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ON THE INVERSE OF THE *LQG* HOMING PROBLEM*

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Abstract

The inverse of the problem of minimizing (or maximizing) the time spent by a diffusion process in a given region is considered. A proposition that shows how the Laplace transform of the probability density function of a first hitting time can be used to obtain the optimal control in a related *LQG* homing problem is proved. The cost criterion takes the risk-sensitivity of the optimizer into account. An example in two dimensions is presented.

1 Introduction.

We consider the controlled stochastic process $x(t)$ defined by the system of stochastic differential equations

$$dx(t) = f(x(t))\,dt + B(x(t))u(t)\,dt + [N(x(t))]^{1/2}\,dW(t), \qquad (1)$$

where $x(t)$ is in $\mathbb{R}^n$, the control variable $u(t)$ is in $\mathbb{R}^m$, f is an n-vector function, B is an $n \times m$ matrix, N is an $n \times n$ symmetric positive definite matrix and $W(t)$ is an n-dimensional standard Brownian motion. Let D be a subset of $\mathbb{R}^n$ and define

$$T(x) = \inf\{t \geq 0 : x(t) \in D | x(0) = x\}.$$

That is, $T(x)$ is the first passage time of the controlled process $x(t)$ into the termination set D. Whittle (1982, p. 289) considered the problem of

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minimizing the time spent by $x(t)$ in the continuation region $C = D^c$ (the complement of D in $\mathbb{R}^n$). Assume that the cost function is given by

$$C(x) = \int_0^{T(x)} [\tfrac{1}{2} u'(t) Q(x(t)) u(t) + \lambda]\, dt, \tag{2}$$

where Q is an $m \times m$ symmetric positive definite matrix and λ is a positive parameter. Let $\xi(t)$ be the uncontrolled process defined by

$$d\xi(t) = f(\xi(t))\, dt + [N(\xi(t))]^{1/2}\, dW(t)$$

and

$$\tau(x) = \inf\{t \geq 0 : \xi(t) \in D | \xi(0) = x\}. \tag{3}$$

Whittle showed that if $P[\tau(x) < \infty] = 1$ and the relation

$$N = \alpha B Q^{-1} B' \tag{4}$$

holds for some positive scalar α, then the control u^* that minimizes the expected value of the cost function $C(x)$ is given by

$$u^* = -Q^{-1} B' F_x' \tag{5}$$

where the function $F(x)$ can be obtained from the formula

$$\exp[-F(x)/\alpha] = E\{\exp[-(\lambda/\alpha)\tau(x)]\}.$$

Thus, Whittle proved that if ultimate entry into D of the *uncontrolled* process $\xi(t)$ that corresponds to $x(t)$ is certain, and if a certain relation holds between the noise and the control matrices, then it is possible to compute the optimal control by simply considering the uncontrolled process.

Notice that if the parameter λ in (2) is negative, then the objective is to maximize the survival time in the continuation region C, taking the quadratic control costs into account.

Next, Kuhn (1985) generalized Whittle's result by using the risk-sensitive cost criterion

$$J(x, \theta) = -\theta^{-1} \log E[e^{-\theta C(x)}],$$

in which θ is the parameter that takes the risk-sensitivity of the optimizer into account. The problem treated by Whittle corresponds to the risk-neutral case $\theta = 0$, since

$$\lim_{\theta \to 0} J(x, \theta) = E[C(x)].$$

Moreover when θ is positive, the optimizer is said to be optimistic or risk-seeking, whereas we say that the optimizer is pessimistic or risk-averse when θ is negative (see Whittle (1990, p. 5)). Kuhn showed that if the relation (4) is replaced by

$$\beta N = BQ^{-1}B' + \theta N, \tag{6}$$

then the optimal control is still given by (5), but with

$$e^{-\beta F(x)} = E[e^{-\lambda \beta \tau(x)}]. \tag{7}$$

Remark. In fact, in Whittle (1990, p. 223) and in Kuhn (1985) the function f and the matrices B, N and Q may also depend on the variable t, the parameter λ is replaced by a function $h(x(t), t)$, and there is a general termination cost function $K[x(T), T]$. Here, we assume that the problem is time-invariant and that $K[x(T), T] \equiv 0$, as did Whittle and Kuhn in particular cases. Moreover, the term *LQG homing* really corresponds to the case when $f(x(t)) = Ax(t)$ and the matrices A, B, N and Q are all constant (see Whittle and Gait (1970)).

In this note, we consider the inverse of the LQG homing problem. More precisely, we show that the moment generating function of a first passage time $\tau(x)$ can sometimes be used to find the optimal control in a related LQG homing problem, even if the relation (6) is not satisfied.

In Section 2, we prove a proposition that gives the LQG homing problems that correspond to the given Laplace transform of the probability density function of a first passage time $\tau(x)$. An example is presented in Section 3. Finally, some concluding remarks are made in Section 4.

2 Theoretical result.

The function $F(x)$ that appears in (5) satisfies the dynamic programming equation (see Kuhn (1985))

$$\inf_u[(1/2)u'Qu + \lambda + F_x(f + Bu) - F_x(\theta N/2)F_x' + (1/2)\mathrm{tr}(NF_{xx})] = 0. \quad (8)$$

This equation is valid in C. We also have the boundary condition

$$F(x) = 0 \quad \text{if } x \in \partial D, \quad (9)$$

where ∂D is the boundary of the stopping region D. Substituting the optimal control u^* given by (5) into the dynamic programming equation, we find that we must solve the partial differential equation

$$\lambda + F_x f - F_x[(1/2)BQ^{-1}B' + (1/2)\theta N]F_x' + (1/2)\mathrm{tr}(NF_{xx}) = 0. \quad (10)$$

Let

$$M := -N[BQ^{-1}B' + \theta N]^{-1}, \quad (11)$$

where we assume that the inverse matrix exists. Next, suppose that there exists a function ϕ such that

$$F_x = \frac{\phi_x}{\phi} M. \quad (12)$$

Then, we have

$$F_{xx} = \frac{\phi_{xx}}{\phi} M - \frac{1}{\phi^2}\phi_x'\phi_x M + \frac{P}{\phi}, \quad (13)$$

where

$$P := \begin{pmatrix} \phi_x M_{x_1} \\ \vdots \\ \phi_x M_{x_n} \end{pmatrix}$$

and $M_{x_i} := \partial M/\partial x_i$ for $i = 1, \ldots, n$. Substituting (12) and (13) into (10) and using the fact that

$$\mathrm{tr}(N\phi_x'\phi_x M) = \mathrm{tr}(\phi_x M N \phi_x') = \phi_x M N \phi_x',$$

we find that Eq. (10) is linearized to

$$\lambda + \frac{\phi_x}{\phi} Mf + \frac{1}{2}\mathrm{tr}\left[N\left(\frac{\phi_{xx}}{\phi}M + \frac{P}{\phi}\right)\right] = 0. \tag{14}$$

Now, we may write that

$$\mathrm{tr}(NP) = \phi_x \sum_{i=1}^{n} M_{x_i} N_i,$$

where N_i is the ith column of the matrix N. It follows that Eq. (14) may be rewritten as

$$\lambda\phi + \phi_x(Mf + (1/2)\sum_{i=1}^{n} M_{x_i} N_i) + (1/2)\mathrm{tr}[MN\phi_{xx}] = 0. \tag{15}$$

If the matrix M is positive definite, Eq. (15) is the Kolmogorov backward equation satisfied by

$$\psi(x) := E[e^{\lambda\sigma(x)}], \tag{16}$$

where

$$\sigma(x) := \inf\{t \geq 0 : \zeta(t) \in \Delta | \zeta(0) = x\},$$

in which Δ is a subset of $\mathbb{R}^n$ and $\zeta(t)$ is the uncontrolled process defined by the system of stochastic differential equations

$$d\zeta(t) = (Mf + (1/2)\sum_{i=1}^{n} M_{\zeta_i} N_i)\, dt + [MN]^{1/2}\, dW(t). \tag{17}$$

If M is negative definite, Equations (16) and (17) become, respectively,

$$\psi(x) := E[e^{-\lambda\sigma(x)}] \tag{18}$$

and

$$d\zeta(t) = -(Mf + (1/2)\sum_{i=1}^{n} M_{\zeta_i} N_i)\, dt + [-MN]^{1/2}\, dW(t). \tag{19}$$

We can prove the following proposition.

Proposition 1 *Suppose that* $P[\sigma(x) < \infty] = 1$. *Then the function* $\phi(x)$ *that appears in (12) (if it exists) is equal to the function* $\psi(x)$ *defined in (16) or (18). Furthermore, the boundary of the termination set* D, *i.e.* ∂D, *is obtained from*

$$\partial D = \{x \in \mathbb{R}^n : F(x) = 0\}. \tag{20}$$

Proof. If $P[\sigma(x) < \infty] = 1$, then the solution of Eq. (15), subject to

$$\psi(x) = 1 \quad \text{if } x \in \Delta,$$

is *unique* (see Whittle (1982, p. 290)). Hence, if there exists a function ϕ such that Eq. (12) is valid, then we may write that $\phi(x) \equiv \psi(x)$ and the function F obtained from (12) will be the unique function that satisfies the partial differential equation (10), subject to the appropriate boundary condition. Finally, the boundary condition (9) implies that ∂D is indeed the set defined in (20). □.

Remarks.

1) Proposition 1 implies that if we have a function $\psi(x)$ such that the random variable $\sigma(x)$ that appears in its definition satisfies the condition $P[\sigma(x) < \infty] = 1$, then we can state that this function can sometimes be used to solve an optimal control problem. There are, in fact, many optimal control problems that can be solved by using the same function $\psi(x)$ since the matrix M defined in (11) depends on the control matrices B and Q, and the noise matrix N (as well as the risk parameter θ). Therefore, different combinations of matrices B, Q and N can lead to the same matrix M and, consequently, to the same function F.

2) If the relation (6) holds, then we can always define the function $\phi(x)$ as follows:

$$\phi(x) = e^{-\beta F(x)} = E[e^{-\lambda\beta\tau(x)}]$$

(see Eq. (7)), where $\tau(x)$ is defined in (3). Furthermore, in this case we will have $D = \Delta$. However, when the relation (6) does not hold, the termination set D obtained from (20) will not, in general, be the same as the set Δ.

In the next section we will give an example, in two dimensions, where a first passage time moment generating function such as $\psi(x)$ can be used to solve explicitly an optimal control problem. Moreover, in this particular example the relation (6) will *not* hold.

3 An example.

Consider the two-dimensional uncontrolled stochastic process $(\zeta_1(t), \zeta_2(t))$ defined by the stochastic differential equations

$$d\zeta_1(t) = [2\zeta_1^2(t)]^{1/2} dW_1(t), \tag{21}$$

$$d\zeta_2(t) = [2\zeta_2^2(t)]^{1/2} dW_2(t), \tag{22}$$

where $W_1(t)$ and $W_2(t)$ are independent one-dimensional standard Brownian motion processes. Let

$$\psi(x_1, x_2; \lambda) := E[e^{-\lambda\sigma(x_1,x_2)}],$$

where λ is a real, non-negative parameter and

$$\sigma(x_1, x_2) := \inf\{t \geq 0 : \zeta_1(t)\zeta_2(t) = 1 | \zeta_1(0) = x_1 > 0, \zeta_2(0) = x_2 > 0\}.$$

That is, $\psi(x_1, x_2; \lambda)$ is the moment generating function (or the Laplace transform of the probability density function) of the first passage time random variable $\sigma(x_1, x_2)$.

Next, using the results in Lefebvre (1995), we can state that

$$\psi(x_1, x_2; \lambda) = (x_1 x_2)^{\nu} \quad \text{for } x_1 > 0, x_2 > 0, x_1 x_2 \geq 1, \tag{23}$$

where

$$\nu := \frac{1}{2} - \frac{[1 + 4\lambda]^{1/2}}{2}. \tag{24}$$

Note that we have

$$\lim_{\lambda \downarrow 0} \psi(x_1, x_2, \lambda) = 1,$$

so that $P[\sigma(x_1, x_2) < \infty] = 1$, as required.

Now, suppose that we choose $N = B = I_2$ in (1), where I_2 is the two-dimensional identity matrix, and

$$Q(x_1(t), x_2(t)) = \begin{pmatrix} 2x_1^2(t) & 0 \\ 0 & 2x_2^2(t) \end{pmatrix}$$

in (2). Furthermore, we assume that the risk parameter θ is equal to 0; that is, we consider the risk-neutral case. Then the matrix M defined in (11) is given by

$$M = -Q(x_1(t), x_2(t)).$$

Remark. Notice that the matrix M is *not* constant (so that the relation (6) does *not* hold) and that we have

$$-M(\zeta_1(t), \zeta_2(t))N(\zeta_1(t), \zeta_2(t)) = \begin{pmatrix} 2\zeta_1^2(t) & 0 \\ 0 & 2\zeta_2^2(t) \end{pmatrix},$$

as should be (see (21) and (22)).

Using Eq. (12) with $\phi(x_1, x_2) = \psi(x_1, x_2; \lambda)$, we obtain that

$$(F_{x_1}, F_{x_2}) = \nu(-2x_1, -2x_2).$$

It follows that

$$F(x_1, x_2) = -\nu(x_1^2 + x_2^2) + k_0,$$

where k_0 is a constant. To complete the example, we must choose the function f in (1) and (19), and we must also find the boundary of the termination set D.

First, using (19), (21) and (22), we can write that

$$-\left(M(\zeta_1, \zeta_2)f(\zeta_1, \zeta_2) + \frac{1}{2}\sum_{i=1}^{2} M_{\zeta_i} N_i\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$N_1 = \begin{pmatrix} 2\zeta_1^2 \\ 0 \end{pmatrix} \quad \text{and} \quad N_2 = \begin{pmatrix} 0 \\ 2\zeta_2^2 \end{pmatrix}.$$

A simple calculation now yields that

$$f(\zeta_1, \zeta_2) = \begin{pmatrix} -1/\zeta_1 \\ -1/\zeta_2 \end{pmatrix}.$$

Moreover, we have

$$F(x_1, x_2) = 0 \iff x_1^2 + x_2^2 = k_0/\nu.$$

Hence, assuming that $k_0 < 0$ (because $\nu < 0$ (see (24)), we may write that

$$\partial D = \{(x_1, x_2) \in I\!R^2 : x_1^2 + x_2^2 = k_0/\nu\}.$$

Summarizing the example presented in this section, we can state that the function $\psi(x_1, x_2; \lambda)$ defined in (23) enables us to solve the following optimal control problem: find the control $u^*(t)$ that minimizes the expected value of the cost function

$$C(x_1, x_2) = \int_0^{T(x_1,x_2)} [\tfrac{1}{2} u'(t) Q(x_1(t), x_2(t)) u(t) + \lambda]\, dt,$$

where

$$T(x_1, x_2) = \inf\{t \geq 0 : x_1^2(t) + x_2^2(t) = k_0/\nu | x_1(0) = x_1 > 0, x_2(0) = x_2 > 0\},$$

$$Q(x_1(t), x_2(t)) = \begin{pmatrix} 2x_1^2(t) & 0 \\ 0 & 2x_2^2(t) \end{pmatrix},$$

and $(x_1(t), x_2(t))$ is the two-dimensional stochastic process defined by the stochastic differential equations

$$\begin{aligned} dx_1(t) &= -\frac{1}{x_1(t)}\, dt + u_1(t)\, dt + dW_1(t), \\ dx_2(t) &= -\frac{1}{x_2(t)}\, dt + u_2(t)\, dt + dW_2(t), \end{aligned}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motion processes.

Finally, from (5) we deduce that the optimal control is given by

$$u^*(t) = \nu \begin{pmatrix} 1/x_1(t) \\ 1/x_2(t) \end{pmatrix}.$$

4 Conclusion.

In this note, we have shown, in Section 2, how the moment generating function of a first passage time random variable for an uncontrolled stochastic process can sometimes be used to obtain the exact solution to related optimal control problems. An explicit example to illustrate the theoretical result was presented in Section 3.

If the relation (6) holds between the control and the noise matrices (and if ultimate entry of the uncontrolled process into the termination region is certain), then it is always possible to obtain the optimal control, in the LQG homing problems studied in this note, by considering the corresponding uncontrolled process.

Similarly, when this special relation (6) holds, the stopping regions in the controlled and the uncontrolled cases will be the same. However, when the relation (6) is not valid, the stopping regions will, in general, be different. Therefore, we do not know in advance what the stopping region in the optimal control problem will be.

Furthermore, when the special relation (6) does not hold, it is generally not possible to reduce the optimal control problems to first passage time problems. However, in another paper (Lefebvre (1996)), the author has shown that in the one-dimensional case we can indeed obtain the solution to the optimal control problem from a mathematical expectation for a related uncontrolled process, even when the relation (6) is not satisfied. For this reason, we wanted to present here an example in two or more dimensions.

In conclusion, in the present paper we considered the inverse LQG homing problem. The last step would now be to try to solve a given LQG homing problem in two or more dimensions, for which the relation (6) does not hold, by considering a related uncontrolled process. As mentioned above, this is not possible in general. Therefore, one or more additional conditions must be imposed.

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