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Semi-classical Lagrangian intersections

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Abstract: We investigate normal forms of germs of glancing Lagrangian intersections, and their semi-classical counterparts. This is motivated by singularities of Bessel beams, or diffraction problems by an obstacle with a non convex boundary.

0/ Introduction

Let $M = \mathbf{R}^n$, we restrict essentially to the case $n = 2$.

Parametrices of linear PDE's on M make use of symbolic calculus adapted to Lagrangian intersection, see [MelUhl] for high frequency asymptotics, and [AnDoNaRo] for semi-classics in the framework of Maslov canonical operator.

Let $H(x, hD_x; h)$ be a h-PDO whose symbol has the asymptotic expansion $H(x, p; h) \sim H_0(x, p) + hH_1(x, p) + \cdots$, and assume that $H_0(x, p) = 0$ is a smooth, non critical hypersurface, fibrated by Hamiltonian curves. To start with, we shall denote H_0 simply by H.

On the other hand, let $\Lambda \to T^*M$ be a smooth embedded Lagrangian manifold, and f_h be a Lagrangian semi-classical distribution microlocally supported on Λ.

The main concern is to construct the asymptotic outgoing solution of

$$H(x, hD_x; h)u_h(x) = f_h(x),\ u_h(x) = E_+ f_h(x; h) = \int_0^\infty e^{-itH/h} f_h(x)\, dt \tag{0.2}$$

and in particular to solve Cauchy problem

$$hD_t v_h + H(x, hD_x)v_h = 0,\ v_h|_{t=0} = f_h \tag{0.3}$$

The local theory amounts to study germs of Lagrangian manifolds and hypersurfaces.

At the classical level the question is about a pair consisting of a Lagrangian submanifold Λ (we call the "initial manifold") and a hypersurface $G = \{H = 0\}$. We are interested in the set Λ_1 of characteristics of H intersecting Λ. Global aspects of Lagrangian intersection theory were greatly motivated by Arnold conjectures [Ar] and have been extensively studied (see [ElGr]). Here we simply focus on the germs of the generating functions near an intersection point.

The generic case is when v_H is transverse to Λ at $t = 0$, then Λ_1 is precisely the union of the outgoing/incoming flows $\Lambda_\pm$ of Λ by the Hamilton vector field v_H in G, see Theorem 1.1 below. This is the situation considered in [MelUhl] and [AnDoNaRo] : Λ and Λ_+ intersect cleanly along $\Lambda \cap \Lambda_+ = \partial\Lambda_+$, and (Λ, Λ_+) is called an *intersecting pair* of Lagrangian manifolds.

Here we want to consider a point $z \in \Lambda$ where the intersection is not clean, i.e. $v_G \in T_z\Lambda$. We call such a point a *glancing point*, and denote their set by $\mathcal{G}(\Lambda, G)$. We answer first the question :

• What is the classification of such pairs (Λ, G) up to symplectic diffeomorphisms ?

Normal forms which deals with isolated points of non-transversality, called a *kiss* ([ElGr,p.19]. This is "the generic case", which has been investigated in [ZaMy], using Morse families. We recall the most simple type of kiss in Theorem 1.2 below.

In case G is tangent to Λ along a non-singular hypersurface $\ell \subset \Lambda$, we call (Λ, ℓ, H) a *triad*.

Of course, such a pair uniquely defines the Lagrangian submanifold Λ_1.

Next we consider the semi-classical level, and try to represent the solution of (0.3) and (0.2) near a glancing point. The first step is to construct the solution of Hamilton-Jacobi equation parametrizing Λ_1, which is generically singular.

A similar problem arises in diffraction theory for the wave equation $H = \partial_t^2 - \Delta_x$ in $M = (\mathbf{R}_x^n \setminus \mathcal{O}) \times \mathbf{R}_t$ outside an obstacle $\mathcal{O}$: the question here is about the classification of glancing hypersurfaces (F, G) in T^*M at some point z. Here $G = \{\tau^2 - \xi^2 = 0\}$ and F stands for the lift of the boundary of $\mathcal{O}$ in T^*M. It turns out that F and G play symmetrical roles.

A theorem of Melrose [Hö,Theorem 21.4.8] says that such a pair, near the glancing point z, is symplectically equivalent to $F = \{x_1 = 0\}$ and $G = \{g = \xi_1^2 - x_1 - \xi_2 = 0\}$.

The second main question we adress here is the following:

• Given a pair of glancing hypersurfaces (F, G) in T^*M at some point z, find the germs at z of some Lagrangian manifolds Λ transverse to F at z and such that $v_g \in T_z\Lambda$?

There are indeed such one-parameter families of Λ, for which we compute $A_z(\Lambda, G), B_z(\Lambda, G)$, see Propositions 3.1-3.2 below.

Next we construct solutions of Hamilton-Jacobi equation for $hD_t + (hD_{x_1})^2 - x_1 - hD_{x_2}$ with initial condition on Λ. This, conveniently adapted to dimensions $n \geq 3$, could be applied to the diffractive Cauchy problem above when the obstacle is not necessarily convex at a point of the boundary, see [Le], [Sm].

1/ Lagrangian intersections

First recall the situation where v_H is transverse to Λ.

Let $\iota_0 : \Lambda_0 \to T^*M$ be a smooth embedded Lagrangian manifold, and $\iota_1 : \Lambda_1 \to T^*M$ be a smooth embedded Lagrangian manifold with smooth boundary $\partial\Lambda_1$ (isotropic manifold). Following [MelUhl] we say that (Λ_0, Λ_1) is an *intersecting pair* of Lagrangian manifolds iff $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$ and the intersection is *clean*, i.e.

$$\forall z \in \partial\Lambda_1 \quad T_z\Lambda_0 \cap T_z\Lambda_1 = T_z\partial\Lambda_1 \tag{1.1}$$

(in particular Λ_0 and Λ_1 cannot be transverse). On the set of intersecting pairs of Lagrangian manifolds we define an equivalence relation by saying that $(\Lambda_0, \Lambda_1) \sim (\Lambda_0', \Lambda_1')$ iff near any $z \in \partial\Lambda_1$, $z' \in \partial\Lambda_1'$, there is a symplectic map κ such that $\kappa(z) = z'$, and a neighbhd $V \subset T^*M$ of z such that $\kappa(\Lambda_0 \cap V) \subset \Lambda_0'$, $\kappa(\Lambda_1 \cap V) \subset \Lambda_1'$. We will call the equivalence class a *Lagrangian pair*. The following result readily extends [MelUhl, Prop.1.3] in the special case of homogeneous Lagrangian manifolds :

Theorem 1.1 [MelUhl]: *All intersecting pairs of manifolds in T^*M are locally equivalent. More precisely near each $z \in T^*M$, there exists a canonical map $\kappa : T^*M \to T^*\mathbf{R}^n$ such that $\kappa(z) = (0, 0)$,*

$\kappa(\Lambda_0 \cap V) \subset T_0^*\mathbf{R}^n$ (vertical fiber at 0), and $\kappa(\Lambda_1 \cap V) \subset \Lambda_+^0$, Λ_+^0 being the flow-out of $T_0^*\mathbf{R}^n$ by the Hamilton vector field $v_{\xi_n} = ((0, \cdots, 0, 1), 0)$ of ξ_n, passing through some $(0; \xi) = (0; (\xi', 0))$ i.e.

$$\Lambda_+^0 = \{(x, \xi) \in T^*\mathbf{R}^n : x = (0, x_n), \xi = (\xi', 0), x_n > 0\} \tag{1.5}$$

Lemma 1.1 has its semi-classical counterpart : near a point z of intersecting pair the normal form for Λ is given by $\{x = 0\}$ and this of H is (say) hD_{x_n}. This solves locally problem (0.2). Namely, let f_h be a Lagrangian distribution defined by its amplitude $a(x', x_n - t, \xi; h)$, and supported microlocally on the "vertical plane" $\Lambda = \{x = 0\}$. Let $\theta_T \in C^\infty(\mathbf{R}_+)$ vanish near $+\infty$ and equal to 1 for $t \leq T$, then

$$u(x, h) = \frac{i}{h} \int_0^\infty \theta_T(t)\, dt \int e^{i(x'\xi' + (x_n - t)\xi_n)/h} a(x', x_n - t, \xi; h)\, d\xi \tag{1.6}$$

satisfies (0.2) mod $\mathcal{O}(h^\infty)$ whenever $x_n \leq T/2$.

Assume instead that Hamilton foliation of H is simply tangent to Λ at some $z_0 \in \Lambda \cap \{H = 0\}$. Then Λ_1 is a singular Lagrangian manifold near z_0, and the intersection $\Lambda \cap \Lambda_1$ is not clean. Such a point z_0 of non-transversality is called *glancing*.

Near a glancing point Theorem 1.1 does not apply. So we need to extend (locally) Theorem 1.1 to this situation, by finding the normal form of non Lagrangian (or "glancing") intersection for (Λ, G) near z_0.

We have the following :

Theorem 1.2 [ZaMy]: *Assume the germs (Λ, G) are non-transversal at an isolated point $z_0 \in T^*M$, then (Λ, G) has generically the following normal form, up to a symplectic transformation $\kappa_\pm$*

$$\Lambda = \{\xi = \frac{\partial \phi_\pm}{\partial x}\}, \quad G = \{g = 0\} \tag{1.7}$$

where $g(x, \xi) = \xi_1$ and the generating functions $\phi_\pm(x) = \pm\frac{x_1^3}{3} + x_1 x_2^2$.

The proof is based on the theory of Morse families, see e.g. [BaWe].

In the case of $\phi_-(x)$ we get the two isotropic lines $x_1 = \pm x_2$, $\xi_2 = \pm 2x_2^2$, $\xi_1 = 0$ in the intersection of Λ with the Hamiltonian level $\xi_1 = 0$. Their extension along the characteristics $\xi_1 = \xi_2 = x_2 = 0$ ($v_H = (1, 0; 0, 0)$ at z_0) is the union of two Lagrangian submanifolds $\{\xi_1 = 0, \xi_2 = \pm 2x_2^2\}$.

In the case of $\phi_+(x)$, the intersection of Λ with $\xi_1 = 0$ reduces to $x = 0$.

We are now to define some invariants of (Λ, G) at $z = z_0$.

Definition 1.3: *Define Λ locally near z by $f_1 = f_2 = 0$, with $\{f_i, f_j\} = 0$. Consider the symmetric matrix*

$$A_z = A_z(\Lambda, G) = \begin{pmatrix} \{f_2, \{f_2, g\}\} & -\{f_1, \{f_2, g\}\} \\ -\{f_2, \{f_1, g\}\} & \{f_1, \{f_1, g\}\} \end{pmatrix}(z)$$

and the vector

$$B_z = B_z(\Lambda, G) = \begin{pmatrix} \{g, \{g, f_1\}\} \\ \{g, \{g, f_2\}\} \end{pmatrix}(z)$$

Let z be a glancing point for the pair (Λ, G) i.e.

$$g(z) = f_1(z) = f_2(z) = 0,\ \{g, f_1\}(z) = \{g, f_2\}(z) = 0$$

We distinguish the following 10 possibilities for the 2-jets of $(\Lambda, G)_z$:

(1) $\det A_z > 0,\ B_z \neq 0$

(2) $\det A_z > 0,\ B_z = 0$

(3) $\det A_z < 0,\ {}^tB_zA_zB_z \neq 0$

(4) $\det A_z < 0,\ {}^tB_zA_zB_z = 0,\ B_z \neq 0$

(5) $\det A_z < 0,\ B_z = 0$

(6) $\det A_z = 0,\ {}^tB_zA_zB_z \neq 0$

(7) $\det A_z = 0,\ A_z \neq 0,\ {}^tB_zA_zB_z = 0,\ B_z \neq 0$

(8) $\det A_z = 0,\ A_z \neq 0,\ B_z = 0$

(9) $A_z = 0,\ B_z \neq 0$

(10) $A_z = 0,\ B_z = 0$

Remark 1.1: Of course, A_z and B_z are invariant under symplectic transformations, but they are not uniquely defined. However each of the cases above is preserved by a non-degenerate linear transformation of the type $\binom{f_1'}{f_2'} = P\binom{f_1}{f_2}$, where $P \in \mathrm{GL}(2)$ has variable coefficients. Namely if A_z' is defined as A_z with f_1', f_2' instead of f_1, f_2, then

$$A_z' = (\det P_z)^2\, {}^tP_z^{-1}A_zP_z^{-1},\ B_z' = P_zB_z$$

However A_z and B_z cannot account for higher order terms occurring in the classification of [ZyMa].

Under the hypothesis of Theorem 1.2, an elementary computation shows that $f_1 = \xi_1 \mp x_1^2 - x_2^2, f_2 = \xi_2 - 2x_1x_2, g = \xi_1$:

$$A_0^{\pm} = \begin{pmatrix} 2 & 0 \\ 0 & \pm 2 \end{pmatrix}, \quad B_{\pm}(0) = \begin{pmatrix} \mp 2 \\ 0 \end{pmatrix}$$

so we are in case (1) for $+$, and in case (3) for $-$.

We turn now to the semi-classical level, and try to solve (0.2) or (0.3). Recall $\Lambda_t = \exp tv_H(\Lambda)$ verifies $\Lambda_{t+s} = \exp tv_H(\Lambda_s)$ for all $t, s \geq 0$. Let Γ_t be the set of points of Λ_t such that v_H is tangent to Λ_t, we have also $\Gamma_{t+s} = \exp tv_H(\Gamma_s)$. So the glancing property is invariant under the Hamiltonian flow.

We can trivially solve in the (ξ_1, x_2) coordinates Hamilton-Jacobi equation $\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x_1} = 0$, $\Psi|_{t=0} = \phi(x_1, x_2)$. This gives $\Psi(t, x_1, x_2) = \phi(-t + x_1, x_2)$, so the condition

$$\Psi|_{t=0} = \phi_{\pm}$$

gives

$$\Psi_{\pm}(t, x_1, x_2) = \pm\frac{1}{3}(-t + x_1)^3 + (-t + x_1)x_2^2 \tag{1.10}$$

By definition

$$\Lambda_1 = \{\xi_1 = \frac{\partial \Psi}{\partial x_1} = 0,\ \xi_2 = \frac{\partial \Psi}{\partial x_2}\} = \{\xi_1 = \pm(-t+x_1)^2 + x_2^2 = 0,\ \xi_2 = 2(-t+x_1)x_2\}.$$

For Ψ_+ we get:

$$\Lambda_1 = \{\xi_1 = 0,\ t = x_1,\ x_2 = 0,\ \xi_2 = 0\}$$

This is a one-dimensional isotropic submanifold because the intersection Λ and G is a point. For Ψ_- we get:

$$\Lambda_1 = \{\xi_1 = 0,\ t = x_1 \pm x_2,\ \xi_2 = \mp 2x_2^2\}.$$

This is a Lagrangian submanifold with a singular line.

2/ Examples of non-Lagrangian intersections.

Consider a positively homogeneous Hamiltonian $H(x,p)$ of degree m with respect to p. The energy surface should then be specified as $H(x,p) = E$. We first classify points $z = (x,p) \in G = \{H = E\}$ according to $v_H(z)$.

Definition 2.1 [AnDoNaRo3]: *Let H be positively homogeneous of degree m with respect to p. We call a point $z = (x,p)$ such that $-\partial_x H(z) \neq 0$ an ordinary point if $\langle -\partial_x H(z), p\rangle \neq 0$, and a special point otherwise. If $-\partial_x H(z) = 0$ we call z a residual point.*

Examples 2.1:

(1) For Tricomi Hamiltonian, $H(x,p) = x_2 p_1^2 + p_2^2$, the residual points are those for $p_1 = 0$, the special points those for $p_1 \neq 0$ but $p_2 = 0$, and the ordinary points those for $p_1 p_2 \neq 0$.

(2) For Métivier Hamiltonian, $H(x,p) = p_1^2 + (x_1^2 + x_2^2)p_2^2$, the residual points are given by $p_2 = 0$ or $x = (x_1, x_2) = 0$, the special points by $p_2 \neq 0$ and $x \neq 0$, but $\langle p, x\rangle = 0$, and the ordinary points by $p_2\langle x, p\rangle \neq 0$.

(3) Let H be the "conformal metric" given by

$$H(x,p) = |p|^m \frac{1}{\rho(x)} \tag{2.1}$$

where ρ is a smooth positive function on M, $m \geq 1$. The residual points are the critical points of ρ ; at a special point, $\langle \nabla\rho, p\rangle = 0$, i.e. v_H is tangent to the level curves of ρ. This example of practical interest was extensively studied in [AnDoNaRo2,3].

Together with H positively homogeneous of degree m, consider an initial manifold Λ. In the following we try to link Lagrangian intersections with the classification above.

We take first $\Lambda = T^*_{x_0}M \approx \{x = x_0\}$ (conormal bundle to $\{x_0\}$ in T^*M). Then Λ intersects cleanly Λ_+ along $\partial\Lambda_+$.

Consider next the "initial" Lagrangian manifold

$$\Lambda = \{x = X(\varphi,\psi) = \varphi\omega(\psi), p = P(\varphi,\psi) = \omega(\psi), \varphi \in \mathbf{R}\} \tag{2.3}$$

which is called the "cylinder" ; here $\omega \in \mathbf{S}^{n-1}$ is the unit vector parametrized by ψ. This is the wave-front set of semi-classical distributions related with Bessel functions.

Let us check first the Lagrangian intersection in this case. The tangent space $T_z\Lambda$ has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^{\perp}(\psi)\delta\psi, \quad \delta P = \omega^{\perp}(\psi)\delta\psi, \quad \delta\varphi, \delta\psi \in \mathbf{R}$$

so $v_H \in T_z\Lambda$ iff there exist $(\delta\psi, \delta\varphi)$ such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^{\perp}(\psi)\delta\psi, \quad -\partial_x H = \omega^{\perp}(\psi)\delta\psi$$

Taking scalar products with $\omega(\psi), \omega^{\perp}(\psi)$, and using Euler identity, we get $\delta\varphi = \langle \partial_p H, P(\psi)\rangle = mH = mE$, $\delta\psi = \langle -\partial_x H, \omega^{\perp}(\psi)\rangle$. Then, for $z \in \Sigma_E$,

$$\langle \partial_p H + \varphi\partial_x H, \omega^{\perp}(\psi)\rangle = 0, \quad \langle -\partial_x H, \omega(\psi)\rangle = 0, \quad H(z) = 0 \tag{2.5}$$

Second relation (2.5) means that $z = \big(\varphi\omega(\psi), \omega(\psi)\big)$ is a special point (Definition 1.2). Since $(\omega(\psi), \omega^{\perp}(\psi)$ form a basis of $\mathbf{R}^2$, these relations are necessary and sufficient for $v_H \in T_z\Lambda$, i.e. z to be glancing.

Examples 2.2:

(4) When $H = p^2$, all points are glancing. When $H = \frac{|p|^m}{\rho(x)}$, $z(0)$ is a glancing point iff

$$\text{either}: \ \varphi \neq 0 \text{ and } \nabla\rho = 0, \ \text{or}: \ \varphi = 0 \text{ and } \langle \nabla\rho(0), \omega(\psi)\rangle = 0 \tag{2.6}$$

Second condition means that if $z(0) = (0, \omega(\psi))$ is a special point.

(5) Assume again $M = \mathbf{R}^2$. For $a, b \in \mathbf{R}$, let $\rho(x) = 1 + ax_1^2 + bx_2^2$. Consider Hamiltonian on $T^*\mathbf{R}^2$ of the form

$$H(x,p) = \frac{\langle \mu, p\rangle}{\rho(x)}$$

with $\mu = (1,0)$. A computation shows that, with $x_1 = \varphi\cos\psi, x_2 = \varphi\sin\psi$

$$\begin{aligned} \langle -\partial_x H, \omega(\psi)\rangle &= \frac{2\varphi\cos\psi}{\rho^2(x)}(a\cos^2\psi + b\sin^2\psi) \\ -\rho^2(x)\langle -\partial_x H, \omega(\psi)\rangle &= \big(\rho(x) + 2\varphi^2\cos^2\psi(a-b)\big)\sin\psi \end{aligned}$$

so choosing $a > b > 0$ and $E > 1$, we see that $\mathcal{G}(\Lambda, G) = \emptyset$. Otherwise, there may be a unique glancing point near $x = 0$.

(6) Consider Hamiltonian with constant coefficients $H(x,p) = \langle \mu, p\rangle$, then when $\omega(\psi)$ is parallel to μ, the corresponding ray is contained entirely in Λ, so the intersection is not transverse. This corresponds to a triad.

(7) Here is another example of a triad. Consider the scattering problem by an obstacle, e.g. $H = -\Delta$ on $L^2(\mathbf{R}^2 \setminus \mathcal{O})$ with Dirichlet boundary conditions on $\partial\mathcal{O}$, and $E = 1$ so that the characteristics are parametrized by arc-length.

Let ℓ be a Lagrangian submanifold in $\{T^*\partial\mathcal{O} : p^2 = 1\}$ and Λ be its preimage under the natural projection $T^*_{\partial\mathcal{O}}\mathbf{R}^2 \to T^*\partial\mathcal{O}$. Then $(\Lambda, \ell, \{p^2 = 1\})$ is a triad and the union of all characteristics in $p^2 = 1$ intersecting ℓ is a Lagrangian submanifold Λ_1 with singularities.

For example, Λ_1 has cusps along the ray being tangent to an inflection point of $\partial\mathcal{O}$. If $n = 3$ then Λ_1 can have more interesting singularities: open swallow-tails and open Whitney umbrellas (see [Arn], [Gi], [Sch]).

The submanifold Λ_1 gives a short-wave approximation of solutions for the obstacle problem. When Λ_1 is singular this is of particular interest since we cannot resort to standard Maslov canonical operator.

Remark 2.2: When (Λ, G) are given by (2.3) and (2.1) and $m = 1$, $G = \{H = E\}$ there is a general Morse family parametrizing Λ_t, of the form

$$\Phi(x, t, \psi, \varphi, \lambda) = mEt + \varphi + \lambda\langle P(t, \varphi, \psi), x - X(t, \varphi, \psi)\rangle$$

where $mEt+\varphi$ is an eikonal coordinate, and (X, P) the solution of Hamilton equations $\dot{X} = \partial_x H, \dot{P} = -\partial_x H$ with prescribed value on Λ, see [AnDoNaRo3]. But for a glancing Lagrangian intersection, this formula is of little help.

3/ Glancing hypersurfaces and Lagrangian intersections.

Let F, G be two smooth hypersurfaces of T^*M intersecting transversally at z. Recall from [Hö,Definition 21.4.6] F and G are said to be *glancing* at z iff the Hamilton foliation of $F = \{f = 0\}$ and $G = \{g = 0\}$ (locally near z) are simply tangent at z.

Stated otherwise, we have $f(z) = g(z) = \{f, g\}(z) = 0$ (Poisson bracket), but the second Poisson brackets $\{f, \{f, g\}\}(z), \{g, \{g, f\}\}(z)$ are non zero.

By the theorem of equivalence of glancing hypersurfaces of Melrose [Hö,Theorem 21.4.8] there are local symplectic coordinates (x, ξ) vanishing at z such that F, G are defined resp. by $x_1 = 0$ and $\xi_1^2 - x_1 - \xi_2 = 0$. Then $g = \xi_1^2 - x_1 - \xi_2 = 0$ will be the "normal form" of H in these coordinates. (We use the notation g for the normal form of H, or $H - E$).

We apply this theorem to G being the energy surface $H = E$ (i.e. $g = 0$) and F an auxiliary hypersurface intersecting G transversally at a glancing point z. We want to find germs of Lagrangian manifolds Λ such that Λ is transverse to F at z but (Λ, G) has glancing intersection at z. Again, this means that $T_z\Lambda \cap (T_zF)^\sigma = \{0\}$ and $\mathbf{R}v_H(z) = (T_zG)^\sigma \subset T_z\Lambda$, where superscript σ denotes symplectic orthogonal.

All computations below will be carried out in the local symplectic coordinates (x, ξ) vanishing at z.

3.1 Constructing some phase functions parametrizing Λ.

The Hamiltonian vector field for $g = \xi_1^2 - x_1 - \xi_2$ takes the form $v_g = 2\xi_1\partial_{x_1} - \partial_{x_2} + \partial_{\xi_1}$, or using coordinates $(\delta x, \delta\xi)$ in TT^*M, as

$$v_g = (\delta x_1, \delta x_2; \delta\xi_1, \delta\xi_2) = (2\xi_1, -1; 1, 0) \tag{3.2}$$

We know that a general Lagrangian manifold can be parametrized in the mixed representation, so when $n = 2$ by one of the following cases

$$\begin{aligned} &\text{(I) } \Lambda = \{\xi = \frac{\partial \phi}{\partial x}\},\ \text{(II) } \Lambda = \{x = -\frac{\partial \phi}{\partial \xi}\} \\ &\text{(III) } \Lambda = \{x_1 = -\frac{\partial \phi}{\partial \xi_1},\ \xi_2 = \frac{\partial \phi}{\partial x_2},\ \text{(IV) } \Lambda = \{x_2 = -\frac{\partial \phi}{\partial \xi_2},\ \xi_1 = \frac{\partial \phi}{\partial x_1}\} \end{aligned} \tag{3.3}$$

To determine Λ, we write that the glancing intersection of (Λ, G) at z should take place at $z = (x_1, x_2, \xi_1, \xi_2) = 0$, i.e. for z small enough, $v_g(z) \in T_z\Lambda$ implies $z = 0$. Then we check that Λ is transverse to F at $z = 0$. There will be of course many possibilities for Λ, and we content ourselves with a few significant examples.

The linear situation is the case where ϕ is quadratic. Cases (I),(II),(III) give one-parameter families of Lagrangian manifolds. We show that Case (IV) does not occur.

Proposition 3.1: *Consider first a quadratic phase $\phi = \phi_0$.*

• *In Cases (I),(II),(III) ϕ_0 are one-parameter families taking values*

$$\text{(I) } \phi_0(x) = \frac{1}{2}(ax_1^2 - 2x_1x_2),\ \text{(II) } \phi_0(\xi) = \frac{1}{2}(2\xi_1\xi_2 + c\xi_2^2),\ \text{(III) } \phi_0(x_2, \xi_1) = \frac{1}{2}b(\xi_1 + x_2)^2 \tag{3.4}$$

respectively, where $a, b, c \neq 0$. The manifold Λ is transverse to F at $z = 0$, and the corresponding matrices $A_z(\Lambda, G), B_z(\Lambda, G)$ in Definition 1.3 are then given by

$$\begin{aligned} &\text{(I) } A_z = 2\begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}, B_z = 2\begin{pmatrix} -a \\ 1 \end{pmatrix},\ \text{(II) } A_z = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_z = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\text{(III) } A_z = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_z = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{3.5}$$

So with the notations of Definition 1.3, case (I) is of type (6) or (7) according to $a^3 + 2a^2 + 1 = 0$ or not, while cases (II) and and (III) are of type (6).

• *Case (IV) does not occur.*

Proof:

• Case (I). The quadratic phase takes the form $\phi_0 = \frac{1}{2}(ax_1^2 + 2bx_1x_2 + cx_2^2)$. That v_g be tangent to Λ at z express as

$$(\delta x_1, \delta x_2; a\delta x_1 + b\delta x_2, b\delta x_1 + c\delta x_2) = (2\xi_1, -1; 1, 0) \tag{3.6}$$

By substitution we find $\xi_1 = \frac{b+1}{2a} = \frac{c}{2b}$, so condition $\xi_1 = 0$ gives $b + 1 = c = 0$. Substituting in $\xi_1 = ax_1 + bx_2$ gives $ax_1 - x_2 = 0$. Then the condition $z \in G$, namely $g = 0$ gives $-x_1 - \xi_2 = 0$, so altogether $x_1 = -\xi_2 = \frac{x_2}{a}, \xi_1 = 0$, and $z = (x_1, ax_1, 0, -x_1)$. Finally the condition $z \in F$ gives $x_1 = 0$, hence $z \in F \cap G$ gives $z = 0$. So the phase ϕ_0 defining Λ is given by $\phi_0 = \frac{1}{2}(ax_1^2 - 2x_1x_2)$. It is also clear that Λ is transverse to F at $z = 0$. Let us compute the matrix elements of A_z, B_z. With $f_j = \xi_j - \frac{\partial \phi}{\partial x_j}$, we have

$$\{f_1, \{f_1, g\}\} = 2a^2,\ \{f_2, \{f_2, g\}\} = 2,\ \{f_1, \{f_2, g\}\} = -2a,\ \{g, \{g, f_1\}\} = -2a,\ \{g, \{g, f_2\}\} = 2$$

which gives (3.5)(I).

• Case (II). The phase is of the form $\phi_0 = \frac{1}{2}(a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2)$, Hamilton vector field $v_g = (-a\delta\xi_1 - b\delta\xi_2, -b\delta\xi_1 - c\delta\xi_2; \delta\xi_1, \delta\xi_2) = (2\xi_1, -1; 1, 0)$, so necessary condition $\xi_1 = 0$ gives $a = 0, b = 1$. The condition $z \in G$ gives $z = (-\xi_2, -c\xi_2; 0, \xi_2)$, and the condition $z \in \Lambda$ gives $-x_2 = 0$, so we get $z = 0$. The phase defining Λ is given by $\phi_0 = \frac{1}{2}(2\xi_1\xi_2 + c\xi_2^2)$. Again Λ is transverse to F at $z = 0$. With $f_j = x_j + \frac{\partial\phi}{\partial\xi_j}$, we have

$$\{f_1, \{f_1, g\}\} = 2,\ \{f_2, \{f_2, g\}\} = 0,\ \{f_1, \{f_2, g\}\} = 0,\ \{g, \{g, f_1\}\} = 2,\ \{g, \{g, f_2\}\} = 0$$

which gives (3.5)(II).

• Case (III). The phase is of the form $\phi_0 = \frac{1}{2}(a\xi_1^2 + 2bx_2\xi_1 + cx_2^2)$, and Hamilton vector field $v_g = (-a\delta\xi_1 - b\delta x_2, \delta x_2; \delta\xi_1, b\delta\xi_1 + c\delta x_2) = (2\xi_1, -1; 1, 0)$, so necessary condition $\xi_1 = 0$ gives $a = b = c$. The condition $z \in G$ gives $z = (x_1, -\frac{x_1}{a}; 0, -x_1)$, and the condition $z \in \Lambda$ gives $x_1 = 0$, so we get $z = 0$ as expected. The phase defining Λ is thus given by $\phi_0 = \frac{a}{2}(\xi_1 + x_2)^2$. Again Λ is transverse to F at $z = 0$. With $f_1 = x_1 + \frac{\partial\phi}{\partial\xi_1}$, $f_2 = \xi_2 - \frac{\partial\phi}{\partial x_2}$ we have

$$\{f_1, \{f_1, g\}\} = 2,\ \{f_2, \{f_2, g\}\} = 0,\ \{f_1, \{f_2, g\}\} = 0,\ \{g, \{g, f_1\}\} = 2,\ \{g, \{g, f_2\}\} = 0$$

which gives (3.5)(III).

• Case (IV). The quadratic phase takes the form $\phi = \frac{1}{2}(ax_1^2 + 2bx_1\xi_2 + c\xi_2^2)$, but Hamilton vector field $v_g = (\delta x_1, -b\delta x_1 - c\delta\xi_2; a\delta x_1 + b\delta\xi_2, \delta\xi_2) = (2\xi_1, -1; 1, 0)$ cannot be tangent to Λ near $z = 0$. ♣

Next we consider $\phi = \phi_0 + \phi_1$, where ϕ_1 is a cubic homogeneous polynomial. The role of the cubic term is suggested by the normal form in Theorem 1.2.

Proposition 3.2: *The phase with a cubic term takes the form $\phi = \phi_0 + \phi_1$. The manifold Λ is again transverse to F at $z = 0$.*

• *In Case (I) ϕ_0 fixes entirely the one-parameter family*

$$\text{(I)}\ \phi(x) = \frac{1}{2}(ax_1^2 - 2x_1x_2) - \frac{1}{3}(ax_1 - x_2)^3,\ A_z = B_z = 0 \tag{3.8}$$

which has type (10) in Definition 1.3.

• *In Case (III), $\phi_0(x_2, \xi_1) = \frac{1}{2}b(\xi_1 + x_2)^2$, and ϕ_1 is of the form $\phi_1(x_2, \xi_1) = \frac{1}{3}(a'\xi_1^3 + 3b'\xi_1^2x_2 + 3c'\xi_1x_2^2 + d'x_2^3)$. Let Δ be as in (3.17). If $\Delta \neq 0$ we have*

$$A_z = 2\begin{pmatrix} d' & -b' + c' \\ -b' + c' & 1 + b' - a' \end{pmatrix},\ B_z = 2\begin{pmatrix} 1 - c' + 2b' \\ b' + d' - c \end{pmatrix}$$

which has type (1) in Definition 1.3. If $\Delta = 0$, $z = 0$ is glancing iff either $(1-c')b' = 0$ or $(1-b')c' = 0$.

• *In Case (II)*

$$\text{(II)}\ \phi(\xi) = \frac{1}{2}(2\xi_1\xi_2 + c\xi_2^2) + \frac{1}{3}(-\xi_1^3 + 3c'\xi_1\xi_2^2 + d'\xi_2^3) \tag{3.10}$$

• *Case (IV) does not occur.*

Proof:

• Case (I). The phase defining Λ is $\phi = \phi_0 + \mathcal{O}(|x|^3)$ with $\phi_0 = \frac{1}{2}(ax_1^2 - 2x_1x_2)$, and

$$\phi_1 = \frac{1}{3}(a'x_1^3 + 3b'x_1^2x_2 + 3c'x_1x_2^2 + d'x_2^3)$$

To this order, the condition $v_g(z) \in T_z\Lambda$ takes the form

$$\begin{aligned}\big(\delta x_1, \delta x_2; (a + 2a'x_1 + 2b'x_2)\delta x_1 + (-1 + 2b'x_1 + 2c'x_2)\delta x_2,\\ (-1 + 2b'x_1 + 2c'x_2)\delta x_1 + (2c'x_1 + 2d'x_2)\delta x_2\big) = (2\xi_1, -1; 1, 0)\end{aligned}$$

which yields the system

$$-a\xi_1 + b'x_1 + c'x_2 = 2\xi_1(a'x_1 + b'x_2), \quad \xi_1 + c'x_1 + d'x_2 = 2\xi_1(b'x_1 + c'x_2) \tag{3.12}$$

Eliminating ξ_1 on the LHS gives

$$(b' + ac')x_1 + (c' + ad')x_2 = 2\xi_1\big((a' + ab')x_1 + (b' + ac')x_2\big) \tag{3.13}$$

The condition $z \in F$ gives $x_1 = 0$. Substituting into (3.13) we get $x_2 = 0$ or $c' + ad' = 2\xi_1(b' + ac')$. Assume $x_2 = 0$, then the condition $z \in G$ gives $\xi_1^2 = x_1 + \xi_2 = b'x_1^2 + 2c'x_1x_2 + d'x_2^2 = 0$, so $z = 0$. Assume instead $c' + ad' = 2\xi_1(b' + ac')$. Then as before $c' + ad' = 0$, so either $\xi_1 = 0$ and (3.12) gives again $x_2 = 0$, so $z = 0$. Finally we are left with the condition $b' + ac' = 0$. Altogether we find $\phi = \phi_0 + \phi_1 = \frac{1}{2}(ax_1^2 - 2x_1x_2) - \frac{1}{3}(ax_1 - x_2)^3$. Again Λ is transverse to F at $z = 0$. With $f_j = \xi_j + \frac{\partial\phi}{\partial x_j}$, a little computation shows that all second brackets vanish at $z = 0$, which gives (3.8).

• Case (III). We recall ψ_0 from (3.4)(III), and try $\phi_1 = \frac{1}{3}(a'\xi_1^3 + 3b'\xi_1^2x_2 + 3c'\xi_1x_2^2 + d'x_2^3)$. With $\phi = \phi_0 + \phi_1$, we have $x_1 = -\frac{\partial\phi}{\partial\xi_1}, \xi_2 = \frac{\partial\phi}{\partial x_2}$, i.e.

$$\begin{aligned}x_1 &= -b(\xi_1 + x_2) - a'\xi_1^2 - 2b'\xi_1x_2 - c'x_2^2\\ \xi_2 &= b(\xi_1 + x_2) + b'\xi_1^2 + 2c'\xi_1x_2 + d'x_2^2\end{aligned} \tag{3.15}$$

with $b \neq 0$. The condition for $v_g(z) = (2\xi_1, -1; 1, 0) = (\delta x_1, \delta x_2; \delta\xi_1, \delta\xi_2) \in T_z\Lambda$ gives $\delta\xi_1 = 1$ and $\delta x_2 = -1$, together with the linear system in (x_2, ξ_1)

$$\begin{aligned}(1 + a' - b')\xi_1 + b'(1 - c')x_2 &= 0\\ (b' - c')\xi_1 + (c' - d')x_2 &= 0\end{aligned} \tag{3.16}$$

with determinant

$$\Delta = (1 + a')(c' - d') + b'(d' - b') + b'c'(b' - c') \tag{3.17}$$

If $\Delta \neq 0$, which holds when (a', b', c', d') vary in an open set of $\mathbf{R}^4$, then (3.15) has the trivial solution $x_2 = \xi_1 = 0$. First Eq. (3.15) then gives $x_1 = 0$ (which means $z \in F$) and second Eq. (3.15) shows that $\xi_1 = 0$ (which means $z \in G$). So $z = 0$ as is claimed.

Assume instead $\Delta = 0$. Then (3.16) reduces to the first Eq. which has the general solution

$$x_2 = \lambda(1 + a' - b'),\ \xi_1 = -\lambda b'(1 - c') \tag{3.18}$$

with $\lambda \in \mathbf{R}$. Substituting into $\xi_1^2 - x_1 - x_2 = 0$ ($z \in G$) and $x_1 = 0$ ($z \in F$) leads to the system

$$\begin{aligned} (1 - b')\xi_1^2 - 2c'\xi_1 x_2 - d'x_2^2 &= b(\xi_1 + x_2) \\ a'\xi_1^2 + 2b'\xi_1 x_2 + c'x_2^2 &= -b(\xi_1 + x_2) \end{aligned} \tag{3.19}$$

Summing up Eq. (3.19), we get rid of the linear term, and substituting (3.17) gives (provided $1 + a' - b' \neq 0$)

$$(1 + a' - b')(c' - d') + b'(1 - c')(-b'c' - b' + 2c') = 0$$

which combines with $\Delta = 0$ to

$$(1 - c')b'(1 - b')c' = 0$$

We discuss according to the cases $(1 - c')b' = 0$ and $(1 - b')c' = 0$.

In the first case, (3.17) gives $\xi_1 = 0$. The condition $z \in F \cap G$ is then equivalent to $\xi_1 = \xi_2 = x_1 = 0$. Substituting into Eq. (3.19) eventually gives $x_2 = 0$, so $z = 0$ as claimed.

Assume next $(1-c')b' \neq 0$, but $(1-b')c' = 0$. When $c' = 0$, $\Delta = 0$ reduces to $(1+a'-b')d'+b'^2 = 0$, and Eq. (3.19) to

$$\begin{aligned} a'\xi_1^2 + 2b'\xi_1 x_2 &= -b(\xi_1 + x_2) \\ (b' - 1)\xi_1^2 + d'x_2^2 &= -b(\xi_1 + x_2) \end{aligned} \tag{3.20}$$

Summing up again Eq.(3.20) gives the quadratic equation

$$b'^2\xi_1^2 - 2b'd'\xi_1 x_2 + d'^2 x_2^2 = (b'\xi_1 - d'x_2)^2 = 0$$

or since $b' \neq 0$, $\xi_1 = \frac{d'}{b'}x_2 = 0$. Substituting into first Eq.(3.20) yields

$$x_2\big(d'(a'd' + 2b'^2)x_2 + bb'(b' + d')\big) = 0$$

When $x_2 = 0$, (3.20) gives $\xi_1(b + a'\xi_1) = 0$. Since $b \neq 0$, we must have (ξ_1 is small) $\xi_1 = 0$, so $\xi_1 = x_2 = 0$, and the condition $z \in F \cap G$ is equivalent to $z = 0$, which proves our claim.

We are left with the case $c' \neq 0$, but $b' = 1$.

- Case (II).
- Case (IV).

♣

Let's summarize our construction so far:

Theorem 3.3: Let $M = \mathbf{R}^2$. Given a point of glancing intersection for the pair of hypersurfaces (F, G), where G takes the form $\{g = 0\}$ with $g(x, \xi) = \xi_1^2 - x_1 - \xi_2$, and $F = \{x_1 = 0\}$ we can choose Λ transverse to F and such that (Λ, G) has a glancing intersection at $z = 0$. It is parametrized by phase

functions $\phi = \phi_0 + \phi_1$ of inequivalent types (I)-(III) (case (IV) cannot occur), ϕ_0, ϕ_1 are quadratic and cubic polynomials respectively. Moreover all cases (1)-(10) in Definition 1.3 can occur.

3.2 Solving Hamilton-Jacobi (HJ) equation in (ξ_1, x_2) coordinates.

As before our purpose is to parametrize Λ_1 by a phase function $\Psi(x_1, x_2)$ with prescribed value on Λ. This is the solution of HJ equation for $(hD_{x_1})^2 - x_1 - hD_{x_2}$, namely

$$\Big(\frac{\partial \Psi}{\partial x_1}\Big)^2 - x_1 - \frac{\partial \Psi}{\partial x_2} = 0 \tag{3.25}$$

This equation cannot be solved directly since the Hamilton vector field v_g is not transversal to $\Lambda 1$. We search Ψ as the critical value of another phase function $\Phi(x_1, x_2, t)$. This will lead to multivalued solutions of (3.25).

Instead of solving directly HJ for $hD_t + g(x, hD_x) = hD_t + (hD_{x_1})^2 - x_1 - hD_{x_2}$, we make a partial Fourier transform quantizing the metaplectic transformation $(x_1, x_2; \xi_1, \xi_2) \mapsto (\xi_1, x_2, -x_1, \xi_2)$, and consider instead HJ equation for $hD_t + hD_{\xi_1} - hD_{x_2} + \xi_1^2$. A first change variables $(t, x_2, \xi_1) \mapsto (s' = \frac{1}{2}(t - x_2), r' = \frac{1}{2}(t + x_2), \xi_1)$, is followed by $(s', r', \xi_1) \mapsto (s = \frac{1}{2}(s' - \xi_1), r = \frac{1}{2}(s' + \xi_1), r')$. The inverse map is given by

$$x_2 = r' - r - s, \quad \xi_1 = r - s, \quad t = r' + r + s \tag{3.30}$$

and operator $hD_t + hD_{\xi_1} - hD_{x_2} + \xi_1^2$ is changed to $hD_{s'} + hD_{\xi_1} + \xi_1^2$, then to $hD_r + (r - s)^2$. In the new variables, HJ equation takes the form

$$\frac{\partial \Phi}{\partial r} + (r - s)^2 = 0 \tag{3.31}$$

and its general solution is given by

$$\Phi(r, r', s) = -\frac{1}{3}(r - s)^3 + \Phi_0(r', s) \tag{3.32}$$

where $\Phi_0(r', s)$ is determined by the initial value at $t = 0$, corresponding to $s = -(r + r')$. This yields the condition

$$\Phi|_{t=0} = \Phi\big(r, r', -(r + r')\big) = -\frac{1}{3}(2r + r')^3 + \Phi_0(r', -r - r') \tag{3.33}$$

so we need to express the initial value $\Phi|_{t=0}$ on Λ in terms of variables r, r', s.

We discuss again according to Cases (I)-(III) listed in Proposition 3.1.

• Case (I). We start with the linear case $\phi(x) = \phi_0(x) = \frac{1}{2}(ax_1^2 - 2x_1x_2)$. Consider partial Legendre transform $x_1 \mapsto \phi_0(x) - x_1\xi_1$, the critical point is given by $x_1 = \frac{1}{a}(x_2 + \xi_1)$, and the critical value $\phi_{\text{crit}} = -\frac{1}{2a}(\xi_1 + x_2)^2$. Substituting (3.30) at $t = 0$ (i.e. $s = -r - r'$) gives

$$\Phi|_{t=0} = \phi = -\frac{1}{2a}(3r' + 2r)^2 \tag{3.34}$$

with

$$\Phi_0(r',s) = -\frac{1}{3}(2s+r')^3 - \frac{1}{2a}(-2s+r')^2 \tag{3.35}$$

We use (3.32)-(3.33) to express Φ in the original variables as

$$\Phi(t,\xi_1,x_2) = -\frac{1}{3}(t-\xi_1)^3 - \frac{1}{3}\xi_1^3 - \frac{1}{2a}(\xi_1+x_2)^2 \tag{3.36}$$

The map $t \mapsto \Phi(t,\xi_1,x_2)$ has a degenerate critical point at $t=\xi_1$, with $\partial_t\Phi = \partial_t^2\Phi = 0$, $\partial_t^3\Phi = -2$. So the situation is similar with the diffraction by an obstacle for the wave equation [Sj], [Ta], [Ro] but Φ does not arise from a Morse family (with an auxiliary θ-parameter). We add now the cubic term $\phi_1(x) = -\frac{1}{3}(ax_1-x_2)^3$. Consider partial Legendre transform $x_1 \mapsto \phi_0(x)+\phi_1(x)-x_1\xi_1$, the critical point is given by the quadratic equation $a^3x_1^2 - 2a(ax_2+\frac{1}{2})x_1 + ax_2^2 - x_2 - \xi_1 = 0$ which yields

$$x_1 = \big(1+2ax_2-\sqrt{1-4a\xi_1}\,\big)/(2a^2) = \frac{1}{a}(x_2+\xi_1) + \xi_1^2 - \frac{a}{4}\xi_1^3 + \mathcal{O}(\xi_1^4)$$

The corresponding critical value of Legendre transform is simply $\phi_{\text{crit}} = -\frac{1}{2a}(\xi_1+x_2)^2 - \frac{1}{3}\xi_1^3 + \mathcal{O}(\xi_1^4)$. Substituting (3.30) at $t=0$ gives as before $\Phi_0(r',s) = -\frac{1}{2a}(r'-2s)^2 + \mathcal{O}\big((r'+2s)^4\big)$. We use (3.32)-(3.33) to express Φ in the original variables as

$$\Phi(t,\xi_1,x_2) = -\frac{1}{3}\xi_1^3 - \frac{1}{2a}(\xi_1+x_2)^2 + \mathcal{O}(\xi_1^4) \tag{3.38}$$

which turns out to be independent of t, and equals, mod $\mathcal{O}(\xi_1^4)$, the critical value of (2.54) when the cubic term ϕ_1 is absent.

• Case (II). We consider only the linear case. We make Legendre transform with respect to ξ_2 variable. The critical point is $\xi_2 = -\frac{1}{c}(x_2+\xi_1)$, and it turns out that the critical value of Legendre transform is $-\frac{1}{2c}(x_2+\xi_1)^2$ which gives again the same result as in Case (I) (linear case).

• Case (III). So $\phi(x_2,\xi_1) = \phi_0(x_2,\xi_1) = \frac{1}{2}b(\xi_1+x_2)^2$ is expressed already in the right coordinates. A simple calculation shows

$$\Phi(t,x_2,\xi_1) = -\frac{1}{3}\xi_1^3 - \frac{1}{3}(t-\xi_1)^3 + \frac{b}{2}(x_2+\xi_1)^2 \tag{3.39}$$

which gives the same result as in Case (I) (linear case).

We add now the cubic term. A computation similar to Case (I) (with cubic term) gives

$$\Phi(x_2,\xi_1,t) = -\frac{1}{3}\xi_1^3 + \frac{b}{2}(x_2+\xi_1)^2 + \frac{b'}{3}(x_2+\xi_1)^3 - \frac{1}{2}(t-\xi_1)^3 - \frac{1}{2}(t+x_2)^2\big(\frac{t-x_2}{2}-\xi_1\big) + \frac{1}{12}(x_2+\xi_1)^3$$

The critical point of $t \mapsto \Phi(t,\xi_1,x_2)$ is given by

$$(t-\xi_1)^2 + \frac{1}{2}(x_2+\xi_1)^2 = 0$$

which has a real solution iff $q = x_2 + \xi_1 = 0$. Then $t = \xi_1 = -x_2$ is a degenerate critical point, with $\partial_t \Phi = \partial_t^2 \Phi = 0$, $\partial_t^3 \Phi = -3/2$, and the critical value is

$$\Phi_{\text{crit}} = -\frac{1}{3}\xi_1^3 = \frac{1}{3}x_2^3 \tag{3.40}$$

Summing up, we have proved so far:

Proposition 3.4: *Let Λ be paramerized by a quadratic phase ϕ_0 or with a cubic term $\phi_0 + \phi_1$. The critical point of $t \mapsto \Phi(t, \xi_1, x_2)$ is given by $t = \xi_1$.*

In Case (I) (quadratic phase ϕ_0) the critical value of $t \mapsto \Phi(t, \xi_1, x_2)$ given in (3.36) is

$$\Phi_{\text{crit}}(\xi_1, x_2) = -\frac{1}{3}\xi_1^3 - \frac{1}{2a}(\xi_1 + x_2)^2 \tag{3.51}$$

For $\phi = \phi_0 + \phi_1$, $\Phi(t, x_1, x_2)$ is independent of t, and equal to (3.51) mod $\mathcal{O}(\xi_1^4)$.

In Case (II) (quadratic phase ϕ_0) the critical value of $t \mapsto \Phi(t, \xi_1, x_2)$ is again (3.51), with another constant a.

In Case (III) (quadratic phase ϕ_0) the critical value of $t \mapsto \Phi(t, \xi_1, x_2)$ is again (3.51), with another constant a. For $\phi = \phi_0 + \phi_1$, $t \mapsto \Phi(t, \xi_1, x_2)$ has a real critical point iff $q = x_1 + \xi_2 = 0$, and the critical value is given by (3.40).

Case (IV) does not occur.

Remarks:

1) Let $J = F \cap G = \{x_1 = 0, \xi_2 = \xi_1^2\}$, $Q = \{q = 0\}$. We note that $q = x_2 + \xi_1$ Poisson commutes with $g = \xi_1^2 - x_1 + \xi_2$, so $G = \{g = 0\}$ is the flow out of J through v_q.

2) In relevant Cases (I)-(III), $\Phi_{\text{crit}} = -\frac{1}{3}\xi_1^3$ on Q.

3.3 The phase function Φ in (x_1, x_2) coordinates.

As in [Sj] we pass from (ξ_1, x_2) to (x_1, x_2) coordinates by Legendre transform, so Ψ is the critical value of $\xi_1 \mapsto x_1\xi_1 + \Phi_{\text{crit}}(\xi_1, x_2)$, so within hypotheses of Proposition 3.4, and in general case (3.51) we have

$$\xi_1 = ax_1 - x_2 + \mathcal{O}\big((ax_1 - x_2)^2\big) \tag{3.52}$$

which gives Ψ by substituting in $\Phi_{\text{crit}}(\xi_1, x_2)$ ...

In case (3.40) we have the system $\xi_1^2 - x_1 = 0$, $x_2 + \xi_1 = 0$. or since the critical points belong to energy surface $G = \{g = 0\}$

$$x_1 = x_2^2, \quad \xi_2 = 0 \tag{3.53}$$

The caustic is then given by the parabola $x_1 = x_2^2$.

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