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# Abstraction refinement for attractivity controllers using quantitative synthesis

W. Alejandro Apaza-Perez, Antoine Girard

**Abstract**—Attractivity specifications consist in driving the state of a system to a target region and to keep it in that region afterwards. In this paper, we develop an approach to controller synthesis for attractivity specifications based on iterative refinements of symbolic abstractions. We compute iteratively sequences of least-violating attractivity controllers and of associated attractors. The current controller is used to restrict possible control actions at the next iteration where a new abstraction is used following a local refinement applied inside the current attractor. The approach results in a nested sequence of attractors which are closer to the target region at each iteration. To illustrate the effectiveness of our approach, we show an application to a marine vessel system.

**Index Terms**—Symbolic control, Attractivity specification, Formal methods.

## I. INTRODUCTION

Recently the use of digital (discrete) controllers has been growing to control cyber-physical systems (CPS), which has implied a growing need to provide formal guarantees of correctness of such controllers. One way to solve this problem relies on the so-called symbolic or abstraction-based controller [3], [17]. Abstraction-based control is a computational approach to controller synthesis for general nonlinear systems with state and input constraints which is generally done in three steps. First, a finite state abstraction is computed from a given nonlinear continuous dynamic system by partitioning the state and discretizing the input space. Second, given this abstraction it is possible to use algorithmic techniques for the automatic synthesis of controllers to enforce many types of formal specifications (e.g. linear temporal logic [11]). Finally, the discrete control strategy is refined to a continuous controller for the given system.

Since the introduction of symbolic control, much research has gone dealing with different classes of systems, such as the following. Construction of symbolic models for incrementally stable nonlinear control systems is proposed in [12]. Abstraction of nonlinear control systems without stability assumptions is proposed in [21]. In [4], an efficient computational procedure is proposed for abstraction of discrete-time mixed monotone systems. Until now, it has been shown that abstraction based control can be used for a wide range of systems. However, the construction of finite abstractions and the synthesis of controllers suffer from the so-called curse of dimensionality: the complexity grows exponentially with the dimensions of the state and input sets. This curse of dimensionality in abstractions has motivated the investigation of different approaches such as optimizing the abstraction by minimizing the number of transitions [15], [20], using of multiple refinements for designing abstractions [7], [9], [16], [19], compositional approaches for the control synthesis in [1], [2], [5], [10] and for the construction of abstractions in [14], [18]. The

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abstraction procedure provided in [8] can complement compositional methods by also taking advantage of sparsity within a component.

In this paper, we propose an approach for the synthesis of abstraction-based controllers for attractivity specification, using quantitative synthesis and iterative refinement. We compute iteratively sequences of least-violating attractivity controllers [6] and of associated attractors. The controller computed in one iteration is used to restrict possible control actions in the next iteration where a new local refinement is used within the current attractor. This setting results in a nested sequence of attractors that are closer to the target region at each iteration.

**Related works:** In the context of abstraction refinement methods, [19] proposes a technique where a coarse abstraction is refined locally to bound accumulated error with respect to the underlying system dynamics. Non-uniform state space partitioning has also been presented for stochastic systems [16]. Both methods are different from our approach in that refinement is constrained on attractors and controllers computed for the attractivity control problem. For uniformly coarse abstractions, [20] presents a technique to optimize the partition based on hypercubes in order to minimize the number of outgoing transitions from constructed abstract states. This technique could be incorporated in our method for choosing the partition based on hypercubes of the coarse abstraction. [9] provides an approach for the controller synthesis based in several layers of abstractions, while [7] includes varying time sampling periods, in our approach we avoid to pay the up-front cost of construction multiple abstract transition relations for whole system.

The structure of this paper is as follows. In Section II, we provide a formal problem formulation and provide the necessary theoretical background. In Section III-A, we describe symbolic systems based on partitions of state space, and in Section III-B, we briefly recall the main results of [6] on the synthesis of least-violating controllers. Section IV contains the main contribution of the paper, which is an algorithm to synthesize an attractivity controller based on iterative refinements of symbolic abstraction inside of attractors. Finally, in Section V, numerical simulations of a marine vessel system are used to illustrate our approach.

## II. BACKGROUND AND PROBLEM STATEMENT

**Notations.**  $\mathbb{R}$ ,  $\mathbb{R}_{>0}$ ,  $\mathbb{R}_{\geq 0}$  and  $\mathbb{N}$  denote the set of real, positive real, non negative real numbers, and non negative integers, respectively. For  $J \subseteq \mathbb{R}$  and  $K \in \mathbb{R}$ , we define the following sets  $J_{<K} = \{k \in J | k < K\}$  and  $J_{\leq K} = \{k \in J | k \leq K\}$ .  $\overline{\mathbb{R}}$  denotes the set of extended real numbers, i.e.  $\overline{\mathbb{R}} = [-\infty, +\infty]$ . Given two sets  $X$  and  $Y$ ,  $f : X \rightrightarrows Y$  and  $f : X \rightarrow Y$  denote a set-valued and ordinary map, respectively. Given an extended real-valued function  $V : X \rightarrow \overline{\mathbb{R}}$ , the lower level sets of function  $V$  are defined as  $L_\delta(V) = \{x \in X | V(x) \leq \delta\}$  where  $\delta \in \mathbb{R}$ .

### A. Transition systems

We consider a class of transition systems formally defined as follows:

*Definition 1:* A transition system  $\Sigma$  is a triple  $\Sigma = (X, U, F)$  where  $X$  is the set of states;  $U$  is the set of control inputs; and  $F$  is the transition map  $F : X \times U \rightrightarrows X$ .

The system  $\Sigma = (X, U, F)$  is called *finite* if  $X$  and  $U$  are finite sets. A transition  $x' \in F(x, u)$  means that  $\Sigma$  can evolve from state  $x$  to state  $x'$  under control input  $u$ . We define the set of *enabled control inputs* at a state  $x \in X$  and transition map  $F$  as

$$\text{enab}_F(x) = \{u \in U \mid F(x, u) \neq \emptyset\}.$$

If  $\text{enab}_F(x) = \emptyset$ , then the state  $x$  is called *blocking*, otherwise it is *non-blocking*. The set of non-blocking states is denoted  $\text{nbs}_F$ .

Within the framework of transition systems, we can define (memoryless state-feedback) controllers as follows:

**Definition 2:** A *controller* for system  $\Sigma = (X, U, F)$  is a set-valued map  $C : X \rightrightarrows U$  such that  $C(x) \subseteq \text{enab}_F(x)$ , for all  $x \in X$ .

The domain of  $C$  is  $\text{dom}(C) = \{x \in X \mid C(x) \neq \emptyset\}$ . Given a system and a controller, we can define closed-loop trajectories as follows:

**Definition 3:** A sequence  $(x_t)_{t=0}^T$ , where  $T \in \mathbb{N} \cup \{+\infty\}$ ,  $x_t \in X$ , for  $t \in \mathbb{N}_{\leq T}$ , is called a *closed-loop trajectory* of system  $\Sigma$  with controller  $C$  if and only if

$$\forall t \in \mathbb{N}_{< T}, \exists u \in C(x_t), x_{t+1} \in F(x_t, u).$$

A trajectory is called *maximal* if either  $T = +\infty$  or  $C(x_T) = \emptyset$ , it is *complete* if  $T = +\infty$ . The set of maximal closed-loop trajectories starting from a given initial state  $x_0 \in X$  is denoted by  $\mathcal{T}_{\max}(\Sigma, C, x_0)$ . The *closed-loop dynamics* can be represented by a transition system  $\Sigma_C = (X, U, F_C)$  where  $x' \in F_C(x, u)$  if and only if there exists  $u \in C(x)$ ,  $x' \in F(x, u)$ .

In this paper, we consider uniform attractivity specifications, which are defined as follows:

**Definition 4:**  $X^* \subseteq X$  is said to be *uniformly attractive* from initial state  $x_0 \in X$  for system  $\Sigma$  and controller  $C$  if there exists  $T_0 \in \mathbb{N}$ , such that all maximal trajectories  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma, C, x_0)$  are complete and satisfy  $x_t \in X^*$ , for all  $t \geq T_0$ .

## B. Problem statement

Consider the following system

$$\Sigma_{\text{cont}} = (\mathcal{X}, \mathcal{U}, f), \quad (1)$$

where  $\mathcal{X} \subseteq \mathbb{R}^n$  is the set of states,  $\mathcal{U} \subseteq \mathbb{R}^p$  is the set of inputs, and  $f : \mathcal{X} \times \mathcal{U} \rightrightarrows \mathcal{X}$  is the state transition function. This system represents a discrete-time continuous-state system where the state and the input of the system at time  $t \in \mathbb{N}$  are expressed as  $x_t \in \mathcal{X}$ ,  $u_t \in \mathcal{U}$ , respectively. The state transition at time  $t$  is expressed as

$$x_{t+1} \in f(x_t, u_t). \quad (2)$$

The system (1) is discrete, however, it encompasses sampled versions of continuous-time systems, possibly subject to disturbances.

**Problem 5:** Given a target set  $\mathcal{X}^* \subseteq \mathcal{X}$ , the goal is to synthesize a controller  $C^* : \mathcal{X} \rightrightarrows \mathcal{U}$  for system (1) and a set of initial states  $\mathcal{B}^* \subseteq \mathcal{X}$  such that  $\mathcal{X}^*$  is uniformly attractive from all initial states  $x_0 \in \mathcal{B}^*$ .

## III. FINITE ABSTRACTIONS AND LEAST-VIOLATING ATTRACTIVITY CONTROLLERS

In the following, we introduce symbolic systems based on partitions of the state space of system (1). This allows us to use some results of least-violating controller presented in [6], which are also briefly described in this section.

### A. Finite abstractions based on state-set partitions

The system (1) corresponds to a discrete-time continuous-state system, which is considered as an infinite system. In this case, our approach is based on partitions of the set of states.

**Definition 6:** A collection  $\{X_q\}_{q \in Q}$  of subsets of  $X$  indexed by a set  $Q$  is a partition of  $X$  if  $\cup_{q \in Q} X_q = X$  and  $X_q \cap X_{q'} = \emptyset$  when  $q \neq q'$ . Let  $\{X_q\}_{q \in Q}$  and  $\{X_{q'}\}_{q' \in Q'}$  be partitions of a set  $X$ , where  $\{X_q\}_{q \in Q}$  is a refinement of  $\{X_{q'}\}_{q' \in Q'}$  i.e.  $\forall q' \in Q', \exists q \in Q$  such that  $X_{q'} \subseteq X_q$ , we denote that partition refinement by  $\{X_q\}_{q \in Q} \sqsubseteq \{X_{q'}\}_{q' \in Q'}$ .

Given a finite index set  $Q$  of a partition  $\{X_q\}_{q \in Q}$  of  $X$  and a finite set  $P \subseteq \mathcal{U}$ , we define the transition system

$$\Sigma^Q = (Q, P, G), \quad (3)$$

where the transition relation  $G : Q \times P \rightrightarrows Q$  is defined as:

$$\forall q \in Q, p \in \text{enab}_G(q), q' \in G(q, p) \iff f(X_q, p) \cap X_{q'} \neq \emptyset \quad (4)$$

and  $\text{enab}_G(q) := \{p \in P \mid \forall x \in X_q, p \in \text{enab}_f(x)\}$ .

Note that although  $Q$  denotes an index set of a partition  $\{X_q\}_{q \in Q}$  of the state space  $X$  of  $\Sigma_{\text{cont}}$  it allows us to define a system  $\Sigma^Q$  which is related to  $\Sigma_{\text{cont}}$  by a feedback refinement relation [13], according to the next Definition 7 and Proposition 8.

**Definition 7:** Given two transition systems  $\Sigma_a = (X_a, U_a, F_a)$  and  $\Sigma_b = (X_b, U_b, F_b)$  with  $U_b \subseteq U_a$ . A relation  $R : X_a \rightrightarrows X_b$  is a *feedback refinement relation* from  $\Sigma_a$  to  $\Sigma_b$  if for all  $x_a \in X_a$ , there exists  $x_b \in X_b$  such that  $x_b \in R(x_a)$  and the following holds for all  $x_a \in X_a$  and  $x_b \in R(x_a)$ :

$$\text{enab}_{F_b}(x_b) \subseteq \text{enab}_{F_a}(x_a);$$

$$u \in \text{enab}_{F_b}(x_b) \implies \cup_{x \in F_a(x_a, u)} R(x) \subseteq F_b(x_b, u).$$

A feedback refinement relation guarantees that a controller designed for  $\Sigma_b$  can be also used for  $\Sigma_a$ . The next result can be obtained from [13].

**Proposition 8:** Let  $\Sigma_{\text{cont}} = (\mathcal{X}, \mathcal{U}, f)$  and  $\Sigma^Q = (Q, P, G)$  be transition systems given by (1) and (3)-(4), respectively. Consider a quantization function  $\theta : \mathcal{X} \rightarrow Q$  defined as  $\theta(x) = q$  if  $x \in X_q$ . Then, the relation  $R : \mathcal{X} \rightrightarrows Q$  defined as  $R(x) = \{\theta(x)\}$  for all  $x \in \mathcal{X}$  is a feedback refinement relation from  $\Sigma_{\text{cont}}$  to  $\Sigma^Q$ .

### B. Least-violating attractivity controllers

In this section, we briefly recall results of [6] on the synthesis of attractivity controllers. Consider a finite system  $\Sigma^Q = (Q, P, G)$  based on a partition  $\{X_q\}_{q \in Q}$  of the state space  $X$ , and a finite set  $P \subseteq \mathcal{U}$  as in (3)-(4), and a target set  $\mathcal{X}^* \subseteq \mathcal{X}$ . Let us consider a function

$$h : Q \rightarrow \mathbb{R}_{\geq 0} \quad (5)$$

such that  $h(q) = 0$  if and only if  $X_q \subseteq \mathcal{X}^*$ . An example of such function is given by

$$h(q) = \max_{x \in X_q} \min_{x' \in \mathcal{X}^*} d(x, x') \quad (6)$$

where  $d$  is a metric on  $\mathcal{X}$ .

Let us consider the following sequence of dynamic programming fixed-point iterations:

$$W_S^0(q) = h(q), \quad (7)$$

$$W_S^{k+1}(q) = \begin{cases} \max \left( h(q), \min_{p \in \text{enab}_G(q)} \max_{q' \in G(q, p)} W_S^k(q') \right) & \text{if } \text{enab}_G(q) \neq \emptyset; \\ +\infty & \text{if } \text{enab}_G(q) = \emptyset; \end{cases} \quad (8)$$

for  $q \in Q$ ,  $k \in \mathbb{N}$ . We denote the fixed-point of (7)-(8) by  $W_S^*$ . Then, let

$$W_A^0(q) = W_S^*(q), \quad (9)$$

$$W_A^{k+1}(q) = \begin{cases} \min \left( W_S^*(q), \min_{p \in \text{enab}_G(q)} \max_{q' \in G(q,p)} W_A^k(q') \right) & \text{if } \text{enab}_G(q) \neq \emptyset; \\ W_S^*(q) & \text{if } \text{enab}_G(q) = \emptyset; \end{cases} \quad (10)$$

for  $q \in Q$ ,  $k \in \mathbb{N}$ . We denote the fixed-point of (9)-(10) by  $W_A^*$ , see [6] for details of proofs.

It has been shown in [6] that for a finite system  $\Sigma = \Sigma^Q$ , there exists  $K \in \mathbb{N}$ , such that for all  $q \in Q$ , for all  $k \geq K$ ,  $W_S^k(q) = W_S^*(q)$  and  $W_A^k(q) = W_A^*(q)$ . Hence,  $W_S^*$  and  $W_A^*$  can be computed in practice. Then, let the function  $k^* : Q \rightarrow \mathbb{N}$  be defined as follows for all  $q \in Q$

$$k^*(q) = \min\{k \in \mathbb{N} \mid W_A^k(q) = W_A^*(q)\},$$

and for  $\delta \in \mathbb{R}$ , let us consider the next controller for  $\Sigma^Q = (Q, P, G)$

$$C^\delta(q) = \begin{cases} \arg \min_{p \in \text{enab}_G(q)} \left( \max_{q' \in G(q,p)} W_A^{k^*(q)-1}(q') \right) & \text{if } W_A^*(q) \leq \delta < W_S^*(q); \\ \left\{ p \in \text{enab}_G(q) \mid \max_{q' \in G(q,p)} W_S^*(q') \leq \delta \right\} & \text{if } W_S^*(q) \leq \delta; \\ \text{enab}_G(q) & \text{if } \delta < W_A^*(q). \end{cases} \quad (11)$$

We state the following claim that will be useful in further discussions and that follows directly from (8) and (11):

*Claim 9:* For all  $q \in Q$ ,  $C^\delta(q) = \emptyset$  if and only if  $\text{enab}_G(q) = \emptyset$ .

Therefore,  $\text{dom}(C^\delta) = \text{nbs}_G$ . We recall the following result of [6]:

*Theorem 10:* There exists  $T_0 \in \mathbb{N}$ , such that for all  $q_0 \in Q$  with  $W_A^*(q_0) \leq \delta$ , all maximal trajectories  $(q_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma^Q, C^\delta, q_0)$  are complete and satisfy

$$\forall t \geq T_0, W_S^*(q_t) \leq \delta.$$

The controller  $C^\delta$  is *least-violating* in the sense that  $W_A^*(q_0) \leq \delta$  if and only if there exists a controller for  $\Sigma^Q$  making the set  $\{q \in Q \mid h(q) \leq \delta\}$  uniformly attractive from  $q_0$ . Note that the control  $C^\delta$  is a set-valued map, where any control value  $u \in C^\delta(q)$  can be chosen when applying it to the system  $\Sigma^Q$ .

Hence, the set  $\mathcal{A} = \{q \in Q \mid W_S^*(q) \leq \delta\}$  is uniformly attractive for system  $\Sigma$  and controller  $C^\delta$  from all initial states  $q_0 \in \mathcal{B} = \{q \in Q \mid W_A^*(q) \leq \delta\}$ . We refer to  $\mathcal{A}$  as the *attractor* and to  $\mathcal{B}$  as the *basin of attraction*.

We end the section by pointing out that the controller  $C^\delta$  is set-valued and may provide more than one input, in particular for states belonging to the attractor.

#### IV. CONTROL SYNTHESIS

In this section, we present an approach for synthesizing controllers solving Problem 5. The main idea of our approach is to compute iteratively refinements of least violating attractivity controllers defined in the previous section, resulting in a nested sequence of attractors which are closer to the target set in each iteration. Let  $\{Q_\alpha\}_{\alpha \in \mathbb{N}_{<N}}$  be a sequence of index sets of partitions of  $\mathcal{X}$  and  $\{P_\alpha\}_{\alpha \in \mathbb{N}_{<N}}$  a sequence of finite subsets of  $\mathcal{U}$ , respectively, such that  $\{X_q\}_{q \in Q_{\alpha+1}} \sqsubseteq \{X_q\}_{q \in Q_\alpha}$  and  $P_\alpha \subseteq P_{\alpha+1}$ ,  $\alpha \in \mathbb{N}_{<N}$ . Refinements can be characterized in different ways, e. g. the refinements can be characterized by uniform partitions in such a way that at each iteration the length of the elements of the partition is reduced uniformly. It is also possible to include partition optimization by minimizing the number of transitions as [15], [20].

For each  $\alpha \in \mathbb{N}_{<N}$ , the system  $\Sigma_\alpha^Q = (Q_\alpha, P_\alpha, G_\alpha)$  is defined as in (3)-(4).

#### A. Initialization of controllers

For the system  $\Sigma_0^Q$ , we start by computing  $W_{0,S}^*$  and  $W_{0,A}^*$ , the fixed-point of equations (7)-(8) and (9)-(10), respectively, for  $Q = Q_0$ ,  $P = P_0$ , and  $G = G_0$ . Then, let us define

$$\ell_0 = \min_{q \in Q_0} W_{0,S}^*(q) \quad (12)$$

$$\mathcal{A}_0 = \left\{ q \in Q_0 \mid W_{0,S}^*(q) \leq \ell_0 \right\} \quad (13)$$

$$\mathcal{B}_0 = \left\{ q \in Q_0 \mid W_{0,A}^*(q) \leq \ell_0 \right\} \quad (14)$$

and let  $C_0$  be the controller for  $\Sigma_0^Q$  defined as in (11) for  $\delta = \ell_0$ . Let us define a quantization function associated to  $\Sigma_0^Q$  as

$$\theta_0 : \mathcal{X} \rightarrow Q_0 \quad (15)$$

where  $\theta_0(x) = q$  if  $x \in X_q$ .

The following result is obtained from Theorem 10.

*Proposition 11:* There exists  $T_0 \in \mathbb{N}$ , such that for all  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$ , all maximal trajectories  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\text{cont}}, C_0 \circ \theta_0, x_0)$  are complete and satisfy  $x_t \in \bigcup_{q \in \mathcal{A}_0} X_q$ , for all  $t \geq T_0$ .

*Proof:* Let us consider  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$  and a maximal trajectory  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\text{cont}}, C_0 \circ \theta_0, x_0)$ . Note that  $W_{0,A}^*(\theta_0(x_0)) \leq \ell_0$  by (14). According to Proposition 8, we have  $(\theta_0(x_t))_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_0^Q, C_0, \theta_0(x_0))$  and  $T = +\infty$  from Theorem 10. Let  $T_0 \in \mathbb{N}$  be given as in Theorem 10 for  $\delta = \ell_0$ , then for all  $t \geq T_0$ ,  $\theta_0(x_t) \in \mathcal{A}_0$ , implying  $x_t \in \bigcup_{q \in \mathcal{A}_0} X_q$ . ■

Notice that the controller  $C_0$  is set-valued and we can exploit this non-determinism for further design. Let us remark that the composed controller  $C^* := C_0 \circ \theta_0 : \mathcal{X} \rightrightarrows \mathcal{U}$  with  $\mathcal{B}^* = \bigcup_{q \in \mathcal{B}_0} X_q$  solves Problem 5 if  $\bigcup_{q \in \mathcal{A}_0} X_q \subseteq \mathcal{X}^*$  (that is if  $\ell_0 = 0$ ). If this is not the case, we can use a partition refinement inside the attractor  $\mathcal{A}_0$ , add control inputs that keep the attractor states inside the attractor, and refine the least-violating attractivity controller  $C_0$  to improve the closed loop behavior. This procedure can then be repeated iteratively, as shown in the next section.

#### B. Iterative refinement of controllers

For  $\alpha \in \mathbb{N}_{<N}$ , starting from  $\alpha = 0$  with  $\Sigma_{0,r}^Q = \Sigma_0^Q$ , let us define the systems

$$\Sigma_{\alpha+1,r}^Q = (Q_{\alpha+1}^r, P_{\alpha+1}^r, G_{\alpha+1}^r) \quad (16)$$

where  $Q_{\alpha+1}^r = (Q_\alpha^r \setminus \mathcal{A}_\alpha) \cup \mathcal{A}_\alpha^r$  with  $\mathcal{A}_\alpha^r = \{q' \in Q_{\alpha+1} \mid X_{q'} \subseteq \bigcup_{q \in \mathcal{A}_\alpha} X_q\}$ , and functions  $\rho_{\alpha+1} : Q_{\alpha+1}^r \rightarrow Q_\alpha^r$ ,  $G_{\alpha+1}^r : Q_{\alpha+1}^r \times P_{\alpha+1}^r \rightrightarrows Q_{\alpha+1}^r$  defined as

$$\rho_{\alpha+1}(q) = q' \text{ if } X_q \subseteq X_{q'}, \quad (17)$$

$$G_{\alpha+1}^r(q,p) = \begin{cases} G_{\alpha+1}(q,p) & \text{if } q \in \mathcal{A}_\alpha^r, G_{\alpha+1}(q,p) \subseteq \mathcal{A}_\alpha^r; \\ (G_\alpha^r(q,p) \setminus \mathcal{A}_\alpha) \cup \rho_{\alpha+1}^{-1}(G_\alpha^r(q,p) \cap \mathcal{A}_\alpha) & \text{if } q \in Q_\alpha^r \setminus \mathcal{A}_\alpha, p \in C_\alpha(q); \\ \emptyset; & \text{otherwise.} \end{cases} \quad (18)$$

Intuitively,  $\Sigma_{\alpha+1,r}^Q$  describes the dynamics  $\Sigma_{\alpha+1}^Q$  which is restricted to the attractor  $\mathcal{A}_\alpha$  where control inputs are restricted by the set  $P_{\alpha+1}$  and the controller  $C_\alpha$ . We then compute least violating attractivity controllers for systems  $\Sigma_{\alpha+1,r}^Q$  where distance functions

$$h_{\alpha+1} : Q_{\alpha+1}^r \rightarrow \mathbb{R}_{\geq 0} \quad (19)$$

proposed as (5) satisfy  $h_{\alpha+1}(q) \leq h_{\alpha}(\rho_{\alpha+1}(q))$  for all  $q \in Q_{\alpha+1}^r$ . In particular, the function  $h$  defined in (6) satisfies this condition. Now, let  $W_{\alpha+1,S}^*$  and  $W_{\alpha+1,A}^*$  be the fixed-point of equations (7)-(8) and (9)-(10), respectively, for system  $\Sigma_{\alpha+1,r}^Q$ . Then, let us define

$$\ell_{\alpha+1} = \max_{q \in \mathcal{A}_{\alpha}^r} W_{\alpha+1,A}^*(q) \quad (20)$$

$$\mathcal{A}_{\alpha+1} = \left\{ q \in \mathcal{A}_{\alpha}^r \mid W_{\alpha+1,S}^*(q) \leq \ell_{\alpha+1} \right\} \quad (21)$$

$$\mathcal{B}_{\alpha+1} = \left\{ q \in Q_{\alpha+1}^r \mid W_{\alpha+1,A}^*(q) \leq \ell_{\alpha+1} \right\} \quad (22)$$

and let  $C_{\alpha+1}$  be the controller for  $\Sigma_{\alpha+1,r}^Q$  defined as in (11) for  $\delta = \ell_{\alpha+1}$ .

We first prove the following instrumental properties:

*Lemma 12:* For  $\alpha \in \mathbb{N}_{<N}$ , the following assertions hold:

- (i)  $\mathcal{A}_{\alpha+1} \subseteq \mathcal{A}_{\alpha}^r$
- (ii)  $\forall q \in Q_{\alpha+1}^r, C_{\alpha}(\rho_{\alpha+1}(q)) \subseteq \text{enab}_{G_{\alpha+1}^r}(q)$ ,
- (iii)  $\forall q \in Q_{\alpha+1}^r, \forall p \in C_{\alpha}(\rho_{\alpha+1}(q)), \rho_{\alpha+1}(G_{\alpha+1}^r(q,p)) \subseteq G_{\alpha}^r(\rho_{\alpha+1}(q),p)$ ,
- (iv) the following relation  $R_{\alpha} : Q_{\alpha}^r \rightrightarrows Q_{\alpha+1}^r$  defined as  $R_{\alpha}(q) = \{q' \in Q_{\alpha+1}^r \mid q = \rho_{\alpha+1}(q')\}$  is a feedback refinement relation from  $\Sigma_{\alpha+1,r}^Q$  to  $\Sigma_{\alpha,r,C_{\alpha}}^Q$ ,
- (v)  $\forall q \in Q_{\alpha+1}^r, W_{\alpha+1,A}^*(q) \leq W_{\alpha,A}^*(\rho_{\alpha+1}(q))$ ;
- (vi)  $\rho_{\alpha+1}(\mathcal{B}_{\alpha+1}) = \mathcal{B}_{\alpha}$ ;
- (vii)  $\bigcup_{q \in \mathcal{B}_0} X_q = \bigcup_{q \in \mathcal{B}_{\alpha+1}} X_q$ .

*Proof:* Assertion (i) is a direct consequence of (21). Let us prove the assertion (ii). By (18), we have for all  $q \in Q_{\alpha}^r \setminus \mathcal{A}_{\alpha}, \rho_{\alpha+1}(q) = q$  and  $C_{\alpha}(q) \subseteq \text{enab}_{G_{\alpha+1}^r}(q)$ . Now, let us prove that for all  $q \in \mathcal{A}_{\alpha}^r, C_{\alpha}(\rho_{\alpha+1}(q)) \subseteq \text{enab}_{G_{\alpha+1}^r}(q)$ . Let  $q \in \mathcal{A}_{\alpha}^r$  and  $p \in C_{\alpha}(\rho_{\alpha+1}(q))$ , then  $\rho_{\alpha+1}(q) \in \mathcal{A}_{\alpha}$ . By (11), (20) and (21) for the system  $\Sigma_{\alpha,r}^Q$ , we have  $G_{\alpha}^r(\rho_{\alpha+1}(q),p) \subseteq \mathcal{A}_{\alpha}$ . For  $\alpha = 0$ , we have  $G_0^r = G_0$  and  $G_1(q,p) \subseteq \mathcal{A}_0^r$  because  $f(X_q,p) \subseteq f(X_{\rho_1(q)},p)$ . Therefore  $G_1^r(q,p) = G_1(q,p)$  and  $p \in \text{enab}_{G_1^r}(q)$ . For  $\alpha > 0$  and from (i), we have  $\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\alpha-1}^r$  which implies  $\rho_{\alpha+1}(q) \in \mathcal{A}_{\alpha-1}^r$ . From first condition in (18), we have  $G_{\alpha}(\rho_{\alpha+1}(q),p) \subseteq \mathcal{A}_{\alpha-1}^r$  which implies  $G_{\alpha}^r(\rho_{\alpha+1}(q),p) = G_{\alpha}(\rho_{\alpha+1}(q),p)$ . This implies  $G_{\alpha}(\rho_{\alpha+1}(q),p) \subseteq \mathcal{A}_{\alpha}$ , from (4), we have  $G_{\alpha+1}(q,p) \subseteq \mathcal{A}_{\alpha}^r$  because  $f(X_q,p) \subseteq f(X_{\rho_{\alpha+1}(q)},p)$ . Therefore,  $p \in \text{enab}_{G_{\alpha+1}^r}(q)$  by (18).

We now prove assertion (iii), let remark that for all  $q \in Q_{\alpha}^r \setminus \mathcal{A}_{\alpha}$ , we have  $\rho_{\alpha+1}(q) = q$ . Let  $q \in Q_{\alpha}^r \setminus \mathcal{A}_{\alpha}$  and  $p \in C_{\alpha}(q)$ , then we have  $G_{\alpha+1}^r(q,p) = (G_{\alpha}^r(q,p) \setminus \mathcal{A}_{\alpha}) \cup \rho_{\alpha+1}^{-1}(G_{\alpha}^r(q,p) \cap \mathcal{A}_{\alpha})$  by (ii) and (18). Then, we have  $\rho_{\alpha+1}(G_{\alpha+1}^r(q,p)) = (G_{\alpha}^r(q,p) \setminus \mathcal{A}_{\alpha}) \cup (G_{\alpha}^r(q,p) \cap \mathcal{A}_{\alpha}) = G_{\alpha}^r(q,p)$  because  $\rho_{\alpha}$  is a surjective function. Therefore,  $\rho_{\alpha+1}(G_{\alpha+1}^r(q,p)) = G_{\alpha}^r(\rho_{\alpha+1}(q),p)$ . Now, let  $q \in \mathcal{A}_{\alpha}^r$  and  $p \in C_{\alpha}(\rho_{\alpha+1}(q))$ , then  $G_{\alpha+1}^r(q,p) = G_{\alpha+1}(q,p)$  by (18). Note that  $f(X_q,p) \subseteq f(X_{\rho_{\alpha+1}(q)},p)$ , and from (18), we have  $\rho_{\alpha+1}(G_{\alpha+1}^r(q,p)) \subseteq G_{\alpha}(\rho_{\alpha+1}(q),p)$ . For  $\alpha = 0$ , we have  $G_0^r = G_0$ . For  $\alpha > 0$  and from (i), we have  $\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\alpha-1}^r$  which implies  $\rho_{\alpha+1}(q) \in \mathcal{A}_{\alpha-1}^r$ . From first condition in (18), we have  $G_{\alpha}(\rho_{\alpha+1}(q),p) \subseteq \mathcal{A}_{\alpha-1}^r$  which implies  $G_{\alpha}^r(\rho_{\alpha+1}(q),p) = G_{\alpha}(\rho_{\alpha+1}(q),p)$ . Then, we have  $\rho_{\alpha+1}(G_{\alpha+1}^r(q,p)) \subseteq G_{\alpha}^r(\rho_{\alpha+1}(q),p)$  and (iii) is proved.

Assertion (iv) is a direct consequence of (ii) and (iii). Let  $q_0 \in Q_{\alpha+1}^r$  such that  $W_{\alpha,A}^*(\rho_{\alpha+1}(q_0)) < +\infty$  and consider a maximal trajectory  $(q_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\alpha+1,r}^Q, C_{\alpha} \circ \rho_{\alpha+1}, q_0)$  by (ii). From assertion (iv), we have  $(\rho_{\alpha+1}(q_t))_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\alpha,r,C_{\alpha}}^Q, C_{\alpha}, \rho_{\alpha+1}(q_0)) = \mathcal{T}_{\max}(\Sigma_{\alpha,r}^Q, C_{\alpha}, \rho_{\alpha+1}(q_0))$ . From Theorem 10 and (19), we have that there exists  $T_0 > 0$  such that for all  $t \geq T_0, h_{\alpha+1}(q_t) \leq h_{\alpha}(\rho_{\alpha+1}(q_t)) \leq W_{\alpha,S}^*(\rho_{\alpha+1}(q_t)) \leq W_{\alpha,A}^*(\rho_{\alpha+1}(q_0))$ . This implies, by the second part of Theorem 10,  $W_{\alpha+1,A}^*(q_0) \leq W_{\alpha,A}^*(\rho_{\alpha+1}(q_0))$

because  $C_{\alpha+1}$  is a least-violating controller for  $\Sigma_{\alpha+1}^r$ . So, the assertion (v) is proved.

Note that for all  $q \in Q^r \setminus \mathcal{A}_{\alpha}$  we have  $\rho_{\alpha+1}(q) = q$  and  $\text{enab}_{G_{\alpha+1}^r}(q) = C_{\alpha}(q)$  by (18). To prove the assertion (vi), we prove two inclusions. Let's first prove  $\rho_{\alpha+1}(\mathcal{B}_{\alpha+1}) \subseteq \mathcal{B}_{\alpha}$ . for all  $q \in \mathcal{B}_{\alpha+1} \cap \mathcal{A}_{\alpha}$ , we have  $\rho_{\alpha+1}(q) \in \mathcal{A}_{\alpha}$ , and from (13)-(14), (21)-(22), we have  $\mathcal{A}_{\alpha} \subseteq \mathcal{B}_{\alpha}$  because  $W_S^* \leq W_A^*$  holds. Now, let  $q_0 \in \mathcal{B}_{\alpha+1} \cap (Q_{\alpha}^r \setminus \mathcal{A}_{\alpha})$  and  $(q_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\alpha+1,r}^Q, C_{\alpha+1}, q_0)$ . Then, there exists  $T' > 0$  such that for all  $t < T', q_t \in Q_{\alpha}^r \setminus \mathcal{A}_{\alpha}, C_{\alpha+1}(q_t) = C_{\alpha}(q_t)$  and  $q_{T'} \in \mathcal{A}_{\alpha}^r$ , because  $\mathcal{A}_{\alpha+1} \subseteq \mathcal{A}_{\alpha}^r$  by (i). As  $C_{\alpha}$  is a least violating controller for  $\Sigma_{\alpha}^r$  we obtain  $W_{\alpha,A}^*(q_0) \leq \ell_{\alpha}$ . Therefore, for all  $q \in \mathcal{B}_{\alpha+1}, \rho_{\alpha+1}(q) \in \mathcal{B}_{\alpha}$ . Now let us prove the inclusion  $\mathcal{B}_{\alpha} \subseteq \rho_{\alpha+1}(\mathcal{B}_{\alpha+1})$ . From (ii), let  $q'_0 \in \mathcal{B}_{\alpha}$  and  $(q_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\alpha+1,r}^Q, C_{\alpha} \circ \rho_{\alpha+1}, q_0)$  such that  $\rho_{\alpha+1}(q_0) = q'_0$ . From assertion (iv), we have  $(\rho_{\alpha+1}(q_t))_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{\alpha,r,C_{\alpha}}^Q, C_{\alpha}, \rho_{\alpha+1}(q_0)) = \mathcal{T}_{\max}(\Sigma_{\alpha,r}^Q, C_{\alpha}, \rho_{\alpha+1}(q_0))$ . Then there exists  $T_0 \geq 0$  such that for all  $t \geq T_0, \rho_{\alpha+1}(q_t) \in \mathcal{A}_{\alpha}^r$ . This implies for all  $t \geq T_0, q_t \in \mathcal{A}_{\alpha}^r$  and  $W_{\alpha+1,A}^*(q_t) \leq \ell_{\alpha+1}$  by (20). As  $C_{\alpha+1}$  is a least violating controller for  $\Sigma_{\alpha+1,r}^Q$ , we have that  $W_{\alpha+1,A}^*(q_0) \leq \ell_{\alpha+1}$  and the inclusion holds. Therefore, the assertion (vi) is proved. Assertion (vii) is a direct consequence of (vi). ■

The next result shows that the construction of the systems  $\Sigma_{\alpha,r}^Q$ 's preserves a feedback refinement relation in each iteration, which allows the designer to work with  $\Sigma_{\alpha,r}^Q$  instead of the system  $\Sigma_{cont}$ .

*Proposition 13:* Let  $\Sigma_{cont}$  and  $\Sigma_{\alpha,r}^Q$  be transition systems as in (1) and (16) for  $\alpha \in \mathbb{N}_{<N}$ . Then, the relation  $R : \mathcal{X} \rightrightarrows Q_{\alpha}^r$  defined as  $R(x) = \{q \in Q_{\alpha}^r \mid x \in X_q\}$  for all  $x \in \mathcal{X}$  is a feedback refinement relation from  $\Sigma_{cont}$  to  $\Sigma_{\alpha,r}^Q$ .

*Proof:* Let us remark that we have for all  $\alpha \in \mathbb{N}_{<N}, \bigcup_{q \in \mathcal{A}_{\alpha}} X_q = \bigcup_{q \in \mathcal{A}_{\alpha}^r} X_q$ . From the partition representations on  $\mathcal{X}$  given by the sets  $Q_{\alpha}^r$ 's, we have that for all  $x \in \mathcal{X}$ , there exists  $q \in Q_{\alpha}^r$  such that  $x \in X_q$ , therefore  $q \in R(x)$ . Now, let  $x \in \mathcal{X}$  and  $q \in R(x)$ , from (4), we have  $\text{enab}_{G_{\alpha}^r}(q) \subseteq \text{enab}_f(x)$ . Let  $u \in \text{enab}_{G_{\alpha}^r}(q), x' \in f(x,u)$  and  $q' \in Q_{\alpha}^r$  such that  $x' \in X_{q'}$ , from (4), we have  $q' \in R(x')$ . ■

### C. Algorithm of refinement attractors and controllers

Our overall approach to synthesize attractivity controllers can be summarized in Algorithm 1.

*Theorem 14:* For  $\alpha \in \mathbb{N}_{<N}$ , there exists  $T^{\alpha} \in \mathbb{N}$ , such that for all  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$ , all maximal trajectories  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{cont}, C_{\alpha} \circ \theta_{\alpha}, x_0)$  are complete, where  $\theta_{\alpha} : \mathcal{X} \rightarrow Q_{\alpha}^r$  is a quantization function defined as  $\theta_{\alpha}(x) = q$  if  $x \in X_q$ , and the following holds  $x_t \in \bigcup_{q \in \mathcal{A}_{\alpha}} X_q$ , for all  $t \geq T^{\alpha}$ .

*Proof:* We proceed by induction. The case  $\alpha = 0$  corresponds to Proposition 11. Then, let us assume that the property is true for some  $\alpha \in \mathbb{N}_{<N}$ , that is that there exists  $T^{\alpha} \in \mathbb{N}$ , such that for all  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$ , all maximal trajectories  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{cont}, C_{\alpha} \circ \theta_{\alpha}, x_0)$  are complete and satisfy  $x_t \in \bigcup_{q \in \mathcal{A}_{\alpha}} X_q$ , for all  $t \geq T^{\alpha}$ . By (18), we have for all  $q \in Q_{\alpha}^r \setminus \mathcal{A}_{\alpha}, C_{\alpha+1}(q) \subseteq C_{\alpha}(q)$  and from Lemma 12, for all  $q \in Q_{\alpha+1}^r = Q_{\alpha}^r \setminus \mathcal{A}_{\alpha} \cup \mathcal{A}_{\alpha}^r, C_{\alpha}(\rho_{\alpha+1}(q)) \subseteq \text{enab}_{G_{\alpha+1}^r}(q)$ . Then it follows that  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$ , all maximal trajectories  $(x_t)_{t=0}^T \in \mathcal{T}_{\max}(\Sigma_{cont}, C_{\alpha+1} \circ \theta_{\alpha+1}, x_0)$  are complete and satisfy  $x_t \in \bigcup_{q \in \mathcal{A}_{\alpha}} X_q$ , for all  $t \geq T^{\alpha}$ . Note that  $\bigcup_{q \in \mathcal{A}_{\alpha}} X_q = \bigcup_{q \in \mathcal{A}_{\alpha}^r} X_q$ . Moreover, from Proposition 13, it follows that  $(\theta_{\alpha+1}(x_t))_{t=0}^{T^{\alpha}} \in \mathcal{T}_{\max}(\Sigma_{\alpha+1,r}^Q, C_{\alpha+1}, \theta_{\alpha+1}(x_0))$  and  $\theta_{\alpha+1}(x_t) \in \mathcal{A}_{\alpha}^r$  for  $t \geq T^{\alpha}$ . Note that  $\theta_{\alpha+1}(x_{T^{\alpha}}) \in \mathcal{A}_{\alpha}^r$  implies that  $W_{\alpha+1,A}^*(\theta_{\alpha+1}(x_{T^{\alpha}})) \leq \ell_{\alpha+1}$  by (20). Therefore it follows that  $W_{\alpha+1,S}^*(\theta_{\alpha+1}(x_t)) \leq \ell_{\alpha+1}$ , for all  $t \geq T^{\alpha} + T'$ , where  $T'$  is the uniform time bound given by Theorem 10.

**Algorithm 1:** Synthesis of attractivity controllers

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**Input:** Sequence  $\Sigma_\alpha^Q = (Q_\alpha, P_\alpha, G_\alpha)$ ,  $\alpha = 0 \dots N$  such that  $\{X_q\}_{q \in Q_{\alpha+1}} \sqsubseteq \{X_q\}_{q \in Q_\alpha}$ ,  $P_\alpha \subseteq P_{\alpha+1}$ ; target set  $\mathcal{X}^* \subseteq \mathcal{X}$ ;

**Output:** Controller  $C$ ; attractor  $\mathcal{A}$ ; basin of attraction  $\mathcal{B}$ ;

// Initialization

- 1 Compute  $W_{0,S}^*$  and  $W_{0,A}^*$ , the fixed-points of (7)-(8) and (9)-(10) for system  $\Sigma_0$ ;
- 2 Compute  $\ell_0, \mathcal{A}_0, \mathcal{B}_0$  given by (12), (13), (14);
- 3 Compute  $C_0$  the controller for  $\Sigma_0^Q$  given by (11) with  $\delta = \ell^0$ ;
- 4  $\Sigma_{0,r}^Q = \Sigma_0^Q$ ;
- // Iterative refinement
- 5 **for**  $\alpha = 0, \dots, N-1$  **do**
- 6     Compute system  $\Sigma_{\alpha+1,r}^Q = (Q_{\alpha+1}^r, P_{\alpha+1}^r, G_{\alpha+1}^r)$  as in (16) where  $G_{\alpha+1}^r$  is given by (18);
- 7     Compute  $W_{\alpha+1,S}^*$  and  $W_{\alpha+1,A}^*$ , the fixed-points of (7)-(8) and (9)-(10) for system  $\Sigma_{\alpha+1,r}^Q$ ;
- 8     Compute  $\ell_{\alpha+1}, \mathcal{A}_{\alpha+1}$  given by (20), (21);
- 9     Compute  $C_{\alpha+1}$  the controller for  $\Sigma_{\alpha+1,r}^Q$  given by (11) with  $\delta = \ell_{\alpha+1}$ ;
- 10  $C := C_N, \mathcal{A} := \mathcal{A}_N, \mathcal{B} := \mathcal{B}_0$ ;

---

Letting  $T^{\alpha+1} = T^\alpha + T'$ , we get from (21) that  $x_t \in \bigcup_{q \in \mathcal{A}_{\alpha+1}} X_q$ , for all  $t \geq T^{\alpha+1}$ . ■

We can now state the main result of the paper:

*Theorem 15:* Algorithm 1 terminates in a finite number of steps and the following properties hold:

- $\mathcal{A}^* = \bigcup_{q \in \mathcal{A}_N} X_q \subseteq \mathcal{X}$  is attractive for system  $\Sigma_{cont}$  defined in (1) and controller  $C_N \circ \theta_N$  from all initial states  $x_0 \in \bigcup_{q \in \mathcal{B}_0} X_q$ .
- If  $\bigcup_{q \in \mathcal{A}_N} X_q \subseteq \mathcal{X}^*$ , the controller  $C^* := C_N \circ \theta_N$  is solution to Problem 5 with  $\mathcal{B}^* := \bigcup_{q \in \mathcal{B}_0} X_q$ .

*Proof:* Since for all  $\alpha \in \mathbb{N}_{\leq N}$   $Q_\alpha$  and  $Q_\alpha^r$  are finite sets because these are index sets of partitions on  $\mathcal{X}$ , and  $P_\alpha \subseteq \mathcal{U}$  is a finite set, the fix point computations converge in a finite number of steps according to [6]. It follows that Algorithm 1 terminates in a finite number of iterations. The first item is a direct consequence of Theorem 14 and the second item follows from the first item. ■

Our approach thus computes successive refinements of least-violating controllers  $C_\alpha$  associated to a sequence of nested attractors  $\mathcal{A}_\alpha$ . This allows us to build attractors that get closer to the target set  $\mathcal{X}^*$ , and that may eventually be contained in  $\mathcal{X}^*$ .

## V. EXAMPLE: REACH-AVOID PROBLEM FOR A MARINE VESSEL

In this section, we show an illustrative application of our results to a model of a marine vessel. The kinematic model on continuous time of a marine vessel is given by:

$$\begin{cases} \dot{x}_1 = u_1 \cos(x_3) - u_2 \sin(x_3) \\ \dot{x}_2 = u_1 \sin(x_3) + u_2 \cos(x_3) \\ \dot{x}_3 = u_3 \end{cases} \quad (23)$$

As sketched in Fig. 1, the state  $x \in \mathbb{R}^3$  describes the planar position  $(x_1, x_2)$  and the heading  $x_3$  of the marine vessel, the control inputs  $u \in \mathbb{R}^3$  are the surge velocity, sway velocity, and yaw rate.

The continuous-time dynamics of (23) is periodically sampled with period 1 min. We impose the state constraint  $\mathcal{X} = [-3.5 \ 3.5] \times [-2.6 \ 2.6] \times [-\pi \ \pi]$ , the input constraint  $\mathcal{U} = [-0.7 \ 2] \times [-1 \ 1] \times [-1 \ 1]$

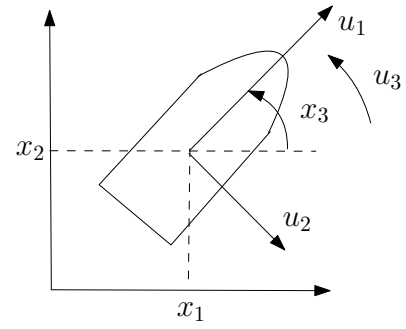


Fig. 1. States and control inputs of a kinematic ship model

and two static obstacles  $Obst_1 = [0.7 \ 1] \times [0.7 \ 2.6] \times [-\pi \ \pi]$ ,  $Obst_2 = [-1.5 \ -1.3] \times [-2.6 \ -1] \times [-\pi \ \pi]$ . The control objective is to stabilize the system in the set  $\mathcal{X}^* = [-1.2 \ 0.1] \times [-1.2 \ 0.1] \times [-\pi/5 \ \pi/5]$  in finite time while staying within the safe space  $\mathcal{X} \setminus (Obst_1 \cup Obst_2)$ . For the abstraction and control synthesis, we used the approach presented in Section III to compute attractors and least-violating controllers. For that propose, we use uniform partitions of the state intervals in  $[-3.5 \ 3.5] \times [-2.6 \ 2.6] \times [-\pi \ \pi]$  in 15, 11 and 10 sub-intervals, respectively. We select 6, 5, 5 input elements uniformly distributed in each intervals of  $\mathcal{U}$ . We synthesize a least-violating controller for the system to obtain an attractor  $\mathcal{A}_0$ , see green set in Figure 2 and Figure 3. Note that  $\mathcal{A}_0$  is not inside the target set, blue rectangle in Figure 3. Then, we refine the partition inside  $\mathcal{A}_0$  partitioning each hyper rectangle into  $4^4$  sub hyper rectangles and keeping the discretizations of the input. We synthesize a least-violating controller for the obtained system and an attractor  $\mathcal{A}_1$ , see red set in Figure 2 and Figure 3.

The overall computation took about 81.42 seconds (CPU: 2.3 GHz Intel Core i7, RAM: 16 Go 3733 MHz LPDDR4X, Matlab R2019b), with 69.04 seconds spent on computing the abstraction and the control synthesis for the first partition, 12.38 seconds spent on computing the abstraction and the control synthesis for the system with a refinement of partition. We show on Figure 2 the sets that have been computed, where the yellow and green regions represent the basin of attraction and attractor of a first abstraction, and the red region corresponds to the attractor in a partition refinement.

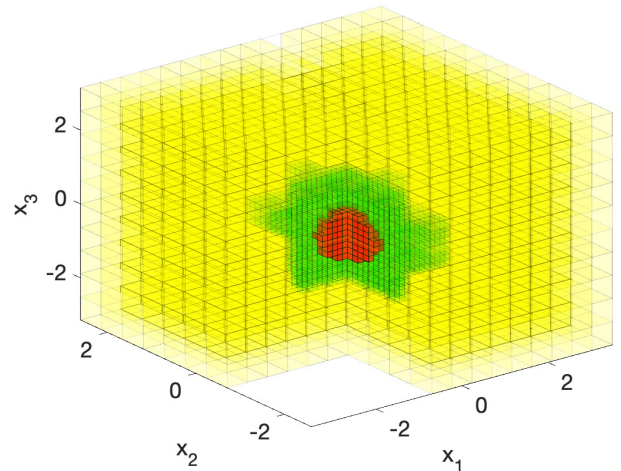


Fig. 2. State space decomposition for system (23): (yellow) the basin of abstraction; (green) attractor for the first partition; (red) attractor in a partition refinement.

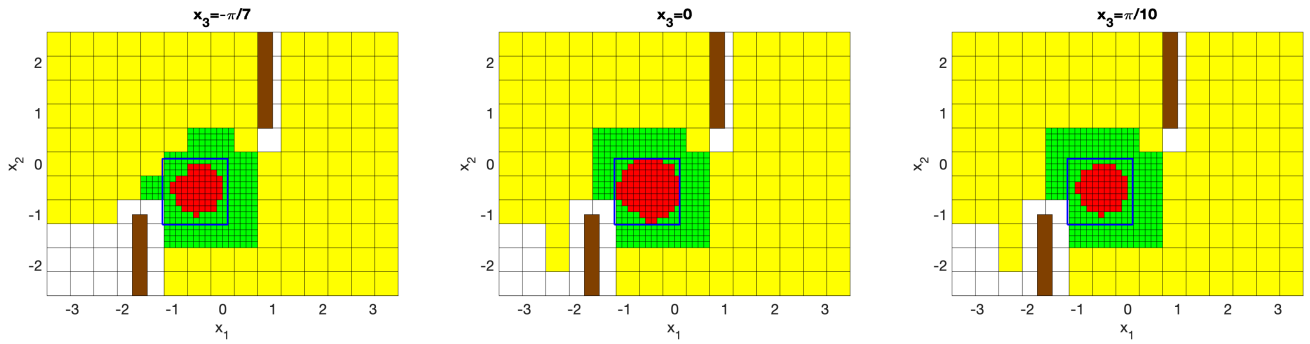


Fig. 3. White set consists of the states from which the safety requirements can not be guaranteed; yellow set contains the states which will reach the first attractor (green); green set contains the states which will reach the attractor based on a partition refinement (red); the brown rectangles represent the obstacles; the blue square describes the boundary of the target set.

It is shown in Figure 3, where the slices are computed at different values of  $x_3$ :

- The white set consists of the states from which the safety requirements can not be guaranteed.
- The yellow represents the basin of attraction  $\mathcal{B}$  described in (14). All trajectories starting in this set reach the green set which represents the initial attractor  $\mathcal{A}_0$  in (13).
- The red set represents the attractor  $\mathcal{A}_1$  described in (21) which is based on a refinement of partition. All trajectories starting in  $\mathcal{A}_0$  reach the set  $\mathcal{A}_1$  and all trajectories starting in this set stay there forever.

The trajectories in closed-loop dynamics are the closer to the target set as measured by distance  $h$  in (6). For the simulation in closed-loop system, we consider the following scenario, the initial value of  $(x_1, x_2, x_3)$  is  $(2, 2, \pi/3)$ . Figure 4 shows a simulation in the state space where the trajectory (black line) reaches the target set (inside blue rectangle) avoiding the obstacles (brown rectangles). In Figure

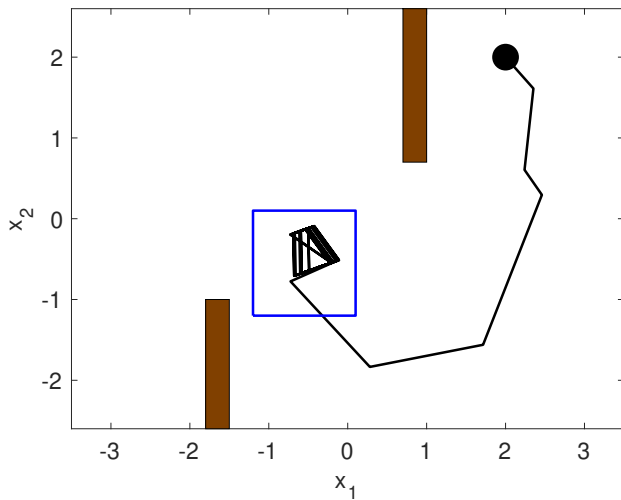


Fig. 4. Simulation of the closed-loop system with initial condition  $x_0 = [2 \ 2 \ \pi/3]^T$  plotted in the state space. The blue rectangle represents the target set.

5, we show a simulation in closed-loop of the system (23) using Algorithm 1 and the attractivity controller given in Theorem 15. We can see on this simulation, that the system behaves as expected.

Table I illustrates a runtime comparison of control abstraction and control synthesis based on [6] and the refinement approach proposed in this work. For the marine vessel system, our partition refinement

(R) approach uses two partitions (fine and coarse partitions) which demands 81.42 seconds, while using results in [6] with the fine partition as uniform partition (U) demands 5592.59 seconds (68 times more than the time for (R)).

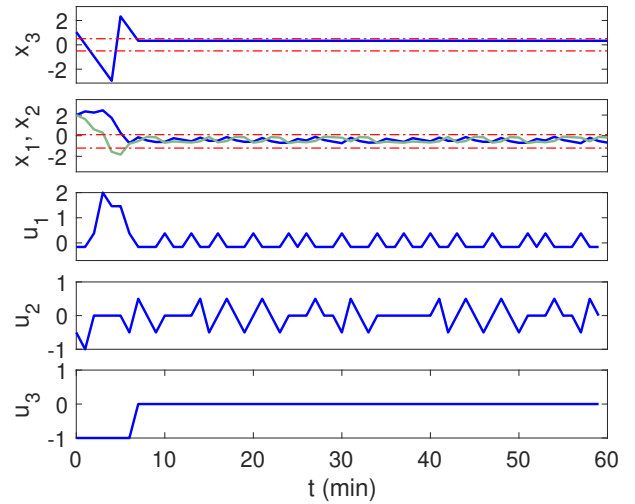


Fig. 5. Simulation (state and input evolutions) of the closed-loop system with initial condition  $x_0 = [2 \ 2 \ \pi/3]^T$ , the dashed line denotes the edge of the target set.

Method	Abstraction	Control synthesis	Total
Uniform (U)	154.03 s	5438.56 s	5592.59 s
Refinements (R)	3.51 s	77.91 s	81.42 s
Comparison U/R	43.88	69.81	68.69

TABLE I

RUNTIME COMPARISON WHEN USING UNIFORM PARTITION AND PARTITION REFINEMENT FOR THE SYSTEM (23).

## VI. CONCLUSIONS

In this work, we presented a novel nested approach based on partitions of the state and input spaces. This approach is based on iterative refinements of the partitions and of least-violating attractivity controllers constrained to the associated attractor. We presented an algorithm for systems with finite state and input spaces and we have shown, using how our approach can be used to design controllers for

systems with infinite state and input spaces using nested symbolic abstractions.

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