

Symétries et Instabilités dans les matériaux

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Summary

1. Observations

2. What now?

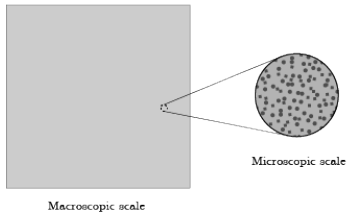
3. Bifurcation Theory

4. Conclusion

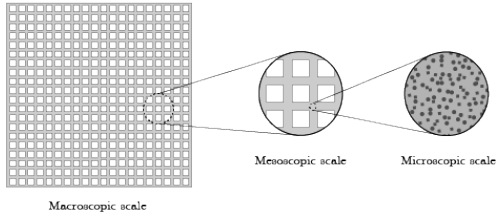
Architected Materials

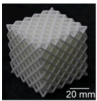
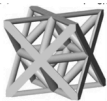
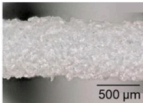
Definition

Scale separation in conventional materials



Scale separation in architected materials



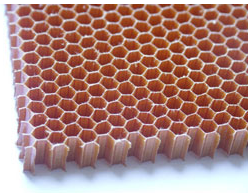
Macroscopic	Mesoscopic	Microscopic
$L \sim 10 \text{ cm}$	$l \sim 5 \text{ mm}$	$\lambda \sim 100 \mu\text{m}$
Structure	Cell	Material
		

Observations

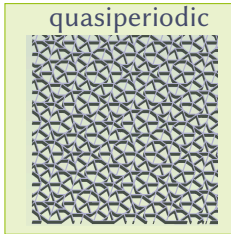
Classification

Mesoscopic scale organization

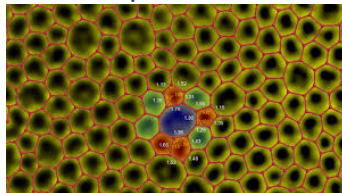
Periodic



quasiperiodic



aperiodic



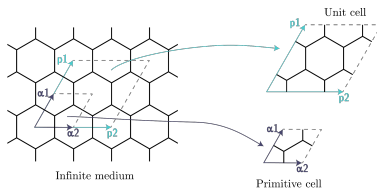
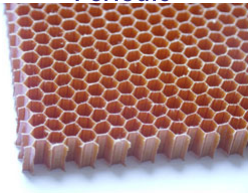
↑ WIP higher order homogenization of QP structures

Architected Materials

Classification

Mesoscopic scale organization

Periodic

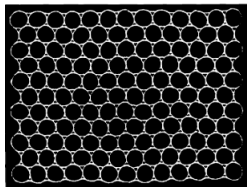


Observations

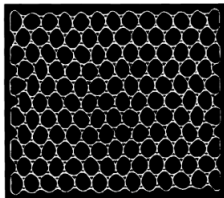
Literature review

Compression – [Papka & Kyriakides 1998]

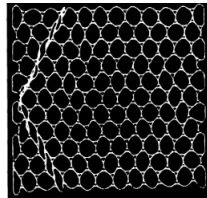
uniaxial horizontal



①



①



②

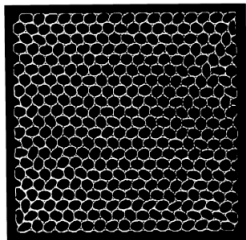
macroscopic localization

Observations

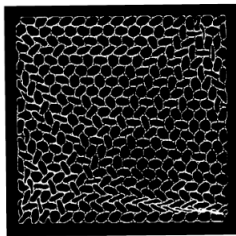
Literature review

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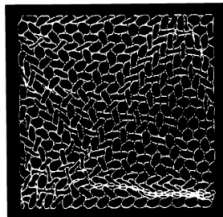
equibiaxial



①



②



③

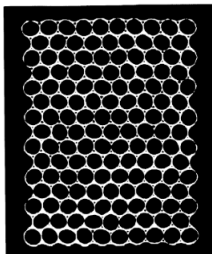
patterning

Observations

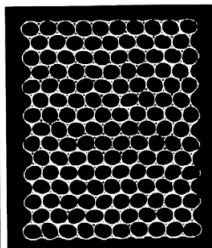
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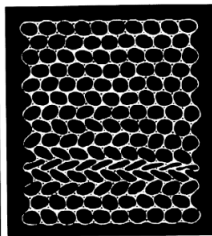
uniaxial vertical



①



②



③

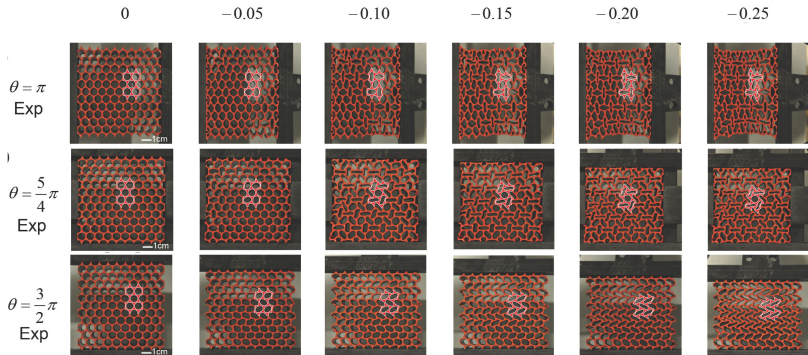
patterning then localization

Observations

Literature review

Compression – [Shan et al. 2014]

\mathcal{E}_{Area}



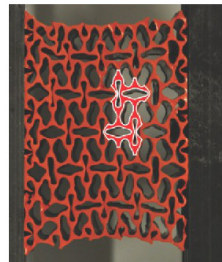
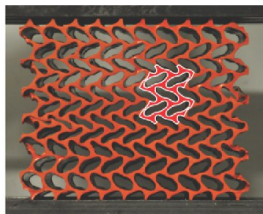
patterning

Observations

Literature review

Compression – [Shan et al. 2014]

Possible patterning buckling

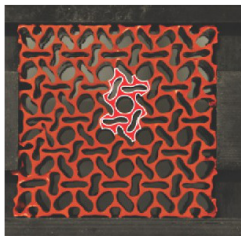
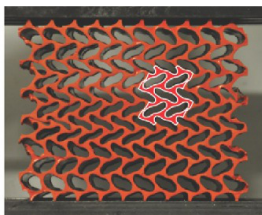


Observations

Literature review

Compression – [Shan et al. 2014]

Possible patterning buckling



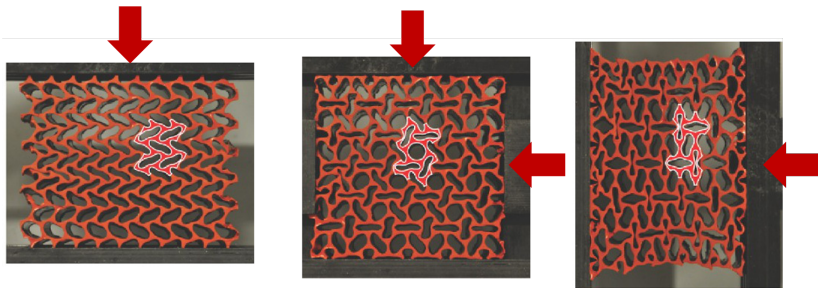
patterning with periodic pattern on larger period

Observations

Literature review

Compression – [Shan et al. 2014]

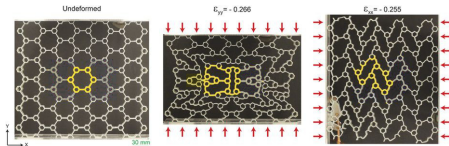
Possible patterning buckling



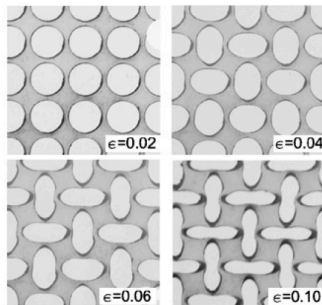
patterning with periodic pattern on larger period depending on loading direction... size of VER ???

Observations

Literature review – other meso-geometries



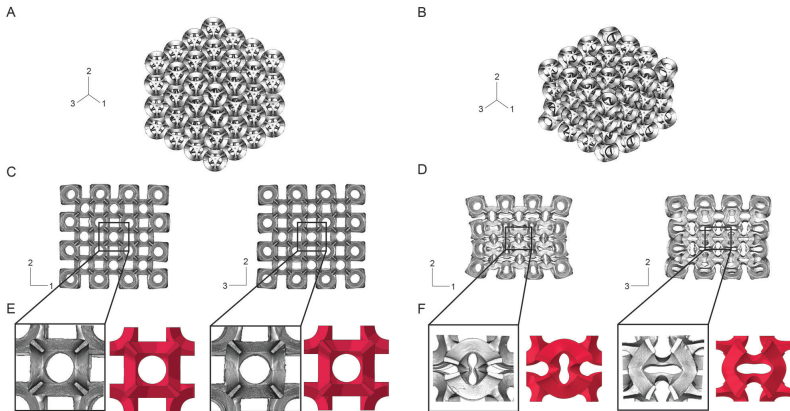
[Haghsanah et al. 2014]



[Bertoldi et al. 2008]

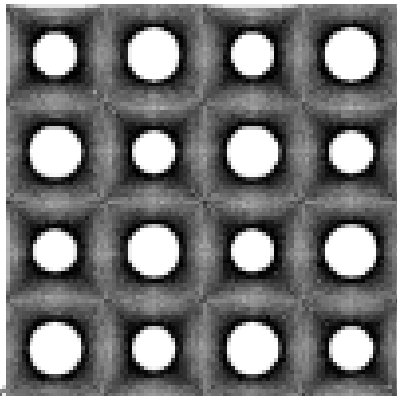
Observations

Literature review – 3D architected material



Observations

Literature review – under tension



[Michel et al. 2006]

Observations

Literature review Instabilities

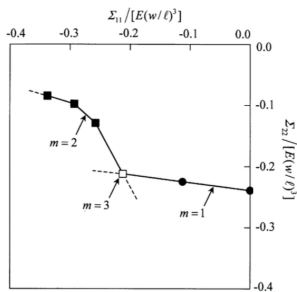
- Patterning can occur under tension or compression
- Macroscopic instabilities are possible
- Patterning can multiply the periodicity
- Patterning depends on loading orientation

What now?

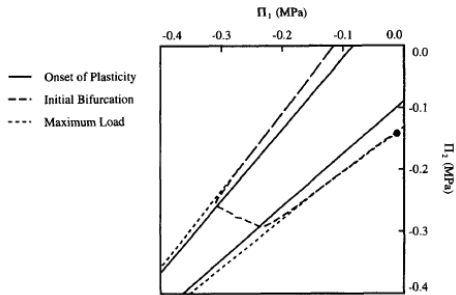
What now?

Avoid buckling

Onset of instability



[Ohno et al. 2002]



[Triantafyllidis & Schraad 1998]

What now?

Avoid buckling

Onset of instability

Tools

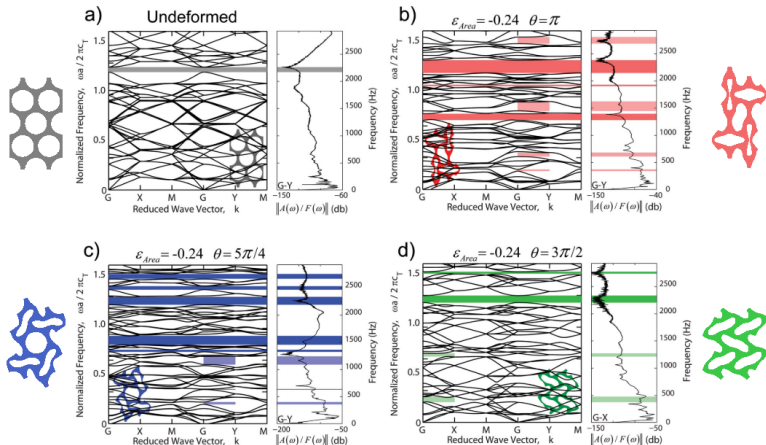
- Bloch-wave theory to predict the size of pattern
- Rank-one convexity for macroscopic buckling
- Computation at the scale of 1 "primitive" cell

[Combescure et al. 2016] [Combescure et al. 2017]

[Combescure et al. 2020]

What now?

Take advantage of buckling



[Shan et al. 2014]

What now?

Take advantage of buckling

Different patterns create different wave-propagation properties

- 1 What are the patterns available for a given mesostructure ?
- 2 Which mesostructure creates the most available patterns ?
- 3 What type of loading can lead to a given pattern ?



What now?

Take advantage of buckling

Different patterns create different wave-propagation properties

- 1 What are the patterns available for a given mesostructure ?
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Post-bifurcation behavior ?



Bifurcation Theory

① the Mechanical way : Analysis

Knowing the system's **energy**, we find **its** post-bifurcated behavior

② the Mathematical way : Design

Knowing the system's **symmetry**, we find **all possible** post-bifurcated behaviors



Bifurcation breaks symmetry

Bifurcation Theory ①

System's Energy $\mathcal{W}(\mathbf{u}, \lambda)$ conservative systems

$$\mathcal{W}(u, \lambda) = \frac{ku^2}{2} - \frac{mu^4}{4} - \lambda Fu$$

Equilibrium of a finite d.o.f structure $\mathcal{W}_{,\mathbf{u}}(\mathbf{u}, \lambda) = 0$

$$\mathcal{W}_{,u} = ku - mu^3 - \lambda F$$

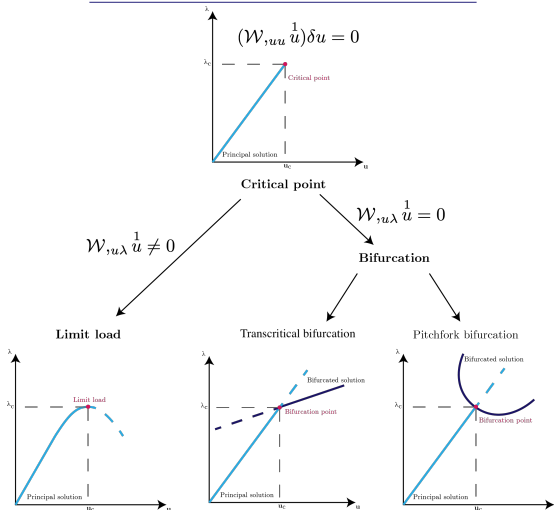
Stability = positive definiteness of tangent matrix $\mathcal{W}_{,\mathbf{uu}}(\mathbf{u}, \lambda)$

$$\mathcal{W}_{,uu} = k - 3mu^2 > 0$$



Principal Path loses Stability at Critical Points

Bifurcation Theory ①



Bifurcation Theory ①

Critical point = for some $\lambda, \exists \delta \mathbf{u}, (\mathcal{W}_{,\mathbf{u}\eta}(\mathbf{u} + \eta \delta \mathbf{u}))|_{\eta=0} = 0$
Eigenvector(s) $\delta \mathbf{u} \in \mathcal{U}^N$ null space

Null space can be multidimensionnal... especially in presence of symmetry

Post-bifurcation = equilibrium branches (=1D lines) in the null space

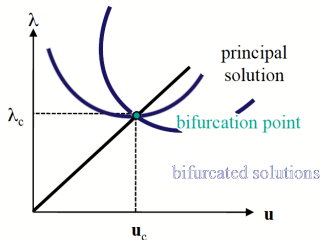
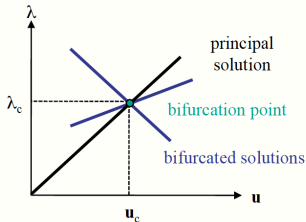


Beware ! Bifurcation = loss of uniqueness:

$$(\mathcal{W}_{,\lambda\eta}(\mathbf{u} + \eta \delta \mathbf{u}))|_{\eta=0} = 0$$

Bifurcation Theory ①

Multiple bifurcations



Number of branches
related to number N of null
EV = $|\mathcal{U}^N|$

- Transcritical bifurcation
 $n_{path} = 2^N - 1$
- Pitchfork bifurcation
 $n_{path} = \frac{3^N - 1}{2}$

[Elliott 2002]

Bifurcation Theory ①

Lyapunov-Schmidt-Koiter decomposition

\mathbf{u} decomposed into 2 components $\mathbf{u} = \mathbf{u}^N + \mathbf{u}^\perp$:

- $\mathbf{u}^N \in$ nullspace
- orthogonal component \mathbf{u}^\perp



With powerful numerical continuation methods, one only need to find tangent to bifurcated branches at the bifurcation point !

Tangents to bifurcated branches:
$$\mathbf{u}^b = \sum_{i=1}^N \alpha_i \phi_i$$

where ϕ_i symmetry adapted basis (\equiv irreducible basis)

Bifurcation Theory ①

Lyapunov-Schmidt-Koiter decomposition

Tangents to bifurcated branches: $\mathbf{u}^b = \sum_{i=1}^N \alpha_i \phi_i$

where ϕ_i symmetry adapted basis (\equiv irreducible basis)

Asymptotic Extension of equilibrium equations in the null space \mathcal{U}^N :

$\mathcal{W}_{,\mathbf{u}}(\mathbf{u}, \lambda) \delta u = 0$ projected on symmetry adapted basis ϕ_i

$$\sum_{j=1}^N \alpha_j \mathcal{W}_{,ij} + \mathcal{W}_{,i\lambda} + \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \alpha_j \alpha_k \alpha_l \mathcal{W}_{,ijkl} + 3\lambda_2 \sum_{j=1}^N \alpha_j \mathcal{W}_{,ij\lambda} = 0$$

Bifurcation Theory ①

Lyapunov-Schmidt-Koiter decomposition

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Bifurcation Theory ①

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Bifurcation Theory ①

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Asymptotic Extension of equilibrium equations in the null space

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Bifurcation Theory ①

Lyapunov-Schmidt-Koiter decomposition

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Bifurcation Theory ①

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Asymptotic Extension of equilibrium equations in the null space

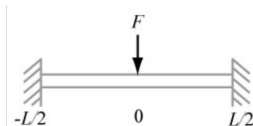
$$\sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \alpha_j \alpha_k \alpha_l \mathcal{W}_{,ijkl} + 3\lambda_2 \sum_{j=1}^N \alpha_j \mathcal{W}_{,ijj} = 0 \quad i = 1, \dots, m$$

Let's use symmetries

Bifurcation Theory ①

Common examples

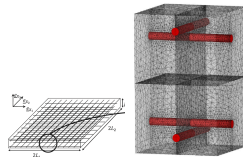
Mirror symmetry



Cyclic symmetry



Periodicity



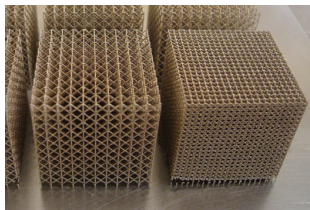
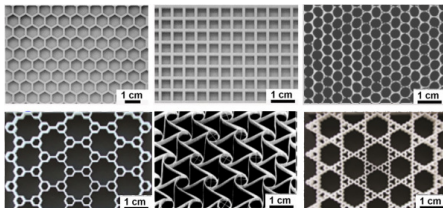
Symmetry of material, geometry and loading

Bifurcation Theory ①

More complex cases

Wallpaper symmetry = 2D
point-group + lattice

Crystal symmetry = 3D point-group
+ lattice

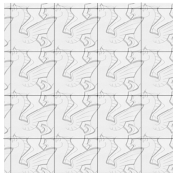


Bifurcation Theory ①

More complex cases

Lattice vs motif

Type de pavage : carré

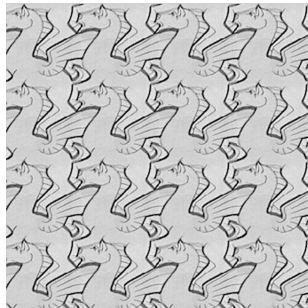


+

motif : pégase carré (pas de symétrie)



matériau architectural périodique



Bifurcation Theory ①

Vector space decomposed into h mutually orthogonal invariant subspaces : $H = \bigoplus_{\mu=1}^{\eta} V^{\mu}$

⇒ block diagonalization of tangent matrix $\mathcal{W}_{,\mathbf{uu}}$

$$\mathcal{W}_{,\mathbf{uu}} = \begin{pmatrix} A_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_h \end{pmatrix}$$

Bifurcation breaks symmetry

⇒ One invariant subspace = the null space

Bifurcation Theory ①

Vector space decomposed into h mutually orthogonal invariant subspaces : $H = \bigoplus_{\mu=1}^{\eta} V^{\mu}$

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Bifurcation breaks symmetry

⇒ One invariant subspace = the null space

$$\delta \mathbf{u}^N \in B_2$$

Bifurcation Theory ①

LSK analysis $\mathcal{W}_{,ijkl}$ and $\mathcal{W}_{,ij\lambda}$ tensors \in null space = B_2

Equivariance of $\mathcal{W}_{,ijkl}$ and $\mathcal{W}_{,ij\lambda}$ tensors in B_2

$$\mathcal{W}_{,ij\lambda} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$\mathcal{W}_{,ijkl} = \begin{pmatrix} X & 0 & 0 & 0 & Y & 0 & 0 & 0 & Y \\ 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & Z & 0 & 0 \\ 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & X & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 & 0 & Z & 0 & Z & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z & 0 & Z & 0 \\ Y & 0 & 0 & 0 & Y & 0 & 0 & 0 & X \end{pmatrix}$$

Bifurcation Theory ①

LSK analysis $\mathcal{W}_{,ijkl}$ and $\mathcal{W}_{,ij\lambda}$ tensors \in null space = B_2

Equivariance of $\mathcal{W}_{,ijkl}$ and $\mathcal{W}_{,ij\lambda}$ tensors in B_2

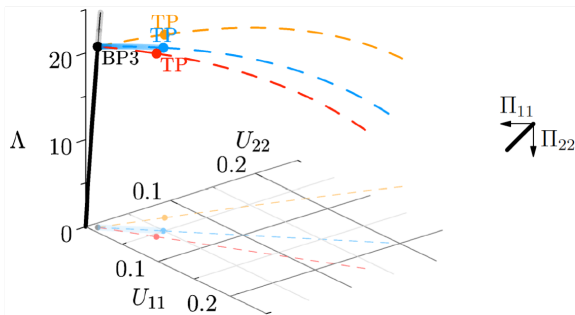
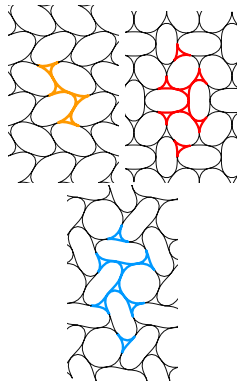
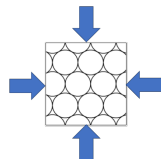
$$\mathcal{W}_{,ij\lambda} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$\mathcal{W}_{,ijkl} = \begin{pmatrix} X & 0 & 0 & 0 & Y & 0 & 0 & 0 & Y \\ 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 & 0 & Z & 0 & Z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z & 0 & Z & 0 \\ Y & 0 & 0 & 0 & Y & 0 & 0 & 0 & X \end{pmatrix}$$

If Lucky we can solve !

Bifurcation Theory ①

Honeycomb-like structures

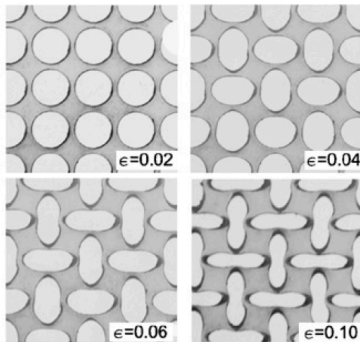


Equibiaxial compression : Triple bifurcation

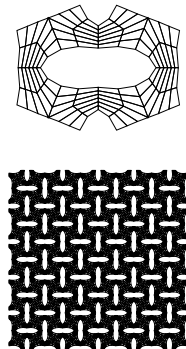
Bifurcation Theory ①

Circular holes, Square tiling

Experiments [Bertoldi 2007]



Simulation: simple mode



Bifurcation Theory ②

Too complicated to find a tangent in a multidimensional space?



Make it one dimensional !

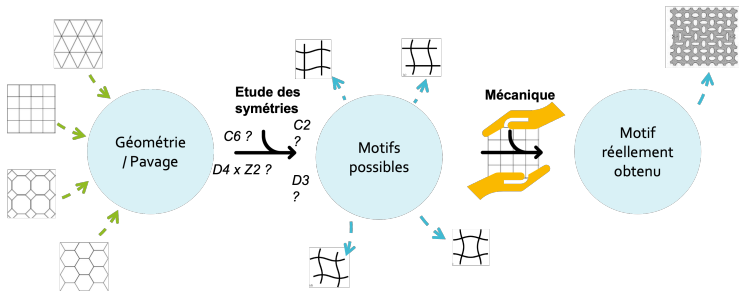
[Golubistky et al. 2000]

Equivariant Branching Lemma : Bifurcated Branches belong to Maximum Isotropy Subgroups where Fixed-point space (=null-space) is 1D

PhD R. Azulay

Bifurcation Theory 2

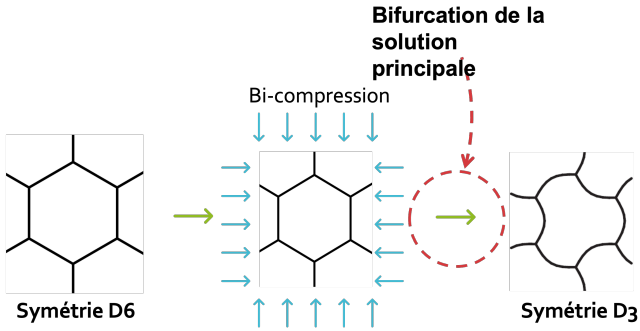
What are the patterns available for a given mesostructure ?



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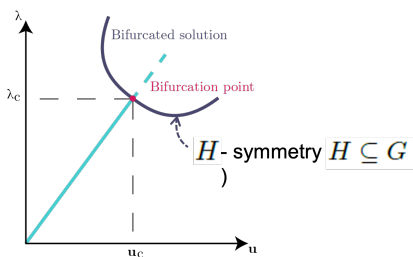
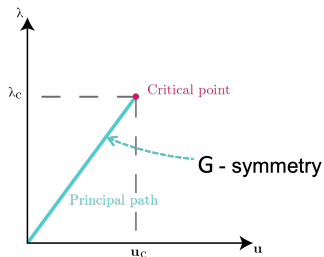
Bifurcation Theory 2

What are the subgroups available for a given mesostructure ?



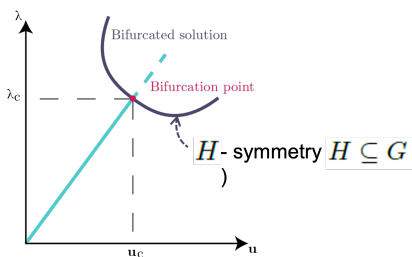
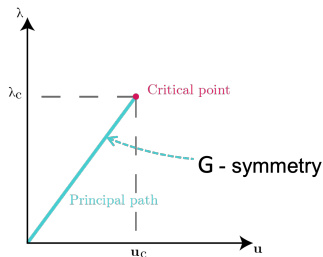
Bifurcation Theory 2

A little bit more on bifurcation theory...



Bifurcation Theory 2

A little bit more on bifurcation theory...



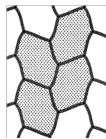
We want to list all the possible subgroups H
[Equivariant Branching Lemma]

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Bifurcation Theory 2

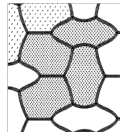
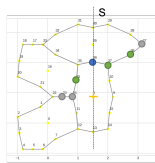
What are the **patterns** available for a given mesostructure ?

• Mode I :



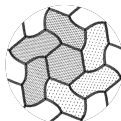
Adapted from [Ohno 2002]

• Mode II :



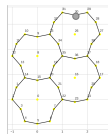
Adapted from [Ohno 2002]

• Mode III :



Adapted from [Ohno 2002]

• D3:



Adapted from [Kang 2013]

Patterns predicted by literature are retrieved

Conclusion

Analysis vs Design

- Different tools to study bifurcation of symmetric systems
- All rely on Group Theory and Representation Theory
- Group theory has still more tools available to help us !

Dangers

- EBL has been proven wrong in some systems
- What happens when the system is not conservative ?

Applications

- Mechanics
- Crystallography
- Economics [Ikeda 2021]



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