(K_X) \right) \Psi(X) \rightarrow \Psi(X)$$

and:

$$\exp \left(- \int^X \nabla_X R(X) H(K_X) \right) \Psi^\dagger(X) \rightarrow \Psi^\dagger(X)$$

so that (142) becomes:

$$\int \Psi^\dagger(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X) H(K_X))^2 + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X) \quad (143)$$

which is of second order in derivatives with a potential:

$$\tau \|\Psi(X)\|^4 + \frac{1}{2\sigma_X^2} \int (\nabla_X R(X) H(K_X))^2 \|\Psi(X)\|^2$$

We assume the number of agents fixed equal to N . We must minimize (143) with the constraint $\|\Psi(X)\|^2 \geq 0$ and $\int \|\Psi(X)\|^2 = N$. We thus replace (143) by:

$$\begin{aligned} & \int \Psi^\dagger(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{(\nabla_X R(X) H(K_X))^2}{2\sigma_X^2} + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X) \\ & + D(\|\Psi\|^2) \left(\int \|\Psi(X)\|^2 - N \right) + \int \mu(X) \|\Psi(X)\|^2 \end{aligned} \quad (144)$$

we have written $D(\|\Psi\|^2)$ the Lagrange multiplier for $\int \|\Psi(X)\|^2$, to keep track of its dependency multiplier in $\|\Psi\|^2$. By a redefinition $D(\|\Psi\|^2) - \frac{\sigma_K^2 - 1}{2\varepsilon} \rightarrow D(\|\Psi\|^2)$, $\frac{D(\|\Psi\|^2)}{D(\|\Psi\|^2) - \frac{\sigma_K^2}{2\varepsilon}} N \rightarrow N$ we can write (144) as:

$$\begin{aligned} & \int \Psi^\dagger(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X) H(K_X))^2 + \frac{H(K_X) \nabla_X^2 R(K_X, X)}{2} + \tau |\Psi(X)|^2 \right) \Psi(X) \\ & + D(\|\Psi\|^2) \left(\int \|\Psi(X)\|^2 - N \right) + \int \mu(X) \|\Psi(X)\|^2 \end{aligned} \quad (145)$$

Introducing the change of variable for $\nabla_X R(X)$ for the sake of simplicity:

$$(\nabla_X R(X))^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \rightarrow (\nabla_X R(X))^2 \quad (146)$$

the minimization of the potential yields, for $\sigma_X^2 \ll 1$:

$$\begin{aligned}
& iD \left(\|\Psi\|^2 \right) + \mu(X) \\
= & 2\tau \|\Psi(X)\|^2 - \frac{H' \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)}{2\sigma_X^2 H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)} \\
& \times \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2 \frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^4} \|\Psi(X)\|^2 \\
& + \frac{1}{2\sigma_X^2} \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2
\end{aligned} \tag{147}$$

Moreover, multiplying (147) by $\|\Psi(X)\|^2$ and integrating yields:

$$\begin{aligned}
D \left(\|\Psi\|^2 \right) N &= 2\tau \int |\Psi(X)|^4 \\
&- \int \frac{H' \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)}{2\sigma_X^2 H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)} \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2 \int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K} \\
&+ \frac{1}{2\sigma_X^2} \int \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2 \|\Psi(X)\|^2 \\
&\simeq 2\tau \int |\Psi(X)|^4 + \frac{1}{2\sigma_X^2} \int \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2 \|\Psi(X)\|^2
\end{aligned} \tag{148}$$

Note that in first approximation, for $H' \ll 1$, (147) and (148) become:

$$D \left(\|\Psi\|^2 \right) + \mu(X) = 2\tau \|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \tag{149}$$

and:

$$ND \left(\|\Psi\|^2 \right) = 2\tau \int |\Psi(X)|^4 + \frac{1}{2\sigma_X^2} \int \left(\nabla_X R(X) H \left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right) \right)^2 \|\Psi(X)\|^2 \tag{150}$$

A2.1.1.4 Resolution of (149) and (150) Two cases arise in the resolution:

Case 1: $\|\Psi(X)\|^2 > 0$ For $\|\Psi(X)\|^2 > 0$, (147) writes:

$$D \left(\|\Psi\|^2 \right) = 2\tau \|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\hat{K}_X}{\|\Psi(X)\|^2} \right) \left(1 - \frac{H'(\hat{K}_X)}{H(\hat{K}_X)} \frac{\hat{K}_X}{\|\Psi(X)\|^2} \right) \tag{151a}$$

with:

$$\hat{K}_X = \int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} = K_X \|\Psi(X)\|^2 \quad (152)$$

Note that restoring the initial variable:

$$(\nabla_X R(X))^2 \rightarrow (\nabla_X R(X))^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \quad (153)$$

yields (57) in the text.

Given the setup, we can assume that

$$H^2 \left(\frac{\hat{K}_X}{\|\Psi(X)\|^2} \right) \left(1 - \frac{H'(\hat{K}_X)}{H(\hat{K}_X)} \frac{\hat{K}_X}{\|\Psi(X)\|^2} \right)$$

is a decreasing function of $\|\Psi(X)\|^2$. Assume a minimum $\Psi_0(X)$ for the right hand side of (151a). It leads to a condition for $D(\|\Psi\|^2)$:

$$D(\|\Psi\|^2) > 2\tau \|\Psi_0(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right) \left(1 - \frac{H'(\hat{K}_X)}{H(\hat{K}_X)} \frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right) \quad (154)$$

and the solution of (151a) writes:

$$\left\| \Psi \left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \right\|^2 \quad (155)$$

where $\hat{K}_{X,0}$ is a constant representing some average to normalize $\frac{\hat{K}_X}{\hat{K}_{X,0}}$ as a dimensionless number.

Case 2 $\|\Psi(X)\|^2 = 0$ On the other hand, if:

$$D(\|\Psi\|^2) < 2\tau \|\Psi_0(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right) \left(1 - \frac{H'(\hat{K}_X)}{H(\hat{K}_X)} \frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right) \quad (156)$$

the solution of (151a) is $\|\Psi(X)\|^2 = 0$

Gathering both cases The value of $\|\Psi\|^2$ thus depends on the conditions (154) and (156). To compute the value of $D(\|\Psi\|^2)$ we integrate (151a) over V/V_0 with V_0 locus where $\|\Psi(X)\|^2 = 0$. V_0 will be then defined by (156) once $D(\|\Psi\|^2)$ found. For H slowly varying, we can replace $\frac{\hat{K}_X}{\|\Psi(X)\|^2}$ by:

$$\frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{\int \|\Psi(X)\|^2 dX} = \frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{N}$$

so that the integration of (156) over X yields:

$$\begin{aligned}
D\left(\|\Psi\|^2\right)(V-V_0) &\simeq 2\tau N + \frac{1}{2\sigma_X^2} \int (\nabla_X R(X))^2 H^2 \left(\frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{N} \right) \\
&\times \left(1 - \frac{H' \left(\frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{N} \right)}{H \left(\frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{N} \right)} \frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} dX}{N} \right) \\
&= 2\tau N + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\langle \hat{K} \rangle}{N} \right) \left(1 - \frac{H' \left(\frac{\langle \hat{K} \rangle}{N} \right)}{H \left(\frac{\langle \hat{K} \rangle}{N} \right)} \frac{\langle \hat{K} \rangle}{N} \right)
\end{aligned}$$

Consequently:

$$D\left(\|\Psi\|^2\right) \simeq 2\tau \frac{N}{V-V_0} + \frac{1}{2\sigma_X^2} \left\langle (\nabla_X R(X))^2 \right\rangle_{V/V_0} H^2 \left(\frac{\langle \hat{K} \rangle}{N} \right) \left(1 - \frac{H' \left(\frac{\langle \hat{K} \rangle}{N} \right)}{H \left(\frac{\langle \hat{K} \rangle}{N} \right)} \frac{\langle \hat{K} \rangle}{N} \right)$$

and V_0 is defined by (156):

$$\begin{aligned}
&2\tau \frac{N}{V-V_0} + \frac{1}{2\sigma_X^2} \left\langle (\nabla_X R(X))^2 \right\rangle_{V/V_0} H^2 \left(\frac{\langle \hat{K} \rangle}{N} \right) \left(1 - \frac{H' \left(\frac{\langle \hat{K} \rangle}{N} \right)}{H \left(\frac{\langle \hat{K} \rangle}{N} \right)} \frac{\langle \hat{K} \rangle}{N} \right) \\
&< 2\tau \|\Psi_0(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right) \left(1 - \frac{H' \left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right)}{H \left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right)} \frac{\hat{K}_X}{\|\Psi_0(X)\|^2} \right)
\end{aligned} \tag{157}$$

On V/V_0 , $\|\Psi\|^2$ is given by (155) and on V_0 , $\|\Psi\|^2 = 0$.

Below, we give explicitaly the form of $\Psi(X)$ form two different form of the function H .

A2.1.2 Introducing the K dependency

A2.1.2.1 First order condition To go beyond approximation (136) and solve for the field $\Psi(K, X)$ that minimizes (134), we come back to the full system for K and X :

$$\begin{aligned}
&\int \Psi^\dagger(K, X) \left(\left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \left(\frac{\nabla_X R(K, X)}{\|\nabla_X R(K, X)\|} \right) H(K) + \tau |\Psi(K, X)|^2 \right) \right) \right. \\
&\left. - \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u(K, X, \Psi, \hat{\Psi}) \right) - \frac{1}{2} \nabla_K u(K, X, \Psi, \hat{\Psi}) \right) \Psi(K, X)
\end{aligned} \tag{158}$$

with $u(K, X, \Psi, \hat{\Psi})$ given by (135). We then look for a minimum of (158) of the form:

$$\Psi(K, X) = \Psi(X) \Psi_1(K - K_X) \tag{159}$$

with K_X given in (141):

$$K_X = \frac{\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{160}$$

and Ψ_1 peaked around 0 and of norm 1. When $H(K)$ is slowly varying around K_X , the minimization of (158) for $\Psi_1(K - K_X)$ writes:

$$\nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u(K, X, \Psi, \hat{\Psi}) + \frac{1}{2} \nabla_K u(K, X, \Psi, \hat{\Psi}) \right) \Psi_1(K - K_X) = 0 \quad (161)$$

Then, using that, in first approximation:

$$\int F_2(R(K', X)) \|\Psi(K', X)\|^2 dK' \simeq F_2(R(K_X, X)) \|\Psi(X)\|^2$$

Equation (161) becomes:

$$\nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))} \right) \Psi_1(K - K_X) = 0 \quad (162)$$

A2.1.2.2 Solving (162) To solve the first order condition (162) we perform the change of variable:

$$\Psi_1(K - K_X) \rightarrow \exp \left(\frac{1}{\sigma_K^2} \int \left[K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))} \right] dK \right) \Psi_1(K - K_X)$$

and (162) is transformed into

$$-\frac{\sigma_K^2}{2} \nabla_K^2 \Psi_1(K - K_X) + \frac{1}{2\sigma_K^2} \left(K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))} \right)^2 \Psi_1(K - K_X) = 0 \quad (163)$$

This equation can be solved by implementing the constraint:

$$\int \|\Psi_1(K - K_X)\|^2 = 1$$

and we find:

$$\begin{aligned} \Psi_1(K - K_X) &\simeq \mathcal{N} \exp \left(-\frac{1}{\sigma_K^2} \left(K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))} \right)^2 \right) \\ &\simeq \mathcal{N} \exp \left(-\frac{1}{\sigma_K^2} \left(K - K_X - (K - K_X) \frac{\partial_K R(K_X, X) F_2'(R(K_X, X))}{F_2(R(K_X, X))} K_X \right)^2 \right) \\ &= \mathcal{N} \exp \left(-\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K_X, X))}{F_2(R(K_X, X))} K_X \right)^2 (K - K_X)^2 \right) \end{aligned}$$

with the normalization factor \mathcal{N} given by:

$$\mathcal{N} = \sqrt{\frac{c}{\sigma_K^2 \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K_X, X))}{F_2(R(K_X, X))} K_X \right)^2}}$$

A2.1.2.3 Expression for the density of firms $\|\Psi(K, X)\|^2$ Having found Ψ_1 , and using (155) and (159) we obtain the expression for $\|\Psi(K, X)\|^2$:

$$\begin{aligned} \|\Psi(K, X)\|^2 &= \mathcal{N} \|\Psi\|^2 \left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \\ &\times \exp \left(-\frac{1}{\sigma_K^2} \left(K - \frac{F_2(R(K, X))}{F_2(R(K_X, X)) \|\Psi(X)\|^2} \int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K} \right)^2 \right) \\ &= \|\Psi\|^2 \left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \frac{c \exp \left(-\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K_X, X))}{F_2(R(K_X, X))} K_X \right)^2 (K - K_X)^2 \right)}{\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K_X, X))}{F_2(R(K_X, X))} K_X \right)^2} \end{aligned} \quad (164)$$

for $X \in V/V_0$ and $\|\Psi(K, X)\|^2 = 0$ otherwise.

As stated in the text, note that the form of the exponential in (164) implies that:

$$\int K \|\Psi(K, X)\|^2 d\hat{K} = \int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K}$$

A2.2 Examples

We solve the minimization for $\Psi(K, X)$ for two particular forms of the function $H(K)$.

A2.2.1 Example 1

We compute $\Psi(K, X)$ for the specific function:

$$H(y) = \left(\frac{y}{1+y} \right)^\varsigma, \quad H'(y) = \varsigma \frac{\left(\frac{y}{y+1} \right)^\varsigma}{y(y+1)}$$

We use the simplified equations (149) and (150) that yield:

$$D\left(\|\Psi\|^2\right) + \mu(X) = \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \left(\left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)^\varsigma \right)^2 \left(1 - \varsigma \frac{1}{\left(\frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} + \langle \hat{K} \rangle \right)} \right)}{\left(\langle \hat{K} \rangle + \frac{\int \hat{K} \|\hat{\Psi}(\hat{K}, X)\|^2 d\hat{K}}{\|\Psi(X)\|^2} \right)^{2\varsigma}}$$

or equivalently:

$$\begin{aligned} D\left(\|\Psi\|^2\right) + \mu(X) &= \tau \|\Psi(X)\|^2 \\ &+ \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \left(\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} \right)^{2\varsigma}}{\left(\langle \hat{K} \rangle \|\Psi(X)\|^2 + \int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} \right)^{2\varsigma+1}} \\ &\times \left(\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} + (1 - \varsigma) \langle \hat{K} \rangle \|\Psi(X)\|^2 \right) \end{aligned}$$

For $\varsigma \simeq \frac{1}{2}$, this reduces to:

$$D\left(\|\Psi\|^2\right) + \mu(X) = \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \hat{K}_X \left(\hat{K}_X + \frac{1}{2} \langle \hat{K} \rangle \|\Psi(X)\|^2 \right)}{\left(\langle \hat{K} \rangle \|\Psi(X)\|^2 + \hat{K}_X \right)^2}$$

and for $\langle \hat{K} \rangle \|\Psi(X)\|^2 \ll \hat{K}_X$ this becomes:

$$D\left(\|\Psi\|^2\right) + \mu(X) \simeq \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \hat{K}_X}{\left(\langle \hat{K} \rangle \|\Psi(X)\|^2 + \hat{K}_X \right)} \quad (165)$$

Two cases arise.

When $\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 << \tau$:

$$\begin{aligned} \|\Psi(X)\|^2 &= \frac{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right) + \sqrt{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)^2 - 4\tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{(\nabla_X R(X))^2}{\sigma_X^2} - D(\|\Psi\|^2)\right)}}{2\tau} \\ &= \frac{4\tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2)\right)}{2\tau \left(\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right) - \sqrt{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)^2 - 4\tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{(\nabla_X R(X))^2}{\sigma_X^2} - D(\|\Psi\|^2)\right)}\right)} \end{aligned} \quad (166)$$

This is positive on the set:

$$\left\{ \left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right) > 0 \right\} \cup \left\{ \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) < 0 \right\} \quad (167)$$

To detail these two conditions, we write (165) for $\|\Psi(X)\|^2 > 0$:

$$D(\|\Psi\|^2) \simeq \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \frac{\hat{K}_X}{\langle \hat{K} \rangle}}{\left(\|\Psi(X)\|^2 + \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)}$$

which is equivalent to:

$$\frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2)}{\left(\|\Psi(X)\|^2 + \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)} \frac{\hat{K}_X}{\langle \hat{K} \rangle} = \frac{-\tau \|\Psi(X)\|^2 + D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}}{\left(\|\Psi(X)\|^2 + \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)} \|\Psi(X)\|^2$$

Then, we have the implication:

$$\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) > 0 \Rightarrow D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} > 0 \quad (168)$$

This implies that (167) is always satisfied, and formula (166) is valid for all X .

The second case arises when $\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 << \tau$. In this case, the solution is:

$$\|\Psi(X)\|^2 = \frac{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right) - \sqrt{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)^2 - 4\tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{(\nabla_X R(X))^2}{\sigma_X^2} - D(\|\Psi\|^2)\right)}}{2\tau}$$

This solution is valid, i.e. $\|\Psi(X)\|^2 > 0$, under the conditions:

$$\left\{ D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} > 0 \right\} \cap \left\{ \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) > 0 \right\} \quad (169)$$

and $\|\Psi\|^2 = 0$ for:

$$\left\{ D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} < 0 \right\} \cup \left\{ \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) < 0 \right\}$$

To detail these two conditions, we use the implication (168) that is equivalent to:

$$D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} < 0 \Rightarrow \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D\left(\|\Psi\|^2\right) < 0$$

Consequently, $\|\Psi(X)\|^2 = 0$ only if:

$$\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D\left(\|\Psi\|^2\right) < 0 \quad (170)$$

We find $D\left(\|\Psi\|^2\right)$ by integration of:

$$D\left(\|\Psi\|^2\right) + \mu(X) \simeq \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \hat{K}_X}{\left(\langle \hat{K} \rangle \|\Psi(X)\|^2 + \hat{K}_X\right)} \quad (171)$$

and this leads to:

$$\begin{aligned} \int_{V/V_0} D\left(\|\Psi\|^2\right) &\simeq \tau N + \int_{V/V_0} \frac{\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \hat{K}_X}{\left(\|\Psi(X)\|^2 + \frac{\hat{K}_X}{\langle \hat{K} \rangle}\right)} \\ &\simeq \tau N + \frac{1}{2} \int_{V/V_0} \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 = \tau N + \frac{1}{2} (V - V_0) \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \end{aligned}$$

we thus have:

$$D\left(\|\Psi\|^2\right) \simeq \frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \quad (172)$$

and V_0 is defined using (170). It is the set of points X such that:

$$\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} - \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 > 0 \quad (173)$$

Similarly, the set V/V_0 is defined by:

$$\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} - \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 < 0 \quad (174)$$

To each function $R(X)$ and any $d > 0$, we associate two functions that depend on the form of $\frac{1}{\sigma_X^2} (\nabla_X R(X))^2$ over the whole space. First, $v(V - V_0)$ is a decreasing function of $V - V_0$, defined by:

$$V \left(\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 > v(V - V_0) \right) = V - V_0 \quad (175)$$

Second, for every $d \geq 0$, the function $h(d)$ is given by:

$$h(d) = \frac{1}{\int_{\nabla_X R(X) > d} dX} \int_{\nabla_X R(X) > d} \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 dX \quad (176)$$

This is an increasing function of d .

Thus, we can rewrite (174) as:

$$\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} = v(V - V_0) \quad (177)$$

and moreover, by integration of (174) over V/V_0 :

$$\left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} = h \left(\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \right) \quad (178)$$

Equations (177) and (178) combine as:

$$2 \left(v(V - V_0) - \frac{\tau N}{V - V_0} \right) = h(v(V - V_0)) \quad (179)$$

which is an equation depending on the form of $R(X)$. If it has a solution, the set on which $\|\Psi(X)\|^2 = 0$ is defined by:

$$\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 < v(V - V_0)$$

and $D(\|\Psi\|^2)$ is given by

$$D(\|\Psi\|^2) \simeq v(V - V_0)$$

Once the solution of (179) is known, the constant $D(\|\Psi\|^2)$ is given by (172) and:

$$\|\Psi(X)\|^2 = \frac{2 \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) \right)}{D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} + \sqrt{\left(D(\|\Psi\|^2) - \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \right)^2 - 4 \tau \frac{\hat{K}_X}{\langle \hat{K} \rangle} \left(\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 - D(\|\Psi\|^2) \right)}} \quad (180)$$

for $X \in V/V_0$.

A2.2.2 Example 2

We choose $H(y) = y$ and equations (149) and (150) yield:

$$D(\|\Psi\|^2) \simeq \tau \|\Psi(X)\|^2 + \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \frac{\hat{K}_X}{\|\Psi(X)\|^2}$$

If:

$$D(\|\Psi\|^2) > 2 \sqrt{\tau \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \hat{K}_X} \quad (181)$$

then:

$$\|\Psi(X)\|^2 = \frac{1}{2\tau} \left(D(\|\Psi\|^2) - \sqrt{\left(D(\|\Psi\|^2) \right)^2 - 4 \hat{K}_X \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \tau} \right) > 0$$

To solve (172) and to find V_0 , we compute $D(\|\Psi\|^2)$ by integrating (171) and (172) is still valid:

$$D(\|\Psi\|^2) \simeq \frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \quad (182)$$

We proceed as in the previous paragraph to find $D(\|\Psi\|^2)$ and V_0 . Using (182), (181) becomes:

$$\frac{1}{4\tau \hat{K}_X} \left(\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \right)^2 > \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \quad (183)$$

Definitions (175) and (176) allow to rewrite (182) and (183):

$$\frac{1}{4\tau\hat{K}_X} \left(\frac{\tau N}{V - V_0} + \frac{1}{2} \left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} \right)^2 = v(V - V_0)$$

$$\left\langle \frac{1}{\sigma_X^2} (\nabla_X R(X))^2 \right\rangle_{V/V_0} = h(v(V - V_0))$$

that reduce to an equation for $V - V_0$:

$$2 \left(2\sqrt{\tau v(V - V_0) \hat{K}_X} - \frac{\tau N}{V - V_0} \right) = h(v(V - V_0))$$

If it has a solution, the set on which $\|\Psi(X)\|^2 = 0$ is defined by:

$$\frac{1}{\sigma_X^2} (\nabla_X R(X))^2 < v(V - V_0)$$

and $D(\|\Psi\|^2)$ is given by

$$D(\|\Psi\|^2) \simeq 2\sqrt{\tau v(V - V_0) \hat{K}_X}$$

Appendix 3. Computation of the background field $\hat{\Psi}(\hat{K}, \hat{X})$ and average capital \hat{K}_X

A3.1 System for $\hat{\Psi}(\hat{K}, \hat{X})$

A3.1.1 Replacing quantities depending on (K, X)

Having found $\Psi(K, X)$, we can rewrite an action functional for $\hat{\Psi}(\hat{K}, \hat{X})$. To do so, we first replace the quantities depending on $\Psi(K, X)$ in the action (38). Given the form of this function we can use the approximation $K \simeq K_X$: at the collective level, the relevant quantity, from the point of view of investors are the sectors.

Using that:

$$\frac{R(K, X)}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')} \simeq \frac{R(K, X)}{\int R(K'_X, X') \|\Psi(X')\|^2 dX'}$$

we first start by rewriting F_1 and we have:

$$F_1 \left(\frac{R(K, X)}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')}, \Gamma(K, X) \right) \simeq F_1 \left(\frac{R(K_X, X)}{\int R(K'_X, X') \|\Psi(X')\|^2 dX'}, \Gamma(K, X) \right)$$

As explained in appendix 1, when $K \simeq K_X$, we also have:

$$\Gamma(K, X) = \int \frac{F_2(R(K, X))}{K_X F_2(R(K_X, X)) \|\Psi(K_X, X)\|} \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K} - 1 = 0$$

Then, we rewrite the expression involving F_2 in (38):

$$\frac{F_2(R(K, \hat{X}))}{\int F_2(R(K', \hat{X})) \|\Psi(K', \hat{X})\|^2 dK'} \left\| \Psi(K, \hat{X}) \right\|^2 \simeq \frac{F_2(R(K, \hat{X}))}{F_2(R(K_{\hat{X}}, \hat{X})) \left\| \Psi(\hat{X}) \right\|^2} \left\| \Psi(K, \hat{X}) \right\|^2$$

$$= \frac{F_2(R(K, \hat{X})) \left\| \Psi_0(K - K_{\hat{X}}) \right\|^2}{F_2(R(K_{\hat{X}}, \hat{X}))}$$

and the $\hat{\Psi}(\hat{K}, \hat{X})$ part of the action functional (38) writes:

$$S_3 + S_4 = - \int \hat{\Psi}^\dagger(\hat{K}, \hat{X}) \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f(K, X, \Psi, \hat{\Psi}) \right) + \nabla_{\hat{X}} \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}} - g(K, X, \Psi, \hat{\Psi}) \right) \right) \hat{\Psi}(\hat{K}, \hat{X}) \quad (184)$$

where:

$$f(\hat{X}, \Psi, \hat{\Psi}) = \frac{1}{\varepsilon} \int \left(\nabla_K R(K, X) - \gamma \frac{\int K' \|\Psi(K', X)\|^2}{K} + F_1 \left(\frac{R(K, X)}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')} \right) \right) \times \frac{F_2(R(K, \hat{X})) \|\Psi_0(K - K_{\hat{X}})\|^2}{F_2(R(K_{\hat{X}}, \hat{X}))} dK \quad (185)$$

$$g(\hat{X}, \Psi, \hat{\Psi}) = \int \left(\frac{\nabla_{\hat{X}} F_0(R(K, \hat{X}))}{\|\nabla_{\hat{X}} R(K, \hat{X})\|} + \nu \nabla_{\hat{X}} F_1 \left(\frac{R(K, \hat{X})}{\int R(K', X') \|\Psi(K', X')\|^2 d(K', X')}, \Gamma(K, X) \right) \right) \times \frac{\|\Psi(K, \hat{X})\|^2 dK}{\int \|\Psi(K', \hat{X})\|^2 dK'} \quad (186)$$

Another simplification arises for the function $F_2(R(K, \hat{X}))$. Actually:

$$\begin{aligned} & \frac{F_2(R(K, \hat{X}))}{\int F_2(R(K', \hat{X})) \|\Psi(K', \hat{X})\|^2 dK'} \|\Psi(K, \hat{X})\|^2 \\ & \simeq \frac{F_2(R(K, \hat{X}))}{\int F_2(R(K_{\hat{X}}, \hat{X})) \|\Psi(\hat{X})\|^2} \|\Psi(K, \hat{X})\|^2 \\ & \simeq \frac{F_2(R(K, \hat{X}))}{F_2(R(K_{\hat{X}}, \hat{X}))} \|\Psi(K - K_{\hat{X}})\|^2 \end{aligned}$$

and by integration in (185) and (186), we have:

$$\begin{aligned} f(\hat{X}, \Psi, \hat{\Psi}) &= \frac{1}{\varepsilon} \left(r(K_{\hat{X}}, \hat{X}) - \gamma \|\Psi(\hat{X})\|^2 + F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\int R(K'_{\hat{X}}, X') \|\Psi(X')\|^2 dX'} \right) \right) \\ g(\hat{X}, \Psi, \hat{\Psi}) &= \frac{\nabla_{\hat{X}} F_0(R(K_{\hat{X}}, \hat{X}))}{\|\nabla_{\hat{X}} R(K_{\hat{X}}, \hat{X})\|} + \nu \nabla_{\hat{X}} F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\int R(K'_{\hat{X}}, X') \|\Psi(X')\|^2 dX'} \right) \end{aligned} \quad (187)$$

In the sequel, for the sake of simplicity, we will write $f(\hat{X})$ and $g(\hat{X})$ for $f(\hat{X}, K_{\hat{X}})$ and $g(\hat{X}, K_{\hat{X}})$ respectively. We then perform the following change of variable in (184):

$$\begin{aligned} \hat{\Psi} &\rightarrow \exp \left(\frac{1}{\sigma_{\hat{X}}^2} \int g(\hat{X}) d\hat{X} \right) \hat{\Psi} \\ \hat{\Psi}^\dagger &\rightarrow \exp \left(\frac{1}{\sigma_{\hat{X}}^2} \int g(\hat{X}) d\hat{X} \right) \hat{\Psi}^\dagger \end{aligned}$$

so that (184) becomes:

$$S_3 + S_4 = - \int \hat{\Psi}^\dagger \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 - \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi} \quad (188)$$

$$- \int \hat{\Psi}^\dagger \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f(\hat{X}, K_{\hat{X}}) \right) \right) \hat{\Psi}$$

This action functional for $\hat{\Psi}$ will be minimized in the next paragraph. Note that we should also include to (188), the action functional $S_1 + S_2$ evaluated at the background field Ψ , since this one depends on $\hat{\Psi}$. However, we have seen that at the background field Ψ , for $K \simeq K_X$, $u(K, X, \Psi, \hat{\Psi}) \simeq 0$ and the action functional $S_1 + S_2$ defined in (134) reduces to:

$$S_1 + S_2 \simeq \int \Psi^\dagger(X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - (\nabla_X R(X) H(K_X)) \right) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X) \quad (189)$$

and this depends on through K_X . Then, due to the first order condition for $\Psi(X)$, one has:

$$\frac{\delta}{\delta \hat{\Psi}} (S_1 + S_2) = \frac{\delta K_X}{\delta \hat{\Psi}} \frac{\partial}{\partial K_X} (S_1 + S_2)$$

We have assumed previously that $H(K_X)$ is slowly varying. Moreover, due to its definition:

$$\frac{\delta K_X}{\delta \hat{\Psi}(\hat{K}, X)} = \frac{\hat{K}}{\|\Psi(X)\|^2}$$

In most of the cases, this reduces to:

$$\frac{\delta K_X}{\delta \hat{\Psi}(\hat{K}, X)} \simeq \frac{\hat{K}}{D(\|\Psi\|^2)} \ll \hat{K}$$

Consequently, we can assume that $\frac{\delta}{\delta \hat{\Psi}} (S_1 + S_2)$ will be negligible with respect to the other quantities in the minimization with respect to $\hat{\Psi}(\hat{K}, X)$. The rationale for this approximation is the following. The field action $S_1 + S_2$ for $\Psi(X)$ depends on the global quantity $\int \hat{K} \left\| \hat{\Psi}(\hat{K}, X) \right\|^2 d\hat{K}$ that represents the total investment in sector X . While minimizing the field action $S_1 + S_2$ with respect to $\hat{\Psi}(\hat{K}, X)$, we compute the change in this action with respect to an individual variation $\hat{\Psi}(\hat{K}, X)$, and the impact of this variation is, consequently, negligible.

A3.1.2 Minimization for $\hat{\Psi}(\hat{K}, \hat{X})$

Adding the Lagrange multiplier $\hat{\lambda}$ implementing the constraint $\int \left\| \hat{\Psi}(\hat{K}, \hat{X}) \right\|^2 = \hat{N}$, the minimization of (188) with the functions given by (187) leads to the first order conditions:

$$0 = \left(\frac{\sigma_{\hat{X}}^2 \nabla_{\hat{X}}^2}{2} - \frac{\left(g(\hat{X}, K_{\hat{X}}) \right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})}{2} \right) \hat{\Psi} + \nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2 \nabla_{\hat{K}}}{2} - \hat{K} f(\hat{X}, K_{\hat{X}}) - \hat{\lambda} \right) \hat{\Psi} \quad (190)$$

$$- \left(\int \hat{\Psi}^\dagger \frac{\delta}{\delta \hat{\Psi}^\dagger} \left(\frac{1}{2\sigma_{\hat{X}}^2} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi} \right) - \left(\int \hat{\Psi}^\dagger \nabla_{\hat{K}} \frac{\delta}{\delta \hat{\Psi}^\dagger} \left(\hat{K} f(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi} \right)$$

Using that:

$$\frac{\delta}{\delta \hat{\Psi}^\dagger} K_{\hat{X}} = \frac{\hat{K}}{\|\Psi(\hat{X})\|^2} \hat{\Psi}$$

equation (190) becomes:

$$\begin{aligned} 0 = & \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 - \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi} \\ & + \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f(\hat{X}, K_{\hat{X}}) \right) - \hat{\lambda} \right) \hat{\Psi} - F(\hat{X}, K_{\hat{X}}) \hat{K} \hat{\Psi} \end{aligned} \quad (191)$$

with:

$$F(\hat{X}, K_{\hat{X}}) = \frac{\left\langle \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right\rangle}{\|\Psi(\hat{X})\|^2} + \frac{\left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right) \right\rangle}{\|\Psi(\hat{X})\|^2} \quad (192)$$

The brackets in (192) are given by:

$$\begin{aligned} & \left\langle \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right\rangle \\ = & \int \hat{\Psi}^\dagger(\hat{X}, \hat{K}) \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi}(\hat{X}, \hat{K}) d\hat{K} \\ \equiv & \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \|\hat{\Psi}(\hat{X})\|^2 \\ & \left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right) \right\rangle \\ = & \int \hat{\Psi}^\dagger(\hat{X}, K_{\hat{X}}) \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi}(\hat{X}, K_{\hat{X}}) d\hat{K} \\ = & -\nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \int \left(\hat{K} \nabla_{\hat{K}} \|\hat{\Psi}(\hat{X}, K_{\hat{X}})\|^2 - \frac{2\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \|\hat{\Psi}(\hat{X}, K_{\hat{X}})\|^2 \right) d\hat{K} \\ = & \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \|\hat{\Psi}(\hat{X})\|^2 + \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \langle \hat{K}^2 \rangle_{\hat{X}} \end{aligned} \quad (193)$$

Where the average $\langle \hat{K}^2 \rangle_{\hat{X}}$ is defined by:

$$\langle \hat{K}^2 \rangle_{\hat{X}} = \int \|\hat{\Psi}(\hat{X}, \hat{K})\|^2 d\hat{K}$$

The previous expression (193) for $F(\hat{X}, K_{\hat{X}})$ can also be rewritten as:

$$\begin{aligned}
F(\hat{X}, K_{\hat{X}}) &= \frac{\left\langle \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right\rangle}{\|\Psi(\hat{X})\|^2} + \frac{\left\langle \nabla_{\hat{K}} (\hat{K} \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}})) \right\rangle}{\|\Psi(\hat{X})\|^2} \\
&= \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \frac{\|\hat{\Psi}(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\
&\quad + \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2 \|\Psi(\hat{X})\|^2} \langle \hat{K}^2 \rangle_{\hat{X}}
\end{aligned} \tag{194}$$

It will be useful to rewrite the last term as:

$$\frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2 \|\Psi(\hat{X})\|^2} \langle \hat{K}^2 \rangle_{\hat{X}} \simeq \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \langle \hat{K} \rangle_{\hat{X}}^2 = \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \tag{195}$$

Consequently:

$$\begin{aligned}
F(\hat{X}, K_{\hat{X}}) &= \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \frac{\|\hat{\Psi}(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\
&\quad + \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2}
\end{aligned} \tag{196}$$

We also have an equation for $\hat{\Psi}^\dagger$ similar to (191):

$$\begin{aligned}
0 &= \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} (g(\hat{X}, K_{\hat{X}}))^2 - \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi}^\dagger \\
&\quad + \left(\left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} + \hat{K} f(\hat{X}, K_{\hat{X}}) \right) \nabla_{\hat{K}} - \hat{\lambda} \right) \hat{\Psi} - F(\hat{X}, K_{\hat{X}}) \hat{K} \hat{\Psi}^\dagger
\end{aligned} \tag{197}$$

A3.1.3 Resolution of (191)

A3.1.3.1 zeroth order in $\sigma_{\hat{X}}^2$ We consider $\sigma_{\hat{X}}^2 \ll 1$ (which means that fluctuation in $X \ll$ fluctuation in K). Thus (191) writes at the lowest order:

$$\left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f(\hat{X}, K_{\hat{X}}) \right) - \frac{(g(\hat{X}))^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})}{2} - F(\hat{X}, K_{\hat{X}}) \hat{K} - \hat{\lambda} \right) \hat{\Psi} = 0 \tag{198}$$

Performing the change of variable:

$$\hat{\Psi} \rightarrow \exp \left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \hat{\Psi}$$

leads to the equation for \hat{K} :

$$\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}}^2 \hat{\Psi} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} f^2(\hat{X}) + F(\hat{X}, K_{\hat{X}}) \hat{K} + \frac{1}{2} f(\hat{X}, K_{\hat{X}}) + \frac{(g(\hat{X}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + \hat{\lambda} \right) \hat{\Psi} \simeq 0 \quad (199)$$

This equation can be normalized by dividing by $f^2(\hat{X})$:

$$\frac{\sigma_{\hat{K}}^2 \nabla_{\hat{K}}^2 \hat{\Psi}}{2f^2(\hat{X})} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} + \frac{F(\hat{X}, K_{\hat{X}}) \hat{K}}{f^2(\hat{X})} + \frac{\frac{f(\hat{X}, K_{\hat{X}})}{2} + \frac{(g(\hat{X}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + \hat{\lambda}}{f^2(\hat{X})} \right) \hat{\Psi} \simeq 0$$

We then define:

$$y = \frac{\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}{\sqrt{\sigma_{\hat{K}}^2}} \left(f^2(\hat{X}) \right)^{\frac{1}{4}}$$

and (191) is transformed into:

$$\nabla_y^2 \hat{\Psi} - \left(\frac{y^2}{4} + \frac{(g(\hat{X}))^2 + \sigma_{\hat{X}}^2 \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda} \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2(\hat{X})}} \right) \hat{\Psi} \simeq 0 \quad (200)$$

Solutions of (200) are obtained by rewriting (200):

$$\hat{\Psi}'' + \left(p(\hat{X}, \hat{\lambda}) + \frac{1}{2} - \frac{1}{4} y^2 \right) \hat{\Psi}$$

where:

$$p(\hat{X}, \hat{\lambda}) = - \frac{(g(\hat{X}))^2 + \sigma_{\hat{X}}^2 \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda} \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2(\hat{X})}} - \frac{1}{2} \quad (201)$$

The solution of (200) is thus:

$$\hat{\Psi}_{\hat{\lambda}, C}^{(0)}(\hat{X}, \hat{K}) = \sqrt{C} D_{p(\hat{X}, \hat{\lambda})} \left(\left(|f(\hat{X})| \right)^{\frac{1}{2}} \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)}{\sigma_{\hat{K}}} \right) \quad (202)$$

where D_p denotes the parabolic cylinder function with parameter p and C is a normalization constant that will be computed as a function of λ using the constraint $\int \|\hat{\Psi}(\hat{K}, \hat{X})\|^2 = \hat{N}$.

A similar equation to (198) can be obtained for $\hat{\Psi}^\dagger$. The equivalent of (190) is (197):

$$\begin{aligned} 0 = & \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 - \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \hat{\Psi} \\ & + \left(\left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} + \hat{K} f(\hat{X}, K_{\hat{X}}) \right) \nabla_{\hat{K}} - \hat{\lambda} \right) \hat{\Psi} - F(\hat{X}, K_{\hat{X}}) \hat{K} \hat{\Psi} \end{aligned} \quad (203)$$

The change of variable:

$$\hat{\Psi}^\dagger \rightarrow \exp \left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \hat{\Psi}^\dagger$$

and the approximation $\sigma_{\hat{X}}^2 \ll 1$ lead ultimately to:

$$\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}}^2 \hat{\Psi}^\dagger - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} f^2(\hat{X}) + \frac{1}{2} \nabla_{\hat{X}} f(\hat{X}, K_{\hat{X}}) + \frac{1}{2\sigma_{\hat{X}}^2} \left(g(\hat{X}) \right)^2 + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + F(\hat{X}, K_{\hat{X}}) + \hat{\lambda} \right) \hat{\Psi}^\dagger \simeq 0 \quad (204)$$

which is the same equation as (199). Consequently, the solutions of (204) write:

$$\hat{\Psi}_{\lambda, C}^{(0)\dagger}(\hat{X}, \hat{K}) = \hat{\Psi}_{\lambda, C}^{(0)}(\hat{X}, \hat{K}) = \sqrt{C} D_{p(\hat{X}, \hat{\lambda})} \left(\left(|f(\hat{X})| \right)^{\frac{1}{2}} \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)}{\sigma_{\hat{K}}} \right) \quad (205)$$

To conclude this section, we detail the expressions for $\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}$ and $\frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})}$. Given the expression for $F(\hat{X}, K_{\hat{X}})$ in (196), the term $\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}$ arising in (202) and (205)

$$\begin{aligned} \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X}, K_{\hat{X}})} &= \frac{\sigma_{\hat{K}}^2}{f^2(\hat{X})} \nabla_{K_{\hat{X}}} \left(\frac{\left(g(\hat{X}, K_{\hat{X}}) \right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \frac{\|\hat{\Psi}(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\ &\quad + \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{f^2(\hat{X}, K_{\hat{X}})} \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \\ &\simeq \frac{\nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}})}{f(\hat{X}, K_{\hat{X}})} \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \end{aligned} \quad (206)$$

$\frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})}$ arising in the definition (201) of $p(\hat{X}, \hat{\lambda})$ is equal to:

$$\begin{aligned} \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} &= \frac{\sigma_{\hat{K}}^2}{2} \left(\left(\frac{\nabla_{K_{\hat{X}}} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 + \sigma_{\hat{X}}^2 \left(\nabla_{\hat{X}}^2 g(\hat{X}, K_{\hat{X}}) + \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right)}{2\sigma_{\hat{X}}^2 f(\hat{X}, K_{\hat{X}})} \right) \frac{\|\hat{\Psi}(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \right. \\ &\quad \left. + 2\nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \right)^2 \end{aligned} \quad (207)$$

and this simplifies as:

$$\frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \simeq 2\sigma_{\hat{K}}^2 \left(\nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \right)^2 \quad (208)$$

since:

$$\begin{aligned} &\frac{\nabla_{K_{\hat{X}}} \left(g(\hat{X}, K_{\hat{X}}) \right)^2 + \sigma_{\hat{X}}^2 \left(\nabla_{\hat{X}}^2 g(\hat{X}, K_{\hat{X}}) + \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right)}{2\sigma_{\hat{X}}^2 f(\hat{X}, K_{\hat{X}})} \frac{\|\hat{\Psi}(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\ &\sim \frac{\left(g(\hat{X}, K_{\hat{X}}) \right)^2 + \sigma_{\hat{X}}^2 \left(\nabla_{\hat{X}}^2 g(\hat{X}, K_{\hat{X}}) + \nabla_{K_{\hat{X}}} f(\hat{X}, K_{\hat{X}}) \right)}{2\sigma_{\hat{X}}^2 f(\hat{X}, K_{\hat{X}})} \left(\frac{\|\hat{\Psi}(\hat{X})\|^2}{K_{\hat{X}} \|\Psi(\hat{X})\|^2} \right) \ll 1 \end{aligned}$$

A3.1.3.2 Corrections in σ_X^2 : To introduce the corrections in σ_X^2 in (191) we factor the solution as:

$$\begin{aligned}\hat{\Psi}_{\lambda,C}(\hat{K},\hat{X}) &= \sqrt{C} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) D_{p(\hat{X},\hat{\lambda})} \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \hat{\Psi}^{(1)}(\hat{K},\hat{X}) \\ &\equiv \hat{\Psi}_{\lambda,C}^{(0)}(\hat{K},\hat{X}) \hat{\Psi}^{(1)}(\hat{K},\hat{X})\end{aligned}$$

and we look for $\hat{\Psi}^{(1)}$ of the form:

$$\hat{\Psi}^{(1)} = \exp(\sigma_X^2 h(K, X)) \quad (209)$$

Introducing the postulated form in (191) we are led to:

$$\frac{\sigma_X^2}{2} \nabla_{\hat{X}}^2 (\hat{\Psi}^{(1)} \hat{\Psi}_{\lambda,C}^{(0)}) + \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}}^2 \hat{\Psi}^{(1)} \right) \hat{\Psi}_{\lambda,C}^{(0)} + (\nabla_{\hat{K}} \hat{\Psi}^{(1)}) (\sigma_{\hat{K}}^2 \nabla_{\hat{K}} \hat{\Psi}_{\lambda,C}^{(0)} - \hat{K} f(\hat{X}) \hat{\Psi}_{\lambda,C}^{(0)}) = 0$$

Written in terms of $h(\hat{K}, \hat{X})$, this equation becomes at the first order in σ_X^2 :

$$\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} + \sigma_{\hat{K}}^2 \nabla_{\hat{K}}^2 h(\hat{K}, \hat{X}) + 2 (\nabla_{\hat{K}} h(\hat{K}, \hat{X})) \left(\sigma_{\hat{K}}^2 \frac{\nabla_{\hat{K}} \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} - \hat{K} f(\hat{X}) \right) = 0 \quad (210)$$

The solution of (210) is of the type:

$$\nabla_{\hat{K}} (h(K, X)) = C(\hat{K}, X) \exp \left(-2 \int \left(\frac{\nabla_{\hat{K}} \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} - \frac{\hat{K} f(\hat{X})}{\sigma_{\hat{K}}^2} \right) d\hat{K} \right) = C(\hat{K}, X) \exp \left(- \left(2 \ln \hat{\Psi}_{\lambda,C}^{(0)} - \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \right)$$

where $C(X)$ satisfies:

$$C'(\hat{K}, X) = - \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)} \sigma_{\hat{K}}^2} \exp \left(2 \ln \hat{\Psi}_{\lambda,C}^{(0)} - \frac{\hat{K}^2 f(\hat{X})}{\sigma_{\hat{K}}^2} \right) = - \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} \left(\hat{\Psi}_{\lambda,C}^{(0)} \right)^2 \exp \left(- \frac{\hat{K}^2 f(\hat{X})}{\sigma_{\hat{K}}^2} \right)$$

and the solution of (210) is:

$$\nabla_{\hat{K}} (h(K, X)) = \exp \left(- \left(2 \ln \hat{\Psi}_{\lambda,C}^{(0)} - \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \right) \left(C - \int \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)} \sigma_{\hat{K}}^2} \left(\hat{\Psi}_{\lambda,C}^{(0)} \right)^2 \exp \left(- \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) d\hat{K} \right)$$

letting $C = 0$, we obtain:

$$\nabla_{\hat{K}} (h(K, X)) = - \frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda,C}^{(0)} \right)^2} \exp \left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \left(\int \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} \left(\hat{\Psi}_{\lambda,C}^{(0)} \right)^2 \exp \left(- \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) d\hat{K} \right) \quad (211)$$

To compute $h(K, X)$, we must estimate $\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}}$ in (211). To do so, we write, for $\varepsilon \ll 1$, i.e. $|f(\hat{X})| \gg 1$:

$$\begin{aligned}
& \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) D_{p(\hat{X}, \hat{\lambda})} \left(\left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \right) \\
& \simeq \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) - \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 |f(\hat{X})|}{4\sigma_{\hat{K}}^2} \right) \left(\left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \right)^{p(\hat{X}, \hat{\lambda})} \\
& = \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) - \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 |f(\hat{X})|}{4\sigma_{\hat{K}}^2} \right) \\
& \quad \times \exp\left(\left(p(\hat{X}, \hat{\lambda}) \right) \ln \left(\left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \right) \right)
\end{aligned}$$

which allows to compute the successive derivatives of $\hat{\Psi}$. We find, for $f > 0$:

$$\begin{aligned}
\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda, C}^{(0)}}{\hat{\Psi}_{\lambda, C}^{(0)}} & \simeq \left(\frac{-f' \sigma_{\hat{X}}^2 \hat{\lambda} - g^2 f' + 2f g g'}{\sigma_{\hat{X}}^2 f^2} \ln \left(\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \left(\frac{f(\hat{X})}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right) \right. \\
& \quad + \frac{1}{2} \left(\frac{\left(g(\hat{X}) \right)^2 + \sigma_{\hat{X}}^2 \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda} \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2(\hat{X})}} + \frac{1}{2} \right) \frac{f'}{f} \\
& \quad \left. + \frac{\hat{K}^2 - \left(\frac{\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}{2} \right)^2}{\sigma_{\hat{K}}^2} f' \right) \\
& \simeq \left(\frac{\left(4\hat{K}^2 - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2
\end{aligned} \tag{212}$$

The same approximation is valid for $f < 0$ and we find for this case:

$$\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda, C}^{(0)}}{\hat{\Psi}_{\lambda, C}^{(0)}} \simeq \left(\frac{\left(4\hat{K}^2 + \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2$$

Then, introducing \mp to account for the sign of $-f$, (211) becomes:

$$\nabla_{\hat{K}}(h(K, X)) = -\frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda, C}^{(0)}\right)^2} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) \int \left(\frac{\left(4\hat{K}^2 \mp \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2 \quad (213)$$

$$\begin{aligned} & \times \left(\hat{\Psi}_{\lambda, C}^{(0)}\right)^2 \exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) d\hat{K} \\ & \simeq -\frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda, C}^{(0)}\right)^2} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) \int \left(\frac{\left(4\hat{K}^2 \mp \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2 \\ & \times \exp\left(\frac{\hat{K}^2 f(\hat{X}) - \frac{1}{2} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2 |f(\hat{X})|}{\sigma_{\hat{K}}^2}\right) d\hat{K} \end{aligned} \quad (214)$$

$$\begin{aligned} & \simeq -\frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda, C}^{(0)}\right)^2} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) \int \left(\frac{\sigma_{\hat{K}}^2 \left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)}{\left(2\hat{K} f(\hat{X}) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right) |f(\hat{X})|\right)^2} \right)^2 \\ & \times \partial_{\hat{K}}^4 \exp\left(\frac{\hat{K}^2 f(\hat{X}) - \frac{1}{2} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2 |f(\hat{X})|}{\sigma_{\hat{K}}^2}\right) d\hat{K} \end{aligned} \quad (215)$$

Assuming $\frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \ll 1$, we have ultimately:

$$\begin{aligned}
\nabla_{\hat{K}}(h(K, X)) &\simeq -\frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda, C}^{(0)}\right)^2} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X})\right) \left(\frac{\sigma_{\hat{K}}^2 \left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X)}{2\hat{K}f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})|} \right)^2 \right. \\
&\quad \times \partial_{\hat{K}}^3 \exp\left(\frac{\left(\hat{K}^2 f(\hat{X}) - \frac{1}{2} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 |f(\hat{X})| \right)}{\sigma_{\hat{K}}^2} \right) \\
&\quad \left. \frac{\left(\frac{\left(\hat{K}^2 - \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X)}{\sigma_{\hat{K}}^2} \right)^2}{2\hat{K}f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})|} \right) \\
&= -\frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\left(\sigma_{\hat{K}}^2 \right)^2 \left(2\hat{K}f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} \\
&= -\frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\left(\sigma_{\hat{K}}^2 \right)^2 \left(2\hat{K}f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)}
\end{aligned}$$

Replacing in first approximation \hat{K} by $\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2}$ in (212), and using (211) and (209) leads to:

$$\hat{\Psi}^{(1)}(\hat{X}) = \sqrt{C} \exp \left(- \int \frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\left(\sigma_{\hat{K}}^2 \right)^2 \left(2\hat{K}f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right)$$

with C a constant to be computed using the normalization condition.

To find $\hat{\Psi}^\dagger$, we need also $\hat{\Psi}^{(1)\dagger}$. Writing:

$$\hat{\Psi}^{(1)\dagger} = \exp(\sigma_X^2 g(K, X))$$

with a function $g(K, X)$ that satisfies:

$$\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda, C}^{(0)\dagger}}{\hat{\Psi}_{\lambda, C}^{(0)}} + \sigma_{\hat{K}}^2 \nabla_{\hat{K}}^2 g(\hat{K}, \hat{X}) + 2 \left(\nabla_{\hat{K}} g(\hat{K}, \hat{X}) \right) \left(\sigma_{\hat{K}}^2 \frac{\nabla_{\hat{K}} \hat{\Psi}_{\lambda, C}^{(0)\dagger}}{\hat{\Psi}_{\lambda, C}^{(0)}} + \hat{K} f(\hat{X}) \right) = 0$$

with:

$$\hat{\Psi}_{\lambda, C}^{(0)\dagger} = \exp \left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) D_{p(\hat{X}, \hat{\lambda})} \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right)$$

we find:

$$\nabla_{\hat{K}}(g(K, X)) = -\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda, C}^{(0)\dagger}}{\hat{\Psi}_{\lambda, C}^{(0)}} \exp \left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) \left(\int \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda, C}^{(0)\dagger}}{\hat{\Psi}_{\lambda, C}^{(0)\dagger}} \left(\hat{\Psi}_{\lambda, C}^{(0)\dagger} \right)^2 \exp \left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f(\hat{X}) \right) d\hat{K} \right)$$

and:

$$\hat{\Psi}^{(1)\dagger}(\hat{X}) = \sqrt{C} \exp \left(\int \frac{\left(\left(\hat{K}^2 \pm \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\sigma_K^2 \left(2\hat{K} f(\hat{X}) + \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right)$$

where \pm accounts for the sign of f .

Ultimately, coming back to the initial definition of the fields we obtain for $\hat{\Psi}_{\lambda,C}(\hat{K}, \hat{X})$ and $\hat{\Psi}_{\lambda,C}^\dagger(\hat{K}, \hat{X})$:

$$\begin{aligned} \hat{\Psi}_{\lambda,C}(\hat{K}, \hat{X}) &= \sqrt{C} \exp \left(-\sigma_X^2 \int \frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{(\sigma_K^2)^2 \left(2\hat{K} f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right) \\ &\quad \times \exp \left(\frac{1}{\sigma_X^2} \int g(\hat{X}) d\hat{X} + \frac{\hat{K}^2}{\sigma_K^2} f(\hat{X}) \right) D_{p(\hat{X}, \lambda)} \left(\hat{K} \left(\frac{|f(\hat{X})|}{\sigma_K^2} \right)^{\frac{1}{2}} \right) \\ \hat{\Psi}_{\lambda,C}^\dagger(\hat{K}, \hat{X}) &= \sqrt{C} \exp \left(\sigma_X^2 \int \frac{\left(\left(\hat{K}^2 \pm \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{(\sigma_K^2)^2 \left(2\hat{K} f(\hat{X}) + \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right) \\ &\quad \times \exp \left(-\left(\frac{1}{\sigma_X^2} \int g(\hat{X}) d\hat{X} + \frac{\hat{K}^2}{\sigma_K^2} f(\hat{X}) \right) \right) D_{p(\hat{X}, \lambda)} \left(\hat{K} \left(\frac{|f(\hat{X})|}{\sigma_K^2} \right)^{\frac{1}{2}} \right) \end{aligned}$$

A3.1.3.3 Computation of $\|\hat{\Psi}(\hat{K}, \hat{X})\|^2$ As a consequence of the previous result, we can compute $\|\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})\|^2$. We start with $\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)}$. We have:

$$\begin{aligned}\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} &= C \exp \left(-\sigma_X^2 \int \left(\frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X)}{\left(\sigma_K^2 \right)^2 \left(2\hat{K} f(\hat{X}) - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} \right)^2 \right. \\ &\quad \left. - \frac{\left(\left(\hat{K}^2 \pm \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\left(\sigma_K^2 \right)^2 \left(2\hat{K} f(\hat{X}) + \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right) \\ &= C \exp \left(-\sigma_X^2 \int \left(\frac{\left(\left(\hat{K}^2 - \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\sigma_K^2 \left(2\hat{K} |f(\hat{X})| - \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} \right. \right. \\ &\quad \left. \left. - \frac{\left(\left(\hat{K}^2 + \frac{1}{4} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \right) f'(X) \right)^2}{\left(\sigma_K^2 \right)^2 \left(2\hat{K} |f(\hat{X})| + \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) |f(\hat{X})| \right)} d\hat{K} \right) \right)\end{aligned}$$

And for $\frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \ll 1$:

$$\begin{aligned}\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} &\simeq C \exp \left(-\sigma_X^2 \int \left(\frac{\left(\frac{3}{4} \hat{K}^2 f'(X) \right)^2}{\left(\sigma_K^2 \right)^2 \hat{K} |f(\hat{X})|} - \frac{\left(\frac{5}{4} \hat{K}^2 f'(X) \right)^2}{3 \left(\sigma_K^2 \right)^2 \hat{K} f(\hat{X})} \right) d\hat{K} \right) \\ &= C \exp \left(-\frac{\sigma_X^2 \hat{K}^4 (f'(X))^2}{96 \left(\sigma_K^2 \right)^2 |f(\hat{X})|} \right)\end{aligned}$$

Gathering the previous results, we obtain the norm of $\|\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})\|^2$:

$$\|\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})\|^2 \simeq C \exp \left(-\frac{\sigma_X^2 \hat{K}^4 (f'(X))^2}{96 \left(\sigma_K^2 \right)^2 |f(\hat{X})|} \right) D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_K^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_K^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \quad (216)$$

with:

$$f(\hat{X}, K_{\hat{X}}) = \left(r(K_{\hat{X}}, \hat{X}) - \gamma \|\Psi(\hat{X})\|^2 + F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\int R(K'_{\hat{X}}, X') \|\Psi(X')\|^2 dX'} \right) \right) \quad (217)$$

$$g(\hat{X}, K_{\hat{X}}) = \left(\frac{\nabla_{\hat{X}} F_0(R(K_{\hat{X}}, \hat{X}))}{\|\nabla_{\hat{X}} R(K_{\hat{X}}, \hat{X})\|} + \nu \nabla_{\hat{X}} F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\int R(K'_{\hat{X}}, X') \|\Psi(X')\|^2 dX'} \right) \right) \quad (218)$$

The solutions are parametrized by C and $\hat{\lambda}$ and $\hat{K}_{\hat{X}}$. Using the constraint $\left\| \hat{\Psi}(\hat{K}, \hat{X}) \right\|^2 = \hat{N}$ will reduce the solutions to a one-parameter set of solutions. The computation of the average capital over this set will lead to the defining equation for $\hat{K}_{\hat{X}}$.

Replacing in first approximation \hat{K} by its average $\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2}$ in the first term yields:

$$\left\| \hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X}) \right\|^2 \simeq C \exp \left(- \frac{\sigma_{\hat{X}}^2 \left(\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2} \right)^4 (f'(\hat{X}))^2}{96 (\sigma_{\hat{K}}^2)^2 |f(\hat{X})|} \right) D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) \quad (219)$$

A3.1.4 Estimation of $S_3(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) + S_4(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}))$

For later purposes, we compute an estimation of $S_3(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) + S_4(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}))$ for any background field $\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})$. We multiply (64) by $\hat{\Psi}_{\hat{\lambda}}^\dagger(\hat{K}, \hat{X})$ on the left and integrate the equation over \hat{K} and \hat{X} . It yields:

$$0 = S_3(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) + S_4(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) - \hat{\lambda} \int \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} d\hat{X} - \int F(\hat{X}, K_{\hat{X}}) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} d\hat{X}$$

Using the constraint about the number of investors:

$$\int \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} = \hat{N}$$

we find:

$$S_3(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) + S_4(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) = \hat{\lambda} \hat{N} + \int F(\hat{X}, K_{\hat{X}}) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} d\hat{X}$$

Moreover, equation (66) implies³⁴:

$$\begin{aligned} & \int F(\hat{X}, K_{\hat{X}}) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} d\hat{X} \\ &= \int K_{\hat{X}} \nabla_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \left\| \hat{\Psi}(\hat{X}) \right\|^2 d\hat{X} \\ & \quad + \int K_{\hat{X}} \frac{\nabla_{K_{\hat{X}}} f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \langle \hat{K}^2 \rangle_{\hat{X}} d\hat{X} \end{aligned} \quad (220)$$

In our applications the involved functions are roughly power functions in $K_{\hat{X}}$, and consequently, the integral $\int F(\hat{X}, K_{\hat{X}}) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} d\hat{X}$ is of order:

$$\int \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \left\| \hat{\Psi}(\hat{X}) \right\|^2 d\hat{X} + \int \frac{f^2(\hat{X}, K_{\hat{X}})}{\sigma_{\hat{K}}^2} \langle \hat{K}^2 \rangle_{\hat{X}} d\hat{X} \quad (221)$$

³⁴ All averages in the next formula are computed in state $\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})$.

Since $\langle \hat{K}^2 \rangle_{\hat{X}} \simeq K_{\hat{X}}^2 \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2}$, the second term in (??) is negligible if we assume $\frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} \ll 1$, i.e. the number of firms is smaller than the number of investors. Consequently, (??) reduces to:

$$\int \left(\frac{\left(g(\hat{X}, K_{\hat{X}}) \right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) + f(\hat{X}, K_{\hat{X}}) \right) \|\hat{\Psi}(\hat{X})\|^2 d\hat{X} \lesssim \int M \|\hat{\Psi}(\hat{X})\|^2 d\hat{X} = M\hat{N}$$

where M is the lowest bound for $|\hat{\lambda}|$, computed below in (242) and (243). Our previous estimation relies on $\frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \ll 1$, which is true for $f^2(\hat{X}) \gg 1$. As a consequence:

$$S_3(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) + S_4(\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})) = (\hat{\lambda} + M) \hat{N} = -(|\hat{\lambda}| - M) \hat{N} \quad (222)$$

A3.1.4 Identification of $K_{\hat{X}}$ and $\|\Psi(\hat{X})\|^2$:

A3.1.4.1 Formula depending on $\hat{\lambda}$ and C In this paragraph, we compute the average capital $K_{\hat{X}}$ and the density of investors $\|\hat{\Psi}(\hat{X})\|^2$ at \hat{X} that are defined by using (152):

$$K_{\hat{X}} \|\Psi(\hat{X})\|^2 = \int_0^\infty \hat{K} C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{(\sigma_{\hat{K}}^2)^2} \right) \times D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) d\hat{K} \quad (223)$$

and:

$$\|\hat{\Psi}(\hat{X})\|^2 = C \int_0^\infty \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{(\sigma_{\hat{K}}^2)^2} \right) \times D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) d\hat{K}$$

with:

$$u(\hat{X}, \hat{K}_{\hat{X}}) = \frac{\left(\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2} \right)^4 (f'(\hat{X}))^2}{96 |f(\hat{X})|} \quad (224)$$

Note that in these formulas, $K_{\hat{X}}$ and $\|\hat{\Psi}(\hat{X})\|^2$ depend implicitly of $\hat{\lambda}$ since they have been computed in the state defined by the background field $\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})$. In the sequel, for the sake of simplicity, $\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})$, the indices λ and C may be omitted.

We will also need $\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2}$ that arises in (224):

$$\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} = \frac{\int_0^\infty \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) d\hat{K}}{\int_0^\infty \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K}}$$

By a change of variable $\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \rightarrow \hat{K}$ we can also write:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 \simeq C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2} \right) \int_{\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}^{\infty} \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K}$$

$$\left\| \hat{\Psi}(\hat{X}) \right\|^2 \simeq C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16\sigma_{\hat{K}}^2} \right) \int_{\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}^{\infty} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K}$$

and by a zeroth order expansion around 0 of $\hat{K} D_{p(\hat{X}, \hat{\lambda})}^2$ and $D_{p(\hat{X}, \hat{\lambda})}^2$ we have:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 \simeq C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16\sigma_{\hat{K}}^2} \right) \int_0^{\infty} \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K} \quad (225)$$

$$\left\| \hat{\Psi}(\hat{X}) \right\|^2 \simeq C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16\sigma_{\hat{K}}^2} \right) \left(\int_0^{\infty} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K} - \frac{\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} 2^{\frac{p(\hat{X}, \hat{\lambda})}{2}} \sqrt{\pi} \sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{\Gamma\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) f^2(\hat{X})} \right) \quad (226)$$

To compute $\left\| \hat{\Psi}(\hat{X}) \right\|^2$ we use that the function D satisfies:

$$\int D_p^2 = \frac{\sqrt{\pi} \text{Psi}\left(\frac{1}{2} - \frac{p}{2}\right) - \text{Psi}\left(-\frac{p}{2}\right)}{2^{\frac{3}{2}} \Gamma(-p)}$$

The computation of the norm implies a second change of variable $\hat{K} \rightarrow \hat{K} \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}}$ and we obtain for (226):

$$\begin{aligned} \left\| \hat{\Psi}(\hat{X}) \right\|^2 &= \int \left\| \hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X}) \right\|^2 d\hat{K} \\ &= C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16\sigma_{\hat{K}}^2} \right) \left(\int D_{p(\hat{X}, \hat{\lambda})}^2 \left(\hat{K} (f^2(\hat{X}))^{\frac{1}{4}} \right) d\hat{K} - \frac{\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} 2^{\frac{p(\hat{X}, \hat{\lambda})}{2}} \sqrt{\pi} \sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{\Gamma\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) f^2(\hat{X})} \right) \\ &= C \exp \left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16\sigma_{\hat{K}}^2} \right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} \\ &\quad \times \left(\frac{\sqrt{\pi} \text{Psi}\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) - \text{Psi}\left(-\frac{p(\hat{X}, \hat{\lambda})}{2}\right)}{2^{\frac{3}{2}} \Gamma(-p(\hat{X}, \hat{\lambda}))} - \frac{2^{\frac{p(\hat{X}, \hat{\lambda})}{2}} \sqrt{\pi} \sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{\Gamma\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) f^2(\hat{X})} \right) \end{aligned} \quad (227)$$

Expression (225) is computed using that:

$$\begin{aligned} \int_0^{\infty} z D_p^2(z) dz &= \int_0^{\infty} D_{p+1}(z) D_p(z) dz + p \int_0^{\infty} D_{p-1}(z) D_p(z) dz \\ \int_0^{\infty} z D_p^2(z) dz &= \frac{\Gamma\left(-\frac{p+1}{2}\right) \Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right) \Gamma\left(-\frac{p}{2}\right)}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma\left(-\frac{p}{2}\right) \Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right) \Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \end{aligned}$$

and:

$$\int \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\hat{K} \left(f^2(\hat{X}) \right)^{\frac{1}{4}} \right) = \left(f(\hat{X}) \right)^{-1} \int u D_{p(\hat{X}, \hat{\lambda})}^2(u)$$

We obtain:

$$\begin{aligned} K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 &\simeq \exp \left(-\frac{\sigma_X^2 u(\hat{X}, \hat{K}_{\hat{X}})}{16(\sigma_{\hat{K}}^2)^2} \right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-1} C \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \right) \end{aligned} \quad (228)$$

where:

$$p = p(\hat{X}, \hat{\lambda}) \quad (229)$$

Ultimately we can compute $\frac{K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2}{\left\| \hat{\Psi}(\hat{X}) \right\|^2}$:

$$\begin{aligned} \frac{K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2}{\left\| \hat{\Psi}(\hat{X}) \right\|^2} &\simeq \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} \frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \\ &\quad \frac{\sqrt{\pi} \text{Psi}(\frac{1-p}{2}) - \text{Psi}(-\frac{p}{2})}{2^{\frac{3}{2}} \Gamma(-p)} \\ &\equiv \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} h(p) \\ &\simeq \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} \sqrt{p + \frac{1}{2}} \end{aligned}$$

so that:

$$\exp \left(-\frac{\sigma_X^2 u(\hat{X}, \hat{K}_{\hat{X}})}{(\sigma_{\hat{K}}^2)^2} \right) \simeq \exp \left(-\frac{\sigma_X^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \quad (230)$$

We end this section by finding asymptotic form for $\left\| \hat{\Psi}(\hat{X}) \right\|^2$ and $K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2$

For $\varepsilon \ll 1$ an asymptotic form yields that:

$$D_{p(\hat{X}, \hat{\lambda})} \left(\hat{K} \left(f^2(\hat{X}) \right)^{\frac{1}{4}} \right) \simeq \exp \left(-\frac{\hat{K}^2 |f(\hat{X})|}{4\sigma_{\hat{K}}^2} \right) \left(\hat{K} \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right)^{p(\hat{X}, \hat{\lambda})} \quad (231)$$

and we obtain:

$$\begin{aligned} \left\| \hat{\Psi}(\hat{X}) \right\|^2 &= C \exp \left(-\frac{\sigma_X^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2} \right) \\ &\times \int_0^\infty \exp \left(-\frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 |f(\hat{X})|}{2\sigma_{\hat{K}}^2} \right) \left(\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right)^{2p(\hat{X}, \hat{\lambda})} d\hat{K} \end{aligned}$$

A change of variable $w = \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2 |f(\hat{X})|}{2\sigma_{\hat{K}}^2}$ leads to:

$$\|\hat{\Psi}(\hat{X})\|^2 \simeq C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} \left(2^{p(\hat{X}, \hat{\lambda}) - \frac{1}{2}} \Gamma\left(p(\hat{X}, \hat{\lambda}) + \frac{1}{2}\right) - \frac{2^{\frac{p(\hat{X}, \hat{\lambda})}{2}} \sqrt{\pi}}{\Gamma\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right)} \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right) \quad (232)$$

By the same token we can use the asymptotic form (231) to find $K_{\hat{X}}$:

$$\begin{aligned} K_{\hat{X}} \|\Psi(\hat{X})\|^2 &\simeq C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right) \int \hat{K} \exp\left(-\frac{\hat{K}^2 |f(\hat{X})|}{2\sigma_{\hat{K}}^2}\right) \left(\hat{K} \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)^{2p(\hat{X}, \hat{\lambda})} d\hat{K} \\ &= \frac{\sigma_{\hat{K}}^2 C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right)}{|f(\hat{X})|} \int y \exp\left(-\frac{y^2}{2}\right) y^{2p(\hat{X}, \hat{\lambda})} dy \end{aligned}$$

We set $y = \sqrt{2w}$ and we obtain:

$$\begin{aligned} K_{\hat{X}} \|\Psi(\hat{X})\|^2 &\simeq C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right) 2^{p(\hat{X}, \hat{\lambda})} \frac{\sigma_{\hat{K}}^2}{|f(\hat{X})|} \int \exp(-w) w^{p(\hat{X}, \hat{\lambda})} dw \\ &= C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right) 2^{p(\hat{X}, \hat{\lambda})} \frac{\sigma_{\hat{K}}^2}{|f(\hat{X})|} \Gamma\left(p(\hat{X}, \hat{\lambda}) + 1\right) \end{aligned}$$

A3.1.4.2 Computation of C as a function of $\hat{\lambda}$: Ultimately, we need to determine the value of the Lagrange multiplier $\hat{\lambda}$ and of the associated value of C . We do so by integrating (216) and the result is constrained to be \hat{N} , the total number of agents:

$$\hat{N} = \int \|\hat{\Psi}_{\lambda, C}(\hat{K}, \hat{X})\|^2 d\hat{K} d\hat{X} = \int \|\hat{\Psi}(\hat{X})\|^2 d\hat{X}$$

Using (227) and (230), we have:

$$\begin{aligned} \hat{N} &= \int \|\hat{\Psi}(\hat{X})\|^2 \simeq \int C \exp\left(-\frac{\sigma_{\hat{X}}^2 u(\hat{X}, \hat{K}_{\hat{X}})}{\sigma_{\hat{K}}^2}\right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} \\ &\quad \times \left(\frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \frac{\text{Psi}\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) - \text{Psi}\left(-\frac{p(\hat{X}, \hat{\lambda})}{2}\right)}{\Gamma(-p(\hat{X}, \hat{\lambda}))} - \frac{2^{\frac{p(\hat{X}, \hat{\lambda})}{2}} \sqrt{\pi}}{\Gamma\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right)} \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right) d\hat{X} \\ &\simeq \int C \exp\left(-\frac{\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3}\right) \left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} \frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \frac{\text{Psi}\left(\frac{1-p(\hat{X}, \hat{\lambda})}{2}\right) - \text{Psi}\left(-\frac{p(\hat{X}, \hat{\lambda})}{2}\right)}{\Gamma(-p(\hat{X}, \hat{\lambda}))} d\hat{X} \end{aligned} \quad (233)$$

with f and g given by (217) and (218). We thus obtain C as a function of $\hat{\lambda}$. For $f(\hat{X})$ slowly varying around its average we can replace $|f(\hat{X})|$ and $f'(X)$ by $\langle |f(\hat{X})| \rangle$ and $\langle f'(X) \rangle$, where the bracket $\langle A(\hat{X}) \rangle$

represents the average of the quantity $A(\hat{X})$ over the sectors space. Given that the integrated function is of order $\Gamma(p)$, we can replace the integral by the maximal values of the integrand. Consequently, we have:

$$C(\bar{p}(\hat{\lambda})) \simeq \frac{\exp\left(-\frac{\sigma_X^2 \sigma_K^2 \left(\frac{(\bar{p}(\hat{\lambda}) + \frac{1}{2})f'(X_0)}{f(X_0)}\right)^2}{96|f(\hat{X}_0)|}\right) \hat{N}\Gamma(-\bar{p}(\hat{\lambda}))}{\left(\frac{\langle |f(\hat{X})| \rangle}{\sigma_K^2}\right)^{-\frac{1}{2}} V_r \left(\text{Psi}\left(-\frac{\bar{p}(\hat{\lambda})-1}{2}\right) - \text{Psi}\left(-\frac{\bar{p}(\hat{\lambda})}{2}\right)\right)} \quad (234)$$

where:

$$\bar{p}(\hat{\lambda}) = \left(-\frac{\frac{(g(\hat{X}_0))^2}{\sigma_X^2} + \left(f(\hat{X}_0) + \frac{1}{2}|f(\hat{X}_0)| + \nabla_{\hat{X}}g(\hat{X}_0, K_{\hat{X}_0}) - \frac{\sigma_K^2 F^2(\hat{X}_0, K_{\hat{X}_0})}{2f^2(\hat{X}_0)} + \hat{\lambda}\right)}{|f(\hat{X}_0)|} \right) \quad (235)$$

and:

$$\hat{X}_0 = \arg \min_{\hat{X}} \left(\frac{\sigma_X^2 \sigma_K^2 \left(\frac{(p(\hat{\lambda}) + \frac{1}{2})f'(X)}{f(\hat{X})}\right)^2}{96|f(\hat{X})|} \right) \quad (236)$$

and V_r is the volume of the reduced space where the maximum is reached defined by:

$$V_r = \sum_{\hat{X}/p(\hat{X}, \hat{\lambda})=\bar{p}(\hat{\lambda})} \frac{1}{\left| \frac{(\|\hat{\Psi}(\hat{X})\|^2)''}{C} \right|}$$

We thus can replace C by $C(\hat{\lambda})$ and we are left with an infinite number of solutions of (198) parametrized by $\hat{\lambda}$ and given by (216). We write $\|\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})\|^2$ the solution for $\hat{\lambda}$.

A3.1.4.2 Identification equation for $K_{\hat{X}}$ To each state $\|\hat{\Psi}_{\hat{\lambda}}(\hat{K}, \hat{X})\|^2$, we can associate an average level of $K_{\hat{X}, \hat{\lambda}}$ satisfying (228) rewritten as a function of $\hat{\lambda}$. Using (230) we find:

$$\begin{aligned} K_{\hat{X}, \hat{\lambda}} \|\hat{\Psi}_{\hat{\lambda}}(\hat{X})\|^2 &= \hat{K}_{\hat{X}, \hat{\lambda}} \\ &= \exp\left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96|f(\hat{X})|^3}\right) \left(\frac{|f(\hat{X})|}{\sigma_K^2}\right)^{-1} \\ &\quad \times C(\bar{p}(\hat{\lambda})) \left(\frac{\Gamma(-\frac{p+1}{2})\Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2})\Gamma(\frac{-p}{2})}{2^{p+2}\Gamma(-p-1)\Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2})\Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2})\Gamma(\frac{-p-1}{2})}{2^{p+1}\Gamma(-p)\Gamma(-p+1)}\right) \end{aligned} \quad (237)$$

where:

$$p(\hat{X}, \hat{\lambda}) = -\frac{\left(g(\hat{X})\right)^2 + \sigma_X^2 \left(f(\hat{X}) + \nabla_{\hat{X}}g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda}\right)}{\sigma_X^2 \sqrt{f^2(\hat{X})}} - \frac{1}{2} \quad (238)$$

As explained in the core of the paper, to compute $K_{\hat{X}}$ we must average (237) over $\hat{\lambda}$ with the weight $\exp(-(S_3 + S_4))$. Given equation (191), a solution (216) for a given $\hat{\lambda}$ and taking into account the constraint $\|\hat{\Psi}(\hat{K}, \hat{X})\|^2 = \hat{N}$, has the associated normalized weight (see (222)):

$$w(|\hat{\lambda}|) = \frac{\exp(-(|\hat{\lambda}| - M)\hat{N})}{\int_{|\hat{\lambda}| > M} \exp(-(|\hat{\lambda}| - M)\hat{N}) d|\hat{\lambda}|}$$

with M is the lower bound for $|\hat{\lambda}|$.

This lower bound is found by considering (199) and adding the term proportional to $\frac{\sigma_{\hat{X}}^2}{2}$:

$$\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 \hat{\Psi} + \nabla_y^2 \hat{\Psi} - \left(\sqrt{f^2(\hat{X})} \frac{y^2}{4} + \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda} \right) \right) \Psi \quad (239)$$

multiplying (239) by $\hat{\Psi}^\dagger$ and integrating. It yields:

$$\begin{aligned} 0 &= -\frac{\sigma_{\hat{X}}^2}{2} \int (\nabla_{\hat{X}} \hat{\Psi}^\dagger) (\nabla_{\hat{X}} \hat{\Psi}) \\ &\quad - \frac{1}{2} \int \sqrt{f^2(\hat{X})} \left((\nabla_y \hat{\Psi}^\dagger) (\nabla_y \hat{\Psi}) + \hat{\Psi}^\dagger \frac{y^2}{4} \hat{\Psi} \right) + \int \hat{\Psi}^\dagger_{y=0} (\nabla_y \hat{\Psi})_{y=0} \\ &\quad - \int \hat{\Psi}^\dagger \left(\sqrt{f^2(\hat{X})} \frac{y^2}{4} + \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} + \hat{\lambda} \right) \right) \Psi \end{aligned} \quad (240)$$

The first part of the right hand side in (240):

$$-\frac{\sigma_{\hat{X}}^2}{2} \int (\nabla_{\hat{X}} \hat{\Psi}^\dagger) (\nabla_{\hat{X}} \hat{\Psi}) - \int \sqrt{f^2(\hat{X})} \left(\frac{1}{2} (\nabla_y \hat{\Psi}^\dagger) (\nabla_y \hat{\Psi}) + \hat{\Psi}^\dagger \frac{y^2}{4} \hat{\Psi} \right) \quad (241)$$

includes the hamiltonian of a sum of harmonic oscillators, and thus (241) is lower than $-\frac{\int \hat{\Psi}^\dagger \sqrt{f^2(\hat{X})} \hat{\Psi}}{2}$. Consequently, we have the inequality for all \hat{X} :

$$\begin{aligned} &\hat{\Psi}^\dagger_{y=0} (\nabla_y \hat{\Psi})_{y=0} + \int \hat{\Psi}^\dagger \left(|\hat{\lambda}| - \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} - \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \right) \Psi d\hat{K} \\ &> \frac{\int \hat{\Psi}^\dagger \sqrt{f^2(\hat{X})} \hat{\Psi} d\hat{K}}{2} \end{aligned}$$

Since:

$$|\hat{\lambda}| \int |\Psi|^2 d\hat{K} = |\hat{\lambda}| \left\| \hat{\Psi}(\hat{X}) \right\|^2$$

and $\hat{\Psi}^\dagger_{y=0} (\nabla_y \hat{\Psi})_{y=0}$ is of order $1 \ll \left\| \hat{\Psi}(\hat{X}) \right\|^2$ since it is integrated over \hat{X} only. Consequently, the condition reduces to:

$$|\hat{\lambda}| \left\| \hat{\Psi}(\hat{X}) \right\|^2 > \int \hat{\Psi}^\dagger \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \Psi d\hat{K}$$

that is:

$$|\hat{\lambda}| > \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})}$$

for each \hat{X} , and we have:

$$M = \max_{\hat{X}} \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \quad (242)$$

Note that in general, for $\varepsilon \ll 1$, $f(\hat{X}) \gg 1$ and:

$$\frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \ll \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})$$

so that:

$$M \simeq \max_{\hat{X}} \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \quad (243)$$

Having found M , this yields:

$$w(|\hat{\lambda}|) = \hat{N} \exp(-(|\hat{\lambda}| - M) \hat{N}) \quad (244)$$

Consequently, averaging equation (237) yields:

$$K_{\hat{X}} = \int K_{\hat{X}, \hat{\lambda}} \hat{N} \exp(-(|\hat{\lambda}| - M) \hat{N}) d\hat{\lambda}$$

$$\begin{aligned} K_{\hat{X}} \|\Psi(\hat{X})\|^2 &= \int C(\hat{\lambda}) w(|\hat{\lambda}|) \exp\left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3}\right) \left(\frac{|f(\hat{X})|}{\sigma_K^2}\right)^{-1} \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)}\right) d\hat{\lambda} \end{aligned} \quad (245)$$

with $C(\bar{p}(\hat{\lambda}))$ given by (234). Given (244), the average value of $|\hat{\lambda}|$ is $M + \frac{1}{\hat{N}}$ and have:

$$\begin{aligned} K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})| &= C\left(\bar{p}\left(-\left(M - \frac{1}{\hat{N}}\right)\right)\right) \sigma_K^2 \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)}\right) \end{aligned} \quad (246)$$

with:

$$p = -\frac{(g(\hat{X}))^2 + \sigma_{\hat{X}}^2 \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} - \left(M - \frac{1}{\hat{N}}\right)\right)}{\sigma_{\hat{X}}^2 \sqrt{f^2(\hat{X})}} - \frac{1}{2}$$

We can consider that $\frac{1}{\hat{N}} \ll 1$ so that $C\left(\bar{p}\left(-\left(M - \frac{1}{\hat{N}}\right)\right)\right) \simeq C(\bar{p}(-M))$. It amounts to consider $|\hat{\lambda}| = M$. We will also write $\bar{p}(-M) = \bar{p}$ and given (??) we have:

$$\bar{p} = \left(\frac{M - \frac{(g(\hat{X}_0))^2}{\sigma_{\hat{X}}^2} + \left(f(\hat{X}_0) + \frac{1}{2} |f(\hat{X}_0)| + \nabla_{\hat{X}} g(\hat{X}_0, K_{\hat{X}_0}) - \frac{\sigma_K^2 F^2(\hat{X}_0, K_{\hat{X}_0})}{2f^2(\hat{X}_0)}\right)}{|f(\hat{X}_0)|} \right) \quad (247)$$

and:

$$p = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \left(f(\hat{X}) + \frac{\sqrt{f^2(\hat{X})}}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \right)}{\sqrt{f^2(\hat{X})}} \quad (248)$$

Equation (234) rewrites:

$$C(\bar{p}) \simeq \frac{\exp \left(-\frac{\sigma_X^2 \sigma_K^2 \left(\frac{(\bar{p}(\lambda) + \frac{1}{2}) f'(X_0)}{f(X_0)} \right)^2}{96 |f(\hat{X}_0)|} \right) \hat{N}\Gamma(-\bar{p})}{\left(\frac{\langle |f(\hat{X})| \rangle}{\sigma_K^2} \right)^{-\frac{1}{2}} V_r(\text{Psi}(-\frac{\bar{p}-1}{2}) - \text{Psi}(-\frac{\bar{p}}{2}))} \quad (249)$$

and (246) reduces to:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| = C(\bar{p}) \sigma_K^2 \hat{\Gamma} \left(p + \frac{1}{2} \right) \quad (250)$$

with:

$$\begin{aligned} \hat{\Gamma} \left(p + \frac{1}{2} \right) &= \exp \left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(\frac{-p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \right) \end{aligned} \quad (251)$$

We note that, asymptotically:

$$\hat{\Gamma} \left(p + \frac{1}{2} \right) \sim_{\infty} \exp \left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \Gamma \left(p + \frac{3}{2} \right) \quad (252)$$

A3.1.4.3 Replacing $\|\Psi(X)\|^2$ in the $K_{\hat{X}}$ equation We can isolate $K_{\hat{X}}$ in (246) by using (166) and (180)

to rewrite $\left\| \Psi(\hat{X}) \right\|^2$:

Using (151a):

$$\begin{aligned} D(\|\Psi\|^2) &= 2\tau \|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2 \left(\frac{\hat{K}_X}{\|\Psi(X)\|^2} \right) \left(1 - \frac{H'(\hat{K}_X)}{H(\hat{K}_X)} \frac{\hat{K}_X}{\|\Psi(X)\|^2} \right) \\ &= 2\tau \|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2(K_X) \left(1 - \frac{H'(K_X)}{H(K_X)} K_X \right) \end{aligned}$$

We rewrite $\|\Psi(X)\|^2$ as a function of K_X :

$$\|\Psi(X)\|^2 = \frac{D(\|\Psi\|^2) - \frac{1}{2\sigma_X^2} (\nabla_X R(X))^2 H^2(K_X) \left(1 - \frac{H'(K_X)}{H(K_X)} K_X \right)}{2\tau} \equiv D - \bar{H}(X, K_X) \quad (253)$$

Ultimately, the equation (250) for $K_{\hat{X}}$ can be rewritten:

$$K_{\hat{X}} |f(\hat{X})| = \frac{C(\bar{p}) \sigma_K^2 \hat{\Gamma} \left(p + \frac{1}{2} \right)}{\|\Psi(X)\|^2} = \frac{C(\bar{p}) \sigma_K^2}{D - \bar{H}(X, K_X)} \hat{\Gamma} \left(p + \frac{1}{2} \right) \quad (254)$$

with $C(\bar{p})$ given by (249), $\hat{\Gamma}(p + \frac{1}{2})$ defined in (251) and p given by (248).

A3.2 Approaches to solutions for $K_{\hat{X}}$

We detail some computations of the three approaches detailed in the core of the paper.

A3.2.1 First approach: Differential form of (75)

To understand the behavior of the solutions of (75), we can write its differential version. Assume a variation $\delta Y(\hat{X})$ for any parameter of the system at point \hat{X} . This parameter $Y(\hat{X})$ can be either $R(X)$, its gradient, or any parameter arising in the definition of f and g . This induces a variation $\delta K_{\hat{X}}$ for the average capital. The equation for $\delta K_{\hat{X}}$ is obtained by differentiation of (75):

$$\begin{aligned} \delta K_{\hat{X}} = & \left(- \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) + k(p) \frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}} \delta K_{\hat{X}} \\ & + \frac{\partial}{\partial Y(\hat{X})} \left(\frac{\sigma_K^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y(\hat{X}) \end{aligned} \quad (255)$$

where we define:

$$\begin{aligned} l(\hat{X}, K_{\hat{X}}) = & \frac{\sigma_X^2 \sigma_K^2 \left(\nabla_{K_{\hat{X}}} (f'(\hat{X}))^2 |f(\hat{X})| - 3 (\nabla_{K_{\hat{X}}} |f(\hat{X})|) (f'(\hat{X}))^2 \right) (p + \frac{1}{2})^2}{120 |f(\hat{X})|^4} \\ & + \frac{\partial p}{\partial K_{\hat{X}}} \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \\ k(p) = & \frac{\frac{d}{dp} \hat{\Gamma}(p + \frac{1}{2})}{\hat{\Gamma}(p + \frac{1}{2})} \sim_{\infty} \sqrt{\frac{p - \frac{1}{2}}{2}} - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \end{aligned} \quad (256)$$

and:

$$\frac{\partial p}{\partial K_{\hat{X}}} = \frac{\partial}{\partial K_{\hat{X}}} \frac{M - A(\hat{X}, K_{\hat{X}})}{|f(\hat{X}, K_{\hat{X}})|} = - \frac{\partial_{K_{\hat{X}}} |f(\hat{X}, K_{\hat{X}})| p + \partial_{K_{\hat{X}}} A(\hat{X}, K_{\hat{X}})}{|f(\hat{X}, K_{\hat{X}})|}$$

with:

$$\begin{aligned} A(\hat{X}, K_{\hat{X}}) = & \frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + \left(f(\hat{X}, K_{\hat{X}}) + \frac{|f(\hat{X}, K_{\hat{X}})|}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X}, K_{\hat{X}})} \right) \\ \simeq & \frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + f(\hat{X}, K_{\hat{X}}) + \frac{|f(\hat{X}, K_{\hat{X}})|}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \end{aligned}$$

A. 3.2.1.1 Expanded form of (255) In an expanded form (255) writes:

$$\begin{aligned}
\delta K_{\hat{X}} &= (k(p) \partial_{K_{\hat{X}}}(p) \\
&\quad - \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}(p + \mathcal{H}(f(\hat{X}, K_{\hat{X}})) + \frac{1}{2})k(p)}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) K_{\hat{X}} \delta K_{\hat{X}} \\
&\quad + \frac{\partial}{\partial Y} \left(\frac{\sigma_{\hat{K}}^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y
\end{aligned}$$

with \mathcal{H} the heaviside function. Moreover:

$$\begin{aligned}
&\frac{\partial}{\partial Y} \left(\frac{\sigma_{\hat{K}}^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y \\
&= (k(p) \partial_Y p \\
&\quad - \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y}(p + \mathcal{H}(f(\hat{X}, K_{\hat{X}})) + \frac{1}{2})k(p)}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial Y}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + m_Y(\hat{X}, K_{\hat{X}}) \right) K_{\hat{X}} \delta Y
\end{aligned}$$

with:

$$\begin{aligned}
m_Y(\hat{X}, K_{\hat{X}}) &= \frac{\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 \left(\nabla_Y (f'(\hat{X}))^2 |f(\hat{X})| - 3 (\nabla_Y |f(\hat{X})|) (f'(\hat{X}))^2 \right) (p + \frac{1}{2})^2}{120 |f(\hat{X})|^4} \\
&\quad + \nabla_Y p \frac{\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}
\end{aligned}$$

so that:

$$\begin{aligned}
\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} &= (k(p) \partial_Y(p) \\
&\quad - \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y}(p + \mathcal{H}(f(\hat{X}, K_{\hat{X}})) + \frac{1}{2})k(p)}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial Y}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + m_Y(\hat{X}, K_{\hat{X}}) \right) \frac{\delta Y}{D}
\end{aligned}$$

with:

$$\begin{aligned}
D &= 1 + \left(\left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}(p + \mathcal{H}(f(\hat{X}, K_{\hat{X}})) + \frac{1}{2})k(p)}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) \right. \\
&\quad \left. - k(p) \partial_{K_{\hat{X}}}(p) \right) K_{\hat{X}}
\end{aligned} \tag{257}$$

A. 3.2.1.2 Local stability As explained in the text, equation (77) can be understood as the fixed-point equation of a dynamical system through the following mechanism.

Each variation $\delta Y(\hat{X})$ in the parameters impacts the average capital, which must then be computed with the new parameters. The first change induced is written $\delta K_{\hat{X}}^{(1)}$:

$$\delta K_{\hat{X}}^{(1)} = \frac{\partial}{\partial Y(\hat{X})} \left(\frac{\sigma_{\hat{K}}^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y(\hat{X}) \tag{258}$$

In a second step, the variation $\delta K_{\hat{X}}$ impacts the various functions implied in (75), and indirectly modifies $K_{\hat{X}}$ through the first term in the rhs of (77):

$$\left(- \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) + k(p) \frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}} \delta K_{\hat{X}}^{(1)} \quad (259)$$

These two effects combined, (258) and (259), yield the total variation $\delta K_{\hat{X}}$.

Importantly, note that if we can interpret $\delta K_{\hat{X}}^{(1)}$ as a variation at time t , we can also infer from the indirect effect (259) that $\delta K_{\hat{X}}$ is itself a variation at time $t + 1$. Equation (77) can thus be seen as the fixed point equation of a dynamical system written:

$$\begin{aligned} \delta K_{\hat{X}}(t+1) = & \left(- \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) + k(p) \frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}} \delta K_{\hat{X}}(t) \\ & + \frac{\partial}{\partial Y(\hat{X}, t)} \left(\frac{\sigma_{\hat{K}}^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y(\hat{X}, t) \end{aligned} \quad (260)$$

whose fixed point is the solution of (77):

$$\delta K_{\hat{X}} = \frac{\frac{\partial}{\partial Y(\hat{X})} \left(\frac{\sigma_{\hat{K}}^2 C(\bar{p}) 2\hat{\Gamma}(p + \frac{1}{2})}{|f(\hat{X}, K_{\hat{X}})| \|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right)}{1 + \left(\left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) - k(p) \frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}}} \delta Y(\hat{X}) \quad (261)$$

This solution (??) is stable when:

$$\left| k(p) \frac{\partial p}{\partial K_{\hat{X}}} - \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + l(\hat{X}, K_{\hat{X}}) \right) \right| < 1 \quad (262a)$$

i.e. when D , defined in (??), is positive, and unstable otherwise. So that the stability of this average capital depends, in last analysis, on the sign of D .

A 3.2.1.3 Applications of the differential form: dependency in expected returns The main application of equation (??) is to consider a parameter denoted $Y(\hat{X})$, that encompasses the relative expected returns of sector X vis-à-vis its neighbouring sectors, and defined as:

$$Y(\hat{X}) = p(\hat{X}) \quad (263)$$

Interpretations are given in the text. To compute the dependency of average capital in this parameter, we use (??), and we have:

$$\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} = \frac{\frac{k(p)}{f(\hat{X}, K_{\hat{X}})}}{D} \delta p(\hat{X}) \quad (264)$$

Given equation (??), $k(p)$ is positive at the first order in σ_X^2 . More precisely, using equation (??):

$$k(p) \sim_{\infty} \sqrt{\frac{p - \frac{1}{2}}{2}} - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}$$

along with equation (??), we can infer that $\sqrt{\frac{p-\frac{1}{2}}{2}}$ is of order $\frac{1}{\sigma_X}$ and $\frac{\sigma_X^2 \sigma_K^2 (p+\frac{1}{2})(f'(X))^2}{48|f(\hat{X})|^3} \sim 1$.

Consequently, in a stable equilibrium, i.e. for $D > 0$, equation (79) implies that the dependency of $K_{\hat{X}}$ in the parameter $p(\hat{X})$ is positive:

$$\frac{\delta K_{\hat{X}}}{\delta p(\hat{X})} > 0$$

We have seen above that $p(\hat{X})$ is maximal for a maximum expected long-term return $R(\hat{X}, K_{\hat{X}})$: when the equilibrium is stable, capital accumulation is maximal for sectors that are themselves a local maximum for $R(\hat{X}, K_{\hat{X}})$.

On the other hand, when the equilibrium is unstable, i.e. for $D < 0$, the capital $K_{\hat{X}}$ is minimal for $R(\hat{X}, K_{\hat{X}})$ maximal.

Actually, as seen above, in the instability range $D < 0$, the average capital $K_{\hat{X}}$ acts as a threshold. When, due to variations in the system's parameters, the average capital per firm is shifted above the threshold $K_{\hat{X}}$, capital will either move to the next stable equilibrium, possibly zero, or tend to infinity. Our results show that when the expected long-term return of a sector increases, the threshold $K_{\hat{X}}$ decreases, which favours capital accumulation.

A 3.2.1.3 Applications of the differential form: dependency in short term returns A second use of equation (??) is to consider $Y(\hat{X})$ as any parameter-function involved in the definition of $f(\hat{X}, K_{\hat{X}})$ that may condition either real short-term returns or the price-dividend ratio.

We can see that in this case, $Y(\hat{X})$ only impacts $f(\hat{X}, K_{\hat{X}})$, so that equation (??) simplifies and yields:

$$\begin{aligned} \frac{\delta K_{\hat{X}}}{K_{\hat{X}}} &= -\frac{m_Y(\hat{X}, K_{\hat{X}})}{D} \delta Y \\ &\quad - \frac{1}{D} \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y} \left(1 + \left(p + H(f(\hat{X}, K_{\hat{X}})) + \frac{1}{2} \right) k(p) \right)}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial Y}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right) \delta Y \end{aligned} \quad (265)$$

Incidentally, note that p being proportional to $f^{-1}(\hat{X})$, $m_Y(\hat{X}, K_{\hat{X}})$ rewrites:

$$\begin{aligned} -m_Y(\hat{X}, K_{\hat{X}}) &= \frac{\sigma_X^2 \sigma_K^2 \left(3 \left(\nabla_Y |f(\hat{X})| \right) \left(f'(\hat{X}) \right)^2 - \nabla_Y \left(f'(\hat{X}) \right)^2 |f(\hat{X})| \right) \left(p + \frac{1}{2} \right)^2}{120 |f(\hat{X})|^4} \\ &\quad + \nabla_Y |f(\hat{X})| \frac{\sigma_X^2 \sigma_K^2 p \left(p + \frac{1}{2} \right) (f'(X))^2}{48 |f(\hat{X})|^4} \end{aligned} \quad (266)$$

The first term in the rhs of (265) is the impact of an increase in investors' short-term returns. The second is the variation in capital needed to maintain investors' overall returns.

The sign of $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ given by equation (265) can be studied under two cases: the stable and the unstable equilibrium.

Let us first consider the case of a stable equilibrium, i.e. $D > 0$.

The first term in the rhs of (265), the variation induced by an increase in short-term returns, is in general positive for $f'(\hat{X})$ proportional to $f(\hat{X})$, that is for instance when the function $f(\hat{X})$, that describes short-term returns and prices, depends on the variable $K_{\hat{X}}$ raised to some arbitrary power.

Indeed in that case:

$$3 \left(\nabla_Y |f(\hat{X})| \right) \left(f'(\hat{X}) \right)^2 - \nabla_Y \left(f'(\hat{X}) \right)^2 |f(\hat{X})| = \left(\nabla_Y |f(\hat{X})| \right) \left(f'(\hat{X}) \right)^2$$

The second term in the rhs of (265) is in general negative. When $\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y} > 0$, i.e. when returns are increasing in Y , a rise in Y increases returns and decreases the capital needed to maintain these returns. Similarly, when $\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial Y} > 0$, i.e. when the number of agents in sector \hat{X} is increasing in Y , a rise in Y increases the number of agents that move towards point \hat{X} , and the average capital per firm diminishes.

The net variation (265) of $K_{\hat{X}}$ is the sum of these two contributions. Considering an expansion of (265) in powers of σ_X^2 , the first contribution $-m_Y \left(\hat{X}, K_{\hat{X}} \right)$ is of magnitude $(\sigma_X^2)^{-1}$, whereas the second is proportional to $k(p) \sim (\sigma_X)^{-1}$. The variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is thus positive: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} > 0$. In most cases, a higher short-term return, decomposed as a sum of dividend and price variation, induces a higher average capital. This effect is magnified for larger levels of capital: the third approach will confirm that, in most cases, the return $f(\hat{X})$ is asymptotically a constant $c \ll 1$ when capital is high: $K_{\hat{X}} \gg 1$.

Turning now to the case of an unstable equilibrium, i.e. $D < 0$, the variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is negative: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} < 0$. In the instability range, and due to this very instability, an increase in returns $f(\hat{X})$ reduces the threshold of capital accumulation for low levels of capital. When short-term returns $f(\hat{X})$ increase, a lower average capital will trigger capital accumulation towards an equilibrium. Otherwise, when average capital $K_{\hat{X}}$ is below this threshold, it will converge toward 0.

A3.2.2 Second approach: Expansion around particular solutions

As explained in the text, we choose to expand (250), or equivalently (254), around solutions with $p = 0$.

A3.2.2.1 Equation (75) for $p = 0$ To find the solution with $p = 0$, we maximize the function:

$$A(\hat{X}) = \frac{\left(g(\hat{X})\right)^2}{\sigma_X^2} + f(\hat{X}) + \frac{1}{2} \sqrt{f^2(\hat{X})} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})}$$

We write:

$$M = \max_{\hat{X}} A(\hat{X}) \quad (267)$$

and denote by $(\hat{X}_M, K_{\hat{X}_M})$ the solutions \hat{X}_M of (267) with $K_{\hat{X}_M}$ their associated value of average capital per firm.

Given that

$$\hat{\Gamma}\left(\frac{1}{2}\right) = \exp\left(-\frac{\sigma_X^2 \sigma_K^2 \left(f'(\hat{X}_M, K_{\hat{X}_M})\right)^2}{384 \left|f(\hat{X}_M, K_{\hat{X}_M})\right|^3}\right) \quad (268)$$

(250) becomes at points $(\hat{X}_M, K_{\hat{X}_M})$ and $p = 0$:

$$K_{\hat{X}_M} \left|f(\hat{X}_M, K_{\hat{X}_M})\right| \left\|\Psi(\hat{X}_M, K_{\hat{X}_M})\right\|^2 \simeq \sigma_K^2 C(\bar{p}) \exp\left(-\frac{\sigma_X^2 \sigma_K^2 \left(f'(\hat{X}_M, K_{\hat{X}_M})\right)^2}{384 \left|f(\hat{X}_M, K_{\hat{X}_M})\right|^3}\right) \quad (269)$$

This equation has in general several solutions, depending on the assumptions on $f(\hat{X}_M, K_{\hat{X}_M})$.

Note that once a solution $K_{\hat{X}}$ of (254) is found, the value of $C(\bar{p})$ can be obtained by solving (247) and using (249). These solutions are discussed in the text.

The next paragraph computes the expansion of (250) around these solutions with $p = 0$. Remark that coming back to (250) and (254) for general values of p defined in (248), the value of $C(\bar{p}) \sigma_K^2$ can be replaced by $K_{\hat{X}_M} \left|f(\hat{X}_M, K_{\hat{X}_M})\right| \left\|\Psi(\hat{X}_M, K_{\hat{X}_M})\right\|^2$ for any solution $(\hat{X}_M, K_{\hat{X}_M})$.

A3.2.2.2 Expansion around particular solutions To better understand the behavior of the solutions of equation (75), we expand this equation around the points $(\hat{X}, K_{\hat{X},M})$ that solve equation (75). We can find approximate solutions to (250):

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| = C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma}\left(p + \frac{1}{2}\right) \quad (270)$$

with:

$$\begin{aligned} \hat{\Gamma}\left(p + \frac{1}{2}\right) &= \exp\left(-\frac{\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3}\right) \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)}\right) \end{aligned} \quad (271)$$

for general form of the functions $f(\hat{X})$ and $g(\hat{X})$ by expanding (270), for each \hat{X} , around the closest point \hat{X}_M satisfying (270) with $p = 0$. We use that:

$$\begin{aligned} &\left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(-\frac{p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)}\right) \\ &= 1 - p(\gamma_0 + \ln 2 - 2) + o(p) \end{aligned} \quad (272)$$

with γ_0 the Euler-Mascheroni constant, as well as the following relations:

$$\nabla_{K_{\hat{X}}} \left(-\frac{\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 h(p) (f'(\hat{X}))^2}{96 |f(\hat{X})|^3} \right)_{p=0} \simeq -\frac{\nabla_{K_{\hat{X}}} (f'(\hat{X}))^2 |f(\hat{X})| - 3 (\nabla_{K_{\hat{X}}} |f(\hat{X})|) (f'(\hat{X}))^2}{120 |f(\hat{X})|^4}$$

and:

$$\begin{aligned} &\nabla_{\hat{X}} \left(-\frac{h(p) (f'(\hat{X}))^2}{96 |f(\hat{X})|^3} \right)_{p=0} \\ &\simeq -\frac{2f'(X) f''(\hat{X}) |f(\hat{X})| - 3 (f'(\hat{X}))^3}{120 |f(\hat{X})|^4} = -\frac{f'(X) (3 (f'(\hat{X}))^2 - 2f''(X) |f(\hat{X})|)}{120 |f(\hat{X})|^4} \end{aligned}$$

the expansion of (270) at the lowest order, is:

$$\begin{aligned}
& \left(1 + \frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right)_{K_{\hat{X}, M}} (K_{\hat{X}} - K_{\hat{X}, M}) + \left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial \hat{X}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial \hat{X}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M) \\
& \simeq - \left(\sigma_X^2 \sigma_K^2 \frac{\nabla_{K_{\hat{X}}} (f'(\hat{X}))^2 |f(\hat{X})| - 3 (\nabla_{K_{\hat{X}}} |f(\hat{X})|) (f'(\hat{X}))^2}{120 |f(\hat{X})|^4} \right)_{K_{\hat{X}, M}} (K_{\hat{X}} - K_{\hat{X}, M}) \\
& - \left(\sigma_X^2 \sigma_K^2 \frac{2f'(X) f''(\hat{X}) |f(\hat{X})| - 3 (f'(\hat{X}))^3}{120 |f(\hat{X})|^4} \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M) \\
& - b \frac{\partial_{K_{\hat{X}}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + \left(f(\hat{X}, K_{\hat{X}}) + \frac{|f(\hat{X}, K_{\hat{X}})|}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right)}{|f(\hat{X}, K_{\hat{X}})|} (K_{\hat{X}} - K_{\hat{X}, M}) \\
& - b \frac{\partial_{\hat{X}} \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + \left(f(\hat{X}, K_{\hat{X}}) + \frac{|f(\hat{X}, K_{\hat{X}})|}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right)}{|f(\hat{X}, K_{\hat{X}})|} (\hat{X} - \hat{X}_M)
\end{aligned}$$

Given the maximization (267), the two last terms in the right hand side is equal to 0.

$$\begin{aligned}
(K_{\hat{X}} - K_{\hat{X}, M}) &= \frac{1}{D} \left(\sigma_X^2 \sigma_K^2 \frac{3 (f'(\hat{X}))^3 - 2 f'(X) f''(\hat{X}) |f(\hat{X})|}{120 |f(\hat{X})|^4} \right. \\
& - \frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial \hat{X}}}{f(\hat{X}, K_{\hat{X}})} - \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial \hat{X}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \left. \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M) \\
& - \frac{1}{D} \frac{b}{2} (\hat{X} - \hat{X}_M) \nabla_{\hat{X}}^2 \left(\frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + \frac{3}{2} f(\hat{X}, K_{\hat{X}}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M)
\end{aligned} \tag{273}$$

with:

$$D = \left(1 + \frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}}}{f(\hat{X}, K_{\hat{X}})} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial K_{\hat{X}}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} + \frac{\sigma_X^2 \sigma_K^2 \left(\nabla_{K_{\hat{X}}} (f'(\hat{X}))^2 |f(\hat{X})| - 3 (\nabla_{K_{\hat{X}}} |f(\hat{X})|) (f'(\hat{X}))^2 \right)}{120 |f(\hat{X})|^4} \right)_{K_{\hat{X}, M}}$$

and $K_{\hat{X}, M}$ solution of:

$$\begin{aligned}
& K_{\hat{X}, M} |f(\hat{X}, K_{\hat{X}, M})| \|\Psi(\hat{X}, K_{\hat{X}, M})\|^2 \\
& \simeq \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) C(\bar{p}) \simeq \sigma_K^2 C(\bar{p})
\end{aligned}$$

The maximization condition (267) cancels the contribution due to:

$$\frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_X^2} + \left(f(\hat{X}, K_{\hat{X}}) + \frac{|f(\hat{X}, K_{\hat{X}})|}{2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)$$

To find a contribution due to this term, we must expand (270) to the second order. The second order contributions proportional to $\left(K_{\hat{X}} - K_{\hat{X},M}\right)^2$ modifies slightly (273) and the term $\left(K_{\hat{X}} - K_{\hat{X},M}\right) \left(\hat{X} - \hat{X}_M\right)$ shifts D at the first order. Both modifications do not alter the interpretation for (273). We can thus consider the sole term:

$$\frac{C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma}\left(p + \frac{1}{2}\right)}{\left\|\Psi\left(\hat{X}\right)\right\|^2 \left|f\left(\hat{X}\right)\right|}$$

Due to (254), for $H(K_{\hat{X}})$ slowly varying, the contribution due to the derivatives of $\left\|\Psi\left(\hat{X}\right)\right\|^2$ can be neglected. Moreover the contribution due to the derivative of $\left|f\left(\hat{X}\right)\right|$ are negligible with respect to the first order terms. We can thus consider only the second order contributions due to $\hat{\Gamma}\left(p + \frac{1}{2}\right)$. In the rhs of (271), the second term is dominant. Moreover, we can check that in the second order expansion of (272), the term in p^2 can be neglected compared to $-p(\gamma_0 + \ln 2 - 2)$. Consequently, the relevant second order correction to (273) is :

$$b\left(\hat{X} - \hat{X}_M\right) \nabla_{\hat{X}}^2 p\left(\hat{X} - \hat{X}_M\right) = b\left(\hat{X} - \hat{X}_M\right) \nabla_{\hat{X}}^2 \left(\frac{M - \frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_{\hat{X}}^2} + \frac{3}{2} f\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|} \right) \left(\hat{X} - \hat{X}_M\right)$$

and the relevant contributions to (273) are:

$$\begin{aligned} \left(K_{\hat{X}} - K_{\hat{X},M}\right) &= \frac{1}{D} \left(\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 \frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'(X) f''\left(\hat{X}\right) \left|f\left(\hat{X}\right)\right|}{120 \left|f\left(\hat{X}\right)\right|^4} \right. \\ &\quad \left. - \frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial \hat{X}}}{f\left(\hat{X}, K_{\hat{X}}\right)} - \frac{\frac{\partial \left\|\Psi(\hat{X}, K_{\hat{X}})\right\|^2}{\partial \hat{X}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} \right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right) \\ &\quad + \frac{1}{D} \frac{b}{2} \left(\hat{X} - \hat{X}_M\right) \nabla_{\hat{X}}^2 \left(\frac{M - \frac{(g(\hat{X}, K_{\hat{X}}))^2}{\sigma_{\hat{X}}^2} + \frac{3}{2} f\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|} \right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right) \end{aligned} \quad (274)$$

A3.2.2.3 Interpretation of (274) As in the first approach, $D > 0$ corresponds to a stable equilibrium, and $D < 0$ to an unstable one. The expansion (??) describes the local variations of $K_{\hat{X}}$ in the neighbourhood of the points $K_{\hat{X},M}$. This approximation (??) suffices to understand the role of the parameters of the system.

We consider the case of stable equilibria, i.e. $D > 0$. Note that under unstable equilibria, $D < 0$, the interpretations are inverted, since $K_{\hat{X}}$ is interpreted as a threshold³⁵.

The equation (??), that expands average capital at sector \hat{X}_M , is composed of a first order and a second order contributions.

The first order part in the expansion (??) writes:

$$\frac{1}{D} \left(\sigma_{\hat{X}}^2 \sigma_{\hat{K}}^2 \frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'(X) f''\left(\hat{X}\right) \left|f\left(\hat{X}\right)\right|}{120 \left|f\left(\hat{X}\right)\right|^4} - \frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial \hat{X}}}{f\left(\hat{X}, K_{\hat{X}}\right)} - \frac{\frac{\partial \left\|\Psi(\hat{X}, K_{\hat{X}})\right\|^2}{\partial \hat{X}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} \right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right) \quad (275)$$

It represents the variation of equilibrium capital as a function of its position. It is decomposed in three contributions:

³⁵See the first approach.

For $f'(\hat{X}) > 0$, the second contribution in (275):

$$-\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial \hat{X}} (\hat{X} - \hat{X}_M)}{f(\hat{X}, K_{\hat{X}})}$$

is positive. It represents the decrease in capital needed to reach equilibrium. Actually, the return is higher at point \hat{X} than at \hat{X}_M : a lower capital will yield the same overall return at point \hat{X} . On the contrary, the first contribution in (275):

$$\frac{\sigma_X^2 \sigma_K^2 \frac{3(f'(\hat{X}))^3 - 2f'(X)f''(\hat{X})|f(\hat{X})|}{120|f(\hat{X})|^4} (\hat{X} - \hat{X}_M)}{D}$$

describes the "net" variation of capital due to a variation in $f(X)$. When returns are decreasing, i.e. when $f'(\hat{X}) > 0$ and $f''(\hat{X}) < 0$, this first contribution has the sign of $f'(\hat{X})$. An increase in returns attracts capital.

The third term in (275):

$$-\frac{1}{D} \left(\frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial \hat{X}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M)$$

represents the number effect. Actually, when:

$$\frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial \hat{X}}}{\|\Psi(\hat{X}, K_{\hat{X}})\|^2} > 0$$

the number of agents is higher at \hat{X} than at \hat{X}_M : the average capital per agent is reduced.

The second order contribution in (??) represents the effect of the neighbouring sector space on each sector. Given the first order condition (82):

$$\nabla_{\hat{X}}^2 \left(\frac{M - A(\hat{X})}{f(\hat{X})} \right)_{K_{\hat{X}, M}} = \left(\frac{\nabla_{\hat{X}}^2 (M - A(\hat{X}))}{f(\hat{X})} \right)_{K_{\hat{X}, M}}$$

and since $A(\hat{X}_M)$ is a maximum, we have:

$$(\hat{X} - \hat{X}_M) \nabla_{\hat{X}}^2 \left(\frac{M - A(\hat{X})}{f(\hat{X})} \right)_{K_{\hat{X}, M}} (\hat{X} - \hat{X}_M) > 0$$

When $f(\hat{X})$ is constant, $A(\hat{X}_M)$ is a local maximum, and $K_{\hat{X}_M}$ is a minimum. To put it differently, $K_{\hat{X}}$ is a decreasing function of $A(\hat{X})$. This is in line with the definition of $A(\hat{X})$ ³⁶, which measures the relative attractiveness of sector \hat{X} 's neighbours: the higher $A(\hat{X})$, the lower the incentive for capital to stay in sector \hat{X} .

A3.2.3 Third approach: Resolution for particular form for the functions

As stated in the text, we can find approximate solutions to (254) by choosing some forms for the parameters functions. The solutions are then studied in some ranges for average capital per firm K_X : $K_X \gg 1$, $K_X \gg \gg 1$, $K_X \ll 1$ and the intermediate range $\infty > K_X > 1$. In the case $K_X \gg \gg 1$, the distinction between stable and unstable cases has to be made.

³⁶See discussions after equations (??) and (??).

A3.2.3.1 Function $H^2(K_X)$ We can choose for $H^2(K_X)$ a power function of K_X :

$$H(K_X) = K_X^\eta \quad (276)$$

so that equation (57) rewrites:

$$\|\Psi(X)\|^2 \simeq \frac{D\left(\|\Psi\|^2\right) - \frac{F}{2\sigma_X^2} \left((\nabla_X R(X))^2 + \frac{2\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)} \right) K_X^\eta}{2\tau} \equiv D - L(X) (\nabla_X R(X))^2 K_X^\eta \quad (277)$$

A3.2.3.2 Function f To determine the function f , we must first assume a form for $r(K, X)$, the physical capital marginal returns, and for F_1 , the function that measures the impact of expected long-term return on investment choices.

Assuming the production functions are of Cobb-Douglas type, i.e. $B(X) K^\alpha$ with $B(X)$ a productivity factor, we have for $r(K, X)$:

$$r(K, X) = \frac{\partial r(K, X)}{\partial K} = \alpha B(X) K^{\alpha-1} \quad (278)$$

For function F_1 , the simplest choice would be a linear form:

$$F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\int R(K'_{\hat{X}}, X') \|\Psi(X')\|^2 dX'} \right) \simeq F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} \right) = b \left(\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(X) \rangle} - 1 \right)$$

where, for any function $u(\hat{X})$, $\langle u(\hat{X}) \rangle$ denotes its average over the sector space, and b an arbitrary parameter.

However, when capital $K_X^\alpha \rightarrow \infty$ and is concentrated at \hat{X} , we have $\langle K_X^\alpha \rangle \simeq \frac{K_X^\alpha}{N^\alpha(X)}$, so that $\frac{K_X^\alpha R(\hat{X})}{\langle K_X^\alpha \rangle \langle R(X) \rangle} \rightarrow \frac{N^\alpha(X) R(\hat{X})}{\langle R(X) \rangle} \gg 1$. To impose some bound on moves in the sector space we rather choose:

$$F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} \right) \simeq b \arctan \left(\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(X) \rangle} - 1 \right) \quad (279)$$

so that $F_1 \left(\frac{R(K_{\hat{X}}, \hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} \right) > 0$ when $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(X) \rangle} > 1$.

Given the above assumptions, the general formula for f given in equation (60) rewrites:

$$f(\hat{X}, \Psi, \hat{\Psi}) = \frac{1}{\varepsilon} \left(r(\hat{X}) K_{\hat{X}}^{\alpha-1} - \gamma \|\Psi(\hat{X})\|^2 + b \arctan \left(\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(X) \rangle} - 1 \right) \right) \quad (280)$$

This general formula can be approximated for $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(X) \rangle} \simeq 1$, when average capital in sector \hat{X} is close to the average capital of the whole space, which is usually the case.

Using our choices (87), (278) and (279) for $\|\Psi(X)\|^2 r(\hat{X})$ and F_1 respectively, the equation (60) for $f(\hat{X}, \Psi, \hat{\Psi})$ becomes:

$$f(\hat{X}, \Psi, \hat{\Psi}) = \frac{1}{\varepsilon} \left(\left(r(\hat{X}) + \frac{b R(\hat{X}) K_{\hat{X}}^\alpha}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} \right) + \gamma L(\hat{X}) K_X^\eta - \gamma D - b \right)$$

We may assume without impairing the results that $\eta = \alpha$. We thus have:

$$\begin{aligned} f(\hat{X}, \Psi, \hat{\Psi}) &= \frac{1}{\varepsilon} \left(\left(\frac{r(\hat{X})}{K_{\hat{X}}^\alpha} + \frac{b R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} + \gamma L(\hat{X}) \right) K_{\hat{X}}^\alpha - \gamma D - b \right) \\ &\equiv B_1(\hat{X}) K_{\hat{X}}^{\alpha-1} + B_2(\hat{X}) K_{\hat{X}}^\alpha - C(\hat{X}) \end{aligned} \quad (281)$$

where:

$$\begin{aligned} B_1(\hat{X}) &= \frac{\alpha B(\hat{X})}{\varepsilon} \\ B_2(\hat{X}) &= \frac{bR(\hat{X})}{\varepsilon \langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} + \frac{\gamma}{\varepsilon} \\ C(\hat{X}) &= \gamma D + b \end{aligned}$$

A3.2.3.3 Function g To determine the form of function g , equation (61), we must first choose a form for the function F_0 .

We assume that:

$$F_0(R(\hat{X}, K_{\hat{X}})) = a \arctan(K_{\hat{X}}^\alpha R(\hat{X})) \quad (282)$$

where a is an arbitrary constant.

Combined to our assumption for F_1 , (279), the formula (61) for g can be written:

$$g(\hat{X}, \Psi, \hat{\Psi}) = a \nabla_{\hat{X}} \arctan(K_{\hat{X}}^\alpha R(\hat{X})) + b \nabla_{\hat{X}} \arctan\left(\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} - 1\right) \quad (283)$$

where the arctan function ensures that the velocity in the sector space g increases with capital and is maximal when average capital per firm in sector \hat{X} tends to infinity, i.e. $K_{\hat{X}}^\alpha \rightarrow \infty$.

This general formula, equation (??), can be approximated for $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle} \simeq 1$, when average capital in sector \hat{X} is close to the average capital of the whole space. It then reduces to:

$$g(\hat{X}, \Psi, \hat{\Psi}) \simeq \frac{K_{\hat{X}}^\alpha}{\langle K_{\hat{X}}^\alpha \rangle} \nabla_{\hat{X}} R(\hat{X}) \left(1 + \frac{b}{\langle R(\hat{X}) \rangle}\right) \equiv \nabla_{\hat{X}} R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha \quad (284)$$

which in turn allows to approximate the gradient of g , $\nabla_{\hat{X}} g(\hat{X}, \Psi, \hat{\Psi})$, by:

$$\nabla_{\hat{X}} g(\hat{X}, \Psi, \hat{\Psi}) \simeq \frac{\nabla_{\hat{X}}^2 R(\hat{X})}{\langle K_{\hat{X}}^\alpha \rangle} \left(1 + \frac{b}{\langle K_{\hat{X}}^\alpha \rangle \langle R(\hat{X}) \rangle}\right) K_{\hat{X}}^\alpha \equiv \nabla_{\hat{X}}^2 R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha \quad (285)$$

A3.2.3.4 Solving (254) Equation (254) can be studied by considering five cases presented in the text:

Case 1. Very high capital, $K_{\hat{X}} \gg \gg 1$, stable case In that case, $K_{\hat{X}} \gg \gg 1$, and we assume in first approximation that (discarding the factor $L(\hat{X})$):

$$\|\Psi(\hat{X})\|^2 \simeq D - \left(\nabla_X R(\hat{X})\right)^2 K_{\hat{X}}^\alpha < 1 \quad (286)$$

This corresponds to a very high level of capital. Consequently, equation (280) implies that the function $f(\hat{X})$ can be rewritten:

$$\begin{aligned} f(\hat{X}) &= \frac{1}{\varepsilon} \left(r(\hat{X}) K_{\hat{X}}^{\alpha-1} - \gamma \left\| \Psi(\hat{X}) \right\|^2 + b \arctan \left(\frac{K_{\hat{X}}^{\alpha} R(\hat{X})}{\langle K_{\hat{X}}^{\alpha} \rangle \langle R(X) \rangle} - 1 \right) \right) \\ &\simeq b \left(\frac{\pi}{2} - \frac{\langle K_{\hat{X}}^{\alpha} \rangle \langle R(X) \rangle}{K_{\hat{X}}^{\alpha} R(\hat{X})} \right) \\ &\equiv c - \frac{d}{K_{\hat{X}}^{\alpha} R(\hat{X})} \simeq c > 0 \end{aligned}$$

Consequently, the expressions for $f'(\hat{X})$, $g(\hat{X})$ and $\nabla_{\hat{X}} g(\hat{X})$ (297) and (298) are still valid.

Two different cases arise in the resolution of (250).

First, we assume that $\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 \neq 0$.

In this case, we will solve (250) by using (286) to replace $K_{\hat{X}} \simeq \left(\frac{D}{(\nabla_X R(\hat{X}))^2} \right)^{\frac{1}{\alpha}}$. We also change the variable $\frac{D}{(\nabla_X R(\hat{X}))^2} \rightarrow D$ temporarily for the sake of simplicity.

Inequality (286) along with $K_{\hat{X}} \gg 1$ and (280) implies that only the case $f > 0$ has to be considered.

Note that using our results about stability, it is easy to check that in that case, this solution is locally unstable. A very high level of capital has the tendency to attract more investments.

Given our assumptions, equation (254) becomes:

$$\left(\nabla_X R(\hat{X}) \right)^2 D^{\frac{1}{\alpha}} (D - K_{\hat{X}}^{\alpha}) = C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \frac{\Gamma(p + \frac{3}{2})}{|f(\hat{X})|} \quad (287)$$

or equivalently:

$$K_{\hat{X}}^{\alpha} = D - \frac{C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \Gamma(p + \frac{3}{2})}{\left(\nabla_X R(\hat{X}) \right)^2 D^{\frac{1}{\alpha}} |f(\hat{X})|} \quad (288)$$

Then, defining $V = \frac{1}{K_{\hat{X}}^{\alpha}}$ as in the first case, we can write (288) as an equation for $V \ll 1$ by replacing all quantities in term of V and then perform a first order expansion.

First, we write (288) as:

$$V - \frac{1}{D - \frac{C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \Gamma(p + \frac{3}{2})}{\left(\nabla_X R(\hat{X}) \right)^2 D^{\frac{1}{\alpha}} |f(\hat{X})|}} = 0 \quad (289)$$

As in the previous case, the first order expansion in V of $\Gamma(p + \frac{3}{2})$ arising in (289) is given by:

$$\Gamma\left(p + \frac{3}{2}\right) \simeq \Gamma\left(\frac{M}{c}\right) + \frac{MV}{c} \left(\frac{\nabla_X^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \Gamma'\left(\frac{M}{c}\right) \quad (290)$$

Moreover, at the first order:

$$\exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \simeq 1$$

and (289) becomes:

$$V - \frac{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} |f(\hat{X})|}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} |f(\hat{X})| - C(\bar{p}) \sigma_K^2 \Gamma(p + \frac{3}{2})} = 0$$

that is:

$$V - \frac{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} \left(c - \frac{dV}{R(\hat{X})}\right)}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} \left(c - \frac{dV}{R(\hat{X})}\right) - C(\bar{p}) \sigma_K^2 \Gamma(p + \frac{3}{2})} = 0 \quad (291)$$

Using (290) the first order expansion of the dominator in (291) is:

$$\begin{aligned} & \left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} \left(c - \frac{dV}{R(\hat{X})}\right) - C(\bar{p}) \sigma_K^2 \Gamma\left(p + \frac{3}{2}\right) \\ &= \left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c - C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right) \\ & \quad - \left(\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} \frac{d}{R(\hat{X})} + \frac{C(\bar{p}) \sigma_K^2 M}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \Gamma'\left(\frac{M}{c}\right) \right) V \end{aligned}$$

so that (291) writes:

$$\begin{aligned} & \frac{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} c}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c - C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)} \\ &= \left(1 - \frac{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} c \left(\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} \frac{d}{R(\hat{X})} + \frac{C(\bar{p}) \sigma_K^2 M}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \Gamma'\left(\frac{M}{c}\right) \right)}{\left(\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c - C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right) \right)^2} \right) V \\ & \quad + \frac{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} \frac{d}{R(\hat{X})}}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c - C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)} V \end{aligned} \quad (292)$$

Equation (292) can be solved for V with solution:

$$\frac{1}{V} = D - \frac{C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{\frac{1}{\alpha}} c} + \frac{d}{cR(\hat{X})} \left(1 - \frac{\left(1 + \frac{C(\bar{p}) \sigma_K^2 M \Gamma\left(\frac{M}{c}\right)}{c \left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}}} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{Md} + \frac{1}{c} \right) \text{Psi}\left(\frac{M}{c}\right) \right)}{\left(1 - \frac{C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c} \right)} \right)$$

Ultimatly, restoring the variable:

$$D \rightarrow \frac{D}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2}$$

we obtain the solution $K_{\hat{X}}^\alpha = \frac{1}{V}$:

$$K_{\hat{X}}^\alpha = \frac{D}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2} - \frac{C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha} c}} \quad (293)$$

$$+ \frac{d}{cR(\hat{X})} \left(1 - \frac{\left(1 + \frac{C(\bar{p})(\nabla_{\hat{X}} R(\hat{X}))^{\frac{2}{\alpha}} \sigma_K^2 \left(\frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{d}\right) \Gamma'\left(\frac{M}{c}\right)\right)}{\left(1 - \frac{(\nabla_{\hat{X}} R(\hat{X}))^{\frac{2}{\alpha}} C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M}{c}\right)}{cD^{1+\frac{1}{\alpha}}}\right)} \right)$$

As stated in the text, this is increasing in c , i.e. in $f(\hat{X})$ and in $R(\hat{X})$. This corresponds to a stable level of capital.

Case 2. Very high capital, $K_{\hat{X}} \gg \gg 1$, unstable case In this second case, we consider that $\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 \rightarrow 0$ and formula (286) and (293) are not valid anymore. Coming back to (146) leads rather to replace $(\nabla_X R(X))^2$:

$$(\nabla_X R(X))^2 \rightarrow (\nabla_X R(X))^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} = \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)}$$

Thus, if $\nabla_X^2 R(K_X, X) < 0$, (287) is replaced by:

$$K_{\hat{X}}^\alpha \left(D + \sigma_X^2 |\nabla_X^2 R(K_X, X)| K_{\hat{X}}^{\frac{\alpha}{2}} \right) = C(\bar{p}) \sigma_K^2 \frac{\Gamma\left(p + \frac{3}{2}\right)}{|f(\hat{X})|}$$

with:

$$p + \frac{3}{2} \simeq \frac{M - \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})}{f(\hat{X})}$$

and the equation for K_X writes:

$$\sigma_X^2 |\nabla_X^2 R(K_X, X)| K_{\hat{X}}^{\frac{3}{2}\alpha} = \frac{C(\bar{p}) \sigma_K^2 \Gamma\left(\frac{M - \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})}{f(\hat{X})}\right)}{|f(\hat{X})|}$$

Since, given our assumptions $f(\hat{X}) \rightarrow c$ we find:

$$K_{\hat{X}} = \left(\frac{C(\bar{p}) \sigma_K^2}{|\nabla_X^2 R(K_X, X)| c} \Gamma\left(\frac{M - \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}})}{c}\right) \right)^{\frac{2}{3\alpha}} \quad (294)$$

Note that given (294), an equilibrium in the range $K_{\hat{X}} \gg \gg 1$ is only possible for $c \ll 1$. Otherwise, there is no equilibrium for a maximum of $R(K_X, X)$. This equilibrium value of $K_{\hat{X}}$ decreases with c , which corresponds to an unstable equilibrium, as detailed in the text.

On the other hand, if $\nabla_X R(X) = 0$ and $\nabla_X^2 R(K_X, X) > 0$, expression (286) becomes:

$$\left\| \Psi(\hat{X}) \right\|^2 \simeq D - \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} K_{\hat{X}}^\alpha = D - \sigma_X^2 \nabla_X^2 R(K_X, X) K_{\hat{X}}^{\frac{\alpha}{2}}$$

and thus:

$$K_{\hat{X}}^\alpha \simeq \left(\frac{D}{\sigma_X^2 \nabla_X^2 R(K_X, X)} \right)^2 \quad (295)$$

However, this solution with $K_X \gg 1$ corresponds to points such that $\nabla_X^2 R(K_X, X) > 0$ and $\nabla_X R(X) = 0$. Then, these points are minima of $R(X)$. This equilibrium may exist only if the level of capital (295) is high enough to compensate the weakness of the purely position dependent part of expected return and match the condition:

$$\frac{K_X^\alpha R(\hat{X})}{\langle K_X^\alpha \rangle \langle R(X) \rangle} - 1 > 0$$

This equilibrium is thus unlikely and may be discarded in general.

Case 3. High capital, $K_{\hat{X}} \gg 1$ In that case, we assume $K_{\hat{X}}$ relatively large, but bounded, to ensure that the approximation:

$$\left\| \Psi(\hat{X}) \right\|^2 \simeq D \quad (296)$$

is still valid.

Equations (280) and (283) imply that the function $f(\hat{X})$ is independent of $K_{\hat{X}}$ and that $g(\hat{X})$ is proportional to $\nabla_{\hat{X}} R(\hat{X})$. Given (280), the function $f(\hat{X})$ can be rewritten:

$$\begin{aligned} f(\hat{X}) &= \frac{1}{\varepsilon} \left(r(\hat{X}) K_{\hat{X}}^{\alpha-1} - \gamma \left\| \Psi(\hat{X}) \right\|^2 + b \arctan \left(\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_X^\alpha \rangle \langle R(X) \rangle} - 1 \right) \right) \\ &\simeq b \left(\frac{\pi}{2} - \frac{\langle K_X^\alpha \rangle \langle R(X) \rangle}{K_{\hat{X}}^\alpha R(\hat{X})} \right) - \gamma D \\ &\equiv c - \frac{d}{K_{\hat{X}}^\alpha R(\hat{X})} - \gamma D \end{aligned}$$

Consequently, the expression for $f'(\hat{X})$ is:

$$f'(\hat{X}) \simeq \frac{d \nabla_{\hat{X}} R(\hat{X})}{K_{\hat{X}}^\alpha R^2(\hat{X})} \quad (297)$$

Similarly, we can approximate (283) as:

$$\begin{aligned} g(\hat{X}) &\simeq - \frac{\nabla_{\hat{X}} R(\hat{X}) f}{K_{\hat{X}}^\alpha R(\hat{X})} \\ \nabla_{\hat{X}} g(\hat{X}) &\simeq - \frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{K_{\hat{X}}^\alpha R(\hat{X})} \end{aligned} \quad (298)$$

Given (296), and including the constant α in the definition of $C(\bar{p})$, equation (254) is:

$$K_{\hat{X}} D \left| f(\hat{X}) \right| = C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 \left| f(\hat{X}) \right|^3} \right) \Gamma \left(p + \frac{3}{2} \right) \quad (299)$$

with:

$$p + \frac{1}{2} = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \left(f(\hat{X}) + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \right)}{\sqrt{f^2(\hat{X})}}$$

Defining $V = \frac{1}{K_X^\alpha}$, we can write (299) as an equation for $V \ll 1$ by replacing all quantities in term of V and then perform a first order expansion. To do so, we first, we write (299) as:

$$V - \frac{D \left| f(\hat{X}) \right|}{C(\bar{p}) \sigma_K^2 \exp \left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \Gamma(p + \frac{3}{2})} = 0 \quad (300)$$

and then find an expansion in V for $\Gamma(p + \frac{3}{2})$.

The first order expansion in V of $p + \frac{3}{2}$ is:

$$\begin{aligned} p + \frac{3}{2} &= \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_X^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) - \frac{\sigma_K^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right)}{f(\hat{X})} \\ &\simeq \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_X^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)}{c - \frac{d}{K_X^\alpha R(\hat{X})}} \\ &= \frac{M - \left(\frac{(\nabla_X R(\hat{X}) \left(-\frac{fV}{R(\hat{X})} \right))^2}{\sigma_X^2} + \nabla_{\hat{X}}^2 R(\hat{X}) \left(-\frac{fV}{R(\hat{X})} \right) \right)}{c - \frac{dV}{R(\hat{X})}} \\ &= \frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R(\hat{X}) \frac{fV}{R(\hat{X})}}{c} + \frac{M \frac{dV}{cR(\hat{X})}}{c} \end{aligned}$$

Consequently, $\Gamma(p + \frac{3}{2})$ arising in (289) is given by:

$$\begin{aligned} \Gamma \left(\frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R(\hat{X}) \frac{fV}{R(\hat{X})}}{c} + \frac{M \frac{dV}{cR(\hat{X})}}{c} \right) &= \Gamma \left(\frac{M}{c} \left(1 + \nabla_{\hat{X}}^2 R(\hat{X}) \frac{fV}{MR(\hat{X})} + \frac{dV}{cR(\hat{X})} \right) \right) \\ &\simeq \Gamma \left(\frac{M}{c} \right) + \frac{MV}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \Gamma' \left(\frac{M}{c} \right) \\ &= \Gamma \left(\frac{M}{c} \right) \left(1 + \frac{MV}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi} \left(\frac{M}{c} \right) \right) \end{aligned}$$

Ultimately, using that at the first order:

$$\exp \left(-\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \simeq 1$$

equation (300) for V becomes:

$$V - \frac{D \left| f(\hat{X}) \right|}{C(\bar{p}) \sigma_K^2 \Gamma \left(\frac{M}{c} \right) \left(1 + \frac{MV}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi} \left(\frac{M}{c} \right) \right)} = 0$$

that is:

$$V - \frac{D \left(c - \frac{dV}{R(\hat{X})} - \gamma D \right)}{C(\bar{p}) \sigma_K^2 \Gamma \left(\frac{M}{c} \right) \left(1 + \frac{MV}{c} \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X}) f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi} \left(\frac{M}{c} \right) \right)} = 0$$

And a first order expansion yields:

$$V - \frac{D(c-\gamma D)}{C(\bar{p})\sigma_K^2\Gamma\left(\frac{M}{c}\right)} \left(1 - \frac{dV}{(c-\gamma D)R(\hat{X})} - MV \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi}\left(\frac{M}{c}\right) \right) = 0$$

with solution:

$$V = (K_{\hat{X}}^\alpha)^{-1} = \frac{\frac{D(c-\gamma D)}{C(\bar{p})\sigma_K^2\Gamma\left(\frac{M}{c}\right)}}{1 + \frac{D(c-\gamma D)}{C(\bar{p})\sigma_K^2\Gamma\left(\frac{M}{c}\right)} \left(\frac{d}{(c-\gamma D)R(\hat{X})} + M \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi}\left(\frac{M}{c}\right) \right)}$$

Coming back to $K_{\hat{X}}^\alpha$ we have:

$$\begin{aligned} K_{\hat{X}}^\alpha &= \frac{C(\bar{p})\sigma_K^2\Gamma\left(\frac{M}{c}\right)}{D(c-\gamma D)} + \frac{d}{(c-\gamma D)R(\hat{X})} + M \left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})} \right) \text{Psi}\left(\frac{M}{c}\right) \\ &= \frac{C(\bar{p})\sigma_K^2\Gamma\left(\frac{M}{c}\right)}{D(c-\gamma D)} + \frac{d}{(c-\gamma D)R(\hat{X})} \left(1 + M \text{Psi}\left(\frac{M}{c}\right) \left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)} \right) \right) \end{aligned} \quad (301)$$

This solution satisfies the condition $K_{\hat{X}} >> 1$ only if $\frac{C(\bar{p})\sigma_K^2\sqrt{\frac{M-c}{c}}}{Dc} >> 1$: formula (301) thus shows that the dependency of $K_{\hat{X}}^\alpha$ in the return $R(\hat{X})$ depends on the sign of $1 + M \text{Psi}\left(\frac{M}{c}\right) \left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)} \right)$. If:

$$1 + M \text{Psi}\left(\frac{M}{c}\right) \left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)} \right) > 0$$

then $K_{\hat{X}}^\alpha$ decreases with $R(\hat{X})$. As stated in the text, this corresponds to an unstable equilibrium.

If:

$$1 + M \text{Psi}\left(\frac{M}{c}\right) \left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)} \right) < 0$$

a stable equilibrium is possible and $K_{\hat{X}}^\alpha$ is an increasing function of $R(\hat{X})$ and $f(\hat{X})$. This corresponds to $\nabla_{\hat{X}}^2 R(\hat{X}) < 0$, which arises for instance for a maximum of $R(\hat{X})$. In such case, an increase in $R(\hat{X})$ allows for an increased number $\|\Psi(\hat{X})\|^2$ of firms, without reducing the average capital per firm.

Case 4. Intermediate capital, $\infty > K_{\hat{X}} > 1$: We start with asymptotic form of (250):

$$K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})| = C(\bar{p})\sigma_K^2 \exp\left(-\frac{\sigma_X^2\sigma_K^2(p+\frac{1}{2})^2(f'(X))^2}{96|f(\hat{X})|^3} \right) \Gamma\left(p+\frac{3}{2}\right) \quad (302)$$

Up to a constant that can be absorbed in the definition of $C(\bar{p})$, we have:

$$\Gamma\left(p+\frac{3}{2}\right) \sim_\infty \sqrt{p+\frac{1}{2}} \exp\left(\left(p+\frac{1}{2}\right) \left(\ln\left(p+\frac{1}{2}\right) - 1 \right) \right)$$

and (302) can be rewritten as:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| = C(\bar{p}) \sigma_K^2 \sqrt{p + \frac{1}{2}} \exp \left(- \frac{\sigma_X^2 \sigma_K^2 \left(p + \frac{1}{2}\right)^2 (f'(X))^2}{96 |f(\hat{X})|^3} + \left(p + \frac{1}{2}\right) \left(\ln \left(p + \frac{1}{2}\right) - 1\right) \right) \quad (303)$$

Since we are in an intermediate range for the parameters, we can replace, in first approximation, $\ln \left(p + \frac{1}{2}\right)$ by its average over this range: $\ln \left(\bar{p} + \frac{1}{2}\right)$. The exponential in (303) thus becomes:

$$\exp \left(- \frac{\sigma_X^2 \sigma_K^2 \left(p + \frac{1}{2} - \frac{48 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)\right)^2 (f'(X))^2}{96 |f(\hat{X})|^3} + \frac{24 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)^2 \right)$$

and equation (303) rewrites:

$$\begin{aligned} & K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| \left(\frac{\sigma_X^2 \sigma_K^2 (f'(X))^2 \exp \left(- \frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)^2 \right)}{96 |f(\hat{X})|^3} \right)^{\frac{1}{4}} \\ &= C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 \sigma_K^2 \left(p + \frac{1}{2} - \frac{48 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)\right)^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \sqrt{\left(p + \frac{1}{2}\right)} \sqrt{\frac{\sigma_X^2 \sigma_K^2 (f'(X))^2}{96 |f(\hat{X})|^3}} \end{aligned} \quad (304)$$

To solve (304) for $K_{\hat{X}}$, we proceed in two steps.

We first introduce an intermediate variable W and rewrite (304) as an equation for $K_{\hat{X}}$ and W . We set:

$$\sqrt{\frac{\sigma_X^2 \sigma_K^2 (f'(X))^2}{96 |f(\hat{X})|^3}} \left(p + \frac{1}{2} - \frac{48 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right) \right) = W \quad (305)$$

and rewrite equation (304) partly in terms of W :

$$\begin{aligned} & K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 \left(\frac{\sigma_X^2 (f'(X))^2 |f(\hat{X})| \exp \left(- \frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)^2 \right)}{96 (\sigma_K^2)^3} \right)^{\frac{1}{4}} \\ &= C(\bar{p}) \exp(-W^2) \sqrt{W + 2 \sqrt{\frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2}\right) - 1\right)}} \end{aligned} \quad (306)$$

Note that, as seen from (305), W is a function of p and as such can be seen as a parameter depending on the shape of the sectors space.

Equation (306) both depends on $K_{\hat{X}}$ and W , and in a second step, we use (305) to write $K_{\hat{X}}$ as a function of W . To do so, we use that in the intermediate case $\infty > K_{\hat{X}} > 1$, we can assume that:

$$f(\hat{X}) = B_1(X) K_{\hat{X}}^{\alpha-1} + B_2(X) K_{\hat{X}}^{\alpha} - C(\hat{X}) \simeq B_2(X) K_{\hat{X}}^{\alpha} \quad (307)$$

and that:

$$\frac{M - \left(\frac{(\nabla_{\hat{X}} R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}^2 R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha \right)}{B_1(X) K_{\hat{X}}^{\alpha-1} + B_2(X) K_{\hat{X}}^\alpha - C(\hat{X})} - \frac{3}{2} \simeq \frac{M - \left(\frac{(\nabla_{\hat{X}} R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}^2 R(\hat{X}) A(\hat{X}) K_{\hat{X}}^\alpha \right)}{B_2(X) K_{\hat{X}}^\alpha} - \frac{3}{2} \quad (308)$$

Moreover, we can approximate $\|\Psi(\hat{X})\|^2$:

$$\|\Psi(\hat{X})\|^2 \simeq D \quad (309)$$

Our assumptions (307), (308) and (309) allow to rewrite the relation (305) between $K_{\hat{X}}^\alpha$ and W as:

$$\sqrt{\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2}{96 B_2^3(X) K_{\hat{X}}^\alpha}} \left(p + \frac{1}{2} - \frac{48 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1 \right) \right) = W$$

that is:

$$W = \sqrt{\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2}{96 B_2^5(X) K_{\hat{X}}^{3\alpha}}} \times \left(M - \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B_2'(X))^2} \right) K_{\hat{X}}^{2\alpha} - (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) K_{\hat{X}}^\alpha \right) \quad (310)$$

To solve this equation for $K_{\hat{X}}^\alpha$, we consider M as the dominant parameter and find an approximate solution of (310). At the lowest order, we write:

$$\sqrt{\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2}{96 B_2^5(X) K_{\hat{X}}^{3\alpha}}} M = W$$

with solution:

$$K_{\hat{X}}^\alpha = \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}$$

Considering corrections to this result, the solution to (310) is decomposed as:

$$K_{\hat{X}}^\alpha = \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} + \chi \quad (311)$$

and using the following intermediate results:

$$K_{\hat{X}}^{2\alpha} = \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} \left(1 + 2\chi \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} \right)$$

$$K_{\hat{X}}^{3\alpha} = \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right) \left(1 + 3\chi \left(\frac{\sigma_X^2 \sigma_K^2 (B_2'(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} \right)$$

we are led to rewrite (310) as an equation for χ at first order:

$$\begin{aligned}
& \chi \left(\frac{3}{2} \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} W \right. \\
& \quad \left. + 2 \frac{W}{M} \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} + \frac{W}{M} (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \right) \\
& = - \frac{W}{M} \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} \\
& \quad - \frac{W}{M} (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}
\end{aligned}$$

whose solution is:

$$\chi = - \frac{\left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right) + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}}$$

so that (311) yields $K_{\hat{X}}^\alpha$:

$$\begin{aligned}
& K_{\hat{X}}^\alpha - \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} \\
& = - \frac{\left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right) + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}} \\
& \quad - \frac{\left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right) + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{(\nabla_{\hat{X}} R(\hat{X}))^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) (\ln(\bar{p} + \frac{1}{2}) - 1)}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \right) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + (\nabla_{\hat{X}}^2 R(\hat{X}) + B_2(X)) \left(\frac{\sigma_X^2 \sigma_K^2 (B'_2(X))^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}}
\end{aligned}$$

In a third step, we can use equation (312) to rewrite (306) in an approximate form. Actually, expression (312) implies that in the intermediate case, where $K_{\hat{X}}^\alpha$ is of finite magnitude, we have $W^2 \sim \sigma_X^2 \sigma_K^2 M^2$ and:

$$\begin{aligned}
& \exp \left(-W^2 + \frac{24 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right) \\
& \simeq \exp \left(\frac{24 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right)
\end{aligned}$$

Moreover using that:

$$W + 2 \sqrt{\frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)} \simeq 2 \sqrt{\frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}$$

and that ultimately the left hand side of equation (306) writes at the first order:

$$\begin{aligned}
& K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 \left(\frac{\sigma_X^2 (f'(X))^2 |f(\hat{X})| \exp \left(-\frac{96 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right)}{96 (\sigma_K^2)^3} \right)^{\frac{1}{4}} \\
&= \left(\frac{\sigma_X^2 (B'_2(X))^2 |B_2(X)|}{96 (\sigma_K^2)^3} \right)^{\frac{1}{4}} K_{\hat{X}}^{1+\frac{3\alpha}{4}} \left\| \Psi(\hat{X}) \right\|^2 \exp \left(-\frac{24 |f(\hat{X})|^3}{\sigma_X^2 \sigma_K^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right) \\
&\simeq D \left(\frac{\sigma_X^2 (B'_2(X))^2 |B_2(X)|}{96 (\sigma_K^2)^3} \right)^{\frac{1}{4}} K_{\hat{X}}^{1+\frac{3\alpha}{4}} \exp \left(-\frac{24 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right)
\end{aligned}$$

equation (306) writes:

$$\begin{aligned}
& D \left(\frac{\sigma_X^2 (B'_2(X))^2 |B_2(X)|}{96 (\sigma_K^2)^3} \right)^{\frac{1}{4}} K_{\hat{X}}^{1+\frac{3\alpha}{4}} \exp \left(-\frac{24 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right) \\
&= C(\bar{p}) \sqrt{2 \sqrt{\frac{96 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}}
\end{aligned}$$

that is:

$$K_{\hat{X}}^{1+\frac{\alpha}{2}} \exp \left(-\frac{24 |B_2(X)|^3 K_{\hat{X}}^\alpha}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right) = \frac{8C(\bar{p})}{D} \sqrt{\frac{3\sigma_K^2 |B_2(X)|}{\sigma_X^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)} \quad (313)$$

Equation (313) has the form:

$$x^d \exp(-ax) = c$$

with solution:

$$x = c^{\frac{1}{d}} \exp \left(-W_0 \left(-\frac{a}{d} c^{\frac{1}{d}} \right) \right)$$

where W_0 is the Lambert W function with parameter 0. Applying this result to our case with:

$$\begin{aligned}
d &= \frac{1+\alpha}{2\alpha} \\
x &= K_{\hat{X}}^\alpha \\
a &= \frac{24 |B_2(X)|^3}{\sigma_X^2 \sigma_K^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \\
c &= \frac{8C(\bar{p})}{D} \sqrt{\frac{3\sigma_K^2 |B_2(X)|}{\sigma_X^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}
\end{aligned}$$

we obtain:

$$\begin{aligned}
K_{\hat{X}}^\alpha &= \left(\frac{8C(\bar{p})}{D} \sqrt{\frac{3\sigma_K^2 |B_2(X)|}{\sigma_X^2 (B'_2(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)} \right)^{\frac{2\alpha}{1+\alpha}} \\
&\times \exp \left(-W_0 \left(-\frac{48\alpha}{1+\alpha} \left(\sqrt{\frac{3\sigma_K^2}{\sigma_X^2}} \frac{8C(\bar{p})}{D} \right)^{\frac{2\alpha}{1+\alpha}} \frac{|B_2(X)|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_X^2 \sigma_K^2 (B'_2(X))^{2+\frac{2\alpha}{1+\alpha}}} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^{2+\frac{\alpha}{1+\alpha}} \right) \right)
\end{aligned}$$

As stated in the text, this is an increasing function of $B_2(X)$. Moreover, the corrections to this formula, given in (312) show that K_X^α is a decreasing function of $\left(\nabla_{\hat{X}} R(\hat{X})\right)^2$ and $\nabla_{\hat{X}}^2 R(\hat{X})$.

Case 5. Low capital, $K_{\hat{X}} \ll 1$: When average physical capital per firm in sector \hat{X} is very low, we can use our assumptions about $g(\hat{X}, \Psi, \hat{\Psi})$ and $\nabla_{\hat{X}} g(\hat{X}, \Psi, \hat{\Psi})$, equations (89) and (??), and assume that:

$$f(\hat{X}) \simeq B_1(\hat{X}) K_{\hat{X}}^{\alpha-1} \gg 1 \quad (314)$$

and:

$$g(\hat{X}) \simeq 0$$

and moreover that:

$$\left\| \Psi(\hat{X}) \right\|^2 = D - L(X) (\nabla_X R(X))^2 K_X^\alpha \simeq D$$

For these conditions, the solution of (??) is locally stable.

Moreover, the conditions $K_{\hat{X}} \ll 1$ and the defining equation (281) for f imply that $f > 0$, and that for $\alpha < 1$:

$$\frac{\sigma_X^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 \left| f(\hat{X}) \right|^3} \ll 1$$

Under these assumptions, equation (??) reduces to:

$$K_{\hat{X}} D \left| f(\hat{X}) \right| \simeq C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma} \left(p + \frac{1}{2} \right) \quad (315)$$

This equation (315) can be approximated. Actually, using formula (??) for p yields:

$$p + \frac{1}{2} = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)}{\sqrt{f^2(\hat{X})}} - 1 \simeq -1$$

and an expansion of $\hat{\Gamma}(p + \frac{1}{2})$ around the value $p + \frac{1}{2} = -1$ writes:

$$\hat{\Gamma} \left(p + \frac{1}{2} \right) \simeq \hat{\Gamma}(-1) + \hat{\Gamma}'(-1) \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)}{\sqrt{f^2(\hat{X})}}$$

Consequently, when returns are large, i.e. $f(\hat{X}) \gg 1$, equation (??) writes:

$$K_{\hat{X}} \left(B_1(\hat{X}) K_{\hat{X}}^{\alpha-1} \right) \simeq \frac{C(\bar{p}) \sigma_{\hat{K}}^2}{D} \left(\hat{\Gamma}(-1) + \hat{\Gamma}'(-1) \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)}{B_1(\hat{X}) K_{\hat{X}}^{\alpha-1}} \right)$$

with first order solution³⁷:

$$K_{\hat{X}} = \left(\frac{C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma}(-1)}{D B_1(\hat{X})} \right)^{\frac{1}{\alpha}} + \frac{\frac{C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma}'(-1)}{D} \left(M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right) \right)}{B_1^{\frac{1}{\alpha}}(\hat{X}) \left(\frac{C(\bar{p}) \sigma_{\hat{K}}^2 \hat{\Gamma}(-1)}{D} \right)^{1-\frac{1}{\alpha}}} \quad (316)$$

³⁷Given our hypotheses, $D \gg 1$, which implies that $K_{\hat{X}} \ll 1$, as needed.

Equation (316) shows that average capital $K_{\hat{X}}$ increases with $M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)$: when expected long-term returns increase, more capital is allocated to the sector. Equation (98) also shows that average capital $K_{\hat{X}}$ is maximal when returns $R(\hat{X})$ are at a local maximum, i.e. when $\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} = 0$ and $\nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) < 0$.

Inversely, the same equations (316) and (98) show that average capital $K_{\hat{X}}$ is decreasing in $f(\hat{X})$. The equilibrium is unstable. When average capital is very low, i.e. $K_{\hat{X}} \ll 1$, which is the case studied here, marginal returns are high. Any increase in capital above the threshold widely increases returns, which drives capital towards the next stable equilibrium, with higher $K_{\hat{X}}$. Recall that in this unstable equilibrium, $K_{\hat{X}}$ must be seen as a threshold. The rise in $f(\hat{X})$ reduces the threshold $K_{\hat{X}}$, which favours capital accumulation and increases the average capital $K_{\hat{X}}$.

This case is thus an exception: the dependency of $K_{\hat{X}}$ in $R(\hat{X})$ is stable, but the dependency in $f(\hat{X})$ is unstable. This saddle path type of instability may lead the sector, either towards a higher level of capital (case 4 below) or towards 0, where the sector disappears.

A 3.3 Instability and modification of sectors' space

Disappearance of Low average capital sectors

Average capital is unstable when $B(\hat{X}) < -1$. A shock on average capital can either drive the equilibrium to some stable value, or worsen the sector's capital landscape.

In the latter case, investors tend to desert the sector, so that both the average capital and the density of investors tend to 0: $K_{\hat{X}} \rightarrow 0$ and $|\hat{\Psi}(\hat{X}, \hat{K})|^2 \rightarrow 0$. Producers remain in the sector but with a very low capital on average. The very lack of capital prevents these firms to shift towards more attractive sectors in the long run. Assuming physical capital returns are Cobb-Douglas, marginal productivity is mathematically high for a very low capital. Thus, short-term returns are very large: $f(\hat{X}) \rightarrow \infty$.

Note that this type of instability only applies to very low level of average capital, so that the total capital involved is negligible, and this instability does not impact the system globally.

Very high level of average capital and modification of space

Average capital is also unstable when $B(\hat{X}) > 1$. However, in this case investors are lured in the sector, so that average capital in the sector increases quickly $K_{\hat{X}} \rightarrow \infty$, and short-term returns tend to be small: $f(\hat{X}) \rightarrow c$ for some constant $c \ll 1$. Consequently, for $K_{\hat{X}} \rightarrow \infty$, $\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial K_{\hat{X}}} \rightarrow 0$, which translates decreasing marginal returns. Similarly, the expected long-term returns will be capped, and $\frac{\partial p}{\partial K_{\hat{X}}} \rightarrow 0$, and $l(\hat{X}, K_{\hat{X}}) \rightarrow 0$.

The instability condition (106) turns out to be a lower bound for the sensitivity of firms density relative to average capital:

$$\frac{\partial \ln |\Psi(\hat{X}, K_{\hat{X}})|^2}{\partial K_{\hat{X}}} > 1 \quad (317)$$

This lower bound creates a herd effect: the number of firms in sector \hat{X} could grow indefinitely with capital: $|\Psi(\hat{X}, K_{\hat{X}})|^2 \rightarrow \infty$.

However, the fixed number of firms implies that this shift towards sector \hat{X} will necessarily reach a maximum $|\hat{\Psi}(\hat{X}, \hat{K})|_{\max}^2 \gg 1$. For this maximum density, the corresponding level of average capital at

sector \hat{X} will be approximatively:

$$K_{\hat{X}} \simeq K_{\max} = \frac{\ln \left(\left| \hat{\Psi}(\hat{X}, \hat{K}) \right|_{\max}^2 \right)}{r}$$

This concentration of capital in some sectors directly impacts the amount of disposable capital along with the instability condition (106) for the rest of the system. This occurs in several steps.

First, the disposable average capital for the rest of the system reduces to $\langle K \rangle - \frac{K_{\max}}{V}$, with V , the volume of the sector space and $\langle K \rangle$, the average physical capital in the whole space.

Second, this reduction of average capital negatively impacts the growth prospects $R(\hat{X})$, the stock prices $F_1(X)$, and consequently the short term returns $f(\hat{X})$.

In turn, this modifies the stability condition $|B(\hat{X})|$ over the whole space. Consequently, some sectors will move over the instability threshold $B(\hat{X}) > 1$, while others will move below $B(\hat{X}) < -1$. Some sectors will experience a capital increase, others will disappear.

If a stable situation finally emerges, the resulting sectors' space will be reduced: some sectors will have disappeared, and only sectors with positive capital will have remained.

A.3.4 Global instability

This appendix completes the analysis of the solutions of (75) for average capital. We have studied the local instability of solutions previously. However, a second source of instability of the system arises outside of the equations for average capital per firm per sector, (75), and its differential version, (77). It stems from the sectors' space expected long-term returns. It is induced by the minimization equations (64) and (65), and is a source of global instability for the background field.

A 3.4.1 Mechanism of global instability:

In these equations, the Lagrange multiplier $\hat{\lambda}$ is the eigenvalue of a second-order differential equation. Because there exist an infinite number of eigenvalues $\hat{\lambda}$, there are an infinite number of local minimum background fields $\Psi(\hat{X}, K_{\hat{X}})$. But the most likely minimum, given in (??), is obtained for $\hat{\lambda} = M$ (see appendix 2).

Yet $\hat{\lambda}$ is also the Lagrange multiplier that implements the constraint of a fixed number N of agents.

Since the number of investors is computed by:

$$\int \left| \Psi(\hat{X}, K_{\hat{X}}) \right|^2 d(\hat{X}, K_{\hat{X}})$$

the constraint implemented by $\hat{\lambda}$ is:

$$\hat{N} = \int \left| \Psi(\hat{X}, K_{\hat{X}}) \right|^2 d(\hat{X}, K_{\hat{X}}) \quad (318)$$

since this constraint runs over the whole space, it is a global property of the system.

Yet equations (64) and (65), the minimization equations defining $\Psi(\hat{X}, K_{\hat{X}})$, may also be viewed as a set of local minimization equations at each point \hat{X} of the sector space. Considered individually, each provide a lower minimum that could be reached separately for each \hat{X} . In other words, provided each sector's number of agents is fixed independently from the rest of the system, a stable background field could be reached at every point.

However, our global constraint rules out this set of local minimizations. The solutions of (64) and (65) are thus a local minimum for the sole points \hat{X} such that the lowest value of $\hat{\lambda}$ is reached at \hat{X} , i.e. points

such that³⁸:

$$A(\hat{X}) \equiv \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2}\sqrt{f^2(\hat{X})} + \nabla_{\hat{X}}g(\hat{X}) - \frac{\sigma_K^2 F^2(\hat{X})}{2f^2(\hat{X})} = M \quad (319)$$

For points \hat{X} that do not satisfy (319), the solutions $\Psi(\hat{X}, K_{\hat{X}})$ and $\Psi^\dagger(\hat{X}, K_{\hat{X}})$ of (64) and (65), with $\hat{\lambda} = -M$ are not global minima, but merely a local one. Any perturbation $\delta\Psi(\hat{X}, K_{\hat{X}})$ due to a change of parameters destabilizes the whole system: the equilibrium is unstable.

The stability of both the background field and the potential equilibria are thus determined by $A(\hat{X})$, the sector space's overall shape of returns and expectations. An homogeneous shape, a space such that $A(\hat{X})$, presents small deviations around M and is more background-stable than an heterogeneous space.

More importantly, the background fields and associated average capital must be understood as potential, not actual long-run equilibria: the whole system is better described as a dynamical system, which is defined in section 12, between potential backgrounds where time enters as a macro-variable. We consider the results of the background field's dynamical behavior in section 17.

Removing global instability As mentioned above, an homogeneous shape is a space such that the parameter $A(\hat{X})$ presents small deviations around M . In an heterogeneous shape, the space presents large differences in $A(\hat{X})$. We find that homogeneous shapes are more background-stable than heterogeneous ones. This partly results from the global constraint (318) imposed on the number of agents in the model, which ensures that the number of financial agents in the system is fixed over the whole sector space.

Relaxing this constraint fully would render the number of agents in sectors independent. The associated background field of each sector could, at each point, adjust to be minimum and stabilize the system.

To do so, we replace equation (64), the minimization equation, by a set of independent equations with independent Lagrange multipliers $\hat{\lambda}_{\hat{X}}$ for each sector \hat{X} , so that for each \hat{X} , the minimum configuration is reached by setting:

$$\hat{\lambda}_{\hat{X}} = \frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + f(\hat{X}) + \frac{1}{2}\sqrt{f^2(\hat{X})} + \nabla_{\hat{X}}g(\hat{X}) - \frac{\sigma_K^2 F^2(\hat{X})}{2f^2(\hat{X})}$$

This is similar to the Lagrange multiplier of the minimization equation for the background field, stripped of the maximum condition $\hat{\lambda} = -M$ ³⁹. This \hat{X} dependency of the Lagrange multiplier implies that the average capital equation (75) is replaced by⁴⁰:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| = C(\bar{p}) \sigma_K^2 \hat{\Gamma}\left(\frac{1}{2}\right) = C(\bar{p}) \sigma_K^2 \exp\left(-\frac{\sigma_{\hat{X}}^2 \sigma_K^2 (f'(\hat{X}))^2}{384 |f(\hat{X})|^3}\right) \quad (320)$$

This equation is identical to (83) and has thus at least one locally stable solution. The solutions are computed in (85) and (86).

Solutions to (320) do no longer directly depend on the relative characteristics of a particular sector, but rather on the returns at point $f(\hat{X})$ and on the number of firms in the sector, $\left\| \Psi(\hat{X}) \right\|^2$. Yet this dependency is only indirect, through the firms' density at sector \hat{X} , $\left\| \Psi(\hat{X}) \right\|^2$, and this quantity does not vary much in the sector space.

³⁸See definition (82) and section 8.2. for a study of such points.

³⁹see discussion following equation (69).

⁴⁰Expression (268) is used to compute $\hat{\Gamma}(\frac{1}{2})$.

An intermediate situation between (75) and (320) could also be considered: it would be to assume a constant number of agents in some regions of the sector space.

Alternatively, limiting the number of investors per sector can be achieved through some public regulation to maintain a constant flow of investment in the sector.

Appendix 4. Dynamics for $K_{\hat{X}}$

A4.1 Variation of the defining equation for $K_{\hat{X}}$

A4.1.1 Compact formulation

As claimed in the text, we consider the dynamics for $K_{\hat{X}}$ generated by modification of the parameters. To do so, we compute the variation of equation (254). We need the variations of the functions involved in (254) with respect to two dynamical variables $K_{\hat{X}}$ and $R(X)$. Starting with (254):

$$K_{\hat{X}} \left(D - L(\hat{X}) K_{\hat{X}}^\eta \right) = \frac{C(\bar{p}) \sigma_{\hat{K}}^2}{|f(\hat{X})|} \hat{\Gamma} \left(p + \frac{1}{2} \right) \quad (321)$$

where:

$$\begin{aligned} \hat{\Gamma} \left(p + \frac{1}{2} \right) &= \exp \left(- \frac{\sigma_X^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \\ &\times \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(\frac{-p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \right) \end{aligned} \quad (322)$$

We first compute the variations of the right hand side and use that, in first approximation:

$$\frac{d}{dp} \left(\ln \left(\frac{\Gamma(-\frac{p+1}{2}) \Gamma(\frac{1-p}{2}) - \Gamma(-\frac{p}{2}) \Gamma(\frac{-p}{2})}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma(-\frac{p}{2}) \Gamma(\frac{2-p}{2}) - \Gamma(-\frac{p-1}{2}) \Gamma(\frac{-p-1}{2})}{2^{p+1} \Gamma(-p) \Gamma(-p+1)} \right) \right) \simeq \ln \left(p + \frac{1}{2} \right) \quad (323)$$

and:

$$\frac{d}{dp} \left(- \frac{\sigma_X^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) = - \frac{\sigma_X^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}$$

so that:

$$\frac{\frac{d}{dp} \hat{\Gamma} \left(p + \frac{1}{2} \right)}{\hat{\Gamma} \left(p + \frac{1}{2} \right)} \simeq \ln \left(p + \frac{1}{2} \right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \quad (324)$$

Assuming that $C(\bar{p})$ is constant, (324) allows to rewrite the variation of of equation (321):

$$\begin{aligned} \nabla_\theta \left(K_{\hat{X}} \left(D - L(\hat{X}) K_{\hat{X}}^\eta \right) \right) &= K_{\hat{X}} \left(D - L(\hat{X}) K_{\hat{X}}^\eta \right) \\ &\times \left(- \frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} + \left(\ln \left(p + \frac{1}{2} \right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \right) \nabla_{\theta p} \right) \\ &+ \frac{\sigma_X^2 \sigma_{\hat{K}}^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \left(\frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} - \frac{\nabla_\theta \left(\frac{f'(X)}{f(\hat{X})} \right)^2}{\left(\frac{f'(X)}{f(\hat{X})} \right)^2} \right) \end{aligned}$$

and we deduce from this equation, that the dynamic version of equation (321) is:

$$\begin{aligned} \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - \frac{\nabla_{\theta} \left(L(\hat{X}) K_{\hat{X}}^{\eta} \right)}{D - L(\hat{X}) K_{\hat{X}}^{\eta}} &= \left(\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} - 1 \right) \frac{\nabla_{\theta} |f(\hat{X})|}{|f(\hat{X})|} \\ &+ \left(\ln \left(p + \frac{1}{2} \right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \right) \nabla_{\theta} p \\ &- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2}{96 |f(\hat{X})|} \left(\nabla_{\theta} \left(\frac{f'(X)}{f(\hat{X})} \right)^2 \right) \end{aligned} \quad (325)$$

A4.1.2 Expanded form of (325)

To find the dynamic equation for $K_{\hat{X}}$ we expand each side of (325).

The left hand side of (325) can be developed as:

$$\begin{aligned} &\left(1 - \eta \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{D - L(\hat{X}) K_{\hat{X}}^{\eta}} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{D - L(\hat{X}) K_{\hat{X}}^{\eta}} \frac{\nabla_{\theta} L(\hat{X})}{L(\hat{X})} \\ &= \left(1 - \eta \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{\|\Psi(\hat{X})\|^2} \frac{\nabla_{\theta} L(\hat{X})}{L(\hat{X})} \\ &= \left(1 - \eta \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{\|\Psi(\hat{X})\|^2} \frac{\nabla_{\theta} \left((\nabla_X R(X))^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \right)}{(\nabla_X R(X))^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)}} \\ &\simeq \left(1 - \eta \frac{L(\hat{X}) K_{\hat{X}}^{\eta}}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - 2 \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \frac{\nabla_{\theta} (\nabla_X R(X))}{\nabla_X R(X)} \end{aligned}$$

and (325) becomes:

$$\begin{aligned} &\left(1 - \eta \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - 2 \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \frac{\nabla_X (\nabla_{\theta} R(X))}{\nabla_X R(X)} \\ &= \left(\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} - 1 \right) \frac{\nabla_{\theta} |f(\hat{X})|}{|f(\hat{X})|} + \left(\ln \left(p + \frac{1}{2} \right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} \right) \nabla_{\theta} p \\ &- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2}{96 |f(\hat{X})|} \nabla_{\theta} \left(\frac{f'(X)}{f(\hat{X})} \right)^2 \end{aligned} \quad (326)$$

To compute the right hand side of (326). We use that:

$$p = - \frac{M - \left(g(\hat{X}) \right)^2 + \sigma_{\hat{X}}^2 \left(\nabla_{\hat{X}} g(\hat{X}, K_{\hat{X}}) \right)}{\sigma_{\hat{X}}^2 f(\hat{X})} - \frac{3}{2}$$

so that, the variation $\nabla_\theta p$ is given by:

$$\nabla_\theta p = -\frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} \left(p + \frac{3}{2}\right) - \left(2 \frac{g(\hat{X}) \nabla_\theta g(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_\theta \nabla_{\hat{X}} g(\hat{X})\right)$$

To compute $\nabla_\theta p$ we must use the form of the functions defined in Appendix 2. We thus obtain:

$$\begin{aligned} \frac{\nabla_\theta g(\hat{X})}{g(\hat{X})} &= \frac{\nabla_\theta \nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \alpha \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \\ \frac{\nabla_\theta \nabla_{\hat{X}} g(\hat{X})}{\nabla_{\hat{X}} g(\hat{X})} &= \frac{\nabla_\theta \nabla_{\hat{X}}^2 R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} + \alpha \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \end{aligned}$$

and as a consequence:

$$\nabla_\theta p = -\frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} \left(p + \frac{3}{2}\right) - \left(2 \frac{g^2(\hat{X}) \left(\frac{\nabla_\theta \nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \alpha \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)}{\sigma_{\hat{X}}^2 |f(\hat{X})|} + \frac{\nabla_{\hat{X}} g(\hat{X})}{|f(\hat{X})|} \left(\frac{\nabla_\theta \nabla_{\hat{X}}^2 R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} + \alpha \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)\right)$$

Ultimately, the right hand side of (326) is given by:

$$\begin{aligned} &\left(\frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3} - 1\right) \frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} + \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}\right) \nabla_\theta p \\ &- \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2}{96 |f(\hat{X})|} \nabla_\theta \left(\frac{f'(X)}{f(\hat{X})}\right)^2 \\ &= -\frac{\nabla_\theta |f(\hat{X})|}{|f(\hat{X})|} \left(\left(1 - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3}\right) + \left(p + \frac{3}{2}\right) \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}\right)\right) \\ &- \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2 |f(\hat{X})|} \frac{\nabla_\theta \nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \frac{\nabla_{\hat{X}} g(\hat{X})}{|f(\hat{X})|} \frac{\nabla_\theta \nabla_{\hat{X}}^2 R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X})\right)}{|f(\hat{X})|} \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right) \\ &\times \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2}) (f'(X))^2}{48 |f(\hat{X})|^3}\right) - \frac{\sigma_X^2 \sigma_K^2 (p + \frac{1}{2})^2}{96 |f(\hat{X})|} \nabla_\theta \left(\frac{f'(X)}{f(\hat{X})}\right)^2 \end{aligned}$$

so that the variational equation for $K_{\hat{X}}$ (326) writes:

$$\begin{aligned}
& \left(1 - \eta \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} - 2 \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \frac{\nabla_X (\nabla_{\theta} R(X))}{\nabla_X R(X)} \\
&= -C_3(p, \hat{X}) \frac{\nabla_{\theta} |f(\hat{X})|}{|f(\hat{X})|} - C_1(p, \hat{X}) \frac{\nabla_{\theta} \left(\frac{f'(X)}{f(\hat{X})} \right)^2}{\left(\frac{f'(X)}{f(\hat{X})} \right)^2} \\
& - C_2(p, \hat{X}) \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2 |f(\hat{X})|} \frac{\nabla_{\theta} \nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \frac{\nabla_{\hat{X}} g(\hat{X})}{|f(\hat{X})|} \frac{\nabla_{\theta} \nabla_{\hat{X}}^2 R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right)}{|f(\hat{X})|} \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} \right)
\end{aligned} \tag{327}$$

with:

$$\begin{aligned}
C_1(p, \hat{X}) &= \frac{\sigma_{\hat{X}}^2 \sigma_K^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \\
C_2(p, \hat{X}) &= \ln \left(p + \frac{1}{2} \right) - \frac{2C_1(p, \hat{X})}{p + \frac{1}{2}} \\
C_3(p, \hat{X}) &= 1 - C_1(p, \hat{X}) + \left(p + \frac{3}{2} \right) C_2(p, \hat{X})
\end{aligned} \tag{328}$$

These term can be reordered and the general dynamic equation for $K_{\hat{X}}$ is ultimately written as:

$$\begin{aligned}
& \left(1 - \eta \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right)}{|f(\hat{X})|} C_2(p, \hat{X}) \right) \frac{\nabla_{\theta} K_{\hat{X}}}{K_{\hat{X}}} \\
& + 2 \left(\frac{g^2(\hat{X}) C_2(p, \hat{X})}{\sigma_{\hat{X}}^2 |f(\hat{X})|} - \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \right) \frac{\nabla_{\theta} \nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \frac{\nabla_{\hat{X}} g(\hat{X}) C_2(p, \hat{X})}{|f(\hat{X})|} \frac{\nabla_{\theta} \nabla_{\hat{X}}^2 R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} \\
&= -C_3(p, \hat{X}) \frac{\nabla_{\theta} |f(\hat{X})|}{|f(\hat{X})|} - C_1(p, \hat{X}) \frac{\nabla_{\theta} \left(\frac{f'(X)}{f(\hat{X})} \right)^2}{\left(\frac{f'(X)}{f(\hat{X})} \right)^2}
\end{aligned} \tag{329}$$

A4.1.3 Dynamic equation for particular forms of $f(\hat{X}, K_{\hat{X}})$ and $\|\Psi(\hat{X})\|^2$

We can put equation (329) in a specific form, by using the explicit formula for $f(\hat{X}, K_{\hat{X}})$ and $\|\Psi(\hat{X})\|^2$ given in appendix 2. We have:

$$\begin{aligned}
\frac{\nabla_\theta f(\hat{X}, K_{\hat{X}})}{f(\hat{X}, K_{\hat{X}})} &\simeq \frac{r(K_{\hat{X}}, \hat{X}) \left(\frac{\nabla_\theta r(K_{\hat{X}}, \hat{X})}{r(K_{\hat{X}}, \hat{X})} + (\alpha - 1) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \right)}{f(\hat{X})} \\
&+ \frac{\gamma \left(\eta L(\hat{X}) K_{\hat{X}}^\eta \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + 2L(\hat{X}) K_{\hat{X}}^\eta \frac{\nabla_\theta (\nabla_X R(X))}{\nabla_X R(X)} \right) + F_1'(R(K_{\hat{X}}, \hat{X})) \frac{\nabla_\theta R(K_{\hat{X}}, \hat{X})}{R(K_{\hat{X}}, \hat{X})}}{f(\hat{X})} \\
&\simeq \frac{r(K_{\hat{X}}, \hat{X}) \left(\frac{\nabla_\theta r(\hat{X})}{r(K_{\hat{X}}, \hat{X})} + (\alpha - 1) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \right)}{f(\hat{X})} \\
&+ \frac{\gamma \left(\eta L(\hat{X}) K_{\hat{X}}^\eta \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + 2L(\hat{X}) K_{\hat{X}}^\eta \frac{\nabla_\theta (\nabla_X R(X))}{\nabla_X R(X)} \right) + \varsigma F_1(R(K_{\hat{X}}, \hat{X})) \frac{\nabla_\theta R(K_{\hat{X}}, \hat{X})}{R(K_{\hat{X}}, \hat{X})}}{f(\hat{X})} \\
&= \frac{r(K_{\hat{X}}, \hat{X}) \frac{\nabla_\theta r(\hat{X})}{r(K_{\hat{X}}, \hat{X})}}{f(\hat{X})} \\
&+ \frac{\left(\gamma \eta L(\hat{X}) K_{\hat{X}}^\eta + (\alpha - 1) \right) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + \varsigma F_1(R(K_{\hat{X}}, \hat{X})) \frac{\nabla_\theta R(K_{\hat{X}}, \hat{X})}{R(K_{\hat{X}}, \hat{X})} + 2\gamma L(\hat{X}) K_{\hat{X}}^\eta \frac{\nabla_\theta (\nabla_X R(X))}{\nabla_X R(X)}}{f(\hat{X})}
\end{aligned}$$

To compute $\nabla_\theta \ln \left(\frac{f'(X)}{f(\hat{X})} \right)^2$ arising in (329), we use that in first approximation, for relatively large $K_{\hat{X}}$:

$$\left(\frac{f'(X)}{f(\hat{X})} \right)^2 \simeq \left(\frac{F_1'(R(K_{\hat{X}}, \hat{X})) \nabla_{\hat{X}} R(K_{\hat{X}}, \hat{X})}{F_1(R(K_{\hat{X}}, \hat{X}))} \right)^2 \simeq (\varsigma \nabla_{\hat{X}} R(K_{\hat{X}}, \hat{X}))^2$$

that can be considered in the sequel negligible at the first order.

Consequently, for the chosen forms of the parameter functions, the dynamics equation (329) becomes ultimately:

$$k \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + l \frac{\nabla_\theta R(\hat{X})}{R(\hat{X})} - 2m \frac{\nabla_{\hat{X}} \nabla_\theta R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + n \frac{\nabla_{\hat{X}}^2 \nabla_\theta R(\hat{X})}{\nabla_{\hat{X}}^2 R(\hat{X})} = -C_3(p, \hat{X}) \frac{\nabla_\theta r(\hat{X})}{f(\hat{X})} \quad (330)$$

with:

$$\begin{aligned}
k &= 1 - \eta \left(1 - \frac{\gamma C_3(p, \hat{X})}{|f(\hat{X})|} \right) \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\
&\quad + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) C_2(p, \hat{X}) - (1 - \alpha) C_3(p, \hat{X})}{|f(\hat{X})|} \\
l &= \frac{{}_\varsigma F_1 \left(R(K_{\hat{X}}, \hat{X}) \right) C_3(p, \hat{X})}{f(\hat{X})} \\
m &= \left(1 - \frac{\gamma C_3(p, \hat{X})}{f(\hat{X})} \right) \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} - \frac{g^2(\hat{X}) C_2(p, \hat{X})}{\sigma_{\hat{X}}^2} \\
n &= \frac{\nabla_{\hat{X}} g(\hat{X}) C_2(p, \hat{X})}{|f(\hat{X})|}
\end{aligned} \tag{331}$$

A4.2 Full dynamical system

To make the system self-consistent, we introduce also a dynamics for R .

We assume that R depends on $K_{\hat{X}}$, \hat{X} and $\nabla_{\theta} K_{\hat{X}}$, that leads to write: $R(K_{\hat{X}}, \hat{X}, \nabla_{\theta} K_{\hat{X}})$. The variation is assumed to follow a diffusion process:

$$\begin{aligned}
\nabla_{\theta} R(\theta, \hat{X}) &= \int_{\theta' < \theta} G_1((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta'} R(\theta', \hat{X}') d(\theta', \hat{X}') \\
&\quad + \int_{\theta' < \theta} G_2((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta'} K_{\hat{X}'} d(\theta', \hat{X}')
\end{aligned}$$

The first orders expansion of the right hand side leads to the following form for $\nabla_{\theta} R(\theta, \hat{X})$:

$$\begin{aligned}
\nabla_{\theta} R(\theta, \hat{X}) &= \int (\hat{X} - \hat{X}') \left(G_1((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\hat{X}} \nabla_{\theta} R(\theta, \hat{X}) + G_2((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\hat{X}} \nabla_{\theta} K_{\hat{X}} \right) \\
&\quad + \frac{1}{2} \int (\hat{X} - \hat{X}')^2 \left(G_1((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\hat{X}}^2 \nabla_{\theta} R(\theta, \hat{X}) + G_2((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\hat{X}}^2 \nabla_{\theta} K_{\hat{X}} \right) \\
&\quad + \int (\theta - \theta') \left(G_1((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta} \nabla_{\theta} R(\theta, \hat{X}) + G_2((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta} \nabla_{\theta} K_{\hat{X}} \right) \\
&\quad + \frac{1}{2} \int (\theta - \theta')^2 \left(G_1((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta}^2 \nabla_{\theta} R(\theta, \hat{X}) + G_2((\theta, \hat{X}), (\theta', \hat{X}')) \nabla_{\theta}^2 \nabla_{\theta} K_{\hat{X}} \right) \\
&\quad + \dots
\end{aligned} \tag{332}$$

where the crossed derivatives have been discarded for the sake of simplicity. We assume $G_1((\theta, \hat{X}), (\theta, \hat{X})) = 0$ to avoid auto-interaction.

Performing the integrals yields:

$$\begin{aligned}
\nabla_\theta R(\theta, \hat{X}) &= a_0(\hat{X}) \nabla_\theta K_{\hat{X}} + a(\hat{X}) \nabla_{\hat{X}} \nabla_\theta K_{\hat{X}} + b(\hat{X}) \nabla_{\hat{X}}^2 \nabla_\theta K_{\hat{X}} \\
&+ c(\hat{X}) \nabla_\theta (\nabla_\theta K_{\hat{X}}) + d(\hat{X}) \nabla_\theta^2 (\nabla_\theta K_{\hat{X}}) \\
&+ e(\hat{X}) \nabla_{\hat{X}} (\nabla_\theta R(\theta, \hat{X})) + f(\hat{X}) \nabla_{\hat{X}}^2 (\nabla_\theta R(\theta, \hat{X})) \\
&+ g(\hat{X}) \nabla_\theta (\nabla_\theta R(\theta, \hat{X})) + h(\hat{X}) \nabla_\theta^2 (\nabla_\theta R(\theta, \hat{X})) \\
&+ u(\hat{X}) \nabla_{\hat{X}} \nabla_\theta (\nabla_\theta K_{\hat{X}}) + v(\hat{X}) \nabla_{\hat{X}} \nabla_\theta (\nabla_\theta R(\theta, \hat{X}))
\end{aligned} \tag{333}$$

We assume that the coefficients are slowly varying, since they are obtained by averages.

Gathering the dynamics (330) and (333) for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R(\theta, \hat{X})$ leads to a matricial system:

$$\begin{aligned}
0 &= \begin{pmatrix} \frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} \\ -a_0(\hat{X}) & 1 \end{pmatrix} \begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} \\
&- \begin{pmatrix} 0 & \frac{2m}{\nabla_{\hat{X}} R(\hat{X})} \nabla_{\hat{X}} \\ a(\hat{X}) \nabla_{\hat{X}} + c(\hat{X}) \nabla_\theta & e(\hat{X}) \nabla_{\hat{X}} + g(\hat{X}) \nabla_\theta \end{pmatrix} \begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} \\
&- \begin{pmatrix} 0 & -\frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})} \nabla_{\hat{X}}^2 \\ d(\hat{X}) \nabla_\theta^2 + b(\hat{X}) \nabla_{\hat{X}}^2 + u(\hat{X}) \nabla_{\hat{X}} \nabla_\theta & e(\hat{X}) \nabla_\theta^2 + f(\hat{X}) \nabla_{\hat{X}}^2 + v(\hat{X}) \nabla_{\hat{X}} \nabla_\theta \end{pmatrix} \begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix}
\end{aligned} \tag{334}$$

A4.3 Oscillatory solutions

We look for a solution of (335) of the form:

$$\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R(\hat{X}) \end{pmatrix} = \exp \left(i\Omega(\hat{X}) \theta + iG(\hat{X}) \hat{X} \right) \begin{pmatrix} \nabla_\theta K_0 \\ \nabla_\theta R_0 \end{pmatrix}$$

with $G(\hat{X})$ and $\Omega(\hat{X})$ slowly varying. Consequently, the system (334) writes:

$$\begin{pmatrix} \frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} - i \frac{2m}{\nabla_{\hat{X}} R(\hat{X})} G - \frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})} G^2 \\ -a_0(\hat{X}) - ia(\hat{X})G - ic(\hat{X})\Omega & 1 - ie(\hat{X})G - ig(\hat{X})\Omega + e\Omega^2 \\ +d\Omega^2 + bG^2 + u\Omega G & +fG^2 + v\Omega G \end{pmatrix} \begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} = 0 \tag{335}$$

By canceling the determinant of the system, we are led to the following relation between $\Omega(\hat{X})$ and $G(\hat{X})$:

$$\begin{aligned}
0 &= \frac{k}{K_{\hat{X}}} (1 - icG - ig\Omega) + \left(\frac{l}{R(\hat{X})} - i \frac{2m}{\nabla_{\hat{X}} R(\hat{X})} G \right) (a_0 + iaG + ic\Omega) \\
&- \frac{l}{R(\hat{X})} (d\Omega^2 + bG^2 + u\Omega G) + \frac{k}{K_{\hat{X}}} (e\Omega^2 + fG^2 + v\Omega G)
\end{aligned}$$

In the sequel, we restrict to the first order terms, which yields the expression for Ω :

$$\begin{aligned}\Omega &= \frac{i}{\left(\frac{lc}{R(\hat{X})} - i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right) - \frac{kg}{K_{\hat{X}}}} \left(\frac{k}{K_{\hat{X}}} (1 - ieG) + \left(\frac{l}{R(\hat{X})} - i\frac{2m}{\nabla_{\hat{X}}R(\hat{X})}G \right) (a_0 + iaG) \right) \\ &= \frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right) + i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \right)^2} \\ &\quad \times \left(\left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})} \right) \right) G + i \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2 \right) \right)\end{aligned}$$

Or equivalently:

$$\begin{aligned}\Omega &= \frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right) \left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})} \right) \right) G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2 \right)}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \right)^2} \\ &\quad + i \frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right) \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2 \right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})} \left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})} \right) \right) G^2}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}} \right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \right)^2}\end{aligned}$$

We focus on the influence of time variations of $\nabla_{\theta}K_{\hat{X}}$ on $\nabla_{\theta}R$, and we can assume $g \simeq 0$ so that there is no self influence of $\nabla_{\theta}R$ on itself: $\nabla_{\theta}R$ depends on the variations of $\nabla_{\theta}K_{\hat{X}}$ as well as the neighborhood sectors variations of $\nabla_{\theta}R$. Moreover, the coefficients e and a , being obtained by integration or first order expansion, can be considered as nul.

Consequently, the equation for Ω reduces to:

$$\Omega = \frac{\frac{lc}{R(\hat{X})} \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} \right) G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} \right)}{\left(\frac{lc}{R(\hat{X})} \right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \right)^2} + i \frac{\frac{lc}{R(\hat{X})} \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} \right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})} \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} \right) G^2}{\left(\frac{lc}{R(\hat{X})} \right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G \right)^2}$$

A4.4 Stability

The system is stable and the dynamics is dampening if:

$$\frac{lc}{R(\hat{X})} \left(\frac{k}{K_{\hat{X}}} + \frac{a_0l}{R(\hat{X})} \right) + \frac{4m^2ca_0}{\left(\nabla_{\hat{X}}R(\hat{X}) \right)^2} G^2 > 0 \quad (336)$$

To study the sign of (336) we need to estimate the coefficient k .

A4.4.1 Estimation of the coefficients k , l and m

We can estimate k and l by computing the factors $C_i(p, \hat{X})$, for $i = 1, 2, 3$.

This is done by estimating $p + \frac{1}{2}$. We start with the asymptotic form of $\hat{\Gamma}(p + \frac{1}{2})$:

$$\hat{\Gamma}\left(p + \frac{1}{2}\right) \simeq \sqrt{p + \frac{1}{2}} \exp \left(-\frac{\sigma_{\hat{X}}^2 \left(p + \frac{1}{2}\right)^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right)$$

and rewriting the equation for $K_{\hat{X}}$ as:

$$K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| \left(\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right)^{\frac{1}{4}} = C(\bar{p}) \sigma_K^2 \exp \left(- \frac{\sigma_X^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right) \sqrt{\left(p + \frac{1}{2} \right)} \sqrt{\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3}} \quad (337)$$

Then, using (328), we set:

$$\left(p + \frac{1}{2} \right) \sqrt{\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3}} = \sqrt{C_1(p, \hat{X})} \quad (338)$$

Equation (337) writes:

$$\frac{K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})| \left(\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3} \right)^{\frac{1}{4}}}{C(\bar{p}) \sigma_K^2} = \exp \left(-C_1(p, \hat{X}) \right) \left(C_1(p, \hat{X}) \right)^{\frac{1}{4}} \quad (339)$$

and the solution to (339) is:

$$\begin{aligned} C_1(p, \hat{X}) &= \frac{\sigma_X^2 (p + \frac{1}{2})^2 (f'(X))^2}{96 |f(\hat{X})|^3} \\ &= C_0(\hat{X}, K_{\hat{X}}) \exp \left(-W(k, -4C_0(\hat{X}, K_{\hat{X}})) \right) \end{aligned} \quad (340)$$

with:

$$C_0(\hat{X}, K_{\hat{X}}) = \left(\frac{K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_K^2} \right)^4 \frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3}$$

and where $W(k, x)$ is the Lambert W function. The parameter $k = 0$ for the stable case with low $K_{\hat{X}}$ and $k = -1$ for the unstable case with $K_{\hat{X}}$ large.

We can deduce $p + \frac{1}{2}$ from (340):

$$p + \frac{1}{2} = \frac{\sqrt{C_1(p, \hat{X})}}{\sqrt{\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3}}} \quad (341)$$

and $2 \frac{C_1(p, \hat{X})}{p + \frac{1}{2}}$:

$$2 \frac{C_1(p, \hat{X})}{p + \frac{1}{2}} = \sqrt{C_1(p, \hat{X})} \frac{\sigma_X^2 (f'(X))^2}{48 |f(\hat{X})|^3}$$

From (341) and (??) we deduce:

$$\begin{aligned} C_2(p, \hat{X}) &= \ln \left(p + \frac{1}{2} \right) - \frac{2C_1(p, \hat{X})}{p + \frac{1}{2}} \\ &= \frac{1}{2} \ln \frac{C_1(p, \hat{X})}{\frac{\sigma_X^2 (f'(X))^2}{96 |f(\hat{X})|^3}} - \sqrt{C_1(p, \hat{X})} \frac{\sigma_X^2 (f'(X))^2}{48 |f(\hat{X})|^3} \simeq \frac{1}{2} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \end{aligned} \quad (342)$$

We can also compute:

$$\begin{aligned}
\left(p + \frac{3}{2}\right) \ln \left(p + \frac{1}{2}\right) &\simeq \frac{48 |f(\hat{X})|^3 C_1(p, \hat{X})}{\sigma_X^2 (f'(X))^2} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \\
&= \frac{1}{2} \left(\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_{\hat{K}}^2} \right)^4 \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \\
&\quad \times \exp \left(-W \left(k, -4C_0(\hat{X}, K_{\hat{X}}) \right) \right)
\end{aligned}$$

so that:

$$\begin{aligned}
C_3(p, \hat{X}) &= 1 - C_1(p, \hat{X}) + \left(p + \frac{3}{2}\right) C_2(p, \hat{X}) \\
&= 1 - C_1(p, \hat{X}) + \left(p + \frac{3}{2}\right) \left(\ln \left(p + \frac{1}{2}\right) - \frac{2C_1(p, \hat{X})}{p + \frac{1}{2}} \right) \\
&\simeq 1 + \left(\frac{48 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} - 1 \right) C_1(p, \hat{X}) \\
&\simeq 1 + \frac{1}{2} \left(\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_{\hat{K}}^2} \right)^4 \exp \left(-W \left(k, -4C_0(\hat{X}, K_{\hat{X}}) \right) \right) \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2}
\end{aligned} \tag{343}$$

Given that our assumptions $\sigma_X^2 < 1$ and in most cases $\frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \gg 1$, then $\frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \gg 1$ and $C_3(p, \hat{X}) \gg 1$.

These computations allow to estimate k and l . We start with k . Given that (see (331)):

$$\begin{aligned}
k &= 1 - \eta \left(1 - \frac{\gamma C_3(p, \hat{X})}{|f(\hat{X})|} \right) \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \\
&\quad + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) C_2(p, \hat{X}) - (1 - \alpha) C_3(p, \hat{X})}{|f(\hat{X})|} \\
l &= \frac{{}_2F_1 \left(R(K_{\hat{X}}, \hat{X}) \right) C_3(p, \hat{X})}{f(\hat{X})} \\
m &= \left(1 - \frac{\gamma C_3(p, \hat{X})}{f(\hat{X})} \right) \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2}
\end{aligned}$$

the sign of k and l depend on the magnitude of $K_{\hat{X}}$.

A4.4.1.1 $K_{\hat{X}} \gg 1$ For $K_{\hat{X}} \gg 1$, using (??) and:

$$\|\Psi(\hat{X})\|^2 = D - \left(\nabla_X R(\hat{X}) \right)^2 K_X^\alpha$$

we have:

$$K_{\hat{X}}^\alpha \simeq \frac{D}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2} - \frac{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^{2(1-\frac{1}{\alpha})} D^{\frac{1}{\alpha} c}}$$

and:

$$\frac{D - \left\| \Psi(\hat{X}) \right\|^2}{\left\| \Psi(\hat{X}) \right\|^2} \simeq \frac{D^{1+\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}}$$

The constant c has been defined in appendix 3, and satisfies $c \ll 1$. As a consequence:

$$\begin{aligned} k &\simeq \eta \frac{\gamma C_3(p, \hat{X})}{|f(\hat{X})|} \frac{D - \left\| \Psi(\hat{X}) \right\|^2}{\left\| \Psi(\hat{X}) \right\|^2} - (1 - \alpha) \frac{C_3(p, \hat{X})}{|f(\hat{X})|} \\ &\simeq \left(\frac{\eta \gamma D^{1+\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}} - (1 - \alpha) \right) \frac{C_3(p, \hat{X})}{|f(\hat{X})|} \end{aligned}$$

This may be negative or positive depending on the relative magnitude of $\frac{\eta \gamma D^{1+\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}}$ and $(1 - \alpha)$. The first case correspond to the stable equilibrium with large $K_{\hat{X}}$ and the second case to the stable case with large $K_{\hat{X}}$ studied in appendix 2.

Unstable case This case corresponds to:

$$\frac{D^{1+\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}} \gg 1$$

Moreover, using (343) and the following estimation, we have:

$$k \simeq \eta \frac{\gamma C_3(p, \hat{X})}{|f(\hat{X})|} \frac{\eta \gamma D^{1+\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}} \gg 1 \quad (344)$$

We can also estimate $\left| \frac{k}{K_{\hat{X}}} \right|$. In this case:

$$\frac{k}{K_{\hat{X}}} \simeq \eta \frac{\gamma C_3(p, \hat{X})}{|f(\hat{X})|} \frac{\eta \gamma D^{\frac{1}{\alpha} c}}{C(\bar{p}) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}} \gg 1 \quad (345)$$

We can estimate l by the same token:

$$l = \frac{\varsigma F_1(R(K_{\hat{X}}, \hat{X})) C_3(p, \hat{X})}{f(\hat{X})} \gg 1$$

and using (345) we have:

$$\left| \frac{k}{K_{\hat{X}}} \right| \gg l$$

The coefficient m is obtained by using that in this case:

$$m \simeq \left(1 - \frac{\gamma C_3(p, \hat{X})}{f(\hat{X})} \right) \frac{D - \left\| \Psi(\hat{X}) \right\|^2}{\left\| \Psi(\hat{X}) \right\|^2} \simeq -\frac{1}{\eta} k$$

Stable case For the stable case we have:

$$\frac{\eta\gamma D^{1+\frac{1}{\alpha}}c}{C(\bar{p})\sigma_K^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}}R(\hat{X})\right)^{\frac{2}{\alpha}}-(1-\alpha)} < 0$$

and we write:

$$k \simeq -(1-\alpha) \frac{C_3(p, \hat{X})}{|f(\hat{X})|} < 0$$

We have:

$$|k| \gg 1$$

and moreover:

$$\left| \frac{k}{K_{\hat{X}}} \right| \simeq (1-\alpha) \frac{C_3(p, \hat{X})}{K_{\hat{X}}|f(\hat{X})|} = \frac{1-\alpha}{\varsigma F_1(R(K_{\hat{X}}, \hat{X}))K_{\hat{X}}} l \ll l \quad (346)$$

The coefficient m is obtained by using that in the stable case:

$$m \simeq -\frac{\gamma}{\varsigma F_1(R(K_{\hat{X}}, \hat{X}))} l$$

A4.4.1.2 $K_{\hat{X}} \ll 1$ On the other hand, for $K_{\hat{X}} \leq 1$, we have:

$$\frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \ll 1 \quad (347)$$

so that:

$$k \simeq 1 + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) C_2(p, \hat{X}) - (1-\alpha) C_3(p, \hat{X})}{|f(\hat{X})|}$$

Given (342) and (343), this yields:

$$k \simeq -\frac{(1-\alpha) C_3(p, \hat{X})}{|f(\hat{X})|} < 0 \quad (348)$$

and, as in the previous case:

$$\begin{aligned} |k| &> 1 \\ l &> 1 \end{aligned}$$

Moreover, given that $K_{\hat{X}} \ll 1$:

$$\left| \frac{k}{K_{\hat{X}}} \right| \gg 1 \quad (349)$$

and:

$$\left| \frac{k}{K_{\hat{X}}} \right| \gg l \quad (350)$$

Moreover, given (347):

$$|m| = \left| 1 - \frac{\gamma C_3(p, \hat{X})}{f(\hat{X})} \right| \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} \ll \left| \frac{\gamma C_3(p, \hat{X})}{f(\hat{X})} \right|$$

and:

$$|m| \ll l$$

A4.4.1.3 Intermediate case In this case, we can consider that $\frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2}$ is of order 1:

$$\frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2} = O(1) \quad (351)$$

Assuming that $\gamma \ll 1$ we have:

$$\begin{aligned} k &\simeq 1 + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) C_2(p, \hat{X}) - (1 - \alpha) C_3(p, \hat{X})}{|f(\hat{X})|} \\ &\simeq 1 + \frac{\frac{\alpha}{2} \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) - \frac{1-\alpha}{2} \left(\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_K^2} \right)^4 \exp(-W(k, -4C_0(\hat{X}, K_{\hat{X}})))}{|f(\hat{X})|} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \end{aligned}$$

Given that the intermediate case is stable (see appendix 2), the relation between $K_{\hat{X}}$ and $R(\hat{X})$ is positive, we can assume that $k < 0$ and:

$$\begin{aligned} k &\simeq 1 + \frac{\alpha \left(2 \frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g(\hat{X}) \right) C_2(p, \hat{X}) - (1 - \alpha) C_3(p, \hat{X})}{|f(\hat{X})|} \\ &\simeq - \frac{\frac{1-\alpha}{2} \left(\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_K^2} \right)^4 \exp(-W(0, -4C_0(\hat{X}, K_{\hat{X}})))}{|f(\hat{X})|} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \end{aligned}$$

and:

$$\begin{aligned} l &= \frac{\varsigma F_1(R(K_{\hat{X}}, \hat{X})) C_3(p, \hat{X})}{f(\hat{X})} \simeq l \\ &= \frac{\varsigma F_1(R(K_{\hat{X}}, \hat{X})) \left(\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2 |f(\hat{X})|}{C(\bar{p}) \sigma_K^2} \right)^4 \exp(-W(0, -4C_0(\hat{X}, K_{\hat{X}})))}{f(\hat{X})} \ln \frac{96 |f(\hat{X})|^3}{\sigma_X^2 (f'(X))^2} \end{aligned}$$

Note that in this case:

$$k \simeq - \frac{1 - \alpha}{\varsigma F_1(R(K_{\hat{X}}, \hat{X}))} l$$

and, given (351):

$$m \simeq -\gamma \frac{D - \|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2 \varsigma F_1(R(K_{\hat{X}}, \hat{X}))} l$$

A4.4.2 Stability conditions

This appendix presents the computations leading to the stability conditions for the three ranges of capital considered. Apart from the intermediate case, interpretations are detailed in the text.

A4.4.2.1 Case $K_{\hat{X}} \gg 1$

Stable case As shown above, $k < 0$, $|k| \gg 1$, $l \gg 1$ and $\left| \frac{k}{K_{\hat{X}}} \right| \ll l$. Coefficients l and m are of the same order. Thus (336) becomes:

$$\frac{l^2 a_0 c}{\left(R(\hat{X}) \right)^2} + \frac{4m^2 c a_0}{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2} G^2 > 0$$

That is, for $c > 0$ the oscillations are stable, whereas for $c < 0$ they are unstable.

Unstable case In this case, $k > 0$, $|k| \gg 1$, $l \gg 1$ and $\left| \frac{k}{K_{\hat{X}}} \right| \gg l$. We have also $m \simeq -\frac{1}{\eta} k$ and (336) writes:

$$\frac{cl}{R(\hat{X})} \frac{k}{K_{\hat{X}}} + \frac{4k^2 c a_0}{\eta^2 \left(\nabla_{\hat{X}} R(\hat{X}) \right)^2} G^2 > 0 \quad (352)$$

That is, for $c > 0$ the oscillations are stable, whereas for $c < 0$ they are unstable.

A4.4.2.1.2 Case $K_{\hat{X}} \ll 1$ Equations (348) and (349) show that $k < 0$, $|k| \gg 1$, $l \gg 1$, $|m| \ll l$ and $\left| \frac{k}{K_{\hat{X}}} \right| \gg l$. Equation (336) thus writes:

$$\frac{cl}{R(\hat{X})} \frac{k}{K_{\hat{X}}} > 0 \quad (353)$$

That is, for $c > 0$ the oscillations are unstable, whereas for $c < 0$ they are stable.

A4.4.2.3 Intermediate case In this case, we have seen above that $k < 0$:

$$k \simeq -\frac{1 - \alpha}{\varsigma F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} l$$

and:

$$m \simeq -\gamma \frac{D - \left\| \Psi(\hat{X}) \right\|^2}{\left\| \Psi(\hat{X}) \right\|^2 \varsigma F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)}$$

Consequently, equation (336) particularizes as:

$$\frac{l^2 c}{R(\hat{X})} \left(\frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} \right) + 4c a_0 \left(\frac{\gamma \left(D - \left\| \Psi(\hat{X}) \right\|^2 \right)}{\varsigma \nabla_{\hat{X}} R(\hat{X}) F_1 \left(R(K_{\hat{X}}, \hat{X}) \right) \left\| \Psi(\hat{X}) \right\|^2} \right)^2 G^2 > 0$$

Given the definition of a_0 and the stability of the intermediate case, we assume $a_0 > 0$. Thus, 2 possibilities arise.

Coefficient $c > 0$ In this case, the oscillations are stable if:

$$\frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} > 0$$

or if:

$$\frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} < 0$$

and:

$$G^2 > \frac{l^2 \left(\nabla_{\hat{X}} R(\hat{X}) \right)^2}{4a_0 R(\hat{X})} \left(\frac{\varsigma \left(\nabla_{\hat{X}} R(\hat{X}) F_1 \left(R(K_{\hat{X}}, \hat{X}) \right) \left\| \Psi(\hat{X}) \right\|^2 \right)}{\gamma \left(D - \left\| \Psi(\hat{X}) \right\|^2 \right)} \right)^2 \left| \frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} \right|$$

Otherwise, the oscillations are unstable.

The constant ς is irrelevant here, although it arises in appendix 3 to estimate short-term returns. The function F_1 , defined in (2), determines the stock's prices evolution. The coefficient α is the Cobb-Douglas power arising in the dividend part of short-term returns. The constant D , defined in (57), determines the relation between number of firms and average capital at sector \hat{X} .

We recover the large average capital case. A relatively high reactivity of expectations to fluctuations in capital allows to maintain the capital at its equilibrium value. This stability is favoured for sectors with large average capital when G is relatively large, i.e. when this sectors present large discrepancies in capital with their neighbours.

Coefficient $c < 0$ The oscillations are stable if:

$$\frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} < 0 \quad (354)$$

and:

$$G^2 < \frac{l^2 \left(\nabla_{\hat{X}} R(\hat{X}) \right)^2}{4a_0 R(\hat{X})} \left(\frac{\varsigma \left(\nabla_{\hat{X}} R(\hat{X}) F_1 \left(R(K_{\hat{X}}, \hat{X}) \right) \left\| \Psi(\hat{X}) \right\|^2 \right)}{\gamma \left(D - \left\| \Psi(\hat{X}) \right\|^2 \right)} \right)^2 \left| \frac{a_0}{R(\hat{X})} - \frac{1 - \alpha}{\varsigma F_1 \left(R(K_{\hat{X}}, \hat{X}) \right)} \right| \quad (355)$$

Conditions (354) and (355) correspond to the case of relatively low capital for which a stability in the oscillations may be reached when expectations are moderately reactive to variation in capital. The condition (355) shows that the stability in oscillations is reached for moderate values of G , i.e. relatively small discrepancy between neighbouring sectors.

We recover the large average capital case. A relatively high reactivity of expectations to fluctuations in capital allows to maintain the capital at its equilibrium value. This stability is favoured for sectors with large average capital when G is relatively large, i.e. when this sectors present large discrepancies in capital with their neighbours.

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