Off-the-grid curve reconstruction through divergence regularisation: an extreme point result

Bastien Laville, Laure Blanc-Féraud, Gilles Aubert

Off-the-grid curve reconstruction through divergence regularisation: an extreme point result

Bastien Laville[1], Laure Blanc-Féraud[1], and Gilles Aubert[2]

[1]Team Morpheme: Inria, CNRS, Université Côte d'Azur
[2]LJAD, CNRS, Université Côte d'Azur

4th May 2022

Abstract

We propose a new strategy for the reconstruction of curves in an image through an off-the-grid variational framework, inspired by the reconstruction of spikes in the literature. We introduce a new functional CROC on the space of 2-dimensional Radon measures with finite divergence denoted $\mathscr{V}$, and we establish several theoretical tools through the definition of a certificate. Our main contribution lies in the sharp characterisation of the extreme points of the unit ball of the $\mathscr{V}$-norm: there are exactly measures supported on 1-rectifiable oriented simple Lipschitz curves, thus enabling a precise characterisation of our functional minimisers and further opening a promising avenue for the algorithmic implementation.

1 Introduction

This work focuses on the definition of a functional designed to recover curves in an off-the-grid fashion, by considering the space of vector Radon measures with finite divergence, namely that their divergence in the distributional sense is a Radon measure. Such choice of space is motivated by multiple works [24, 23, 2] pertaining to Jordan curve or at least Radon measure absolutely continuous with respect to $\mathscr{H}_1$. However, the literature nowadays does not offer a tractable off-the-grid framework for open and closed curves.

Yet such situations arise in several domains such as biomedical imaging (blood vessels, filaments structures), in magnetic imaging through observation of the magnetic field (Scanning Magnetic Microscopy), *etc*. An example of curves encountered in a natural and practical setting is presented in the Figure 1. In the following, we propose a method referred to as *Atomic based Method for Gridless* (AMG), designed for curves reconstruction through off-the-grid variational optimisation. To the best knowledge of the authors, this is the first attempt to recover curves in an off-the-grid manner and one of the few papers to characterise the space of divergence vector field.

1.1 Contributions

Our main contributions in this article are listed below:

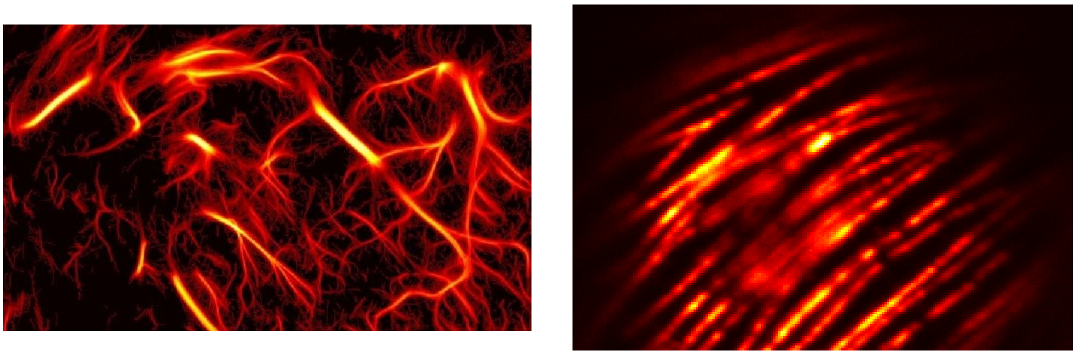

Figure 1: Curves arise genuinely in several biological structures, such as (on the left) blood vessels [1] (on the right) *Ostreopsis* protein samples on the right.

- Propose a new space $\mathscr{V}$ for off-the-grid curve variational analysis, and qualify several of its properties;
- Develop a new functional $(\mathcal{Q}_\alpha(y))$ for curves reconstruction in an off-the-grid variational context called CROC;
- Establish the main results, such as the certificate outlining, for theoretical off-the-grid analysis and future numerical implementation;
- Characterise precisely the extreme points of the $\mathscr{V}$-norm unit ball as 1-rectifiable simple oriented Lipschitz curves, thereby describing the structure of the minimisers of our functional and thus proving the interest of AMG.

1.2 Paper outline

The paper is organized as follows. We present the main definitions and tools in section 2, we propose a functional in 3, our main theorem lies in 4 while the conclusions and the outlook follow in 5.

1.3 Notations

In the following, $\mathcal{X}$ denotes the ambient space where the positions of the curves/spikes live. We suppose $\mathcal{X}$ is a compact subset of $\mathbb{R}^d$ such that its interior $\mathring{\mathcal{X}}$ is a non-empty submanifold of dimension $d \in \mathbb{N}^*$.

Note that this paper makes extensive use of the notion of *measure*, through both functional and set definition: remember that thanks to Riesz-Markov representation theorem and $\mathcal{X}$ compact, these two notions are strictly equivalent. A comprehensive characterisation of measure from a functional standpoint is given in the 2. The interested reader can explore the set point-of-view in several celebrated work such as [21]. Let ρ be a measure, $\rho \llcorner A$ denotes the restriction of the measure ρ to the set A, namely for all measurable sets B: $\rho \llcorner A(B) \stackrel{\text{def.}}{=} \rho(A \cap B)$. The support of ρ denoted by $\operatorname{spt}(\rho)$ is the smallest measurable set $B \subset \mathcal{X}$ such that $\rho(\mathcal{X} \backslash B) = 0$. Let us denote by $\mathscr{H}_1$ the 1-dimensional Hausdorff measure (see [13] for a definition). We say that $E \subset \mathbb{R}^d$ is p-rectifiable if it is the countable union of Lipschitz function images from $\mathbb{R}^p$ to $\mathbb{R}^d$, up to a set of null $\mathscr{H}^p$-measure [18].

2 Off-the-grid framework

This section is divided into two parts: a recall of the classic off-the-grid framework for spike reconstruction, and the new notions needed for curve reconstruction.

2.1 Classical scalar off-the-grid framework

We recall some definitions and handy properties stemming from the off-the-grid literature, the interested reader may take a deeper look in the review [17] for more insights. In this case, the aim is to reconstruct spikes, *i.e.* dots localised in the space $\mathcal{X}$ with some amplitude information encoded. This problem is encountered in many fields such as astronomy for stars imaging enhancement, fluorescence microscopy where one locates dyes, contrast enhanced ultrasound when one performs the tracking of small air bubbles, *etc*.

2.1.1 Quick digest of measure theory

A spike, not constrained to a finite set of positions, can be accurately modelled by a Dirac measure. Loosely speaking, this object allows us to encode both amplitude and spatial information in the same object. However, since the Dirac measure is not a classic continuous function, one needs to consider a more general class of maps called the Radon measures. From a distributional standpoint, it is a subset of the distribution space $\mathscr{D}'(\mathcal{X})$; the latter being the space of linear forms over the space of test functions $\mathscr{C}_0^\infty(\mathcal{X})$ *i.e.* smooth functions (continuous derivatives of all orders) compactly supported. This functional approach[1] is based on the definition of a measure as a linear form on a function space, namely:

Definition 1 (Evanescent continuous function on $\mathcal{X}$)**.** *We call $\mathscr{C}_0(\mathcal{X}, \mathcal{Y})$ the set of evanescent continuous functions, namely all the continuous map $\psi : \mathcal{X} \to \mathcal{Y}$ such that :*

$$\forall \varepsilon > 0, \exists K \subset \mathcal{X} \text{ compact}, \quad \sup_{x \in \mathcal{X} \setminus K} \|\psi(x)\|_{\mathcal{Y}} \leq \varepsilon.$$

We write $\mathscr{C}_0(\mathcal{X})$ when $\mathcal{Y} = \mathbb{R}$. Then we can introduce:

Definition 2 (Set of Radon measures)**.** *We denote by $\mathcal{M}(\mathcal{X})$ the set of real signed Radon measures on $\mathcal{X}$ of finite masses. It is the topological dual of $\mathscr{C}_0(\mathcal{X})$ with* supremum *norm $\|\cdot\|_{\infty,\mathcal{X}}$ by the Riesz-Markov representation theorem [13]. Thus, a Radon measure m is a continuous linear form on functions $f \in \mathscr{C}_0(\mathcal{X})$, with the duality bracket for $m \in \mathcal{M}(\mathcal{X})$ denoted by $\langle f, m \rangle_{\mathcal{M}} = \int_{\mathcal{X}} f \,\mathrm{d}m$.*

A 'signed' measure entails that the quantity $\langle f, m \rangle_{\mathcal{M}}$ can be negative, further generalising the notion of probability (positive) measure. While m is a measure, also remember that it can be evaluated on all measurable sets $A \subset \mathcal{X}$ through:

$$|m|(A) \stackrel{\text{def.}}{=} \sup\left(\int_A f \,\mathrm{d}m, f \in \mathscr{C}_0(A), \|f\|_{\infty,\mathcal{X}} \leq 1 \right).$$

[1] One can then define equivalently the space of Radon measures, either by a set-related approach or by functional analysis approach (thanks to Riesz–Markov theorem). In the more set-related [21] insight, a *measure* is an object which takes sets as an input. A *Borel measure* is a measure defined on all open sets of $\mathcal{X}$, and a *Radon measure* is a Borel measure such that it is finite on all compact sets of $\mathcal{X}$ (by an isomorphism). The functional and the set point-of-views are different approaches to describe the same object.

Classic examples of Radon measures are the Lebesgue measure, the celebrated Dirac measure δ_z centred in $z \in \mathcal{X}$ namely for all $f \in \mathscr{C}_0(\mathcal{X})$ one has $\langle f, \delta_z \rangle_{\mathcal{M}} = f(z)$, *etc*. Eventually, since $\mathscr{C}_0(\mathcal{X})$ is a Banach space, $\mathcal{M}(\mathcal{X})$ is complete [6] if endowed with its dual norm called the total variation (TV) norm, defined for $m \in \mathcal{M}(\mathcal{X})$ by:

$$\|m\|_{\mathrm{TV}} \stackrel{\text{def.}}{=} |m|(\mathcal{X}) = \sup\left(\int_{\mathcal{X}} f \,\mathrm{d}m, f \in \mathscr{C}_0(\mathcal{X}), \|f\|_{\infty,\mathcal{X}} \leq 1\right).$$

2.1.2 Observation model

Let us introduce $\mathcal{H}$ the Hilbert space containing the acquired data. For the instance of images we use a finite dimensional space of acquisition $\mathcal{H} = \mathcal{H}_n \stackrel{\text{def.}}{=} \mathbb{R}^n$, for $n \in \mathbb{N}$. Let $m \in \mathcal{M}(\mathcal{X})$ be a source measure, we call *acquisition* $y \in \mathcal{H}$ the result of the *forward/acquisition map* $\Phi : \mathcal{M}(\mathcal{X}) \to \mathcal{H}$ evaluated on m, with measurement kernel $\varphi : \mathcal{X} \to \mathcal{H}$ continuous and bounded [8, 10]:

$$y \stackrel{\text{def.}}{=} \Phi m = \int_{\mathcal{X}} \varphi(x) \,\mathrm{d}m(x). \tag{1}$$

Also note that the forward operator Φ incorporates a sampling operation, hence $\mathcal{H} = \mathcal{H}_n$. In the following, we impose $\varphi \in \mathscr{C}^2(\mathcal{X}, \mathcal{H})$. Let us also define the adjoint operator of $\Phi : \mathcal{M}(\mathcal{X}) \to \mathcal{H}$ in the weak-$*$ topology, namely the map $\Phi^* : \mathcal{H} \to \mathscr{C}_0(\mathcal{X})$, defined for all $x \in \mathcal{X}$ and $p \in \mathcal{H}$ by $\Phi^*(p)(x) = \langle p, \varphi(x) \rangle_{\mathcal{H}}$. The choice of φ and $\mathcal{H}$ is dictated by the physical process of acquisition, with generic measurement kernels such as convolution, Fourier, Laplace, *etc* [17].

2.1.3 An off-the-grid functional: the BLASSO

Consider the source measure $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^{N} a_{0,i} \delta_{x_{0,i}}$ with amplitudes $\mathbf{a_0} \in \mathbb{R}^N$ and positions $\mathbf{x_0} \in \mathcal{X}^N$, the sparse spike problem aims to recover this measure from the acquisition $y \stackrel{\text{def.}}{=} \Phi m_{a_0,x_0} + w$ where $w \in \mathcal{H}$ is an additive noise. In order to tackle this inverse problem, let us introduce the following convex functional called BLASSO [6, 9], standing for Beurling-LASSO:

$$\underset{m \in \mathcal{M}(\mathcal{X})}{\operatorname{argmin}}\; T_\lambda(m) \stackrel{\text{def.}}{=} \frac{1}{2}\|y - \Phi(m)\|_{\mathcal{H}}^2 + \lambda |m|(\mathcal{X}) \tag{$\mathcal{P}_\lambda(y)$}$$

where $\lambda > 0$ is the regularisation parameter accounting for the trade-off between fidelity and sparsity of the reconstruction. The BLASSO in a noiseless setting writes down:

$$\underset{\Phi m = y_0}{\operatorname{argmin}}\; |m|(\mathcal{X}) \quad \text{with } y_0 = \Phi m_{a_0,x_0}. \tag{$\mathcal{P}_0(y_0)$}$$

BLASSO is genuinely linked with its discrete counterpart, the LASSO [11]: one can formally see BLASSO as the functional limit of LASSO on a finer and finer grid. If the LASSO problem exhibits existence and uniqueness of the solution, is it extendable to its off-the-grid counterpart? Foremost, let us observe that:

- $m \mapsto |m|(\mathcal{X})$ is lower semi-continuous w.r.t. the weak-$*$ convergence;
- Φ is continuous from the weak-$*$ topology of $\mathcal{M}(\mathcal{X})$ to the $\mathcal{H}$ weak topology.

Then thanks to convex analysis results, one can establish the existence of solutions to ($\mathcal{P}_\lambda(y)$) as proved in [6]. The difficulties also lie in the question of uniqueness of the solution and correct support recovery: in order to tackle these concerns, let us introduce several notions of convex analysis in the following subsection.

2.1.4 Dual problems and certificates

The BLASSO problem ($\mathcal{P}_\lambda(y)$) above is convex, thus one can define its dual problem [6, 17] which writes down for $p \in \mathcal{H}$:

$$\underset{\|\phi^* p\|_{\infty,\mathcal{X}} \leq 1}{\operatorname{argmax}} \; \langle y, p \rangle_{\mathcal{H}} - \frac{\lambda}{2}\|p\|_{\mathcal{H}}^2 \tag{$\mathcal{D}_\lambda(y)$}$$

Strong duality between ($\mathcal{P}_\lambda(y)$) and ($\mathcal{D}_\lambda(y)$) is proved in [6]. Hence, any solution m_λ of ($\mathcal{P}_\lambda(y)$) is linked [11] to the unique solution p_λ of ($\mathcal{D}_\lambda(y)$) by the extremality conditions:

$$\begin{cases} \Phi^* p_\lambda \in \partial|m_\lambda|(\mathcal{X}), \\ -p_\lambda = \dfrac{1}{\lambda}(\Phi m_\lambda - y) \end{cases} \tag{2}$$

where $\partial|\cdot|(\mathcal{X})$ is the sub-differential of the TV norm. Indeed and similarly to the ℓ^1 norm, the total variation is not differentiable but lower semi-continuous w.r.t. the weak-$*$-topology. Thus, we use its sub-differential [11] which identifies to, for $m \in \mathcal{M}(\mathcal{X})$:

$$\partial|m|(\mathcal{X}) = \left\{ \eta \in \mathscr{C}_0(\mathcal{X}) \,; \|\eta\|_{\infty,\mathcal{X}} \leq 1 \text{ and } \int_{\mathcal{X}} \eta \,\mathrm{d}m = |m|(\mathcal{X}) \right\}. \tag{3}$$

Elements of this subgradient are called *certificate*: then thanks to strong duality, let us define peculiar certificates named the *dual certificates* [7].

Definition 3. *We call $\eta_\lambda \stackrel{\text{def.}}{=} \Phi^* p_\lambda$ a* dual certificate *of m_λ, where p_λ satisfies* (2).

η_λ is a certificate since $\Phi^* p_\lambda \in \partial|m_\lambda|(\mathcal{X})$, it is called *dual* since it verifies the second extremality (2) condition: indeed it is defined by the dual solution p_λ. Loosely speaking, a dual certificate η_λ is associated to a measure m_λ and it *certifies* that the measure m_λ is a *minimum* of the BLASSO. If there exist for instance solutions of ($\mathcal{P}_\lambda(y)$) of the form $m_\lambda \stackrel{\text{def.}}{=} \sum_{i=1}^N a_i \delta_{x_i}$, the support satisfies [11] for all $0 \leq i \leq N : |\eta_\lambda|(x_i) = 1$. The interested reader can take a glance at [11] for uniqueness and support recovery guarantees.

We now extend this classical scalar off-the-grid formulation to the vector case, and present the space of divergence vector field.

2.2 The space of divergence vector field/charges

Consider the space of *vector* Radon measures:

Definition 4. *We define the set of vector Radon measures $\boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{X}})^2$ as the topological dual of the space of continuous vector functions $\boldsymbol{\mathscr{C}}_0(\boldsymbol{\mathcal{X}})^2 \stackrel{\text{def.}}{=} \mathscr{C}_0(\boldsymbol{\mathcal{X}}, \mathbb{R}^2)$. The properties of the scalar case hold for the vector one, indeed $\boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{X}})^2$ has a natural TV-norm denoted by $\|\cdot\|_{\mathrm{TV}^2}$, a duality bracket $\langle\cdot,\cdot\rangle_{\boldsymbol{\mathcal{M}}^2}$,* etc.

We only carry out $d = 2$, since some (wild) pathological cases appear when $d > 2$, see [24, Section 1.3]. Let us denote by div the divergence operator, which ought to be understood in the distributional sense. Indeed, for all $\boldsymbol{m} \in \boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{X}})^2$:

$$\forall \xi \in \mathscr{C}_0^\infty(\mathcal{X}), \quad \langle \operatorname{div} \boldsymbol{m}, \xi \rangle_{\mathscr{D}'(\mathcal{X}) \times \mathscr{C}_0^\infty(\mathcal{X})} = -\langle \boldsymbol{m}, \boldsymbol{\nabla} \xi \rangle_{\boldsymbol{\mathcal{M}}^2}.$$

We say that a measure $\boldsymbol{m}$ is *of finite divergence* if $\operatorname{div}(\boldsymbol{m}) \in \mathcal{M}(\mathcal{X})$. Let us now introduce the following useful space [24, 23].

Definition 5. *We denote by $\mathscr{V}$ the space of* divergence vector fields *or* charges*, namely the space of vector Radon measures with finite divergence:*

$$\mathscr{V} \stackrel{\text{def.}}{=} \left\{ \boldsymbol{m} \in \boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{X}})^2, \operatorname{div}(\boldsymbol{m}) \in \mathcal{M}(\mathcal{X}) \right\}.$$

It is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}^2} + \|\operatorname{div}(\cdot)\|_{\mathrm{TV}}$ (see Appendix A). A charge with null divergence is called a solenoid*.*

Obviously this topology is not adequate for convergence statement, indeed it does not provide any property for bounded set or sequence. Therefore, the idea is to choose a topology with fewer open sets but more compact sets, thus enabling compactness properties:

Definition 6 (Weak-$*$ convergence)**.** *$\mathscr{V}$ can be endowed with the weak-$*$ topology of the distributions; we say that a sequence of charges $(\boldsymbol{m}_n)_{n\in\mathbb{N}}$ weakly-$*$ converges to $\boldsymbol{m} \in \mathscr{V}$, denoted by $\boldsymbol{m}_n \stackrel{*}{\rightharpoonup} \boldsymbol{m}$, if:*

$$\forall \boldsymbol{g} \in \mathscr{C}_0^\infty(\mathcal{X}), \quad \int_{\mathcal{X}} \boldsymbol{g} \, \mathrm{d}\boldsymbol{m}_n \xrightarrow[n \to +\infty]{} \int_{\mathcal{X}} \boldsymbol{g} \, \mathrm{d}\boldsymbol{m}.$$

Note that we could equivalently define the weak-$*$ convergence with $\boldsymbol{g} \in \mathscr{C}_0(\mathcal{X})$, since $\mathscr{C}_0^\infty(\mathcal{X})$ is dense in $\mathscr{C}_0(\mathcal{X})$. While we have investigated several algebraic facts concerning $\mathscr{V}$, let us precise which maps live in this space.

Claim 1. *The* vector *Dirac measure $\boldsymbol{\delta} = (\delta, \delta)$ does not belong to $\mathscr{V}$, an interesting observation since Dirac measures are extreme points of the TV-norm unit ball.*

Proof. Indeed, let $\xi \in \mathscr{C}_0^\infty(\mathcal{X})$ such that $\partial_1 \xi(0) = \partial_2 \xi(0) = 1$ and the sequence:

$$\forall n \in \mathbb{N}^*, \forall x \in \mathcal{X}, \quad \xi_n(x) = \frac{1}{n} \xi(xn).$$

Then, by the dominated convergence theorem, for all measure $\mu \in \mathcal{M}(\mathcal{X})$ one yield $\lim_{n\to\infty} \int_{\mathcal{X}} \xi_n \, \mathrm{d}\mu = 0$. However, for all $n \in \mathbb{N}^*$ one has:

$$\int_{\mathcal{X}} \xi_n \, \mathrm{d}(\operatorname{div} \boldsymbol{\delta}) = -\int_{\mathcal{X}} \boldsymbol{\nabla} \xi_n \, \mathrm{d}\boldsymbol{\delta} = -\partial_1 \xi(0) - \partial_2 \xi(0) = -2$$

thus reaching a contradiction. □

Here is now an example of an element of $\mathscr{V}$, namely the *curve measure i.e.* a measure supported on a curve and defined through integration:

Definition 7 (Curve measure). *Let $\gamma : [0,1] \to \mathcal{X}$ a parametrised 1-rectifiable Lipschitz curve, we say that $\boldsymbol{\mu}_\gamma \in \mathscr{V}$* is a measure supported on the curve γ *if:*

$$\forall \boldsymbol{g} \in \mathscr{C}_0(\boldsymbol{\mathcal{X}})^2, \quad \langle \boldsymbol{\mu}_\gamma, \boldsymbol{g} \rangle_{\boldsymbol{\mathcal{M}}^2} \stackrel{\text{def.}}{=} \int_0^1 \boldsymbol{g}(\gamma(t)) \cdot \dot{\gamma}(t)\, \mathrm{d}t.$$

We denote by $\Gamma \stackrel{\text{def.}}{=} \gamma([0,1])$ the support of the curve.

Loosely speaking, the braket w.r.t. to a curve measure is the circulation [24] of a test vector field function along the curve γ. The rectifiability hypothesis can be understood as a finite length enforcement, indeed it is equivalent to $\mathscr{H}_1(\Gamma) < +\infty$. Please note that for the sake of conciseness, we indifferently refer to the curve measure $\boldsymbol{\mu}_\gamma$, or to the curve γ the measure is supported on. Let us introduce some properties that a curve might enjoy:

Definition 8 (Several characterisation of curves). *A curve is called* simple *if γ is an injective mapping. A curve is* closed *if $\gamma(0) = \gamma(1)$, it is called a* loop *if it is simple and closed (it is then homotopic to the unit circle of $\mathbb{R}^2$).*

By Sard's theorem for Lipschitz function, one can show that $\boldsymbol{\mu}_\gamma$ is independent of the curve parametrisation [2]; thus we assume that γ has a constant speed parametrisation (unless stated otherwise). Let us precise that a curve measure can either be seen through a rather functional standpoint (in the latter section), or through a set angle [2]. Indeed, for every measurable set $B \subset \mathcal{X}$, one has:

$$\boldsymbol{\mu}_\gamma(B) = \int_{\Gamma \cap B} \left(\sum_{t \in \gamma^{-1}(x)} \dot{\gamma}(t) \right) \mathrm{d}\, \mathscr{H}_1(x) \tag{4}$$

Eventually, here is the simple expression of the divergence of a curve.

Proposition 1 (Curve divergence). *Let $\boldsymbol{\mu}_\gamma$ be a measure supported on a curve γ, then* $\operatorname{div}(\boldsymbol{\mu}_\gamma) = \delta_{\gamma(0)} - \delta_{\gamma(1)}$. *In particular,* $\operatorname{div}(\boldsymbol{\mu}_\gamma) = 0$ *if γ is a closed curve.*

Proof. Let $\xi \in \mathscr{C}_0^\infty(\mathcal{X})$, then:

$$\begin{aligned}
\langle \operatorname{div} \boldsymbol{\mu}_\gamma, \xi \rangle_{\mathscr{D}'(\mathcal{X}) \times \mathscr{C}_0^\infty(\mathcal{X})} &= - \langle \boldsymbol{\mu}_\gamma, \boldsymbol{\nabla} \xi \rangle_{\mathcal{M}} \\
&= \int_0^1 \boldsymbol{\nabla} \xi(\gamma(t)) \cdot \dot{\gamma}(t)\, \mathrm{d}t \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} (\xi(\gamma(t))) \\
&= \xi(\gamma(0)) - \xi(\gamma(1)) \\
&= \langle \delta_{\gamma(0)} - \delta_{\gamma(1)}, \xi \rangle_{\mathscr{D}'(\mathcal{X}) \times \mathscr{C}_0^\infty(\mathcal{X})}.
\end{aligned}$$

Thus $\operatorname{div}(\boldsymbol{\mu}_\gamma) = \delta_{\gamma(0)} - \delta_{\gamma(1)}$ in the sense of distributions. □

Remark (A further insight). *A natural question would be:* 'is a measure supported on a curve always defined through integration, such as Definition (7)?'. *This concern arises also in the De Rham* *theory, where one considers a generalisation of measure supported on some submanifold of $\mathcal{X}$ called a* rectifiable current*: the total variation of a charge is the mass of the associated current, and its divergence amounts to the current manifold boundary [24]. In the same fashion, does a rectifiable current admit a definition by integration? Since we consider charges, so currents whose boundaries are still rectifiable currents, we bring more regularity and in some sense information to the considered measure: the current associated to a charge is then necessarily an integrable rectifiable current [18]. Shorthand: a charge of $\mathscr{V}$ supported on a curve is necessarily representable (up to a multiplicative factor) through integration, such as Definition 7.*

3 A variational problem on the space of charges

Similarly to the scalar case, one would want to define a BLASSO on $\mathscr{V}$. Let $\alpha > 0$, we present the following functional we coined CROC (standing for *Curves Represented On Charges*):

$$\underset{\boldsymbol{m}\in\mathscr{V}}{\operatorname{argmin}}\ T_\alpha(\boldsymbol{m}) \stackrel{\text{def.}}{=} \frac{1}{2}\|y - \boldsymbol{\Phi}\,\boldsymbol{m}\|_{\mathcal{H}}^2 + \alpha\|\boldsymbol{m}\|_{\mathscr{V}}. \tag{$\mathcal{Q}_\alpha(y)$}$$

$\boldsymbol{\Phi} : \mathscr{V} \to \mathcal{H}$ is linear and maps the divergence vector field to the acquisition space $\mathcal{H}$. We recall that $\|\boldsymbol{m}\|_{\mathscr{V}} \stackrel{\text{def.}}{=} \|\boldsymbol{m}\|_{\mathrm{TV}^2} + \|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}$, the several terms of our energy may be interpreted in the following sense:

- $\frac{1}{2}\|y - \boldsymbol{\Phi}\,\boldsymbol{m}\|_{\mathcal{H}}^2$ is the data term;
- $\|\boldsymbol{m}\|_{\mathrm{TV}^2}$ amounts to the length of the curve, indeed it is valued $\mathscr{H}_1(\Gamma)$ with Γ the image of the map γ. It may discard high frequency oscillating behaviour and smooth over the curvature of γ;
- $\|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}$ is the curve's counting term, thus enforcing sparsity of the solution.

A convenient improvement for ($\mathcal{Q}_\alpha(y)$) would be the penalisation of curves length and curves count with different weights, such as $\alpha, \beta > 0$ and the regularisation $\alpha\|\boldsymbol{m}\|_{\mathrm{TV}^2} + \beta\|\operatorname{div}\boldsymbol{m}\|_{\mathrm{TV}}$. This enhancement is out of the scope of this article, but it may be handy for future numerical implementation.

Remark. *Authors believe that there is a connection between the Beckman's problem [22, Section 4.2] in optimal transport theory and the latter optimisation problem. Indeed, the Beckman's problem as well is formulated over the space of Radon measures with a constraint on the divergence. Its solution reads as a transport map from a measure to another: in the case of scalar Dirac measures transport, the transport map is precisely a vector measure supported on a curve.*

Theorem 1. *The problem ($\mathcal{Q}_\alpha(y)$) admits solutions.*

Proof. The functional T_α is proper and coercive on $\mathscr{V}$. Let $(u_n)_{n\in\mathbb{N}}$ be a minimising sequence of T_α. Since T_α is coercive, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{V}$ so it converges in the weak-$*$ topology to some $\overline{u} \in \mathscr{V}$, up to a subsequence. Observe also that:

- $\|\cdot\|_{\mathscr{V}}$ is lower semi-continuous w.r.t. the weak-$*$ convergence;
- $\boldsymbol{\Phi}$ is continuous from $\boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{X}})^2$ weak-$*$ topology to $\mathcal{H}$ weak topology.

Hence, by the direct method of the calculus of variations, it is clear that $\overline{u}$ is a minimiser of T_α, thus proving the existence of a minimiser. $\square$

Remark. *The problem ($\mathcal{Q}_\alpha(y)$) enjoys uniqueness if $\mathbf{\Phi}$ is injective, since it compels T_α strictly convex. Note that this constitutes a rather strong hypothesis, hardly fulfilled in applications, such as for instance super-resolution.*

We have then proved the existence of a minimiser. Also consider:

Theorem 2. *Let $\boldsymbol{m}$ be a minimiser of ($\mathcal{Q}_\alpha(y)$). There exists a dual problem with minimiser $q^* \in \mathscr{C}_0(\mathcal{X})$ optimal, and extremality conditions read:*

$$\begin{cases} -\nabla q^* & \in \partial \|\boldsymbol{m}\|_{\mathrm{TV}^2} + \frac{1}{2}\|y - \mathbf{\Phi}\,\boldsymbol{m}\|_{\mathcal{H}}^2 \\ -q^* & \in \partial \|\operatorname{div} \boldsymbol{m}\|_{\mathrm{TV}}. \end{cases} \tag{5}$$

Proof. Consider the Ekeland-Temam duality from [12, Remark 4.2], with a little caveat: the Banach space $\mathscr{V}$ has to be reflexive, which is clearly not the case here. However, the reflexive hypothesis is only needed for the sake of the existence proof. Since we already proved the solution existence, this reflexivity hypothesis is not required in our case. Back to the Remark 4.2 of [12] stating, for $\Lambda : \mathscr{V} \to Y$ linear, $F : \mathscr{V} \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ convex, that the primal problem:

$$\inf_{\boldsymbol{u} \in \mathscr{V}} F(\boldsymbol{u}) + G(\Lambda \boldsymbol{u})$$

has a dual problem which writes down:

$$\sup_{p^* \in Y^*} -F^*(\Lambda^* p^*) - G^*(-p^*). \tag{6}$$

If $\boldsymbol{m}$ and p^* are respectively solutions of the primal and dual problem, the extremality conditions are:

$$\begin{cases} \Lambda^* p^* & \in \partial F(\boldsymbol{m}) \\ -p^* & \in \partial G(\Lambda \boldsymbol{m}). \end{cases}$$

Here we set the map:

$$\begin{array}{rccc} \Lambda : & \mathscr{V} & \longrightarrow & \mathcal{M}(\mathcal{X}) \\ & \boldsymbol{u} & \longmapsto & \operatorname{div} \boldsymbol{u} \end{array}$$

and its dual

$$\begin{array}{rccc} \Lambda^* : & \mathscr{C}_0(\mathcal{X}) & \longrightarrow & \mathscr{V}^* \\ & p^* & \longmapsto & -\nabla p^*. \end{array}$$

We also set for $\boldsymbol{u} \in \mathscr{V}$, $F(\boldsymbol{u}) \stackrel{\text{def.}}{=} \|\boldsymbol{u}\|_{\mathrm{TV}^2} + \frac{1}{2}\|y - \mathbf{\Phi}\,\boldsymbol{u}\|_{\mathcal{H}}^2$, and $G(q) \stackrel{\text{def.}}{=} \|q\|_{\mathrm{TV}}$ for $q \in \mathcal{M}(\mathcal{X})$. Then the dual problem writes down:

$$\underset{q^* \in \mathscr{C}_0(\mathcal{X})}{\operatorname{argmax}} -F^*(-\boldsymbol{\nabla} q^*) - G^*(-q^*)$$

Extremality conditions then boils down to q^* solution of the dual problem (6):

$$\begin{cases} \Lambda^* q^* & \in \partial F(\boldsymbol{m}) \\ -q^* & \in \partial G(\Lambda \boldsymbol{m}). \end{cases}$$

□

In the off-the-grid literature [11, 6, 10], the dual problem ($\mathcal{D}_\lambda(y)$) of the BLASSO ($\mathcal{P}_\lambda(y)$) is useful to establish both the existence of the solutions and to characterise the so-called certificate. The latter is crucial for the sake of numerical implementation, since it is involved in both measure support estimation step and stopping condition of state-of-the-art greedy algorithms [6, 10].

Corollary 1. *Let $\boldsymbol{m}$ be a minimiser of ($\mathcal{Q}_\alpha(y)$). Then, there exist certificates $\boldsymbol{\eta_1} \in \partial\|\boldsymbol{m}\|_{\mathrm{TV}^2}$ and $\boldsymbol{\eta_2} \in \partial\|\mu\|_{\mathrm{TV}}\big|_{\mu=\operatorname{div}\boldsymbol{m}}$ with $\boldsymbol{\eta_1}, \boldsymbol{\eta_2} \in \mathscr{C}_0(\boldsymbol{\mathcal{X}})^2$ such that:*

$$\langle \boldsymbol{m}, \boldsymbol{\eta_1} + \boldsymbol{\eta_2} \rangle_{\boldsymbol{\mathcal{M}}^2} = \|\boldsymbol{m}\|_{\mathscr{V}}.$$

Proof. Let us investigate the two optimality conditions by probing a bit more the extremality conditions:

- let us consider the former optimality condition $-\boldsymbol{\nabla} q^* \in \partial F(\boldsymbol{m})$. In this case the sub-differential of the sum is the sum of the sub-differential, meaning:

$$\partial F(\boldsymbol{m}) = \partial\|y - \boldsymbol{\Phi}(\cdot)\|^2_{\mathcal{H}}(\boldsymbol{m}) + \partial\|\cdot\|_{\mathrm{TV}^2}(\boldsymbol{m})$$

  by [12]. We reach, using the closed form of the least square norm:

$$\partial\|y - \boldsymbol{\Phi}(\cdot)\|^2_{\mathcal{H}}(\boldsymbol{m}) = \{-\boldsymbol{\Phi}^*(y - \boldsymbol{\Phi}\,\boldsymbol{m})\}$$

  Then :

$$\begin{cases} -\boldsymbol{\nabla} q^* = \boldsymbol{\eta_1} - \boldsymbol{\Phi}^*(y - \boldsymbol{\Phi}\,\boldsymbol{m}) \\ \boldsymbol{\eta_1} \in \partial\|\boldsymbol{m}\|_{\mathrm{TV}^2}. \end{cases}$$

  The latter $\boldsymbol{\eta_1} \in \partial\|\boldsymbol{m}\|_{\mathrm{TV}^2}$, with the result in equation (3) adapted to the vector case, amounts to:

$$\begin{cases} \|\boldsymbol{\eta_1}\|_{\infty,\mathcal{X}} \leq 1 \\ \|\boldsymbol{m}\|_{\mathrm{TV}^2} = \langle \boldsymbol{m}, \boldsymbol{\eta_1} \rangle_{\boldsymbol{\mathcal{M}}^2}. \end{cases}$$

- let us consider the latter optimality condition $-q^* \in \partial G(\operatorname{div} \boldsymbol{m})$. We note $\nu \stackrel{\text{def.}}{=} \operatorname{div} \boldsymbol{m}$. Then, as stated before in equation (3) and since $G \stackrel{\text{def.}}{=} \|\cdot\|_{\mathrm{TV}}$:

$$-q^* \in \partial G(\nu) \iff \begin{cases} q^* \in \mathscr{C}_0(\mathcal{X}) \\ \|q^*\|_{\infty,\mathcal{X}} \leq 1 \\ \langle -q^*, \nu \rangle_{\mathcal{M}} = \|\nu\|_{\mathrm{TV}}. \end{cases}$$

But, while denoting $\boldsymbol{\eta_2} \stackrel{\text{def.}}{=} \boldsymbol{\nabla} q^*$:

$$\|\operatorname{div} \boldsymbol{m}\|_{\mathrm{TV}} = \langle -q^*, \operatorname{div} \boldsymbol{m} \rangle_{\mathcal{M}} = \langle \boldsymbol{\nabla} q^*, \boldsymbol{m} \rangle_{\boldsymbol{\mathcal{M}}^2} = \langle \boldsymbol{\eta_2}, \boldsymbol{m} \rangle_{\boldsymbol{\mathcal{M}}^2}.$$

Finally, if we merge these two results, we yield:

$$\begin{aligned} \|\boldsymbol{m}\|_{\mathrm{TV}^2} + \|\operatorname{div} \boldsymbol{m}\|_{\mathrm{TV}^2} &= \langle \boldsymbol{m}, \boldsymbol{\eta_1} \rangle_{\boldsymbol{\mathcal{M}}^2} + \langle \boldsymbol{\eta_2}, \boldsymbol{m} \rangle_{\boldsymbol{\mathcal{M}}^2} \\ &= \langle \boldsymbol{m}, \boldsymbol{\eta_1} + \boldsymbol{\eta_2} \rangle_{\boldsymbol{\mathcal{M}}^2}, \end{aligned}$$

thus ensuring $\|\boldsymbol{m}\|_{\mathscr{V}} = \langle \boldsymbol{m}, \boldsymbol{\eta_1} + \boldsymbol{\eta_2} \rangle_{\boldsymbol{\mathcal{M}}^2}$. □

We have then specified the optimality conditions and defined the certificate $\boldsymbol{\eta_\alpha} = \boldsymbol{\eta_1} + \boldsymbol{\eta_2} \in \mathscr{C}_0(\boldsymbol{\mathcal{X}})^2$. This vector map has several applications from both theoretical and numerical standpoint: indeed, as one works out an algorithm based on the Frank-Wolfe algorithm [14], one needs for instance a stopping condition based on the criterion $\|\boldsymbol{\eta_\alpha}\|_{\infty,\mathcal{X}} \leq 1$.

However, we still lack a more precise description of ($\mathcal{Q}_\alpha(y)$) minimisers. To tackle this issue, let us introduce the following representation theorem.

We strongly advise the reader to notice that the choice of F and G in the further section will be different. Indeed, the choice in the latter section was only made for the sake of easing the dual problem computation. Let $n \in \mathbb{N}^*$, $\mathcal{H}_n$ be the finite dimensional n space of acquisitions, $G : \mathcal{H}_n \to \mathbb{R}$ an arbitrary data fitting term; suppose that $\Lambda : \mathscr{V} \to \mathcal{H}_n$ is linear and $F : \mathscr{V} \to \mathbb{R}$ is convex. Consider the general setting for $\boldsymbol{m} \in \mathscr{V}$:

$$J(\boldsymbol{m}) \stackrel{\text{def.}}{=} G(\Lambda \boldsymbol{m}) + F(\boldsymbol{m}) \tag{7}$$

The unit ball of F is denoted $\mathcal{B}_F \stackrel{\text{def.}}{=} \{\boldsymbol{u} \in \mathscr{V}, F(\boldsymbol{u}) \leq 1\}$ of the regulariser F. The following theorem, due to [4, 5], establish (up to some hypothesis) the link between the minimisers of the functional J in (7) and the extreme points (see 4 and Definition 9) of $\mathcal{B}_F$ denoted $\operatorname{Ext}(\mathcal{B}_F)$:

Theorem 3 (Representer theorem). *If F is semi-norm, there exists $\overline{u} \in \mathscr{V}$, a minimiser of (7) with the representation:*

$$\overline{u} = \sum_{i=1}^{p} \alpha_i u_i$$

where $p \leq \dim \mathcal{H}_n$, $u_i \in \operatorname{Ext}(\mathcal{B}_F)$ and $\alpha_i > 0$ with $\sum_{i=1}^{p} \alpha_i = F(\overline{u})$.

It amounts to the characterisation of minimisers' structural properties, without actually needing to solve the problem [4]. Moreover, the description of the extreme points of the regulariser's unit ball is fundamental for the numerical implementation: indeed, the Frank-Wolfe algorithm recovers a solution by iteratively adding F unit ball extreme points to the reconstructed measure. In the following, it rather makes sense to consider $F : \boldsymbol{m} \mapsto \|\boldsymbol{m}\|_{\mathscr{V}}$, and obviously the data term $G : p \mapsto \|y - p\|_{\mathcal{H}}^2$ and $\Lambda = \boldsymbol{\Phi}$.

4 Extreme points of the $\mathscr{V}$-norm's unit ball are curves

First, we recall some definitions and convenient results concerning the extreme points.

Definition 9. *Let X be a topological vector space and $K \subset X$. An* extreme point *x of K is a point such that $\forall y, z \in K$:*

$$\forall \lambda \in (0,1), \qquad x = \lambda y + (1-\lambda) z \implies x = y = z$$

The set of extreme points of K is denoted by $\operatorname{Ext} K$.

We further introduce the celebrated Krein-Milman theorem, stating that if K convex and compact, the closed convex hull of the set of the extreme points of K coincides with K.

Theorem 4 (Krein-Milman theorem)**.** *If $K \subset X$ is a non-empty, compact and convex set, then $K = \overline{\operatorname{co}}(\operatorname{Ext}(K))$.*

Also note the following refinement of Krein-Milman, stating the decomposition of any point of the space onto a 'combination' of extreme points, through a concept [19] called the Choquet integral:

Theorem 5 (Choquet theorem)**.** *Let X be a metrisable topological vector space and K a non-empty, convex and compact subset. Then for any $x \in K$ there exists a Borel measure ρ on X, concentrated on $\operatorname{Ext} K$ and satisfying:*

$$x = \int_{\operatorname{Ext} K} y \, \mathrm{d}\rho(y)$$

namely for every $\omega : X \to \mathbb{R}$ linear and continuous, $\omega(x) = \int_{\operatorname{Ext} K} \omega(y) \, \mathrm{d}\rho(y)$

These notions lie in a very general framework, let us precise some theoretical tools pertaining to the space of charges for our main theorem. Consider the following [24]:

Definition 10. *Let $\boldsymbol{T} \in \mathscr{V}$ and $J \subset \mathscr{V}$, we say that $\boldsymbol{T}$* decomposes *into charges lying in J if there exists a finite Borel measure ρ on J such that:*

$$\boldsymbol{T} = \int_J \boldsymbol{R} \, \mathrm{d}\rho(\boldsymbol{R})$$
$$\|\boldsymbol{T}\|_{\mathrm{TV}^2} = \int_J \|\boldsymbol{R}\|_{\mathrm{TV}^2} \, \mathrm{d}\rho(\boldsymbol{R})$$

We say that $\boldsymbol{T}$ completely decomposes *into charges lying in J if the latter conditions hold and moreover if:*

$$\|\operatorname{div} \boldsymbol{T}\|_{\mathrm{TV}} = \int_J \|\operatorname{div} \boldsymbol{R}\|_{\mathrm{TV}} \, \mathrm{d}\rho(\boldsymbol{R})$$

In our case we can use the following sets:

Definition 11 (Curve measures set)**.** *We denote by $\mathfrak{S}$ the space of curve measures, supported on either open or closed simple ones, endowed with weak-$*$ topology:*

$$\mathfrak{S} = \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \, \gamma \textit{ is a 1-rectifiable simple oriented Lipschitz curve} \right\}.$$

It is a metric (non-complete) space for the weak-$*$ topology. We also denote by $\mathfrak{S}_{\text{loop}}$ the space of curve measures, supported on loops:

$$\mathfrak{S}_{\text{loop}} = \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \, \gamma \textit{ is a 1-rectifiable simple closed oriented Lipschitz curve} \right\}.$$

Eventually we note $\mathfrak{S}_{\text{open}} \stackrel{\text{def.}}{=} \mathfrak{S} \backslash \mathfrak{S}_{\text{loop}}$ the space of open curves.

We now recall the following fundamental results of [24, 16]. Remember that a solenoid is a charge of $\mathscr{V}$ with null divergence:

Theorem 6 (Solenoid decomposition theorem [16, Remark 1.17])**.** *Any solenoid $\boldsymbol{T} \in \mathscr{V}$ can be decomposed into elements of $\mathfrak{S}_{\text{loop}}$.*

Let us stress out that this latter theorem only holds in the $d = 2$ case; indeed there exists several counterexamples for $d > 2$, see [24] for various illustrations. The following theorem holds in any dimension:

Theorem 7 (Smirnov's Theorem C [24])**.** *Any charge $\boldsymbol{T} \in \mathscr{V}$ is a sum of two charges $\boldsymbol{P}$ and $\boldsymbol{Q}$ such that* $\operatorname{div} \boldsymbol{P} = 0$ *(Theorem (6) can then be applied) and $\boldsymbol{Q}$ is completely decomposable into measures supported on simple oriented curves.*

We now present the main result of this paper, namely the theorem establishing a link between curve measures and extreme points of the unit ball of the regulariser (*i.e.* the $\mathscr{V}$-norm).

Theorem 8 (Main result)**.** *Let us denote by $\mathcal{B}^1_{\mathscr{V}} \stackrel{\text{def.}}{=} \{\boldsymbol{m} \in \mathscr{V}, \|\boldsymbol{m}\|_{\mathscr{V}} \leq 1\}$ the unit ball of the norm of $\mathscr{V}$, weakly-$*$ compact. Then one has:*

$$\operatorname{Ext}(\mathcal{B}^1_{\mathscr{V}}) = \mathfrak{S} = \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \, \gamma \textit{ is a 1-rectifiable simple oriented Lipschitz curve} \right\}.$$

Proof. We start by proving

$$\operatorname{Ext}(\mathcal{B}^1_{\mathscr{V}}) \supset \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \, \gamma \text{ is a 1-rectifiable simple oriented Lipschitz curve} \right\}.$$

Let $\gamma : [0,1] \to \mathcal{X}$ be a Lipschitz 1-rectifiable simple oriented curve of length $\ell > 0$, with image $\Gamma = \gamma([0,1])$, and $\boldsymbol{\mu}_\gamma$ the measure supported on this curve. Note that it is either an open curve or a closed curve. By contradiction, let us suppose that there exists $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathscr{V}$ such that $\|\boldsymbol{u_1}\|_{\mathscr{V}} \leq 1$, $\|\boldsymbol{u_2}\|_{\mathscr{V}} \leq 1$ and for $\lambda \in (0,1)$:

$$\frac{\mu_\gamma}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} = \lambda \boldsymbol{u_1} + (1-\lambda)\boldsymbol{u_2}. \tag{8}$$

For every $A \subset \mathcal{X}$ measurable, while denoting $|\cdot|_V \overset{\text{def.}}{=} |\cdot|(A) + |\text{div}\cdot|(A)$, one has:

$$\frac{1}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} \left|\mu_\gamma\right|_V(A) = \lambda|\boldsymbol{u_1}|_V(A) + (1-\lambda)|\boldsymbol{u_2}|_V(A). \tag{9}$$

Indeed, following a proof from [5, Theorem 4.7], if there exists $A \subset \mathcal{X}$ such that $\lambda|\boldsymbol{u_1}|_V(A) + (1-\lambda)|\boldsymbol{u_2}|_V(A) > \frac{|\mu_\gamma|_V(A)}{\|\mu_\gamma\|_{\mathcal{V}}}$ then we get

$$\begin{aligned}
1 &= \frac{\left|\mu_\gamma\right|_V(\mathcal{X})}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} \\
&= \frac{\left|\mu_\gamma\right|_V(A)}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} + \frac{\left|\mu_\gamma\right|_V(A^c)}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} \\
&< \lambda|\boldsymbol{u_1}|_V(A) + (1-\lambda)|\boldsymbol{u_2}|_V(A) + \frac{\left|\mu_\gamma\right|_V(A^c)}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} \\
&\leq \lambda|\boldsymbol{u_1}|_V(A) + (1-\lambda)|\boldsymbol{u_2}|_V(A) + \lambda|\boldsymbol{u_1}|_V(A^c) + (1-\lambda)|\boldsymbol{u_2}|_V(A^c) \\
&\leq 1.
\end{aligned}$$

So we reach $1 < 1$ which yields a contradiction, then:

$$\lambda|\boldsymbol{u_1}|_V(A) + (1-\lambda)|\boldsymbol{u_2}|_V(A) \leq \frac{1}{\left\|\mu_\gamma\right\|_{\mathcal{V}}} \left|\mu_\gamma\right|_V(A).$$

Since we always have the opposite inequality, we get (9). We further deduce from equation (9) that:

$$\|\boldsymbol{u_1}\|_{\mathcal{V}} = \|\boldsymbol{u_2}\|_{\mathcal{V}} = 1. \tag{10}$$

Now let $A \subset \mathcal{X}$ be a Borel set such that $A \cap \Gamma = \varnothing$, then from (9) we deduce:

$$0 = \lambda|\boldsymbol{u_1}|_{\mathcal{V}}(A) + (1-\lambda)|\boldsymbol{u_2}|_{\mathcal{V}}(A) \implies \boldsymbol{u_1}(A) = \boldsymbol{u_2}(A) = 0.$$

Let $i \in \{1,2\}$, then if $A_i = \text{spt}(u_i)$ it follows $A_i \cap \Gamma \neq \varnothing$.

Let us study the case of partly disjoint support between u_i and μ_γ. Consider, as illustrated in the Figure 2:

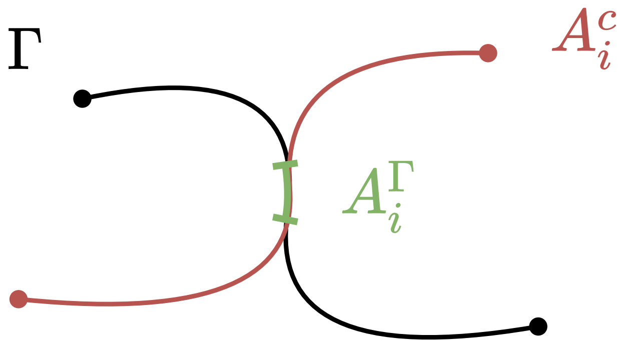

Figure 2: Illustration of $A_i^\Gamma \overset{\text{def.}}{=} A_i \cap \Gamma$ and its counterpart $A_i^c \overset{\text{def.}}{=} A_i \backslash A_i^\Gamma$.

$$\begin{aligned} A_i^\Gamma &= A_i \cap \Gamma \\ A_i^c &= A_i \backslash A_i^\Gamma. \end{aligned}$$

Suppose that $A_i^c \neq \varnothing$. Let $A = A_i^c$ in (9), then:

$$0 = \lambda \|\boldsymbol{u_1}\|_{\mathcal{V}}(A_i^c) + (1-\lambda)\|\boldsymbol{u_2}\|_{\mathcal{V}}(A_i^c).$$

Then in the case $i = 1$,

$$0 = \lambda \underbrace{\|\boldsymbol{u_1}\|_{\mathcal{V}}(A_i^c)}_{>0} + (1-\lambda) \underbrace{\|\boldsymbol{u_2}\|_{\mathcal{V}}(A_i^c)}_{\geq 0}$$

thus reaching a contradiction. Exactly the same applies to $i = 2$, then we deduce that $A_i \subset \Gamma$, *i.e.* $\operatorname{spt}(u_i) \subset \Gamma$. Let us now study the cases where $\operatorname{spt}(u_i) \subsetneq \Gamma$. We give in the Figure 3 the several cases encountered in the following proof.

Let us now discuss each case:

(a) Suppose that $A_1 \cup A_2 \subsetneq \Gamma$; and A_1, A_2 are disjoint, *i.e.* $A_1 \cap A_2 = \varnothing$. Set $A = \Gamma \backslash (A_1 \cup A_2)$ and by (9):

$$0 < \left\| \frac{\boldsymbol{\mu_\gamma}}{\left\|\boldsymbol{\mu_\gamma}\right\|_{\mathcal{V}}} \right\|_{\mathcal{V}} (A) = 0 + 0$$

thus reaching a contradiction. Then, $A_1 \cap A_2 \neq \varnothing$. Consider now the cases where A_1 and A_2 are not disjoint.

(b) Overlap between A_1 and A_2, but $A_1 \cup A_2 \subsetneq \Gamma$. As before, let us choose $A = \Gamma \backslash (A_1 \cup A_2)$. Then the same contradiction is yielded:

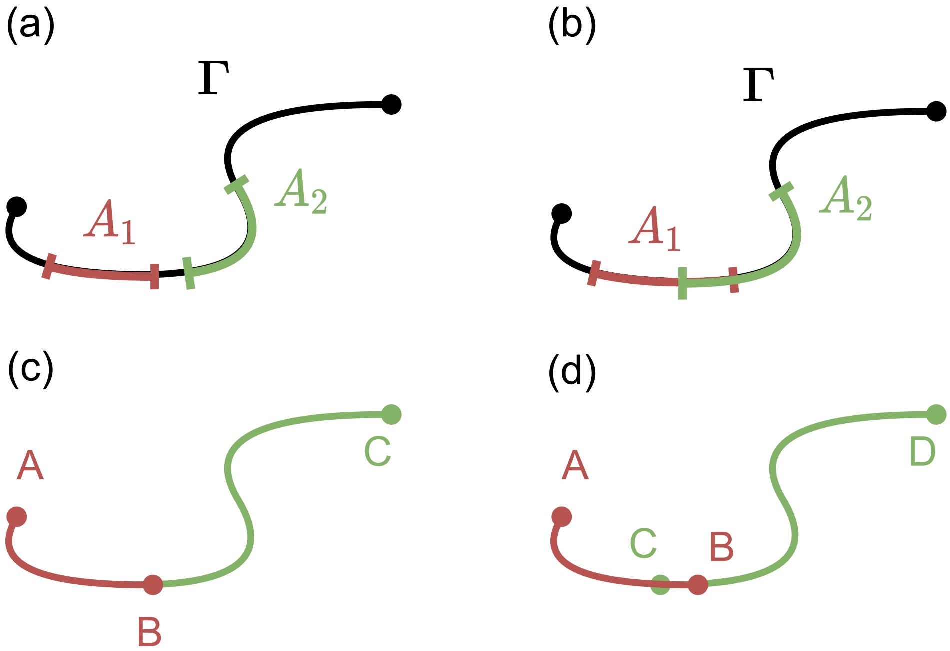

Figure 3: (a) $A_1 \cup A_2 \neq \Gamma$ and no overlap (b) $A_1 \cup A_2 \neq \Gamma$ and overlap (c) $A_1 \cup A_2 = \Gamma$ and no overlap (d) $A_1 \cup A_2 = \Gamma$ and overlap.

$$0 < \left\| \frac{\mu_\gamma}{\left\|\mu_\gamma\right\|_{\mathscr{V}}} \right\|_{\mathscr{V}} (A) = 0 + 0.$$

It follows that $A_1 \cup A_2 = \Gamma$.

(c) No overlap between A_1 and A_2 and $A_1 \cup A_2 = \Gamma$. Suppose that γ is an open curve, let us recall that $\left\|\mu_\gamma\right\|_{\mathscr{V}} = \ell(\gamma) + 2$. Now set $A = A_1$ then $A = A_2$ to reach:

$$\left| \frac{\mu_\gamma}{\left\|\mu_\gamma\right\|_{\mathscr{V}}} \right|_{\mathscr{V}} (A_1) = \frac{\mathscr{H}_1(A_1) + 2}{\ell + 2} = \lambda |\boldsymbol{u_1}|_{\mathscr{V}} (A_1) = \lambda$$

$$\left| \frac{\mu_\gamma}{\left\|\mu_\gamma\right\|_{\mathscr{V}}} \right|_{\mathscr{V}} (A_2) = \frac{\mathscr{H}_1(A_2) + 2}{\ell + 2} = (1-\lambda) |\boldsymbol{u_2}|_{\mathscr{V}} (A_2) = 1 - \lambda.$$

Thus,

$$\begin{aligned}\lambda+1-\lambda &= \frac{(\mathscr{H}_1(A_1)+2)}{\ell+2}+\frac{\mathscr{H}_1(A_2)+2}{\ell+2}\\ \implies 1 &= \frac{\mathscr{H}_1(A_1)+\mathscr{H}_1(A_2)+4}{\ell+2}\\ \implies \ell &= \underbrace{\mathscr{H}_1(A_1)+\mathscr{H}_1(A_2)}_{=\ell}+2.\end{aligned}$$

Thus yielding a contradiction. The same apply for $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ closed or $\boldsymbol{u_1}$ open and $\boldsymbol{u_2}$ closed. Suppose now that $\boldsymbol{\mu_\gamma}$ is closed. One can easily conclude if one of the curves is open, or if both are open, thanks to (9). Suppose now that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are closed. Then we conclude, since $\boldsymbol{\mu_\gamma}$ is simple, thus not allowing self-intersection and thus yielding a contradiction.

(d) Overlap between A_1 and A_2 and $A_1 \cup A_2 = \Gamma$. Let $\Gamma_1 \stackrel{\text{def.}}{=} \widehat{AC}$, $\Gamma_2 \stackrel{\text{def.}}{=} \widehat{CD}$ respectively be the support of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$; for A, B, C and $D \in \mathcal{X}$. Let us evaluate (9) at Γ_1 then Γ_2 to get:

$$\left|\frac{\boldsymbol{\mu_\gamma}}{\left\|\boldsymbol{\mu_\gamma}\right\|_{\mathcal{V}}}\right|_{\mathcal{V}}(\Gamma_1) = \lambda|\boldsymbol{u_1}|_{\mathcal{V}}(\Gamma_1)+0 < \lambda|\boldsymbol{u_1}|_{\mathcal{V}}(\mathcal{X}) = \lambda$$
$$\left|\frac{\boldsymbol{\mu_\gamma}}{\left\|\boldsymbol{\mu_\gamma}\right\|_{\mathcal{V}}}\right|_{\mathcal{V}}(\Gamma_2) = (1-\lambda)|\boldsymbol{u_2}|_{\mathcal{V}}(\Gamma_2)+0 < (1-\lambda)|\boldsymbol{u_2}|_{\mathcal{V}}(\mathcal{X}) = 1-\lambda$$

and by summing up we reach $1 = \lambda|\boldsymbol{u_1}|_{\mathcal{V}}(\Gamma_1)+(1-\lambda)|\boldsymbol{u_2}|_{\mathcal{V}}(\Gamma_2) < 1$, thus reaching a contradiction.

Since all these cases are not possible, one has $\operatorname{spt}(\boldsymbol{u_1}) = \operatorname{spt}(\boldsymbol{u_2}) = \Gamma$. One would now prove that $\boldsymbol{u_1} = \boldsymbol{u_2} = \boldsymbol{\mu_\gamma}$, quite a non-trivial step: indeed think of $\boldsymbol{u_1} = 2\boldsymbol{\mu_\gamma}$; while the support is common, one clearly has $\boldsymbol{u_1} \neq \boldsymbol{\mu_\gamma}$. Consider $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, each with the same support as $\boldsymbol{\mu_\gamma}$. Suppose that either $\boldsymbol{u_1} \neq \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathcal{V}}}$ or $\boldsymbol{u_2} \neq \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathcal{V}}}$. Then, consider both cases:

- Let us suppose that $\boldsymbol{\mu_\gamma} \in \mathfrak{S}_{\text{open}}$. Since $\operatorname{spt}(\boldsymbol{u_i}) = \Gamma$, one has the decomposition of $\boldsymbol{u_i}$ through simple open curves, namely $\boldsymbol{u_i} = \int_{\mathfrak{S}_{\text{open}}} \boldsymbol{R}\,\mathrm{d}\rho_i(\boldsymbol{R})$ with ρ_i zeroing out if evaluated on curves with support not included in Γ. Then look out on the following, deduced by linearity of the divergence:

$$\|\operatorname{div}(\lambda\boldsymbol{u_1}+(1-\lambda)\boldsymbol{u_2})\|_{\mathrm{TV}} = \int_{\mathfrak{S}_{\text{open}}} \|\operatorname{div} R\|_{\mathrm{TV}}\,(\lambda\mathrm{d}\rho_1(R)+(1-\lambda)\mathrm{d}\rho_2(R)). \tag{11}$$

Since $R \in \mathfrak{S}_{\text{open}}$, we deduce $\|\operatorname{div} R\|_{\mathrm{TV}} = 2$. Then one has by application of divergence on both sides of equation (8):

$$\frac{2}{2+\ell} = 2\int_{\mathfrak{S}_{\text{open}}} (\lambda\mathrm{d}\rho_1+(1-\lambda)\mathrm{d}\rho_2)(R) = 2$$

since ρ_i are probability measures[2], thus yielding a contradiction.

- Let us suppose that $\boldsymbol{\mu}_\gamma \in \mathfrak{S}_{\text{loop}}$. In the same fashion, (11) boils down to:

$$\begin{aligned} 0 = 2 \times \int_{\mathfrak{S}_{\text{loop}}} (\lambda \,\mathrm{d}\rho_1(\boldsymbol{R}) + (1-\lambda)\,\mathrm{d}\rho_2(\boldsymbol{R})) \\ + 0 \times \int_{\mathfrak{S}_{\text{open}}} (\lambda \,\mathrm{d}\rho_1(\boldsymbol{R}) + (1-\lambda)\,\mathrm{d}\rho_2(\boldsymbol{R})) > 0 \end{aligned}$$

since $\mathfrak{S}_{\text{open}}$ is not a ρ_1-negligible (resp. ρ_2-negligible) set of $\mathscr{V}$, thus reaching a contradiction.

It ensues that $\lambda = 0$ or $\lambda = 1$, then yielding $\boldsymbol{u_1} = \boldsymbol{u_2} = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}$, and thus concluding this part.

We now prove the converse, namely:

$$\mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}}) \subset \left\{ \frac{\boldsymbol{\mu}_\gamma}{\left\|\boldsymbol{\mu}_\gamma\right\|_{\mathscr{V}}}, \gamma \text{ is a 1-rectifiable simple oriented Lipschitz curve} \right\}.$$

Let $\boldsymbol{T} \in \mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}})$ be an extreme point of the unit ball norm of $\mathscr{V}$, one has $\|\boldsymbol{T}\|_{\mathscr{V}} = 1$ and there exists[3] a finite Borel measure ρ such that

$$\boldsymbol{T} = \int_{\mathfrak{S}} \boldsymbol{R} \,\mathrm{d}\rho(\boldsymbol{R})$$

by Smirnov's decomposition in Theorem (7) [24, 16, 3]. Let us observe that:

$$\|\boldsymbol{T}\|_{\mathscr{V}} = \int_{\mathfrak{S}} \underbrace{\|\boldsymbol{R}\|_{\mathscr{V}}}_{=1} \,\mathrm{d}\rho(\boldsymbol{R}) = \rho(\mathfrak{S}).$$

Since $\boldsymbol{T} \in \mathrm{Ext}(\mathcal{B}^1_{\mathscr{V}})$, one has $\|\boldsymbol{T}\|_{\mathscr{V}} = 1$ thus reaching $\rho(\mathfrak{S}) = 1$. If ρ is an atomic measure, then it is supported on a singleton of $\mathfrak{S}$ [15, Theorem 2] and the proof would be achieved: there would exist a 1-rectifiable simple Lipschitz curve γ such that $\frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}} \in \mathfrak{S}$ and $\rho = \delta_{\mu_\gamma / \|\mu_\gamma\|_{\mathscr{V}}}$ hence $\boldsymbol{T} = \frac{\mu_\gamma}{\|\mu_\gamma\|_{\mathscr{V}}}$. By contradiction, let us suppose that ρ is a non-atomic measure. Then, there exists a measurable set $A \subset \mathfrak{S}$ such that $0 < \rho(A) < 1$ [20, Proposition A.1] and:

$$\rho = |\rho|(A) \left(\frac{1}{|\rho|(A)} \rho \llcorner A \right) + |\rho|(A^c) \left(\frac{1}{|\rho|(A^c)} \rho \llcorner A^c \right).$$

We recall that $\rho \llcorner A$ denotes the restriction of the measure ρ to the set A, namely for all measurable set B: $\rho \llcorner A(B) \overset{\text{def.}}{=} \rho(A \cap B)$. Hence,

[2] Without loss of generality, we can suppose for $i \in \{1, 2\}$ that $\rho_i(\mathfrak{S}_{\text{open}}) = 1$.

[3] Actually, this is allowed by the decomposition $\boldsymbol{T} = \boldsymbol{P} + \boldsymbol{Q}$ from Theorem 7 and $d = 2$. Hence, both $\boldsymbol{P}$ and $\boldsymbol{Q}$ decompose on $\mathfrak{S}$ and their respective Borel measures can be summed up to reach a unique Borel measure ρ concentrated on $\mathfrak{S}$.

$$\boldsymbol{T} = |\rho|(A) \underbrace{\left[\int_{\mathfrak{S}} \frac{1}{|\rho|(A)} \boldsymbol{R} \, \mathrm{d}(\rho \llcorner A)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_1}} + |\rho|(A^c) \underbrace{\left[\int_{\mathfrak{S}} \frac{1}{|\rho|(A^c)} \boldsymbol{R} \, \mathrm{d}(\rho \llcorner A^c)(\boldsymbol{R})\right]}_{\stackrel{\text{def.}}{=} \boldsymbol{u_2}}.$$

Consider now, thanks to Smirnov's decomposition:

$$\begin{aligned} \|\boldsymbol{u_1}\|_{\mathscr{V}} &= \int_{\mathfrak{S}} \frac{1}{|\rho|(A)} \underbrace{\|\boldsymbol{R}\|_{\mathscr{V}}}_{=1} \mathrm{d}(\rho \llcorner A)(\boldsymbol{R}) \\ &= \frac{1}{|\rho|(A)} \times \int_{\mathfrak{S}} \mathrm{d}(\rho \llcorner A)(\boldsymbol{R}) \\ &= \frac{|\rho|(A)}{|\rho|(A)} = 1. \end{aligned}$$

Exactly the same applies to $\boldsymbol{u_2}$, thus ensuring that $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{B}^1_{\mathscr{V}}$. By setting $\lambda \stackrel{\text{def.}}{=} |\rho|(A) = \rho(A)$ since ρ is a probability measure, and noting that $|\rho|(A^c) = 1 - |\rho|(A)$, we yield a non-trivial convex-combination of $\boldsymbol{T}$ through:

$$\boldsymbol{T} = \lambda \boldsymbol{u_1} + (1 - \lambda) \boldsymbol{u_2},$$

thus reaching a contradiction, and therefore concluding the proof. □

Remark. *One can see Smirnov's theorems A and C of [24] as a refined Choquet integral result. Indeed, the article yields the same conclusions (with some extensions) as the application of Choquet theorem,* i.e. *every charge decomposes as a weighted average of the set of extreme points $\mathfrak{S}$.*

We have then successfully established a result on extreme points, thus enabling a promising avenue for numerical implementation. Indeed, the state-of-the-art algorithms in the off-the-grid literature boil down to the iterative reconstruction of a linear sum of extreme points of the energy regulariser.

5 Outlook

This article performed the analysis of the space of divergence vector fields $\mathscr{V}$, and proposed a new optimisation problem ($\mathcal{Q}_\alpha(y)$) based on an off-the-grid functional called CROC. Existence of CROC minimisers and characterisation of the associated certificates have been established. Moreover, the extreme point theorem 8, coupled with the representer theorem 3, allows a precise analysis of the structure of ($\mathcal{Q}_\alpha(y)$) minimisers.

In further works, we plan to exploit this theoretical result through an adaptation of the (Sliding) Frank-Wolfe algorithm [6, 10] suited for curves. Indeed, this greedy algorithm reconstructs an iterated linear combination of extreme points, hence offering a tractable approach for curves reconstruction in an off-the-grid fashion.

A $\mathscr{V}$ is a Banach space

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{V}$. By definition, it is also a Cauchy sequence in $\mathcal{M}(\mathcal{X})^2$, which is complete therefore it admits a limit $u \in \mathcal{M}(\mathcal{X})^2$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded (as a Cauchy sequence) for the TV-topology in $\mathscr{V}$ and by lower semi-continuity of $\|\mathrm{div}(\cdot)\|_{\mathrm{TV}}$, then:

$$\|\mathrm{div}\, u\|_{\mathrm{TV}} \leq \liminf_{n\to+\infty} \|\mathrm{div}\, u_n\|_{\mathrm{TV}} < +\infty$$

and since $u \in \mathcal{M}(\mathcal{X})^2$, one has $\|u\|_{\mathscr{V}} = \|\mathrm{div}\, u\|_{\mathrm{TV}} + \|u\|_{\mathrm{TV}^2} < +\infty$. Hence $u \in \mathscr{V}$, thus ensuring that $\mathscr{V}$ is a Banach space.

Acknowledgments

The work of Bastien Laville has been supported by the French government, through the UCA DS4H Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-17-EURE-0004. The work of Laure Blanc-Féraud has been supported by the French government, through the 3IA Cote d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA-0002. We would like to thank both Ms Vasiliki Stregipoulou and Mr Sébastien Schaub for their help on the *Ostreopsis* SOFI imaging; Mr Théo Bertrand for his helpful remarks on the draft and his advice on the blood vessel ULM imaging.

References

[1] C. Arthur, B. Heiles, H. Vincent, Lopez Pauline, Eliott Teston, and C. Olivier, *Opulm pala*, 2020, https://doi.org/10.5281/ZENODO.4343435.

[2] L. Baratchart, D. Hardin, and C. Villalobos-Guillén, *Divergence-free measures in the plane and inverse potential problems in divergence form*, https://arxiv.org/abs/2006.09072.

[3] P. Bonicatto and N. A. Gusev, *On the structure of divergence-free measures on* $\mathbb{R}^2$, arXiv:1912.10936 [math], (2020), http://arxiv.org/abs/1912.10936 (accessed 2021-12-13). arXiv: 1912.10936.

[4] C. Boyer, A. Chambolle, Y. D. Castro, V. Duval, F. de Gournay, and P. Weiss, *On representer theorems and convex regularization*, SIAM Journal on Optimization, 29 (2019), pp. 1260–1281, https://doi.org/10.1137/18m1200750, https://arxiv.org/abs/1806.09810.

[5] K. Bredies and M. Carioni, *Sparsity of solutions for variational inverse problems with finite-dimensional data*, Calculus of Variations and Partial Differential Equations, 59 (2019), https://doi.org/10.1007/s00526-019-1658-1, https://arxiv.org/abs/1809.05045.

[6] K. Bredies and H. K. Pikkarainen, *Inverse problems in spaces of measures*, ESAIM: Control, Optimisation and Calculus of Variations, 19 (2012), pp. 190–218, https://doi.org/10.1051/cocv/2011205.

[7] E. J. Candès and C. Fernandez-Granda, *Towards a mathematical theory of super-resolution*, Communications on Pure and Applied Mathematics, 67 (2013), pp. 906–956, https://doi.org/10.1002/cpa.21455, https://arxiv.org/abs/1203.5871.

[8] L. Chizat and F. Bach, *On the global convergence of gradient descent for over-parameterized models using optimal transport*, Advances in Neural Information Processing Systems (NIPS), (2018), https://arxiv.org/abs/1805.09545.

[9] Y. de Castro and F. Gamboa, *Exact reconstruction using beurling minimal extrapolation*, Journal of Mathematical Analysis and Applications, 395 (2012), pp. 336–354, https://doi.org/10.1016/j.jmaa.2012.05.011, https://arxiv.org/abs/1103.4951.

[10] Q. Denoyelle, V. Duval, G. Peyré, and E. Soubies, *The sliding frank–wolfe algorithm and its application to super-resolution microscopy*, Inverse Problems, 36 (2019), p. 014001, https://doi.org/10.1088/1361-6420/ab2a29, https://arxiv.org/abs/1811.06416.

[11] V. Duval and G. Peyré, *Exact support recovery for sparse spikes deconvolution*, Foundations of Computational Mathematics, 15 (2014), pp. 1315–1355, https://doi.org/10.1007/s10208-014-9228-6, https://arxiv.org/abs/1306.6909.

[12] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Society for Industrial and Applied Mathematics, Jan 1999, https://doi.org/10.1137/1.9781611971088, https://doi.org/10.1137/1.9781611971088.

[13] H. Federer, *Geometric measure theory*, Springer, Berlin, 1996.

[14] M. Frank and P. Wolfe, *An algorithm for quadratic programming*, Naval Research Logistics Quarterly, 3 (1956), pp. 95–110, https://doi.org/10.1002/nav.3800030109.

[15] V. Kadets, *A Course in Functional Analysis and Measure Theory*, Springer International Publishing, 2018, https://doi.org/10.1007/978-3-319-92004-7, https://www.ebook.de/de/product/39459884/vladimir_kadets_a_course_in_functional_analysis_and_measure_theory.html.

[16] V. P. Khavin and S. K. Smirnov, *Approximation and extension problems for some classes of vector fields*, St. Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences, 10 (1998), pp. 507–528, https://mathscinet.ams.org/mathscinet-getitem?mr=1628034 (accessed 2022-02-07).

[17] B. Laville, L. Blanc-Féraud, and G. Aubert, *Off-The-Grid Variational Sparse Spike Recovery: Methods and Algorithms*, Journal of Imaging, 7 (2021), p. 266, https://doi.org/10.3390/jimaging7120266, https://www.mdpi.com/2313-433X/7/12/266 (accessed 2021-12-07).

[18] F. Morgan, *Geometric measure theory : a beginner's guide*, Academic Press, Boston, 1988.

[19] R. R. Phelps, ed., *Lectures on Choquet's Theorem*, Springer Berlin Heidelberg, 2001, https://doi.org/10.1007/b76887.

[20] R. N. R. M. Dudley, *Concrete Functional Calculus*, Springer-Verlag GmbH, 2010, https://www.ebook.de/de/product/16200145/r_m_dudley_r_norvaisa_concrete_functional_calculus.html.

[21] W. RUDIN, *Real and Complex Analysis*, MCGRAW HILL BOOK CO, 1986, https://doi.org/10.2307/2348852, https://www.ebook.de/de/product/4028964/walter_rudin_real_and_complex_analysis.html.

[22] F. SANTAMBROGIO, *Optimal Transport for Applied Mathematicians*, Springer International Publishing, 2015, https://doi.org/10.1007/978-3-319-20828-2.

[23] M. ŠILHAVÝ, *Divergence measure vectorfields: their structure and the divergence theorem*, Mathematical modelling of bodies with complicated bulk and boundary behavior, 20 (2008), pp. 214–237.

[24] S. K. SMIRNOV, *Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows*, St. Petersburg Department of Steklov Institute of Mathematics, Russian Academy of Sciences, 5 (1993), pp. 206–238, https://mathscinet.ams.org/mathscinet-getitem?mr=1246427 (accessed 2022-02-07).