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PROBLEMS OF BUFFON TYPE FOR A LATTICE OF PARALLELEPIPEDS

ELENA BOSETTO

ABSTRACT. The aim of this paper is to solve some problems of Buffon type for different convex test bodies (namely a semisphere, a spherical segment and a disk) and a lattice of parallelepipeds in the euclidean space E_3 . The independence question for three lattices of parallel planes at different distances apart is also studied.

AMS 1980 Subject Classification: Geometric probability, stochastic geometry, random sets; random convex sets and integral geometry.

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1. INTRODUCTION

Let \mathcal{R} be a lattice of rectangle parallelepipeds, of sides a , b and c . We fix an orthogonal frame of reference in E_3 , and we suppose, without loss of generality, that one of the elementary tiles of \mathcal{R} is the parallelepiped

$$\mathcal{P} = \{(x, y, z) \in E_3 : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}.$$

We are interested in the probability p_T that a test body T (i.e. a semisphere $H\Sigma$, a spherical segment S_α and a disk D), dropped at random in E_3 , meets some of the boundary points of at least one of the tiles of \mathcal{R} .

If we denote by \mathcal{M} the set of all congruent copies of T whose barycentres (just to make a choice) are in the interior of \mathcal{P} , and by \mathcal{N}_T the set of all those bodies that are congruent to T and completely contained in \mathcal{P} , we have

$$(1) \quad p_T = 1 - \frac{\mu(\mathcal{N}_T)}{\mu(\mathcal{M})}$$

where $\mu(\mathcal{N}_T)$ and $\mu(\mathcal{M})$ are the measures of the sets \mathcal{N}_T and \mathcal{M} . As usual we compute $\mu(\mathcal{N}_T)$ and $\mu(\mathcal{M})$ by means of the kinematic density ([2])

$$(2) \quad dK_3 = |\sin \vartheta| d\vartheta \wedge d\varphi \wedge d\psi \wedge dx \wedge dy \wedge dz$$

(where $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ are the spherical coordinates on the unit sphere in E_3 and $\psi \in [0, 2\pi]$ is an angle of rotation), for the use of this density provides the invariance of all our probability statements under the group of rigid motions.

The computation of $\mu(\mathcal{M})$ is trivial:

$$(3) \quad \mu(\mathcal{M}) = \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^\pi |\sin \vartheta| d\vartheta \int_{\{(x,y,z) \in \mathcal{P}\}} dx dy dz = 8\pi^2 abc.$$

In order to compute $\mu(\mathcal{N}_T)$ we choose a line s , which is steadily jointed to T , and we denote by ϑ the angle between s and the z axis and by φ the angle between the projection of s onto the xy plane and the x axis. For any fixed direction (ϑ, φ) of s , we move T inside \mathcal{P} and we consider the limiting positions when T is simultaneously tangent to three faces of \mathcal{P} . Let $\mathcal{P}_{T,(\vartheta,\varphi)}$ be the parallelepiped with sides parallel to those of \mathcal{P} and vertices determined by the coordinates of the barycentre P of T in each one of the limiting positions. Then we can write

$$(4) \quad \begin{aligned} \mu(\mathcal{N}_T) &= \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^\pi |\sin \vartheta| d\vartheta \int_{\{(x,y,z) \in \mathcal{P}_{T,(\vartheta,\varphi)}\}} dx dy dz \\ &= \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^\pi \text{vol}(\mathcal{P}_{T,(\vartheta,\varphi)}) |\sin \vartheta| d\vartheta. \end{aligned}$$

2. HITTING PROBABILITIES

For the test bodies semisphere and disk the probabilities of intersection are given by

Theorem 1. *Let $H\Sigma$ and D be a random semisphere, of constant base radius r , and a random disk, of constant radius r . Suppose that*

$$(5) \quad r < \min\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$$

Then we have

$$(6) \quad p_{H\Sigma} = \left(1 + \frac{\pi}{4}\right) \frac{ab + bc + ac}{abc} r - \left(1 + \frac{\pi}{2} + \frac{2}{\pi} I\right) \frac{a + b + c}{abc} r^2 + \left(1 + \frac{3\pi}{4} + \frac{6}{\pi} I + \frac{2}{\pi} J\right) \frac{1}{abc} r^3$$

and

$$(7) \quad p_D = \frac{\pi(ab + bc + ac)}{2abc} r - \frac{8I(a + b + c)}{\pi abc} r^2 + \frac{16J}{\pi abc} r^3.$$

The constants I and J are given by

$$(8) \quad I = \int_0^{\frac{\pi}{2}} \mathcal{E}(\sin \vartheta) \sin^2 \vartheta d\vartheta \simeq 0.932$$

$$(9) \quad J = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos^4 \vartheta) \mathcal{E}\left(\frac{\sin^2 \vartheta}{1 + \cos^2 \vartheta}\right) d\vartheta \simeq 0.663$$

where $\mathcal{E}(\mu) := \int_0^{\frac{\pi}{2}} \sqrt{1 - \mu^2 \sin^2 x} dx$ is the complete elliptic function of second kind.

Proof. Let Γ be the base circumference of $H\Sigma$ and let C be the center of Γ . In order to apply formula (4) for the computation of the measure of the set $\mathcal{N}_{H\Sigma}$, we consider the line s through C and orthogonal to the support plane of Γ . The direction of s is given by the

versor $(\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$. For any (ϑ, φ) fixed in $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ we start from the initial position when $C \equiv (r, r, r)$ and $H\Sigma$ is tangent to the coordinate planes, having equation

$$\begin{cases} (x-r)^2 + (y-r)^2 + (z-r)^2 = r^2 \\ (x-r) \sin \vartheta \cos \varphi + (y-r) \sin \vartheta \sin \varphi + (z-r) \cos \vartheta \leq 0 \end{cases}$$

and we translate $H\Sigma$ parallelly to the x, y and z axes. Then the equation of $H\Sigma$ becomes

$$\begin{cases} (x-h_x)^2 + (y-h_y)^2 + (z-h_z)^2 = r^2 \\ (x-h_x) \sin \vartheta \cos \varphi + (y-h_y) \sin \vartheta \sin \varphi + (z-h_z) \cos \vartheta \leq 0 \end{cases}$$

where (h_x, h_y, h_z) are the coordinates of C after the translation. The semisphere $H\Sigma$ is entirely contained in \mathcal{P} if and only if

$$(10) \quad \begin{aligned} r \leq h_x &\leq a - r \sqrt{1 - \sin^2 \vartheta \cos^2 \varphi} \\ r \leq h_y &\leq b - r \sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} \\ r \leq h_z &\leq c - r \sin \vartheta \end{aligned}$$

where the upper bounds for h_x, h_y and h_z correspond to the cases when Γ is respectively tangent to the planes $x = a, y = b$ and $z = c$. Hence, for $(\vartheta, \varphi) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, the parallelepiped $\mathcal{P}_{H\Sigma, (\vartheta, \varphi)}$ is defined by conditions (10).

When $\frac{\pi}{2} < \vartheta \leq \pi$ or $\frac{\pi}{2} < \varphi \leq 2\pi$ the inequalities (10) should be modified, but because of the symmetry of the problem, the volume of $\mathcal{P}_{H\Sigma, (\vartheta, \varphi)}$ remains unchanged.

Hence formula (4) yields

$$\mu(\mathcal{N}_{H\Sigma}) = 16\pi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{P}_{H\Sigma, (\vartheta, \varphi)}) \sin \vartheta d\vartheta$$

where

$$\text{vol}(\mathcal{P}_{H\Sigma, (\vartheta, \varphi)}) = \left[a - r \left(1 + \sqrt{1 - \sin^2 \vartheta \cos^2 \varphi} \right) \right] \left[b - r \left(1 + \sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} \right) \right] [c - r(1 + \sin \vartheta)].$$

Thus we have

$$\begin{aligned} \mu(\mathcal{N}_{H\Sigma}) &= 4\pi^2 abc - (ab + ac + bc)(4 + \pi)\pi^2 r \\ &\quad + \left[\left(4 + 2\pi + \frac{8I_1}{\pi} \right) (a + b) + \left(4 + 2\pi + \frac{8I_2}{\pi} \right) c \right] \pi^2 r^2 \\ &\quad - \left[4 + 3\pi + \frac{8}{\pi} (2I_1 + I_2 + J) \right] \pi^2 r^3 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} \sin^2 \vartheta d\vartheta \\ I_2 &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sqrt{(1 - \sin^2 \vartheta \cos^2 \varphi)(1 - \sin^2 \vartheta \sin^2 \varphi)} \sin \vartheta d\vartheta \\ J &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sqrt{(1 - \sin^2 \vartheta \cos^2 \varphi)(1 - \sin^2 \vartheta \sin^2 \varphi)} \sin^2 \vartheta d\vartheta \end{aligned}$$

are integrals for which no explicit expression of the primitive is available.

At this point, some geometrical considerations are useful to simplify the final form of $\mu(\mathcal{N}_{H\Sigma})$. As a matter of fact, by observing the structure of the elementary tile \mathcal{P} , we note that the probability of intersection $p_{H\Sigma}$ must be symmetric in a , b and c . In terms of $\mu(\mathcal{N}_{H\Sigma})$ this means that $I_1 = I_2$. A numerical integration proves that the previous equality is really true. If we denote by I the common value of the two integrals, we can write

$$I = \int_0^{\frac{\pi}{2}} \mathcal{E}(\sin \vartheta) \sin^2 \vartheta d\vartheta$$

where $\mathcal{E}(\mu) := \int_0^{\frac{\pi}{2}} \sqrt{1 - \mu^2 \sin^2 x} dx$, $0 \leq \mu \leq 1$, is the complete elliptic function of second kind.

Finally we have

$$\begin{aligned} J &= \int_0^{\frac{\pi}{2}} \sin^2 \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \vartheta + \sin^4 \vartheta \sin^2 \varphi \cos^2 \varphi} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin^2 \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 \vartheta + \frac{\sin^4 \vartheta}{4} - \frac{\sin^4 \vartheta \sin^2 \varphi}{4}} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin^2 \vartheta d\vartheta \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1 + \cos^2 \vartheta)^2}{4} - \frac{\sin^4 \vartheta \sin^2 \varphi}{4}} d\varphi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 \vartheta (1 + \cos^2 \vartheta) \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{\sin^2 \vartheta}{1 + \cos^2 \vartheta}\right)^2 \sin^2 \varphi} d\varphi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos^4 \vartheta) \mathcal{E}\left(\frac{\sin^2 \vartheta}{1 + \cos^2 \vartheta}\right) d\vartheta. \end{aligned}$$

By replacing the values of $\mu(\mathcal{N}_{H\Sigma})$ and $\mu(\mathcal{M})$ in formula (1) we get the probability (6).

Suppose now that the test body is a random disk D of constant radius r .

In order to keep the previous notations and results, let us identify D with the base circle of $H\Sigma$. When the angles ϑ and φ are fixed in $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ and $H\Sigma$ is moved by translation within \mathcal{P} , the disk D is entirely contained in \mathcal{P} if and only if the conditions (10) are replaced by

$$(11) \quad \begin{aligned} r\sqrt{1 - \sin^2 \vartheta \cos^2 \varphi} &\leq h_x \leq a - r\sqrt{1 - \sin^2 \vartheta \cos^2 \varphi} \\ r\sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} &\leq h_y \leq b - r\sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} \\ r \sin \vartheta &\leq h_z \leq c - r \sin \vartheta \end{aligned}$$

Thus we have

$$\mu(\mathcal{N}_D) = 16\pi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{P}_{D,(\vartheta,\varphi)}) \sin \vartheta d\vartheta$$

where now

$$\text{vol}(\mathcal{P}_{D,(\vartheta,\varphi)}) = \left[a - 2r\sqrt{1 - \sin^2 \vartheta \cos^2 \varphi} \right] \left[b - 2r\sqrt{1 - \sin^2 \vartheta \sin^2 \varphi} \right] [c - 2r \sin \vartheta]$$

Once the expression of $\mu(\mathcal{N}_D)$ is determined, we get the value of p_D by applying formula (1). ■

Remark. If we let $b, c \rightarrow \infty$ in (6) and (7) we find

$$\lim_{b, c \rightarrow \infty} p_{H\Sigma} = \left(1 + \frac{\pi}{4}\right) \frac{r}{a} \qquad \lim_{b, c \rightarrow \infty} p_D = \frac{\pi r}{2a}$$

The first probability was given by Duma-Stoka in [1], the second one was obtained by Stoka ([3], p.178).

The test body semisphere studied in the previous theorem is a particular example of spherical segment. Now we examine the general situation, but the increasing complexity of the computations forces us to consider only the case when the spherical segment, obtained by cutting a sphere Σ with a plane Π , contains in its interior the center C of Σ .

From now on the spherical segment will be denoted by S_α , where α , $0 < \alpha < \frac{\pi}{2}$, is the angle shown in fig.1

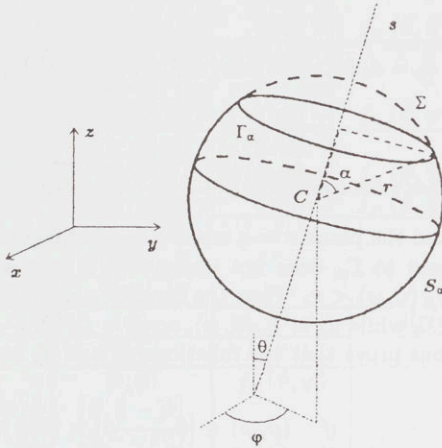


FIGURE 1. Test body: spherical segment S_α

Let s be the line through the center C of Σ and orthogonal to the base circle of S_α . As usual, we denote by ϑ the angle between s and the z axis and by φ the angle between the projection of s onto the xy plane and the x axis. The position of S_α is completely determined by the direction (ϑ, φ) of s and by the coordinates (x_C, y_C, z_C) of C : this will be summed up with the notation $\{S_\alpha; C, s\}$.

Because of the symmetry of S_α with respect to the point C , it is sufficient to study the function $\text{vol}(\mathcal{P}_{S_\alpha, (\vartheta, \varphi)})$ when $(\vartheta, \varphi) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

If $C \in \mathcal{P}$, then $\{S_\alpha; C, s\}$ does not intersect the planes $z = 0$ and $z = c$ if and only if $r \leq z_C \leq c - d_C$, where

$$d_C = d_C(\vartheta, \varphi) = \begin{cases} r \cos(\alpha - \vartheta) & \text{if } 0 \leq \vartheta \leq \alpha \\ r & \text{if } \alpha \leq \vartheta \leq \frac{\pi}{2} \end{cases}$$

Now let us study the position of S_α with respect to the coordinate planes and the planes $x = a$ and $y = b$ delimiting \mathcal{P} . When the position of S_α is $\{S_\alpha; C \equiv (r, r, r), s\}$, the sphere

Σ is tangent to the coordinate planes and the support plane of the base circle of S_α has equation:

$$(x - r) \sin \vartheta \cos \varphi + (y - r) \sin \vartheta \sin \varphi + (z - r) \cos \vartheta = r \cos \alpha.$$

Starting from this position, we move C within \mathcal{P} and we observe that $\{S_\alpha; C, s\}$ belongs to \mathcal{N}_{S_α} if and only if the set $\{S_\alpha; C, s\} \cap \partial \mathcal{P}$ is empty or consists of exactly one point P , i.e. if $\{S_\alpha; C, s\}$ is tangent to $\partial \mathcal{P}$ in P . If $\{S_\alpha; C, s\} \cap \partial \mathcal{P} = \{S_\alpha; C, s\} \cap \{x = 0 \cup y = 0\} = \{P\}$, then necessarily $P \in \Sigma$ and hence we must have $x_C \geq r$ and $y_C \geq r$ for all C such that $\{S_\alpha; C, s\} \in \mathcal{N}_{S_\alpha}$. On the contrary, when $\{S_\alpha; C, s\} \cap \partial \mathcal{P} = \{S_\alpha; C, s\} \cap \{x = a \cup y = b\} = \{P\}$, then $P \in \Sigma$ only if $0 \leq \vartheta \leq \frac{\pi}{2} - \alpha$, otherwise it can be on Σ or on the base circumference Γ_α of S_α . In terms of the coordinates of C this means that $x_C \leq a - d_a$ and $y_C \leq b - d_b$ for all C such that $\{S_\alpha; C, s\} \in \mathcal{N}_{S_\alpha}$, where

$$d_a = d_a(\vartheta, \varphi) = \min(r, f_\alpha(\vartheta, \varphi))$$

and

$$d_b = d_b(\vartheta, \varphi) = \min\left(r, f_\alpha\left(\vartheta, \frac{\pi}{2} - \varphi\right)\right)$$

having denoted by

$$f_\alpha(\vartheta, \varphi) = r \cos \alpha \sin \vartheta \cos \varphi + r \sin \alpha \sqrt{1 - \sin^2 \vartheta \cos^2 \varphi}$$

the distance between C and the plane $x = a$ when $x = a$ is tangent to Γ_α .

Note that $x = a$ tangent to Γ_α does not imply that $x = a$ is also tangent to S_α : it happens precisely when $f_\alpha(\vartheta, \varphi) < r$. Thus the condition $d_a = r$, or $d_b = r$, corresponds to the situation when $P \in \Sigma$, while $d_a = f_\alpha(\vartheta, \varphi)$, or $d_b = f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$ means that $P \in \Gamma_\alpha$. Straightforward calculations prove that the functions d_a and d_b have the following expressions:

$$d_a(\vartheta, \varphi) = \begin{cases} r & \text{if } (\vartheta, \varphi) \in \left[0, \frac{\pi}{2} - \alpha\right] \times \left[0, \frac{\pi}{2}\right] \\ f_\alpha(\vartheta, \varphi) & \text{if } (\vartheta, \varphi) \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right] \times \left[0, \arccos\left(\frac{\cos \alpha}{\sin \vartheta}\right)\right] \\ r & \text{if } (\vartheta, \varphi) \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right] \times \left[\arccos\left(\frac{\cos \alpha}{\sin \vartheta}\right), \frac{\pi}{2}\right] \end{cases}$$

$$d_b(\vartheta, \varphi) = \begin{cases} r & \text{if } (\vartheta, \varphi) \in \left[0, \frac{\pi}{2} - \alpha\right] \times \left[0, \frac{\pi}{2}\right] \\ r & \text{if } (\vartheta, \varphi) \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2} - \arccos\left(\frac{\cos \alpha}{\sin \vartheta}\right)\right] \\ f_\alpha\left(\vartheta, \varphi - \frac{\pi}{2}\right) & \text{if } (\vartheta, \varphi) \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right] \times \left[\frac{\pi}{2} - \arccos\left(\frac{\cos \alpha}{\sin \vartheta}\right), \frac{\pi}{2}\right] \end{cases}$$

In order to compute

$$\mu(\mathcal{N}_{S_\alpha}) = 8 \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \text{vol}(\mathcal{P}_{S_\alpha, (\vartheta, \varphi)}) \sin \vartheta d\vartheta$$

where

$$\text{vol}(\mathcal{P}_{S_\alpha, (\vartheta, \varphi)}) = (a - r - d_a(\vartheta, \varphi))(b - r - d_b(\vartheta, \varphi))(c - r - d_c(\vartheta, \varphi))$$

we must distinguish three different cases, according to the possible orderings of the angles $\tilde{\varphi} := \arccos\left(\frac{\cos \alpha}{\sin \vartheta}\right)$ and $\frac{\pi}{2} - \tilde{\varphi}$ when $\alpha \in (0, \frac{\pi}{2})$:

(1) $0 < \alpha \leq \frac{\pi}{4}$:

ϑ	φ	$d_a(\vartheta, \varphi)$	$d_b(\vartheta, \varphi)$	$d_c(\vartheta, \varphi)$
$[0, \alpha]$	$[0, \frac{\pi}{2}]$	r	r	$r \cos(\alpha - \vartheta)$
$[\alpha, \frac{\pi}{2} - \alpha]$	$[0, \frac{\pi}{2}]$	r	r	r
$[\frac{\pi}{2} - \alpha, \frac{\pi}{2}]$	$[0, \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	r
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	r	$f_\alpha(\vartheta, \varphi - \frac{\pi}{2})$	

(2) $\frac{\pi}{4} < \alpha \leq \arctan \sqrt{2}$:

ϑ	φ	$d_a(\vartheta, \varphi)$	$d_b(\vartheta, \varphi)$	$d_c(\vartheta, \varphi)$
$[0, \frac{\pi}{2} - \alpha]$	$[0, \frac{\pi}{2}]$	r	r	$r \cos(\alpha - \vartheta)$
$[\frac{\pi}{2} - \alpha, \alpha]$	$[0, \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	$r \cos(\alpha - \varphi)$
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	r	$f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$	
$[\alpha, \arcsin(\sqrt{2} \cos \alpha)]$	$[0, \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	r
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	r	$f_\alpha(\vartheta, \varphi - \frac{\pi}{2})$	
$[\arcsin(\sqrt{2} \cos \alpha), \frac{\pi}{2}]$	$[0, \frac{\pi}{2} - \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	r
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	$f_\alpha(\vartheta, \varphi)$	$f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$	

(3) $\arctan \sqrt{2} < \alpha \leq \frac{\pi}{2}$:

ϑ	φ	$d_a(\vartheta, \varphi)$	$d_b(\vartheta, \varphi)$	$d_c(\vartheta, \varphi)$
$[0, \frac{\pi}{2} - \alpha]$	$[0, \frac{\pi}{2}]$	r	r	$r \cos(\alpha - \vartheta)$
$[\frac{\pi}{2} - \alpha, \arcsin(\sqrt{2} \cos \alpha)]$	$[0, \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	$r \cos(\alpha - \varphi)$
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	r	$f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$	
$[\arcsin(\sqrt{2} \cos \alpha), \alpha]$	$[0, \frac{\pi}{2} - \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	$r \cos(\alpha - \vartheta)$
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	$f_\alpha(\vartheta, \varphi)$	$f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$	
$[\alpha, \frac{\pi}{2}]$	$[0, \frac{\pi}{2} - \tilde{\varphi}]$	$f_\alpha(\vartheta, \varphi)$	r	r
	$[\frac{\pi}{2} - \tilde{\varphi}, \frac{\pi}{2}]$	$f_\alpha(\vartheta, \varphi)$	$f_\alpha(\vartheta, \frac{\pi}{2} - \varphi)$	

Once the measure $\mu(\mathcal{N}_{S_\alpha})$ has been determined, we apply formula (1) and we get:

Theorem 2. Let S_α be a spherical segment of base radius $r \sin \alpha$ and height $r(1 + \cos \alpha)$, where $\alpha \in (0, \frac{\pi}{2})$ (see fig.1) and $r < \min(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$. When $0 < \alpha \leq \frac{\pi}{4}$ the probability p_{S_α} that S_α intersects one of the tiles of the lattice \mathcal{R} is:

$$p_{S_\alpha} = \left(1 + \frac{\alpha}{2} \sin \alpha + \cos \alpha\right) \frac{ab + ac + bc}{abc} r - 2(\alpha \sin \alpha + 2 \cos \alpha) \frac{a + b + c}{abc} r^2 + \frac{2(3\alpha \sin \alpha + 6 \cos \alpha - 2)}{abc} r^3$$

for $\frac{\pi}{4} < \alpha \leq \arctan \sqrt{2}$ we have

$$\begin{aligned}
 p_{S_\alpha} = & \left(1 + \frac{\alpha}{2} \sin \alpha + \cos \alpha\right) \frac{ab + ac + bc}{abc} r \\
 & - \left(1 + 2\alpha \sin \alpha + 4 \cos \alpha + f(\alpha) + \frac{2}{\pi} \sin \alpha I(\alpha)\right) \frac{a + b + c}{abc} r^2 \\
 & + \left(2 + 6\alpha \sin \alpha + 12 \cos \alpha + 6f(\alpha) + \frac{12}{\pi} \sin \alpha I(\alpha)\right) \frac{r^3}{abc}
 \end{aligned}$$

while for $\arctan \sqrt{2} < \alpha < \frac{\pi}{2}$ we get

$$\begin{aligned}
 p_{S_\alpha} = & \left(1 + \frac{\alpha}{2} \sin \alpha + \cos \alpha\right) \frac{ab + ac + bc}{abc} r \\
 & - \left(1 + 2\alpha \sin \alpha + 4 \cos \alpha + f(\alpha) + \frac{2}{\pi} \sin \alpha I(\alpha)\right) \frac{a + b + c}{abc} r^2 \\
 & + \left\{2 + \left[\frac{11\alpha}{2} + \left(\frac{1}{2} - \frac{2}{\pi} \cos^2 \alpha\right) \arcsin(\sqrt{2} \cos \alpha)\right] \sin \alpha + 12 \cos \alpha\right. \\
 & \left. + \frac{2 \cos^2 \alpha \left[\alpha \sin \alpha + 3 \cos \alpha - \frac{4 \cos^3 \alpha}{3} + \left(3 - \frac{4 \cos 2\alpha}{3} + \frac{\sin 2\alpha}{\sqrt{2}}\right) \sqrt{-\cos 2\alpha}\right]}{\pi}\right. \\
 & \left. + g(\alpha) + \frac{4}{\pi} \left(3I(\alpha) + H(\alpha) \sin \alpha + \frac{2}{\pi} J(\alpha)\right)\right\} \frac{r^3}{abc}
 \end{aligned}$$

where the functions $f(\alpha)$ and $g(\alpha)$ are defined as:

$$\begin{aligned}
 f(\alpha) = & \frac{1}{\pi} (\arctan u_\alpha + \arctan v_\alpha) - \frac{2}{\pi} (2 + \sin^2 \alpha) \cos \alpha \arccos(\cot \alpha) \\
 & + \frac{2}{\pi} \left(1 - \frac{1}{3} \cos 2\alpha\right) \sqrt{-\cos 2\alpha} \cos^2 \alpha \\
 g(\alpha) = & - \left[1 + \frac{2}{\pi} \left(4 \arccos(\cot \alpha) + \arccos \frac{\sqrt{-\cos 2\alpha}}{\sin \alpha}\right) (2 + \sin^2 \alpha)\right] \cos \alpha \\
 & + \frac{4}{\pi} (\arctan u_\alpha + \arctan v_\alpha) + \frac{2}{\pi} (\arctan w_\alpha + \arctan z_\alpha) \\
 & - \frac{\sqrt{2}}{4} \sqrt{-\cos 2\alpha} \sin 2\alpha
 \end{aligned}$$

with

$$\begin{aligned}
 u_\alpha = & \frac{\sin^2 \alpha - \cos \alpha}{\sqrt{-\cos 2\alpha} \cos \alpha} & v_\alpha = & -\frac{\sin^2 \alpha + \cos \alpha}{\sqrt{-\cos 2\alpha} \cos \alpha} \\
 w_\alpha = & \frac{\sin^2 \alpha - \sqrt{-\cos 2\alpha}}{\cos^2 \alpha} & z_\alpha = & -\frac{\sin^2 \alpha + \sqrt{-\cos 2\alpha}}{\cos^2 \alpha}
 \end{aligned}$$

while

$$\begin{aligned}
 I(\alpha) &= \int_{\frac{\pi}{2}-\alpha}^{\alpha} \left[\mathcal{E}(\sin \vartheta) - \mathcal{E}\left(\frac{\pi}{2} - \tilde{\varphi}; \sin \vartheta\right) \right] [\cos(\alpha - \vartheta) - 1] \sin \vartheta d\vartheta \\
 &\quad + \int_{\frac{\pi}{2}-\alpha}^{\alpha} (\cos \alpha \sin \tilde{\varphi} \sin \vartheta - \tilde{\varphi}) \sin^2 \vartheta d\vartheta \\
 H(\alpha) &= \int_{\arcsin(\sqrt{2} \cos \alpha)}^{\alpha} \left[\mathcal{E}(\sin \vartheta) - \mathcal{E}\left(\frac{\pi}{2} - \tilde{\varphi}; \sin \vartheta\right) \right] [\cos(\alpha - \vartheta) - 1] \sin \vartheta d\vartheta \\
 &\quad + \int_{\arcsin(\sqrt{2} \cos \alpha)}^{\alpha} (\tilde{\varphi} - \cos \alpha \sin \tilde{\varphi} \sin \vartheta) \sin^2 \vartheta d\vartheta \\
 J(\alpha) &= \int_{\arcsin(\sqrt{2} \cos \alpha)}^{\alpha} [\cos(\alpha - \vartheta) - 1] \left[\mathcal{E}(\tilde{\varphi}; \sin \vartheta) - \mathcal{E}\left(\frac{\pi}{2} - \tilde{\varphi}; \sin \vartheta\right) \right] \sin^2 \vartheta d\vartheta
 \end{aligned}$$

are integrals that cannot be computed explicitly for general values of the angle α . We have denoted by $\mathcal{E}(\mu) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \mu^2 \sin^2 \varphi} d\varphi$ and $\mathcal{E}(\varphi_1; \mu) = \int_0^{\varphi_1} \sqrt{1 - \mu^2 \sin^2 \varphi} d\varphi$, $0 \leq \mu \leq 1$ and $0 \leq \varphi_1 \leq \frac{\pi}{2}$, the complete and incomplete elliptic integrals of second kind.

Remark. For $\alpha = \frac{\pi}{4}$ we have

$$I\left(\frac{\pi}{4}\right) = 0 \quad \lim_{\alpha \rightarrow \frac{\pi}{4}^-} u_{\alpha} = \lim_{\alpha \rightarrow \frac{\pi}{4}^+} v_{\alpha} = -\frac{\pi}{2}$$

whereas for $\alpha = \arctan \sqrt{2}$ we get

$$H\left(\arctan \sqrt{2}\right) = J\left(\arctan \sqrt{2}\right) = 0$$

and

$$\begin{aligned}
 \lim_{\alpha \rightarrow \arctan \sqrt{2}^-} u_{\alpha} &= \lim_{\alpha \rightarrow \arctan \sqrt{2}^-} w_{\alpha} = 2 - \sqrt{3} \\
 \lim_{\alpha \rightarrow \arctan \sqrt{2}^+} v_{\alpha} &= \lim_{\alpha \rightarrow \arctan \sqrt{2}^+} z_{\alpha} = -2 - \sqrt{3}
 \end{aligned}$$

hence

$$\lim_{\alpha \rightarrow \frac{\pi}{4}^+} p_{S_{\alpha}} = p_{S_{\frac{\pi}{4}}} \quad \text{and} \quad \lim_{\alpha \rightarrow \arctan \sqrt{2}^+} p_{S_{\alpha}} = p_{S_{\arctan \sqrt{2}}}$$

that is the probability of intersection $p_{S_{\alpha}}$ is a continuous function of $\alpha \in (0, \frac{\pi}{2})$. As a matter of fact the function $p_{S_{\alpha}}$ can be extended to the closed interval $[0, \frac{\pi}{2}]$ since

$$\lim_{\alpha \rightarrow 0^+} p_{S_{\alpha}} = p_{\Sigma} \quad \lim_{\alpha \rightarrow \frac{\pi}{2}^-} p_{S_{\alpha}} = p_{H\Sigma}$$

where p_{Σ} is the probability of intersection for the test body sphere which was determined by Stoka in [4].

Since we are dealing with a regular lattice, it is natural to end the study of \mathcal{R} and the test bodies $H\Sigma$, D and S_{α} by briefly discussing the problem of independence for the so called "hitting events". There are two possible decompositions of \mathcal{R} :

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \quad \mathcal{R} = \mathcal{R}_{(1,2)} \cup \mathcal{R}_3$$

where $\mathcal{R}_1 = \{x = na, n \in \mathbb{Z}\}$, $\mathcal{R}_2 = \{y = nb, n \in \mathbb{Z}\}$ and $\mathcal{R}_3 = \{z = nc, n \in \mathbb{Z}\}$ are lattices of parallel and equidistant planes, and $\mathcal{R}_{(1,2)} = \mathcal{R}_1 \cup \mathcal{R}_2$. Hence we can study the sets of events $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ and $\{\mathcal{I}_{(1,2)}, \mathcal{I}_3\}$ where \mathcal{I}_i ($i = 1, 2, 3$) is the event "the random test body T intersects one of the planes of the lattice \mathcal{R}_i " and $\mathcal{I}_{(1,2)} = \mathcal{I}_1 \cup \mathcal{I}_2$.

As far as the test bodies $H\Sigma$, D , and S_α , $\pi/4 \leq \alpha \leq \pi/2$, is concerned, we find that the conditions of independence are never satisfied for both sets of events. On the contrary, if we choose as test body S_α , $0 \leq \alpha \leq \frac{\pi}{4}$, we get that the events $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ are independent if and only if

$$\begin{cases} h_1(\alpha) = \left(1 + \frac{\alpha}{2} \sin \alpha + \cos \alpha\right)^2 - 2(\alpha \sin \alpha + 2 \cos \alpha) = 0 \\ h_2(\alpha) = \left(1 + \frac{\alpha}{2} \sin \alpha + \cos \alpha\right)^3 - 2(3\alpha \sin \alpha + 6 \cos \alpha - 2) = 0 \end{cases}$$

the first equation being equivalent to $P(\mathcal{I}_i \cap \mathcal{I}_j) = P(\mathcal{I}_i)P(\mathcal{I}_j)$, $i \neq j = 1, 2, 3$, the second to $P(\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3) = P(\mathcal{I}_1)P(\mathcal{I}_2)P(\mathcal{I}_3)$, where

$$\begin{aligned} P(\mathcal{I}_1) &= \lim_{b, c \rightarrow +\infty} p_{S_\alpha} & P(\mathcal{I}_2) &= \lim_{a, c \rightarrow +\infty} p_{S_\alpha} & P(\mathcal{I}_3) &= \lim_{a, b \rightarrow +\infty} p_{S_\alpha} \\ P(\mathcal{I}_1 \cap \mathcal{I}_2) &= \lim_{c \rightarrow +\infty} p_{S_\alpha} & P(\mathcal{I}_1 \cap \mathcal{I}_3) &= \lim_{b \rightarrow +\infty} p_{S_\alpha} & P(\mathcal{I}_2 \cap \mathcal{I}_3) &= \lim_{a \rightarrow +\infty} p_{S_\alpha} \end{aligned}$$

The system has the unique solution $\alpha = 0$ in the interval $[0, \pi/4]$, hence independence holds only when the spherical segment becomes a sphere.

Likewise, when we look for possible values of α such that

$$P(\mathcal{I}_{(1,2)} \cap \mathcal{I}_3) = \lim_{c \rightarrow +\infty} p_{S_\alpha} + \lim_{a, b \rightarrow +\infty} p_{S_\alpha} - p_{S_\alpha} = P(\mathcal{I}_{(1,2)})P(\mathcal{I}_3) = \lim_{c \rightarrow +\infty} p_{S_\alpha} \lim_{a, b \rightarrow +\infty} p_{S_\alpha}$$

we find that this condition is satisfied if and only if $h_1(\alpha) = 0$, so, once again, $\alpha = 0$ is the unique admissible solution, and the events $\{\mathcal{I}_{(1,2)}, \mathcal{I}_3\}$ are independent only if the test body is a sphere.

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